Wick products of complex valued random variables

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ABSTRACT. In this paper we consider Wick products of complex valued random variables. We prove that Wick products of such variables coincide with the ordinary product in a variety of cases. Ordinary SDEs are considered in relation to their Wick versions. We present examples where these notions are equivalent in the complex case.

1. Introduction

The relationship between stochastic integration and complex analysis has been a topic of several authors. Analytic functions are conformal mappings and they will always map Brownian paths into new Brownian paths. The area is thus characterized by a number of phenomena which do not appear in the real case. The basis for many of these issues can be found from the complex version of the Ito formula, see e.g. [14]:

$$df(Z_t) = \frac{\partial f}{\partial z}dZ_t + \frac{\partial f}{\partial \overline{z}}d\overline{Z}_t + \frac{1}{2}\frac{\partial^2 f}{\partial z^2}dZ_t dZ_t + \frac{\partial^2 f}{\partial z \partial \overline{z}}dZ_t d\overline{Z}_t + \frac{1}{2}\frac{\partial^2 f}{\partial \overline{z}^2}d\overline{Z}_t d\overline{Z}_t \tag{1.1}$$

If f happens to be an analytic function, this simplifies to:

$$df(Z_t) = \frac{\partial f}{\partial z}dZ_t + \frac{1}{2}\frac{\partial^2 f}{\partial z^2}dZ_t dZ_t \tag{1.2}$$

Usually one only wants to consider processes Z_t with some kind of holomorphic structure, e.g. conformal martingales see [5] or [18]. In these cases the quadratic variation term $dZ_t dZ_t$ vanish, and we end up with the ordinary chain rule:

$$df(Z_t) = \frac{\partial f}{\partial z} dZ_t \tag{1.3}$$

Once we have a chain rule of this form, we are able to solve various problems in stochastic calculus using simple techniques from classical calculus. The awkward correction terms from the usual Ito calculus are no longer present, and the basic intuition from ordinary differential equation applies without change. By contrast, one can achieve more or less the same effect using Wick products and Wick calculus. For references to the theory of Wick calculus see [6], [8], [10] or [12]. This way of approach applies already in the real variable case.

Some notation

Let $S(\mathbb{R})$ be denote the usual Schwartz space of rapidly decreasing smooth (C^∞) functions on $\mathbb{R}$ with its dual space $S'(\mathbb{R})$ equipped with the weak star topology, and let $S'_{\mathbb{C}}(\mathbb{R})$ denote the complexification of $S'(\mathbb{R})$. On $S'_{\mathbb{C}}(\mathbb{R})$ we define a probability measure μ as the product of two white noise measures, see [7]. To be more precise, the complexification of the real white noise probability space is carried out as follows. Put:

$$S_{\mathbb{C}}(\mathbb{R}) = S(\mathbb{R}) + i\,S(\mathbb{R}) \qquad \text{and:} \qquad S'_{\mathbb{C}}(\mathbb{R}) = S'(\mathbb{R}) + i\,S'(\mathbb{R}) \tag{1.4}$$

By the Bochner-Minlos theorem, define two measures μ_1 and μ_2 on $S'(\mathbb{R})$ with:

$$\int_{S'(\mathbb{R})} \exp(i\langle\omega,\phi\rangle)d\mu_j(\omega) = \exp(-\frac{1}{4}\|\phi\|^2_{L^2(\mathbb{R})}), \quad j = 1,2. \tag{1.5}$$

With $\mathcal{B}$ the Borel σ-algebra on $S'_{\mathbb{C}}(\mathbb{R})$, introduce the product measure $\nu = \mu_1 \times \mu_2$. Then the triplet:

$$(S'_{\mathbb{C}}(\mathbb{R}), \mathcal{B}, \nu) \tag{1.6}$$

is called the complex white noise probability space. From the expression (1.5) we get the familiar isometry $E[| < \cdot, \phi > |^2] = ||\phi||^2_{L^2(\mathbb{R})}$ for all $\phi \in S(\mathbb{R})$ where $< \omega, \phi >= \omega(\phi)$ is the dual action. Using this isometry, we can define $< \omega, \phi >:= \lim_{k\to\infty} < \omega, \phi_k >$ for all $\phi \in L^2(\mathbb{R})$ (ϕ_k is any sequence in $S(\mathbb{R})$ s.t. $\phi_k \to \phi$ in $L^2(\mathbb{R})$). This allows us to define:

$$\tilde{\mathbb{B}}_t(\omega) := \langle\omega, 1_{[0,t)}\rangle = \langle\omega_{real}, 1_{[0,t)}\rangle + i\langle\omega_{imaginary}, 1_{[0,t)}\rangle \tag{1.7}$$

$\tilde{\mathbb{B}}_t$ is then essentially a Brownian motion in the complex plane in the sense that there exist a t-continuous version $\mathbb{B}_t$ of $\tilde{\mathbb{B}}_t$ such that $\mathbb{B}_t$ is a Brownian motion in the complex plane. We let $B_{1\,t}$ and $B_{2\,t}$ denote the real and the imaginary components of $\mathbb{B}_t$. We also need the corresponding white noise processes and indicate these as $\mathbb{W}_t, W_{1t}$ etc. The familiar constructions of white noise analysis now carry over to the complex case with some minor modifications.

Following Hida [7], we introduce the complex Hermite polynomials $H_{n,m}(z,\bar{z})$ as:

$$H_{n,m}(z,\bar{z}) = (-1)^{n+m}\exp(z\bar{z})\frac{\partial^{n+m}}{\partial\bar{z}^n\partial z^m}\exp(-z\bar{z}) \tag{1.8}$$

where n, m are non-negative integers. With this definition, we see that our Brownian motion can be written:

$$\mathbb{B}_t(\omega) = H_{1,0}(<\omega, 1_{[0,t)}>, \overline{<\omega, 1_{[0,t)}>}) \tag{1.9}$$

Denote by $(L^2_{\mathbb{C}}) := L^2(\nu)$, and let $\mathcal{H}_{(n,m)}$ be the subspace spanned by the functions $\{H_{n,m}(<\omega, e_i>, \overline{<\omega, e_i>})\}_i$, where $\{e_i\}_i$ is a CONS in $L^2_{\mathbb{C}}(\mathbb{R})$. We will make the assumption that $e_i \in S_{\mathbb{C}}(\mathbb{R})$ for all $i = 1, 2, \ldots$. From [7], proof of prop. 6.11, we have the orthogonality relation:

$$\begin{aligned} &\int_{S'_{\mathbb{C}}(\mathbb{R})} H_{p,k}(<\omega,\psi>, \overline{<\omega,\psi>})\overline{H_{n,m}(<\omega,\gamma>, \overline{<\omega,\gamma>})}\, d\nu(\omega) \\ =&\delta_{p,n}\delta_{q,m}p!q!(\gamma,\psi)^p\overline{(\gamma,\psi)}^q \end{aligned} \tag{1.10}$$

In [7] it is shown that we have a Wiener-Ito-Segal decomposition for every $\phi \in (L^2_{\mathbb{C}})$:

$$\phi(\omega) = \sum_{n,m} \phi_{n,m}(\omega) \tag{1.11}$$

where $\phi_{n,m} \in \mathcal{H}_{(n,m)}$. We obtain a Fock space structure, i.e. ϕ is in a one-to-one correspondence with a sequence of functions $\{f^{(n,m)}\}_{n,m}$, with $f^{(n,m)} \in L^2_{\mathbb{C}}(\mathbb{R}^{n+m})$. Moreover:

$$\|\phi\|^2_{(L^2_{\mathbb{C}})} = \sum_{n=1}^{\infty}\sum_{m=1}^{n} n!m!|f^{(n,m)}|^2_{L^2_{\mathbb{C}}(\mathbb{R}^{n+m})} \tag{1.12}$$

We introduce the complex Kondratiev spaces of random test functions and distributions. Our construction follows closely the one found in [1]. Let $\mathbf{P} := \mathbf{P}(\{e_i\})$ be the space of polynomials as defined in [7], Ch. 6.3. Every element $\phi \in \mathbf{P}$ is expressible in the form:

$$\phi(\omega) = \sum_{n=0}^{N}\sum_{m=0}^{n} \phi_{n,m}(\omega) \tag{1.13}$$

where $\phi_{n,m} \in \mathcal{H}_{(n,m)}$. Let the space $(S_{\mathbb{C}})^1_p$ be the completion of **P** in the norm:

$$\|\phi\|^2_{2,p,\mathbb{C}} = \sum_{n=0}^{\infty}\sum_{m=0}^{n}(n!m!)^2|f^{(n,m)}|^2_{2,p,\mathbb{C}} \tag{1.14}$$

where $|\cdot|_{2,p,\mathbb{C}}$ is the complexification of the norm $|\cdot|_{2,p} := |A^p\cdot|_2$. A is the harmonic oscillator. The complex Kondratiev space of random test functions is the projective limit of the spaces $(S_{\mathbb{C}})^1_p$, and is denoted $(S_{\mathbb{C}})^1$. Its dual, the space of complex Kondratiev distributions, is denoted $(S_{\mathbb{C}})^{-1}$. All elements $\Phi \in (S_{\mathbb{C}})^{-1}$ is in a one-to-one correspondence with a sequence of functions $\{F^{(n,m)}\}_{n,m}$, with $F^{(n,m)} \in S'_{\mathbb{C}}(\mathbb{R}^{n+m})$, such that for a $p > 0$:

$$\|\Phi\|^2_{-2,-p,\mathbb{C}} := \sum_{n=0}^{\infty}\sum_{m=0}^{n}|F^{(n,m)}|^2_{2,-p,\mathbb{C}} < \infty \tag{1.15}$$

On the space of complex Kondratiev distributions, we introduce the S-transform: For $\Phi \in (S_{\mathbb{C}})^{-1}$ and $\xi \in S_{\mathbb{C}}(\mathbb{R})$, let:

$$S\Phi(\xi) := \langle\overline{\Phi}, \exp(<\cdot,\xi> + \overline{<\cdot,\xi>} - |\xi|^2_2)\rangle \tag{1.16}$$

where $\langle\cdot,\cdot\rangle$ is the dual pairing between $(S_{\mathbb{C}})^{-1}$ and $(S_{\mathbb{C}})^1$. It is easy to see that for $\phi \in (L^2_{\mathbb{C}})$:

$$S\phi(\xi) = \int_{S'(\mathbb{R})}\overline{\phi(\omega)}\exp(<\omega,\xi> + \overline{<\omega,\xi>} - |\xi|^2_2)\,d\nu(\omega) \tag{1.17}$$

From formula A.40 in [7], we have:

$$\exp(<\omega,\xi> + \overline{<\omega,\xi>} - |\xi|^2_2) = \sum_{n,m=0}^{\infty}\frac{|\xi|_2^{n+m}}{n!m!}H_{n,m}(<\omega,\xi\cdot|\xi|_2^{-1}>, \overline{<\omega,\xi\cdot|\xi|_2^{-1}>}) \tag{1.18}$$

Using the orthogonality relation for the complex Hermite polynomials, we obtain:

$$SB_t(\xi) = |\xi|_2(1_{[0,t)}, \xi\cdot|\xi|_2^{-1}) = \int_0^t \overline{\xi(s)}\,ds \tag{1.19}$$

From [1] it is known that the S-transform characterizes the Kondratiev distributions. Consider a function $G : \mathcal{U} \to \mathbb{C}$, where $\mathcal{U}$ is a neighborhood around zero in $S_{\mathbb{C}}(\mathbb{R})$. If G is locally bounded on $\mathcal{U}$, and the mapping $z \to G(\xi + z\eta)$ is analytic in a neighborhood around zero in $\mathbb{C}$ for each pair $\xi, \eta \in \mathcal{U}$, then there exists a $\Phi \in (S_{\mathbb{C}})^{-1}$ such that $S\Phi = G$.

Opposite, every element in $(S_{\mathbb{C}})^{-1}$ has a S-transform which is of this type. We refer the reader to the papers [12] and [1], and the contribution of F. E. Benth in this volume, for more about the Kondratiev distribution space. The Wick product of two complex Kondratiev distributions is defined as follows: Let $\Phi, \Psi \in (S_{\mathbb{C}})^{-1}$, then:

$$\Phi \diamond \Psi = S^{-1}\left(S\Phi \cdot S\Psi\right) \tag{1.20}$$

With this definition, we can easily calculate the S-transform of $\mathbb{B}_t^{\diamond k}$, for a integer k:

$$S\mathbb{B}_t^{\diamond k}(\xi) = \left(\int_0^t \overline{\xi(s)}\, ds\right)^k \tag{1.21}$$

A straightforward calculation shows that:

$$S\left(H_{k,0}(<\omega,\xi>, \overline{<\omega,\xi>})\right)(\xi) = \left(\int_0^t \overline{\xi(s)}\, ds\right)^k \tag{1.22}$$

Hence, we find:

$$\mathbb{B}_t^{\diamond k}(\omega) = H_{k,0}(<\omega, 1_{[0,t)}>, \overline{<\omega, 1_{[0,t)}>}) = <\omega, 1_{[0,t)}>^k = \mathbb{B}_t(\omega)^k \tag{1.23}$$

2. Complex Wick multiplication

We now let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function. If X is a random variable, we define the Wick version of $f(X)$ by the expression:

$$f^{\diamond}(X) = \sum_{n=0}^{\infty} a_n X^{\diamond n} = \lim_{N\to\infty} \sum_{n=0}^{N} a_n X^{\diamond n}$$

the limit being taken in $(S_{\mathbb{C}})^{-1}$, see [10] or [12]. If $X \in (S_{\mathbb{C}})^{-1}$, this limit always exists. With these conventions the following theorem follows trivially from (1.23).

THEOREM 2.1

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function, and let $f^{\diamond}$ denote the Wick version. Then:

$$f(\mathbb{B}_t) = f^{\diamond}(\mathbb{B}_t) \tag{2.1}$$

We have proved that Wick-powers of complex Brownian motion coincide with usual powers. In the following we want to extend this property to other random variables as well. We first observe the following lemma:

LEMMA 2.2

$$\mathbb{B}_s^{\diamond m} \diamond \mathbb{B}_t^{\diamond n} = \mathbb{B}_s^m \cdot \mathbb{B}_t^n \tag{2.2}$$

PROOF

Assume that $t \geq s$, then we have:

$$\begin{aligned}
\mathbb{B}_s^{\diamond m} \diamond \mathbb{B}_t^{\diamond n} &= \mathbb{B}_s^{\diamond m} \diamond (\mathbb{B}_t - \mathbb{B}_s + \mathbb{B}_s)^{\diamond n} \\
&= \mathbb{B}_s^{\diamond m} \diamond \sum_{k=0}^{n} \binom{n}{k} (\mathbb{B}_t - \mathbb{B}_s)^{\diamond k} \diamond \mathbb{B}_s^{\diamond (n-k)} \\
&= \sum_{k=0}^{n} \binom{n}{k} (\mathbb{B}_t - \mathbb{B}_s)^{\diamond k} \diamond \mathbb{B}_s^{\diamond (n-k+m)} \\
&= \sum_{k=0}^{n} \binom{n}{k} (\mathbb{B}_t - \mathbb{B}_s)^{k} \diamond \mathbb{B}_s^{n-k+m} \\
&= \sum_{k=0}^{n} \binom{n}{k} (\mathbb{B}_t - \mathbb{B}_s)^{k} \cdot \mathbb{B}_s^{n-k+m} \\
&= \mathbb{B}_s^m \cdot \mathbb{B}_t^n
\end{aligned} \tag{2.3}$$

In the fifth equality we have used that $\mathbb{B}_t - \mathbb{B}_s$ and $\mathbb{B}_s$ are strongly independent. In this case the Wick product always coincide with the ordinary product, see [6].

□

PROPOSITION 2.3

Let p be a polynomial in k complex variables, then:

$$p^{\diamond}(\mathbb{B}_{t_1}, \mathbb{B}_{t_2}, \ldots, \mathbb{B}_{t_k}) = p(\mathbb{B}_{t_1}, \mathbb{B}_{t_2}, \ldots, \mathbb{B}_{t_k}) \tag{2.4}$$

where $p^{\diamond}$ is interpreted in the sense that all powers are Wick powers.

PROOF

Use lemma 1.4 repeatedly to see that:

$$\mathbb{B}_{t_1}^{\diamond n_1} \diamond \mathbb{B}_{t_2}^{\diamond n_2} \diamond \cdots \diamond \mathbb{B}_{t_k}^{\diamond n_k} = \mathbb{B}_{t_1}^{n_1} \cdot \mathbb{B}_{t_2}^{n_2} \cdots \mathbb{B}_{t_k}^{n_k} \tag{2.5}$$

The general result then follows by linearity.

□

For easy reference we will call any expression on the form $\Pi_k = p(\mathbb{B}_{t_1}, \mathbb{B}_{t_2}, \dots, \mathbb{B}_{t_k})$ a $\mathbb{B}$-analytic polynomial. The multiplicative property (2.4) can now be extended to limits of $\mathbb{B}$-analytic polynomials. A convenient space to work in is then the space $(\mathcal{S}_{\mathbb{C}})^{-1}$ of Kondratiev distributions, see [10] or [12]. We start out with some definitions:

DEFINITION 2.4

$X \in (\mathcal{S}_{\mathbb{C}})^{-1}$ is called $\mathbb{B}$-analytic if there exists a sequence X_n of $\mathbb{B}$-analytic polynomials such that $X_n \to X$ strongly in $(\mathcal{S}_{\mathbb{C}})^{-1}$.

DEFINITION 2.5

$X \in (L^p)$, $p > 1$, is $\mathbb{B}_p$-analytic if there exists a sequence X_n of $\mathbb{B}$-analytic polynomials such that $X_n \to X$ in (L^p).

From these definitions, we have:

COROLLARY 2.6

Let $X \in (L^p)$, $p > 1$, be $\mathbb{B}_p$-analytic. Then X is $\mathbb{B}$-analytic.

PROOF

By assumption we have a sequence X_n of $\mathbb{B}$-analytic polynomials converging to X in (L^p). But convergence in (L^p) implies strong convergence in $(\mathcal{S}_{\mathbb{C}})^{-1}$, see [12]. Hence, X is $\mathbb{B}$-analytic.

□

COROLLARY 2.7

If $\{X_n\}$ is a sequence of $\mathbb{B}$-analytic elements which converges strongly to X in $(\mathcal{S}_{\mathbb{C}})^{-1}$, then X is $\mathbb{B}$-analytic.

PROOF

The proof is straightforward: Since X_n converges strongly to X in $(\mathcal{S}_{\mathbb{C}})^{-1}$, there exists a $p > 0$ such that:

$$\|X_n - X\|_{2,-p,\mathbb{C}} \to 0 \tag{2.6}$$

For each n, let $\{Y_n^m\}$ be a sequence of $\mathbb{B}$-analytic polynomials converging strongly to X_n in $(\mathcal{S}_{\mathbb{C}})^{-1}$. (Such a sequence exists by definition of $\mathbb{B}$-analyticity). Since X_n is an element in $(\mathcal{S}_{\mathbb{C}})^{-1}_{-p}$, we have:

$$\|Y_n^m - X_n\|_{2,-p,\mathbb{C}} \to 0, \quad m \to \infty \tag{2.7}$$

This yields that for each n, there exists a natural number N_n such that:

$$\|Y_n^m - X_n\|_{2,-p,\mathbb{C}} < \frac{1}{n}, \quad \text{for } m \geq N_n \tag{2.8}$$

It is then easy to see that the sequence $\{Y_n^{N_n}\}$ of $\mathbb{B}$-analytic polynomials converges strongly in $(S_{\mathbb{C}})^{-1}$ to X: For a given $\epsilon > 0$, we find a M_ϵ such that $1/n < \epsilon/2$ and:

$$\|X - X_n\|_{2,-p,\mathbb{C}} < \epsilon/2 \tag{2.9}$$

for $n \geq M_\epsilon$. Hence, by the triangle inequality:

$$\|X - Y_n^{N_n}\|_{2,-p,\mathbb{C}} \leq \|X - X_n\|_{2,-p,\mathbb{C}} + \|X_n - Y_n^{N_n}\|_{2,-p,\mathbb{C}} < \epsilon \tag{2.10}$$

□

For $\mathbb{B}_p$-analyticity, we have the same result:

COROLLARY 2.8

If $\{X_n\}$ is a sequence of $\mathbb{B}_p$-analytic elements which converges to X in (L^p), then X is $\mathbb{B}_p$-analytic.

□

PROPOSITION 2.9

If X, Y are $\mathbb{B}$-analytic, then $X \diamond Y$ is $\mathbb{B}$-analytic.

PROOF

Let $\{X_n\}$ and $\{Y_n\}$ be two sequences of $\mathbb{B}$-analytic polynomials converging strongly in $(S_{\mathbb{C}})^{-1}$ to X and Y respectively. That means, for $p, q > 0$:

$$\|X_n - X\|_{2,-p,\mathbb{C}} \to 0, \quad n \to \infty \tag{2.11}$$

and:

$$\|Y_n - Y\|_{2,-q,\mathbb{C}} \to 0, \quad n \to \infty \tag{2.12}$$

Define, for an $\alpha > \frac{1}{2}$:

$$r := \alpha + \max(p, q) \tag{2.13}$$

From the triangle inequality and Corollary 4.22 in [9], we have:

$$\|X \diamond Y - X_n \diamond Y_n\|_{2,-r,\mathbb{C}} \le \|X \diamond (Y - Y_n)\|_{2,-r,\mathbb{C}} + \|Y_n \diamond (X - X_n)\|_{2,-r,\mathbb{C}} \tag{2.14}$$

$$\le K_1 \|X\|_{2,-p\mathbb{C}} \|Y - Y_n\|_{2,-q,\mathbb{C}} + K_2 \|Y_n\|_{2,-q,\mathbb{C}} \|X - X_n\|_{2,-p,\mathbb{C}} \tag{2.15}$$

We see that $X_n \diamond Y_n$ converges strongly to $X \diamond Y$, and since $X_n \diamond Y_n$ is a W-analytic polynomial, the proposition follows.

□

THEOREM 2.10

Let $p > 1$ and $q > p$. Assume that $X \in (L^q)$ is $\mathbb{B}_q$-analytic and that $Y \in (L^{\frac{qp}{q-p}})$ is $\mathbb{B}_{\frac{qp}{q-p}}$-analytic. Then $X \cdot Y \in (L^p)$ is $\mathbb{B}_p$-analytic. Moreover,

$$X \diamond Y = X \cdot Y \tag{2.16}$$

PROOF

Let X_n and Y_n be the $\mathbb{B}$-analytic sequences converging to X and Y respectively. $X_n \cdot Y_n$ is of course again a $\mathbb{B}$-analytic polynomial. Observe that by the Cauchy-Schwartz inequality we have:

$$\|f \cdot g\|_p \le \|f\|_q \|g\|_{\frac{qp}{q-p}}. \tag{2.17}$$

Hence, by the triangle inequality it follows that:

$$X_n \cdot Y_n \to X \cdot Y \tag{2.18}$$

in (L^p). Thus, $X \cdot Y$ is $\mathbb{B}_p$-analytic. Since $X_n \cdot Y_n$ is a $\mathbb{B}$-analytic polynomial, we have:

$$X_n \cdot Y_n = X_n \diamond Y_n. \tag{2.19}$$

Since X_n and Y_n converge in (L^q) and $(L^{\frac{qp}{q-p}})$ respectively, we can show that:

$$S(X_n \diamond Y_n)(\xi) \to SX(\xi) \cdot SY(\xi) \tag{2.20}$$

pointwise, and:

$$|S(X_n \diamond Y_n)(\xi)| \le K, \tag{2.21}$$

uniformly in n. Here, ξ is in a neighbourhood around zero in $S_{\mathbb{C}}(\mathbb{R})$. Hence, by Theorem 5 in [12], it follows that:

$$X_n \diamond Y_n \to X \diamond Y \tag{2.22}$$

weakly in $(S_{\mathbb{C}})^{-1}$. By the corollary above, $X_n \cdot Y_n \to X \cdot Y$ strongly in $(S_{\mathbb{C}})^{-1}$. Hence:

$$X \diamond Y = \lim_n X_n \diamond Y_n = \lim_n X_n \cdot Y_n = X \cdot Y. \tag{2.23}$$

where $\lim_n$ denotes the strong limit in $(S_{\mathbb{C}})^{-1}$.

□

Remark

Note if X is $\mathbb{B}_q$ analytic for some $q > 1$ and Y is $\mathbb{B}_r$-analytic for all $r < \infty$, then it is always possible to find $p > 1$ s.t. the conditions in theorem 2.9 are satisfied.

We now go on to consider stochastic processes. Here we call a stochastic process X_t $\mathbb{B}$-analytic if X_t is $\mathbb{B}$-analytic for every fixed t and similarly for $\mathbb{B}_p$-analyticity. We want to consider certain elementary observations on $\mathbb{B}$-analytic processes and start out with some observations.

PROPOSITION 2.11

Assume X_t is an Itô integrable and that for each t it can be approximated in L^2 by adapted $\mathbb{B}$-analytic polynomials. Then then Itô integral:

$$\int_0^T X_t d\mathbb{B}_t \tag{2.24}$$

is $\mathbb{B}_2$-analytic.

PROOF

By definition of the Itô integral:

$$\int_0^T X_t d\mathbb{B}_t = \lim_j \sum_j X_{t_j} \cdot (\mathbb{B}_{t_{j+1}} - \mathbb{B}_{t_j}) \tag{2.25}$$

where the limit is taken in (L^2). By assumption, X_{t_j} is $\mathbb{B}_2$-analytic. Since X_t is adapted, we have:

$$E[|X_{t_j}|^2 |\mathbb{B}_{t_{j+1}} - \mathbb{B}_{t_j}|^2] = E[|X_{t_j}|^2] \cdot E[|\mathbb{B}_{t_{j+1}} - \mathbb{B}_{t_j}|^2]. \tag{2.26}$$

Hence, we see that $X_{t_j}(\mathbb{B}_{t_{j+1}} - \mathbb{B}_{t_j})$ is $\mathbb{B}_2$-analytic. The Itô integral is then the (L^2)-limit of $\mathbb{B}_2$-analytic elements, which imply the proposition.

□

PROPOSITION 2.12

Let $\{x_i(t)\}_{i=1}^{\infty}$ be any sequence of $L^2(\mathbb{R})$-functions. Then all the integrals $\int x_i(t)d\mathbb{B}_t$, $i = 1, 2, \ldots$ are $\mathbb{B}_q$-analytic for any q and any combination of Wick powers and Wick products of these random variables coincide with the corresponding expressions defined in terms of the ordinary product.

PROOF

It is an easy application of the Burkholder-Gundy inequalities to see that the integrals $\int x_i(t)d\mathbb{B}_t$, $i = 1, 2, \ldots$ are $\mathbb{B}_q$-analytic for any q. The second part follows from theorem 2.10.

□

PROPOSITION 2.13

Let X_t be a $\mathbb{B}$-analytic process where $\langle X_t, \phi\rangle$ is measurable on $[0, T]$ for all $\phi \in (S)^1$. Assume there exists a $p \geq 0$ such that:

$$\int_0^T \|X_t\|_{2,-p,\mathbb{C}} dt < \infty \tag{2.27}$$

Then the Bochner integral $\int_0^T X_t dt \in (S_{\mathbb{C}})^{-1}$ is $\mathbb{B}$-analytic.

PROOF

By Pettis' Theorem (see e.g. [22]), the measurability of $\langle X_t, \phi\rangle$ implies strong measurability, i.e. the existence of a sequence $\{X_t^n\}_n$ in $(S_{\mathbb{C}})^{-1}$ converging strongly to X_t. By inspection of the proof of Pettis' theorem in [22], this sequence can be chosen in the following manner:

$$X_s^n = Y_i, \quad \text{when } s \in B_i^n, \tag{2.28}$$

where $\{B_i^n\}_{i=1}^{N_n}$ are disjoint measurable sets in $[0, T]$, and $Y_i = X_{s_i}$ for some $s_i \in [0, T]$. Hence, by assumption, X_t^n is $\mathbb{B}$-analytic.
By the condition $\int_0^T \|X_t\|_{2,-p} dt < \infty$ Bochner integrability of X_t follows. We have:

$$\int_0^T X_t dt = \lim_{n\to\infty} \int_0^T X_t^n dt = \lim_{n\to\infty} \sum_{i=1}^{\infty} Y_i m(B_i^n). \tag{2.29}$$

The limit is strongly in $(S_{\mathbb{C}})^{-1}$. Hence, the Bochner integral is the strong limit of $\mathbb{B}$-analytic elements, and the proposition follows.

□

In the Kondratiev space we have a generalization of Itô/Skorohod integration. If X_t is an Itô integrable process, then:

$$\int_0^T X_t \diamond \mathbb{W}_t dt = \int_0^T X_t d\mathbb{B}_t \tag{2.30}$$

We have the following result about $\mathbb{B}$-analyticity of this integral:

PROPOSITION 2.14

Let X_t be a $\mathbb{B}$-analytic process such that:

i) $SX_t(\xi)$ is measurable for $\xi \in \mathcal{U}$

ii) $|SX_t(\xi)| \leq C(t)$, where $C(t) \in L^1([0,T], dt)$ for $\xi \in \mathcal{U}$

Then $X_t \diamond \mathbb{W}_t$ is Bochner integrable, and:

$$\int_0^T X_t \diamond \mathbb{W}_t dt \tag{2.31}$$

is $\mathbb{B}$-analytic.

PROOF

We have:

$$\mathbb{W}_t = \lim_{h \to 0} \frac{1}{h}(\mathbb{B}_{t+h} - \mathbb{B}_t), \tag{2.32}$$

where the limit is strong in $(\mathcal{S}_{\mathbb{C}})^{-1}$. Hence, $\mathbb{W}_t$ is a $\mathbb{B}$- analytic process. This implies by proposition (above) that $X_t \diamond \mathbb{W}_t$ is $\mathbb{B}$-analytic. By Theorem 6 in [12], we have the Bochner integrability of $X_t \diamond \mathbb{W}_t$. Hence, the proposition follows.

□

3. Applications to SDEs

We now want to compare SDEs of the from:

$$dZ_t = (X_t \cdot Z_t + Y_t)d\mathbb{B}_t + (U_t \cdot Z_t + V_t)dt \tag{3.1}$$

$$dZ_t = (X_t \diamond Z_t + Y_t)d\mathbb{B}_t + (U_t \diamond Z_t + V_t)dt \tag{3.2}$$

The equation (3.2) can be solved under very mild conditions on the coefficients. We will first consider the properties of this equation. If in addition, the coefficients are $\mathbb{B}$-analytic and sufficiently nice for (3.1) to make sense, we can expect the two solutions to coincide.

PROPOSITION 3.1

Assume that for every $\xi \in S_{\mathbb{C}}(\mathbb{R})$, that the S-transforms $S(X_t)(\xi), S(Y_t)(\xi), S(U_t)(\xi)$ and $S(V_t)(\xi)$ are locally Lipschitz functions (as functions of t). If (3.2) has a $(S_{\mathbb{C}})^{-1}$-valued solution defined for all $t \geq 0$, this solution is unique.

Remark: We call a function $f = f(z)$ locally Lipschitz if there for every z_0 exists a constant $C < \infty$ s.t. $|f(z) - f(z_0)| \leq C|z - z_0|$ for every z in a neighbourhood of z_0.

PROOF

Apply the S-transform to both sides of (3.2) to see that that the S-transform of Z_t is uniquely defined. Since any element in $(S_{\mathbb{C}})^{-1}$ is uniquely defined in terms of its S-transform, the proposition follows.

□

PROPOSITION 3.2

Assume that:

i) $S(X_t)(\xi), S(Y_t)(\xi), S(U_t)(\xi)$ and $S(V_t)(\xi)$ are measurable for $\xi \in \mathcal{U}$

ii) $|S(X_t)(\xi)|, |S(Y_t)(\xi)|, |S(U_t)(\xi)|, |S(V_t)(\xi)| \leq C_T(t)$, where $e^{C_T(t)} \in L^p([0,T], dt)$ for every $\xi \in \mathcal{U}$, all $p > 0$ and all $T < \infty$

iii) $Z_0 \in (S_{\mathbb{C}})^{-1}$ then (3.2) has a $(S_{\mathbb{C}})^{-1}$-valued solution Z_t given by the expression:

$$\begin{aligned} Z_t &= Z_0 \diamond \operatorname{Exp}[\int_0^t X_r d\mathbb{B}_r + \int_0^t U_r dr] \\ &+ \int_0^t \operatorname{Exp}[\int_s^t X_r d\mathbb{B}_r + \int_s^t U_r dr] \diamond Y_s d\mathbb{B}_s \\ &+ \int_0^t \operatorname{Exp}[\int_s^t X_r d\mathbb{B}_r + \int_s^t U_r dr] \diamond V_s ds \end{aligned} \tag{3.3}$$

PROOF

The idea is to use the analogy with the differential equation $y' = f(t)y + g(t)$. This equation has the solution $y = y_0 e^{\int_0^t f(r)dr} + \int_0^t e^{\int_s^t f(r)dr} g(s)ds$. Formally we write that $f(t) = X_t \frac{d\mathbb{B}_t}{dt} + U_t$ and $g(t) = Y_t \frac{d\mathbb{B}_t}{dt} + V_t$. If we insert this in the solution formula and replace all the ordinary products with Wick products, we get (3.3). From the arguments of proposition 2.14, we see that all the necessary expressions are Bochner integrable and that (3.3) makes sense as an element of $(S_{\mathbb{C}})^{-1}$. If we insert this expression in (3.2) all the operations in (3.2) are well defined. Hence the ordinary chain rule applies and Z_t is a solution.

□

As we remarked earlier the interesting question is to compare the equation (3.1) and (3.2). To make sense out of (3.1) we must put quite strong growth conditions on the coefficients. We have to work with elements in (L^p). The idea is then to apply theorem 2.10 to see that the Wick products coincide with the ordinary product.

THEOREM 3.3

Assume that:

i) $X_t = x(t)$ where $x(t) \in L^2(\mathbb{R})$
ii) $U_t = \int_0^t u_1(s)d\mathbb{B}_s + u_2(t)$ where $u_1(t), u_2(t) \in L^2(\mathbb{R})$
iii) Y_t, V_t are adapted and $\mathbb{B}_q$-analytic for some $q > 2$ and satisfies the conditions in 3.2.
iv) Z_0 is $\mathbb{B}_r$-analytic for some $r > 1$ (If Z_0 is non-constant, the meaning of (3.1) is interpreted in the Hitsuda-Skorohod sense, se [8]).

Then the solutions of (3.1) and (3.2) coincide and are both given by the expression:

$$(3.4) \quad Z_t = Z_0 \cdot e^{\int_0^t X_r d\mathbb{B}_r + \int_0^t U_r dr} + \int_0^t e^{\int_s^t X_r d\mathbb{B}_r + \int_s^t U_r dr} \cdot Y_s d\mathbb{B}_s + \int_0^t e^{\int_s^t X_r d\mathbb{B}_r + \int_s^t U_r dr} \cdot V_s ds$$

Before we turn to the proof of this theorem, we need to prove two technical lemmas.

LEMMA 3.4

If $e^{|X|} \in (L^p)$ for all p and X is $\mathbb{B}_p$-analytic for all p, then e^X is also $\mathbb{B}_p$-analytic for all p.

PROOF

Put $f_N = e^{|X|} - \sum_{n=0}^N \frac{1}{n!}|X|^n$. By the monotone convergence theorem $f_N \to 0$ in every (L^p). Observe that:

$$(3.5) \quad \mathrm{E}[|e^X - \sum_{n=0}^N \frac{1}{n!}X^n|^p] = \mathrm{E}[|\sum_{n=N+1}^\infty \frac{1}{n!}X^n|^p] \le \mathrm{E}[|\sum_{n=N+1}^\infty \frac{1}{n!}|X|^n|^p] = \mathrm{E}[|f_N|^p] \to 0$$

Hence e^X can be approximated as well as we please by $\sum_{n=0}^N \frac{1}{n!}X^n$ in any (L^p). Since X is $\mathbb{B}_q$-analytic for every q, clearly each $\sum_{n=0}^N \frac{1}{n!}X^n$ is $\mathbb{B}_p$-analytic and this proves the lemma.

□

LEMMA 3.5

$e^{\int_0^t X_r d\mathbb{B}_r}$ and $e^{\int_0^t U_r dr}$ are adapted and $\mathbb{B}_p$-analytic for every p.

PROOF

The adaptedness is trivial. Fix t and put $X = \int_0^t x(s)d\mathbb{B}_r$. By proposition 2.12 X is $\mathbb{B}_q$-analytic for every q. Choose any p. We must prove that $e^{|X|} \in (L^p)$. For simplicity we will replace x and $\mathbb{B}$ by the corresponding real expressions in the rest of the argument. Since X is gaussian, we then have:

$$\mathrm{E}[|X|^{2n}] = \frac{(2n)!\mathrm{E}[|X|^2]}{2^n n!} = \frac{(2n)! \int_0^t x(s)^2 ds}{2^n n!} \tag{3.6}$$

From this we get using Stirlings formula:

$$\begin{aligned} \mathrm{E}[e^{p|X|}] &= \sum_{n=0}^{\infty} \frac{p^n}{n!}\mathrm{E}[|X|^n] \le \sum_{n=0}^{\infty} \frac{p^n}{n!}\left(\mathrm{E}[|X|^{2n}]\right)^{\frac{1}{2}} \\ &= \sum_{n=0}^{\infty} \frac{p^n}{n!}\left(\frac{(2n)! \int_0^t x(s)^2 ds}{2^n n!}\right)^{\frac{1}{2}} < \infty \end{aligned} \tag{3.7}$$

Hence X satisfies the hypothesis of the previous lemma and the conclusion follows. The proof for the expression $e^{\int_0^t U_r dr}$ is similar.

□

We now turn to the proof of theorem 3.4.

PROOF

We first want to prove that the expression in (3.3) equals the expression in (3.4). Since $\int_0^t X_r d\mathbb{B}_r$ and $\int_0^t U_r dr$ are $\mathbb{B}_q$-analytic for every q, we have:

$$\mathrm{Exp}[\int_s^t X_r d\mathbb{B}_r + \int_s^t U_r dr] = e^{\int_s^t X_r d\mathbb{B}_r + \int_s^t U_r dr} \tag{3.8}$$

It then follows by lemma 3.6, theorem 2.10 and proposition 2.11 that:

$$\int_0^t \mathrm{Exp}[\int_s^t X_r d\mathbb{B}_r + \int_s^t U_r dr] \diamond Y_s ds = \int_0^t e^{\int_s^t X_r d\mathbb{B}_r + \int_s^t U_r dr} \cdot Y_s ds \tag{3.9}$$

The same arguments works to prove that all the terms in (3.3) and (3.4) are equal. If we insert the expression (3.4) on the right side of (3.1), we may replace all the the ordinary products by Wick products. Hence the right side of (3.1) equals dZ_t and this completes the proof of theorem 3.4.

□

LEMMA 3.6

Let f be analytic in a neighbourhood of y_0. Then the differential equation:

$$y' = f(y) \qquad y(z_0) = y_0 \tag{3.10}$$

has a unique solution $y = y(z)$ analytic in a neighbourhood of z_0.

PROOF

Local uniqueness follows from the Lipschitz continuity of f at y_0. If $f(y_0) = 0$, then $y(z) = y_0$ is the solution. If on the other hand $f(y_0) \neq 0$, then $f(z) \neq 0$ in a neighbourhood of y_0 and the function $\frac{1}{f(z)}$ is analytic on this neighbourhood. Hence there exists a neighbourhood of y_0 and an analytic function $g(z)$ on this neighbourhood s.t. $g'(z) = \frac{1}{f(z)}$. Since $g'(z) \neq 0$ it follows from the inverse function theorem, see [16] theorem 1.3.7, that g has an inverse function h which is analytic in some neighbourhood of $g(y_0)$. Put $y(z) = h(z - z_0 + g(y_0))$. Then $y(z_0) = y_0$ and:

$$\begin{aligned} y' &= h'(z - z_0 + g(y_0)) = h'(g(h(z - z_0 + g(y_0))) \\ &= \frac{1}{g'(h(z - z_0 + g(y_0))} = f(h(z - z_0 + g(y_0)) = f(y) \end{aligned} \tag{3.11}$$

Hence $y = y(z)$ is analytic in a neighbourhood of z_0 and is a solution of (3.10).

□

THEOREM 3.7

Let f be analytic in a neighbourhood D and let $z_0 \in D$. Then:

$$dZ_t = f^{\diamond}(Z_t)d\mathbb{B}_t \qquad Z_0 = z_0 \tag{3.12}$$

has a unique $(S_{\mathbb{C}})^{-1}$-valued solution Z_t defined for all $t \geq 0$. Moreover there exists a stopping time $\tau > 0$ s.t. $Z_{t\wedge\tau}$ is a local solution to:

$$dZ_t = f(Z_t)d\mathbb{B}_t \qquad Z_0 = z_0 \tag{3.13}$$

PROOF

Uniqueness. Choose and fix $\xi \in S_{\mathbb{C}}(\mathbb{R})$. Then apply the S-transform to both sides of (3.12) to see that $y = S(Z_t)(\xi)$ is a solution to the ODE:

$$y' = f(y)\overline{\xi}(t) \qquad y(0) = z_0 \tag{3.14}$$

Since (3.14) has a unique solution y and the S-transform uniquely characterizes every element in $(\mathcal{S}_{\mathbb{C}})^{-1}$, the solution Z_t of (3.12) is unique.

Existence. First use the previous lemma to find an analytic function $y(z) = \sum_{k=0}^{\infty} a_k z^k$ s.t. y is a solution to the problem:

$$y' = f(y) \qquad y(0) = z_0 \tag{3.15}$$

Since y is analytic in a neighbourhood of the origin, there exists two positive constants $M < \infty, r < \infty$ s.t. $|a_k| \le Mr^k$. The expression:

$$Z_t = \sum_{k=0}^{\infty} a_k \mathbb{B}_t^{\diamond k} = \sum_{k=0}^{\infty} a_k \mathbb{B}_t^k \tag{3.16}$$

then makes sense as an element of $(\mathcal{S}_{\mathbb{C}})^{-1}$. The ordinary chain rule applies, and hence Z_t is a solution of (3.12). Now let τ_d be the first exit time of $\mathbb{B}_t$ from a small neighbourhood of the origin. If we put $Y_t = Z_{t\wedge\tau_d} = \sum_{k=0}^{\infty} a_k \mathbb{B}_{t\wedge\tau_d}^k$, it follows from the complex Ito formula that the ordinary chain rule applies. Y_t is then a local solution to (3.13) in the sense that it solves the problem $Z_{t\wedge\tau_d} = z_0 + \int_0^{t\wedge\tau_d} f(Z_s)d\mathbb{B}_s$.

□

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REFERENCES

1. S.Albeverio, Y.L.Daletsky, Y.G.Kondratiev, and L.Streit (1994). Non-Gaussian Infinite Dimensional Analysis. Preprint nr. 217, University of Bochum, Germany.
2. H.Arai (1986). On the algebra of bounded holomorphic martingales. *Proc.Amer. Math.Soc. 97*, 616-620.
3. F.E.Benth (1993). Integrals in the Hida distribution space $(S)^*$. T. Lindstrøm, B.Øksendal and A.S.Ustunel (editors): *Stochastic Analysis and Related Topics*, Gordon & Breach, 89-99.
4. M.Fukushima and M.Okada (1984). On conformal martingale diffusions and pluripolar sets. *J.Func.Anal. 55*, 377-388.
5. R.K.Getoor and M.J.Sharpe (1972). Conformal martingales. *Invent.Math. 16*, 271-308.
6. H.Gjessing, H.Holden, T.Lindstrøm, B.Øksendal, J.Ubøe and T.-S.Zhang (1993). Wick multiplication, H.Niemi et al. (editors): *Frontiers in Pure and Applied Probability*, TVP Publishers, Moskow, 29-67.
7. T.Hida (1980), *Brownian Motion*, Springer.

8. T.Hida, H.-H.Kuo, J.Potthoff and L.Streit (1993). *White Noise: An infinite dimensional calculus*, Mathematics and its applications 253, Kluwer.
9. H.Holden, T.Lindstrøm, B.Øksendal, J.Ubøe and T.-S.Zhang (1993). Stochastic boundary value problems. A white noise functional approach", *Probability Theory and Related Fields 95*, 391-419.
10. H.Holden, T.Lindstrøm, B.Øksendal, J.Ubøe and T.-S.Zhang (1995). *Stochastic partial differential equations - White Noise Functional Methods, Models and Applications*, book manuscript.
11. H-H.Kuo (1992). Lectures on white noise analysis, *Soochow J.Math.* 18, no.3, 229-300.
12. Y.G.Kondratiev, P.Leukert and L.Streit (1994). Wick Calculus in Gaussian analysis. Preprint nr. 637/6/94, University of Bielefeld, Germany.
13. T. Lindstrøm, B. Øksendal and J. Ubøe (1992). Wick multiplication and Ito-Skorohod stochastic differential equations. In S. Albeverio et al (editors): *Ideas and Methods in Mathematical Analysis, Stochastics, and Applications.* Cambridge Univ. Press, 183-206.
14. B.Øksendal (1992). *Stochastic Differential Equations*, 3 edn., Springer Verlag.
15. J.Potthoff and L.Streit (1991). A characterization of Hida distributions, *J.Func. Anal. 101*, 212-229.
16. W.Rudin (1980). *Function Theory in the Unit Ball of* $\mathbb{C}^n$, Springer.
17. L.Schwartz (1980). Semi-martingales sur des variétés, et martingales conformes sur des variétés analytiques complexes. *Lecture notes in Mathematics No.780*, Springer Verlag.
18. J.Ubøe (1987). Conformal martingales and analytic functions. *Math.Scand. 60*, 292-309.
19. J.Ubøe (1992). Riemann integration on complex brownian paths. *Stoch.Anal.Appl. 10(3)*, 351-361.
20. J.Ubøe (1994). Complex valued, multiparameter stochastic integrals. To appear in *J.Theor.Prob.*.
21. M.Yor (1977). Étude de certains processus (stochastiquement) différentiables ou holomorphes. *Ann.Inst.Henri Poincare XIII No.1*, 1-25.
22. K.Yosida (1978). *Functional Analysis*, Springer.
23. T.-S.Zhang (1992). Characterizations of white noise test functions and Hida distributions, *Stochastics* 41, 71-87.