Two equations modeling pollution in a stochastic medium
– A white noise approach –

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Abstract

In this paper we look at two models for pollution given by the equations

\[ \frac{\partial u}{\partial t} = \frac{1}{2} \eta^2 \Delta u + \overline{W}_{\phi_\varepsilon} \circ \nabla u - \kappa(t,x) \circ u + g(t,x) \quad (t, x) \in [0, T] \times \mathbb{R}^n \]

\[ u(0, x) = f(x) \quad \forall x \in \mathbb{R}^n \]

and

\[ \frac{\partial u}{\partial t} = \frac{1}{2} \eta^2 \Delta u + \overline{W}_{\phi_\varepsilon} \circ \nabla u - \kappa(t,x) \circ u + g(t,x) \quad (t, x) \in [0, T] \times \mathcal{D} \]

\[ u(0, x) = \phi(x) \quad \forall x \in \mathcal{D} \]

\[ u(t, x) = h(t, x) \quad (t, x) \in [0, T] \times \partial \mathcal{D} \]

where \( T \) and \( \eta \) are constants, \( \mathcal{D} \) is a bounded domain, \( \overline{W}_{\phi_\varepsilon}, \kappa(t,x), f, \phi, h \) and \( g \) are elements in the space \( (\mathcal{S})^{-1} \) of generalized white noise distributions. With suitable conditions on \( \kappa, f, \phi, h \) and \( g \), we show that both equations have unique solutions given by explicit solution formulas.

Keywords: Generalized white noise distributions, Wick product, Hermite transform.

\section{Introduction}

We will consider two stochastic models based on the PDE

\[ \frac{\partial}{\partial t} u(t, x) = \frac{1}{2} \eta^2 \Delta u(t, x) - V \cdot \nabla u(t, x) - \alpha u(t, x) + \xi(t, x) \quad (1) \]

where \( \frac{1}{2} \eta^2 \) is the dispersion coefficient, \( V \) is the water velocity, \( \alpha \) is the leakage rate, \( \xi(t, x) \) is the rate of increase of the chemical concentration at \( (t, x) \) and \( u(t, x) \) is the chemical concentration at time \( t \).
on location \(x\). This work is motivated from the paper of G. Kallianpur et al. [KAL] where the \(\xi(t, x)\) was supposed to be random. We will in addition consider the case where the drift vector \(V\) is modeled as an \(n\)-dimensional white noise, consisting of independent components. We will work in the space \((S)^{-1}\) of generalized white noise distributions, since this space will allow explicit solutions formulas for a wide range of possible choices for random \(\xi\)'s and because the methods are particularly simple.

The methods used to solve the stochastic versions of (1) are the same as those used by Holden et al. in [HLØUZ3] and several other SPDE's are solved in a similarly fashion:

- The transport equation ([GjHØUZ]).
- The pressure equation for fluid flow ([HLØUZ3]).
- The Dirichlet equation ([Gj2]).
- The Burgers equation ([HLØUZ2]).
- The Schrödinger equation ([HLØUZ]).

For more examples and background on white noise, please read [BØ3].

\[ \text{§2 Preliminaries on multidimensional white noise} \]

There are many problems of physical nature where the need for several independent white noise sources arises. For example, given \(m\) independent positive white noise sources in a domain \(D\), one wants to calculate the effect of these on a particle traveling in \(D\). The result should intuitively be given by

\[ \sum_{i=1}^{m} \text{Exp}[W_{\phi}^{(i)}] \]

where \(\{\text{Exp}[W_{\phi}^{(i)}]\}_{i=1}^{m}\) are one dimensional independent positive white noise sources.

We will now give a short introduction of definitions and results from multidimensional Wick calculus, taken mostly from [Gj], [HLØUZ3], [HKPS] and [KLS].

In the following we will fix the parameter dimension \(n\) and space dimension \(m\).

Let

\[ \mathcal{N} := \prod_{i=1}^{m} S(\mathbb{R}^{n}) \]

where \(S(\mathbb{R}^{n})\) is the Schwartz space of rapidly decreasing \(C^\infty\)-functions on \(\mathbb{R}^{n}\), and

\[ \mathcal{N}^* := \left(\prod_{i=1}^{m} S(\mathbb{R}^{n})\right)^* \cong \prod_{i=1}^{m} S'(\mathbb{R}^{n}) \]

where \(S'(\mathbb{R}^{n})\) is the space of tempered distributions.
Let $B := B(\mathcal{N}^*)$ denote the Borel $\sigma$-algebra on $\mathcal{N}^*$ equipped with the weak star topology and set

$$H := \bigoplus_{i=1}^{m} L^2(\mathbb{R}^n)$$

where $\oplus$ denotes orthogonal sum.

Since $\mathcal{N}$ is a countably Hilbert nuclear space (cf. [Gj]) we get, using Minlos' theorem, a unique probability measure $\nu$ on $(\mathcal{N}^*, B)$ such that

$$\int_{\mathcal{N}^*} e^{i\langle \omega, \phi \rangle} \, d\nu(\omega) = e^{-\frac{1}{2} \|\phi\|_H^2} \quad \forall \phi \in \mathcal{N}$$

where $\|\phi\|_H^2 = \sum_{i=1}^{m} \|\phi_i\|_{L^2(\mathbb{R}^n)}^2$.

Note that if $m = 1$ then $\nu$ is usually denoted by $\mu$.

**THEOREM 2.1** [Gj] We have the following

1. $\otimes_{i=1}^{m} B(S'(\mathbb{R}^n)) = B(\prod_{i=1}^{m} S'(\mathbb{R}^n))$
2. $\nu = x_{i=1}^{m} \mu$

**DEFINITION 2.2** [Gj] The triple

$$(\prod_{i=1}^{m} S'(\mathbb{R}^n), B, \nu)$$

is called the (m-dimensional) (n-parameter) white noise probability space.

For $k = 0, 1, 2, \ldots$ and $x \in \mathbb{R}$ let

$$h_k(x) := (-1)^ke^{\frac{x^2}{2}} \frac{d^k}{dx^k}(e^{-\frac{x^2}{2}})$$

be the Hermite polynomials and

$$\xi_k(x) := \pi^{-\frac{1}{2}}((k-1)!)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} h_{k-1}(\sqrt{2}x) \quad k \geq 1$$

the Hermite functions.

It is well known that the family $\{\xi_\alpha\} \subset S(\mathbb{R}^n)$ of tensor products

$$\xi_\alpha := \xi_{\alpha_1} \otimes \cdots \otimes \xi_{\alpha_n}$$

forms an orthonormal basis for $L^2(\mathbb{R}^n)$.

Give the family of all multi-indices $\zeta = (\zeta_1, \ldots, \zeta_n)$ a fixed ordering

$$(\zeta^{(1)}, \zeta^{(2)}, \ldots, \zeta^{(k)}, \ldots)$$

where $\zeta^{(k)} = (\zeta_1^{(k)}, \ldots, \zeta_n^{(k)})$
and define $\tilde{e}_k := \tilde{e}_{\xi(k)}$.

Let $\{e_k\}_{k=1}^\infty$ be the orthonormal basis of $\mathcal{H}$ we get from the collection

$$\{(\tilde{0}, \ldots, \tilde{0}, \tilde{e}_j, \tilde{0}, \ldots, \tilde{0}) \in \mathcal{H} \mid 1 \leq i \leq m, 1 \leq j < \infty\}$$

and let $\gamma : \mathbb{N} \to \mathbb{N}$ be a function such that

$$e_k = (\tilde{0}, \ldots, \tilde{0}, \tilde{e}_{\xi(\gamma(k))}, \tilde{0}, \ldots, \tilde{0}).$$

Finally, let $(\beta^{(1)}, \beta^{(2)}, \ldots, \beta^{(k)}, \ldots)$ with $\beta^{(k)} = (\beta_1^{(k)}, \ldots, \beta_n^{(k)})$ be a sequence such that $\beta^{(k)} = \xi(\gamma(k))$.

If $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a multi-index of non-negative integers we put

$$H_\alpha(\omega) := \prod_{i=1}^k h_{\alpha_i}(\langle \omega, e_i \rangle).$$

From theorem 2.1 in [HLØUZ] we know that the collection

$$\{H_\alpha(\cdot) ; \alpha \in \mathbb{N}_0^k ; k = 0, 1, \ldots\}$$

forms an orthogonal basis for $L^2(N^*, B, \nu)$ with $\|H_\alpha\|_{L^2(\nu)} = \alpha!$ where $\alpha! = \prod_{i=1}^k \alpha_i!$.

This implies that any $f \in L^2(\nu)$ has the unique representation

$$f(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega)$$

where $c_\alpha \in \mathbb{R}$ for each multi-index $\alpha$ and

$$\|f\|^2_{L^2(\nu)} = \sum_\alpha \alpha! c_\alpha^2.$$

**DEFINITION 2.3** [Gj] The $m$-dimensional white noise map is a map

$$W : \prod_{i=1}^m S'(\mathbb{R}^n) \times \prod_{i=1}^m S'(\mathbb{R}^n) \to \mathbb{R}^m$$

given by

$$W^{(i)}(\phi, \omega) := \omega_i(\phi_i) \quad 1 \leq i \leq m$$

**PROPOSITION 2.4** [Gj] The $m$-dimensional white noise map $W$ satisfies the following

1. $\{W^{(i)}(\phi, \cdot)\}_{i=1}^m$ is a family of independent normal random variables.
2. $W^{(i)}(\phi, \cdot) \in L^2(\nu)$ for $1 \leq i \leq m$. 
DEFINITION 2.5 [HLØUZ3] Let \( 0 \leq \rho \leq 1 \).

- Let \((S_n^m)\rho\), the space of \textbf{generalized white noise test functions}, consist of all
  \[ f = \sum_{\alpha} H_\alpha \in L^2(\nu) \]
  such that
  \[ \|f\|_{\rho,k}^2 := \sum_{\alpha} c_\alpha^2 (\alpha!)^{1+\rho} (2N)^{\alpha k} < \infty \quad \forall k \in \mathbb{N} \]

- Let \((S_n^m)^{-\rho}\), the space of \textbf{generalized white noise distributions}, consist of all formal expansions
  \[ F = \sum_{\alpha} b_\alpha H_\alpha \]
  such that
  \[ \sum_{\alpha} b_\alpha^2 (\alpha!)^{1-\rho} (2N)^{-\alpha q} < \infty \text{ for some } q \in \mathbb{N} \]
  where
  \[ (2N)^\alpha := \prod_{i=1}^{k} (2n\beta_1^{(i)} \cdots \beta_n^{(i)})^{\alpha_i} \text{ if } \alpha = (\alpha_1, \ldots, \alpha_k). \]

We know that \((S_n^m)^{-\rho}\) is the dual of \((S_n^m)^{\rho}\) (when the later space has the topology given by the seminorms \(\| \cdot \|_{\rho,k}\)) and if \( F = \sum b_\alpha H_\alpha \in (S_n^m)^{-\rho}\) and \( f = \sum c_\alpha H_\alpha \in (S_n^m)^{\rho}\) then
  \[ \langle F, f \rangle = \sum_{\alpha} b_\alpha c_\alpha \alpha!. \]

It is obvious that we have the inclusions
  \[ (S_n^m)^{1} \subset (S_n^m)^{\rho} \subset (S_n^m)^{-\rho} \subset (S_n^m)^{-1} \quad \rho \in [0, 1] \]
and in the remaining of this paper we will consider the larger space \((S_n^m)^{-1}\).

DEFINITION 2.6 [HLØUZ3] The Wick product of two elements in \((S_n^m)^{-1}\) given by
  \[ F = \sum_{\alpha} a_\alpha H_\alpha, \quad G = \sum_{\beta} b_\beta H_\beta \]

is defined by
  \[ F \circ G = \sum_{\gamma} c_\gamma H_\gamma \]
where
  \[ c_\gamma = \sum_{\alpha + \beta = \gamma} a_\alpha b_\beta \]
LEMMA 2.7 [HLØUZ3] We have the following

1. \( F, G \in (S_n^m)^{-1} \Rightarrow F \circ G \in (S_n^m)^{-1} \)

2. \( f, g \in (S_n^m)^{1} \Rightarrow f \circ g \in (S_n^m)^{1} \)

DEFINITION 2.8 [HLØUZ3] Let \( F = \sum b_{\alpha} H_{\alpha} \) be given. Then the Hermite transform of \( F \), denoted by \( \mathcal{H}F \), is defined to be (whenever convergent)

\[
\mathcal{H}F := \sum_{\alpha} b_{\alpha} z^{\alpha}
\]

where \( z = (z_1, z_2, \ldots) \) and \( z^{\alpha} = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_k^{\alpha_k} \) if \( \alpha = (\alpha_1, \ldots, \alpha_k) \).

LEMMA 2.9 [HLØUZ3] If \( F, G \in (S_n^m)^{-1} \) then

\[
\mathcal{H}(F \circ G)(z) = \mathcal{H}F(z) \cdot \mathcal{H}G(z)
\]

for all \( z \) such that \( \mathcal{H}F(z) \) and \( \mathcal{H}G(z) \) exists.

LEMMA 2.10 [HLØUZ3] Suppose \( g(z_1, z_2, \ldots) \) is a bounded analytic function on \( B_q(\delta) \) for some \( \delta > 0, q < \infty \) where

\[
B_q(\delta) := \{ \zeta = (\zeta_1, \zeta_2, \ldots) \in \mathbb{C}^\mathbb{N}, \sum_{\alpha \neq 0} |\zeta^\alpha|^2 (2N)^{\alpha q} < \delta^2 \}.
\]

Then there exists \( X \in (S_n^m)^{-1} \) such that \( \mathcal{H}X = g \).

LEMMA 2.11 [HLØUZ3] Suppose \( X \in (S_n^m)^{-1} \) and that \( f \) is an analytic function in a neighborhood of \( \mathcal{H}X(0) \) in \( \mathbb{C} \). Then there exists \( Y \in (S_n^m)^{-1} \) such that \( \mathcal{H}Y = f \circ \mathcal{H}X \).

THEOREM 2.12 [KLS] Let \((T, \Sigma, \tau)\) be a measure space and let \( \Phi : T \rightarrow (S_n^m)^{-1} \) be such that there exists \( q < \infty, \delta > 0 \) such that

1. \( \mathcal{H}\Phi_t(z) : T \rightarrow \mathbb{C} \) is measurable for all \( z \in B_q(\delta) \)

2. there exists \( C \in L^1(T, \tau) \) such that \( |\mathcal{H}\Phi_t(z)| \leq C(t) \) for all \( z \in B_q(\delta) \) and for \( \tau \)-almost all \( t \).

Then \( \int_T \Phi_t \, d\tau(t) \) exists as a Bochner integral in \( (S_n^m)^{-1} \). In particular, \( \langle \int_T \Phi_t \, d\tau(t), \phi \rangle = \int_T (\Phi_t, \phi) \, d\tau(t) \); \( \phi \in (S_n^m)^{1} \).

EXAMPLE 2.13 Define the x-shift of \( \phi \), denoted by \( \phi_x \), by \( \phi_x(y) := \phi(y - x) \). Then \( \text{Exp}[W_{\phi_x}^{(i)}] \in (S_n^m)^{-1}, 1 \leq i \leq m, \forall x \in \mathbb{R}^n \)

which is an immediate consequence of proposition 2.4 and lemma 2.11.
§3 The pollution model in $\mathbb{R}^n$

We will in this and the next section assume that $(b^{t,x}_r(\omega), \hat{V}^{t,x})$ is a Brownian motion starting at location $x \in \mathbb{R}^n$ at time $t$, and use the notation

- $\hat{E}^{t,x}$ is expectation w.r.t. the measure $\hat{P}^{t,x}$.
- $C^2(\mathbb{R}^n)$ are the functions in $\mathbb{R}^n$ with continuous derivatives up to order 2.
- $C^2_0(\mathbb{R}^n)$ are the functions in $\mathbb{R}^n$ with compact support and continuous derivatives up to order 2.

**THEOREM 3.1** Let $T > 0$ be given and assume furthermore that we are given functions $\mathbb{R}^n \ni x \mapsto f(x) \in (S^n_\mathbb{N})^{-1}$, $[0, T) \times \mathbb{R}^n \ni (t, x) \mapsto g(t, x) \in (S^n_\mathbb{N})^{-1}$ and $[0, T) \times \mathbb{R}^n \ni (t, x) \mapsto \kappa(t, x) \in (S^n_\mathbb{N})^{-1}$ such that

- $\exists (q_f, N, \delta_f > 0, K_f > 0)$ such that
  1. $\sup_{x \in \mathbb{R}^n, z \in B_{q_f} \delta_f} |\mathcal{H}^f(x, z)| \leq K_f$.
  2. $x \mapsto \mathcal{H}^f(x, z) \in C^2_0(\mathbb{R}^n)$ whenever $z \in B_{q_f} \delta_f$.

- $\exists (q_g, N, \delta_g > 0, K_g > 0)$ such that
  1. $\sup_{(t, x) \in [0, T) \times \mathbb{R}^n, z \in B_{q_g} \delta_g} |\mathcal{H}^g(t, x, z)| \leq K_g$.
  2. $x \mapsto \mathcal{H}^g(t, x, z) \in C^2_0(\mathbb{R}^n)$ whenever $t \in [0, T)$, $z \in B_{q_g} \delta_g$.
  3. $\exists (\alpha(z) > 0 \forall z \in B_{q_g} \delta_g)$ such that $(t, x) \mapsto \mathcal{H}^g(t, x, z)$ is Hölder continuous (exponent $\alpha(z)$) in $x \in \mathbb{R}^n$, uniformly in $t \in [0, T)$.

- $\exists (q_\kappa, N, \delta_\kappa > 0, K_\kappa > 0)$ such that
  1. $\sup_{(t, x) \in [0, T) \times \mathbb{R}^n, z \in B_{q_\kappa} \delta_\kappa} |\mathcal{H}^\kappa(t, x, z)| \leq K_\kappa$.
  2. $x \mapsto \mathcal{H}^\kappa(t, x, z) \in C^2_0(\mathbb{R}^n)$ whenever $t \in [0, T)$, $z \in B_{q_\kappa} \delta_\kappa$.
  3. $\exists (\beta(z) > 0 \forall z \in B_{q_\kappa} \delta_\kappa)$ such that $(t, x) \mapsto \mathcal{H}^\kappa(t, x, z)$ is uniformly Hölder continuous (exponent $\beta(z)$) in $(t, x)$ on compact subsets of $[0, T) \times \mathbb{R}^n$.
  4. $\mathcal{H}^\kappa(t, x, z) \geq 0$ whenever $z \in B_{q_\kappa} \delta_\kappa \cap \mathbb{R}^n_0$.

Then

$$u(t, x) = \hat{E}^{T-t, x}[f(b_T) \circ \text{Exp}(- \int_{T-t}^T \kappa(T - \theta, b_T) \, d\theta) \circ \mathcal{J}_{t,T}]
+ \hat{E}^{T-t, x} \left[ \int_{T-t}^T g(T - s, b_s) \circ \text{Exp}(- \int_{T-t}^s \kappa(T - \theta, b_T) \, d\theta) \, ds \circ \mathcal{J}_{t,T} \right]$$

where

$$\mathcal{J}_{t,T} := \text{Exp} \left( \sum_{i=1}^n \eta^{-1} \int_{T-t}^T [W_{\phi_y}^{(i)}]_{y = \eta b_s} \, dB_s^i - \frac{1}{2} \sum_{i=1}^n \eta^{-2} \int_{T-t}^T [W_{\phi_y}^{(i)}]^2]_{y = \eta b_s} \, ds \right)$$

(2)
is the unique $(S_n^n)^{-1}$-valued process which solves
\[
\frac{\partial u}{\partial t} = \frac{1}{2} \eta^2 \Delta u + \tilde{W}_{\phi_x} \circ \nabla u - \kappa(t, x) \circ u + g(t, x) \quad (t, x) \in [0, T] \times \mathbb{R}^n
\]
\[
u(0, x) = f(x) \quad x \in \mathbb{R}^n
\]
where $\tilde{E}$ and $\int s$ are Bochner integrals in $(S_n^n)^{-1}$.

**REMARK 3.2** If $u(t, x) \in (S_n^n)^{-1}$ and $A(\mathcal{H}u(t, x)) \in A_b(B_{\delta}(\delta))$ for some $q \in \mathbb{N}, \delta > 0$, where $A_b(B_{\delta}(\delta))$ is the space of all bounded analytic functions on $B_{\delta}(\delta)$ and $A := \frac{\partial}{\partial t} - \frac{1}{2} \eta^2 \Delta - \tilde{W}_{\phi} \circ \nabla$, we will use the convention that $A u(t, x) := \mathcal{H}^{-1} A(\mathcal{H}u(t, x))$.

**PROOF:**

We must find $q \in \mathbb{N}$ and $\hat{\delta} > 0$ such that $\tilde{u}(t, x, z) := \mathcal{H}(u(t, x)) \circ \eta \in A_b(B_{\delta}(\hat{\delta}))$ solves the equation
\[
\frac{\partial \tilde{u}}{\partial t} = \frac{1}{2} \eta^2 \Delta \tilde{u} + \tilde{W}_{\phi_x} \circ \nabla \tilde{u} - \kappa(t, x) \circ \tilde{u} + \vec{g}(t, x) \quad (t, x) \in [0, T] \times \mathbb{R}^n
\]
\[
\tilde{u}(0, x) = f(x) \quad x \in \mathbb{R}^n
\]
when $z \in B_{\delta}(\hat{\delta})$.

**LEMMA 3.3** \exists $\delta > 0, \lambda \in \mathbb{N}$ such that $z \mapsto \tilde{u}(t, x, z) \in A_b(B_{\delta}(\hat{\delta})) \quad \forall (t, x) \in [0, T] \times \mathbb{R}^n$.

**PROOF:**

By taking absolute values, we get
\[
|\tilde{u}(t, x, z)| \leq K_f e^{TK_x} + TK_g e^{TK_x}
\]
whenever $z \in B_{\delta}(\hat{\delta})$ where $\hat{\delta} \geq \max\{q, q_\delta, q_x\}$ and $0 < \hat{\delta} \leq \min\{\delta, \delta_g, \delta_x\}$, since, by using [BØ, Corollary 8.23],
\[
\tilde{E}^{t-t,x}[J_{t,T}] = \tilde{E}^{t-t,x}[\exp(\sum_{i=1}^n \eta^{-1} \int_{T-t}^T \mathcal{R} \tilde{W}_{\phi_x}^{(i)}_{y=\eta b_s} \, ds - \frac{1}{2} \sum_{i=1}^n \eta^{-2} \int_{T-t}^T \mathcal{R} \tilde{W}_{\phi_x}^{(i)}_{y=\eta b_s} \, ds)]
\]
\[
= 1.
\]

**LEMMA 3.4** The Bochner integrals in the expression for $u(\lambda)$ are well-defined.

**PROOF:**

This is obvious from the estimates in lemma 3.3
LEMMMA 3.5 \( A\mathcal{u}(t,x) \) is well-defined as an element in \((S^n_\kappa)^{-1}\) \( \forall (t,x) \in [0,T) \times \mathbb{R}^n \).

PROOF:

Since

\[
A\hat{u} = -\kappa \cdot \hat{u} + \hat{g}
\]

it follows from lemma 3.3 that

\[
|A\hat{u}(t,x,z)| \leq K_\kappa (K_\kappa e^{TK_\kappa} + TK_\kappa e^{TK_\kappa}) + K_g
\]

when \( z \in B_\kappa \), i.e. the claim follows.

LEMMMA 3.6 \( \hat{u}(t,x,z) \) is the unique function which solves equation (3) when \( z \in B_\kappa \).

PROOF:

Equation (3) may be written as

\[
\frac{\partial \hat{u}}{\partial t} + \kappa \hat{u} = \mathcal{A}^\xi \hat{u} + \hat{g} \quad (t,x) \in [0,T] \times \mathbb{R}^n \tag{5}
\]

\[
\hat{u}(0,x) = \hat{f} \quad x \in \mathbb{R}^n \tag{6}
\]

where \( \mathcal{A}^{\xi} \) is the second order differential operator given by

\[
\mathcal{A}^{\xi} = \sum_{i=1}^{n} \frac{1}{2} \nabla^2 \phi_i + \sum_{i=1}^{n} \nabla_{\phi_i}(\xi) \frac{\partial}{\partial x_i}.
\]

Assume now that \( \xi \in B_\kappa \cap \mathbb{R}_N^N \).

The operator \( \mathcal{A}^{\xi} \) is clearly uniformly elliptic with drift term which satisfies the linear growth condition

\[
|\hat{W}^{(i)}_{\phi_x} - \hat{W}^{(i)}_{\phi_y}|(\xi) = |\sum_{k=0}^{\infty} (\phi_x - \phi_y, e_k)\xi_k|
\]

\[
\leq \sum_{k=0}^{\infty} |(\phi_x - \phi_y, e_k)||\xi_k|
\]

\[
\leq (M \sum_{k=0}^{\infty} \int_{\mathbb{R}^n} |e_k(x)|dx)|\xi_k|\]

where

\[
M := \max \{ \sup_{1 \leq i \leq n} \sup_{x \in \mathbb{R}^n} |\frac{\partial \phi}{\partial x_i}| < \infty.
\]

It follows by standard results that the stochastic process

\[
dX_t^{\xi} = \mathcal{W}_{\phi^{\xi}_x}(\xi) dt + dB_t \quad ; \quad X_0^{\xi} = x
\]

exists with \( \mathcal{A}^{\xi} \) as generator.
The solution of (5) is given by the Feynman-Kac formula [KS, Theorem 5.7.6]

\[
\tilde{u}(t, x, \xi) = \mathcal{E}^{T-t, x}[f(X^T_t)] \exp\left(-\int_{T-t}^T \tilde{r}(T - \theta, X^T_{\theta}) \, d\theta\right)
+ \mathcal{E}^{T-t, x}\left[\int_{T-t}^T \tilde{g}(T - s, X^T_{s}) \exp\left(-\int_{T-t}^s \tilde{r}(T - \theta, X^T_{\theta}) \, d\theta\right) \, ds\right]
\]

and by a change of measure this may be written as

\[
\tilde{u}(t, x, \xi) = \mathcal{E}^{T-t, x}[f(b_T) \exp\left(-\int_{T-t}^T \tilde{r}(T - \theta, b_{\theta}) \, d\theta\right) \mathcal{M}_{t,T}]
+ \mathcal{E}^{T-t, x}\left[\int_{T-t}^T \tilde{g}(T - s, b^T_{s}) \exp\left(-\int_{T-t}^s \tilde{r}(T - \theta, X^T_{\theta}) \, d\theta\right) \, ds \mathcal{M}_{t,T}\right]
\]

where

\[
\mathcal{M}_{t,T} := \exp\left(\sum_{i=1}^n \eta^{-1} \int_{T-t}^T \tilde{\mathcal{W}}_{\phi^{(i)}}(s) \, ds \right) - \frac{1}{2} \sum_{i=1}^n \eta^{-2} \int_{T-t}^T \tilde{\mathcal{W}}_{\phi^{(i)}}^2(s) \, ds.
\]

This expression is easily seen to have an analytic extension to all \( z \in B_\eta(\delta) \) and by applying the generator of \( b_t \) on both the real and imaginary part of \( \tilde{u}(t, x, z) \) we see that equation (3) also holds in this case.

The theorem now follows from the previous lemmas.

\[\square\]

\section*{§4 The pollution model in a bounded domain}

\textbf{THEOREM 4.1} Let \( T > 0 \) be given and suppose \( D \subset \mathbb{R}^n \) is a bounded domain such that every point on the boundary of \( \delta D \) has the exterior sphere property; i.e. there exists a ball \( B \ni x \) such that \( \bar{B} \setminus \bar{D} = \emptyset, B \setminus \delta D = \{x\} \).

Assume furthermore that we are given functions \([0, T] \times \partial D \ni (t, x) \mapsto h(t, x) \in (S_{n}^1)^{-1}, [0, T] \times D \ni (t, x) \mapsto g(t, x) \in (S_{n}^1)^{-1} \) and \([0, T] \times D \ni (t, x) \mapsto \phi(x) \in (S_{n}^1)^{-1} \) and \([0, T] \times D \ni (t, x) \mapsto \kappa(t, x) \in (S_{n}^1)^{-1} \) such that

- \( \exists (q_h \in \mathbb{N}, \delta_h > 0, K_h > 0) \) such that
  1. \( \sup_{(t,x) \in [0,T] \times \partial D, \xi \in B_{q_h} (\delta_h)} |\mathcal{H}h(t, x, z)| \leq K_h. \)
  2. \( x \mapsto \mathcal{H}h(t, x, z) \in C^2([0, T] \times \partial D) \) whenever \( t \in [0, T], z \in B_{q_h} (\delta_h). \)

- \( \exists (q_\phi \in \mathbb{N}, \delta_\phi > 0, K_\phi > 0) \) such that
  1. \( \sup_{x \in D, \xi \in B_{q_\phi} (\delta_\phi)} |\mathcal{H}\phi(x, z)| \leq K_\phi. \)
  2. \( x \mapsto \mathcal{H}\phi(x, z) \in C^2(D) \) whenever \( z \in B_{q_\phi} (\delta_\phi). \)

- \( \exists (q_\kappa \in \mathbb{N}, \delta_\kappa > 0, K_\kappa > 0) \) such that
1. \( \sup_{(t,x) \in [0,T] \times D, z \in B_{q_{\kappa}}(\delta_{\kappa})} |\mathcal{H}g(t, x, z)| \leq K_{\kappa}. \)

2. \( x \mapsto \mathcal{H}g(t, x, z) \in C^2([0,T] \times D) \) whenever \( t \in [0,T], z \in B_{q_{\kappa}}(\delta_{\kappa}). \)

3. \( \exists (\alpha(z) > 0 \forall z \in B_{q_{\kappa}}(\delta_{\kappa})) \) such that \( (t, x) \mapsto \mathcal{H}g(t, x, z) \) is Hölder continuous (exponent \( \alpha(z) \)) in \( x \in D, \) uniformly in \( t \in [0,T]. \)

- \( \exists (q_{\kappa} \in \mathbb{N}, \delta_{\kappa} > 0, K_{\kappa} > 0) \) such that
  1. \( \sup_{(t,x) \in [0,T] \times D, z \in B_{q_{\kappa}}(\delta_{\kappa})} |\mathcal{H} \kappa(t, x, z)| \leq K_{\kappa}. \)
  2. \( x \mapsto \mathcal{H} \kappa(t, x, z) \in C^2(D) \) whenever \( t \in [0,T], z \in B_{q_{\kappa}}(\delta_{\kappa}). \)
  3. \( \exists (\beta(z) > 0 \forall z \in B_{q_{\kappa}}(\delta_{\kappa})) \) such that \( (t, x) \mapsto \mathcal{H} \kappa(t, x, z) \) is uniformly Hölder continuous (exponent \( \beta(z) \)) in \( (t, x) \) in compact subsets of \( [0,T] \times D. \)
  4. \( \mathcal{H} \kappa(t, x, z) \geq 0 \) whenever \( z \in B_{q_{\kappa}}(\delta_{\kappa}) \cap \mathbb{R}^N. \)

- \( h(0, x) = \phi(x) \quad \forall x \in \partial D \)

Then

\[
\begin{align*}
  u(t, x) &= \mathcal{E}^{t-t,T}[h(b_T, T-t) \circ \text{Exp}(-\int_{T-t}^{T} \kappa(T-s, b_s) \, ds) \chi_{T-t} \circ \mathcal{J}_{t,T}] \\
  &\quad + \mathcal{E}^{t-t,T}[\phi(b_T) \circ \text{Exp}(-\int_{T-t}^{T} \kappa(T-s, b_s) \, ds) \chi_{T-t} \circ \mathcal{J}_{t,T}] \\
  &\quad + \mathcal{E}^{t-t,T}[\int_{T-t}^{T} g(b_s, T-s) \circ \text{Exp}(-\int_{T-t}^{s} \kappa(T-\lambda, b_\lambda) \, d\lambda) \, ds \circ \mathcal{J}_{t,T}]
\end{align*}
\]

where \( \mathcal{J}_{t,T} \) is given by (2) and \( \tau \) is first time \( \lambda \in [t,T] \) that \( X_\lambda \) leaves \( D \) if such a time exists and \( \tau := T \) otherwise, is the unique \( (\mathcal{S}^\kappa)_{t,T}^{-1} \)-valued process which solves

\[
\frac{\partial u}{\partial t} = \frac{1}{2} \eta^2 \Delta u + \nabla \phi_x \circ \nabla u - \kappa(t, x) \circ u + g(t, x) \quad (t, x) \in [0,T] \times D
\]

\[
\begin{align*}
  u(0, x) &= \phi(x) \\
  u(t, x) &= h(t, x) \quad (t, x) \in [0,T] \times \partial D
\end{align*}
\]

\( \mathcal{E} \) and \( \int \cdot \, ds \) are Bochner integrals in \( (\mathcal{S}^\kappa_{t,T})^{-1} \).

**PROOF:**

This follows, since \( \tau \leq T \), as in the proof of theorem 3.1, but instead of using the Feynman-Kac formula, we use [Fri, Theorem 5.2].

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References


