Semi-parallel pseudo-Riemannian submanifolds with non-null principal normals of extremal dimension

Dedicated to 70-th birthday of Professor Katsumi Nomizu

Introduction

Let $N_s^n(c)$ be the standard model of a pseudo-Riemannian space form of curvature c and of signature $(s, n-s)$. If $c = 0$, $c > 0$ or $c < 0$, then $N_s^n(c)$ is, respectively, $\mathbb{R}_s^n$, the quadric $S_s^n \subset \mathbb{R}_s^{n+1}$ or the quadric $H_s^n \subset \mathbb{R}_{s+1}^{n+1}$ (both with centre in the origin and with some specifications in exceptional cases; see [34], Sect. 2.4). If $c \neq 0$, this $N_s^n(c)$ can be considered also by its projective model. The projectivization of $\mathbb{R}^{n+1}$ gives $P^n(\mathbb{R})$ and then the asymptotic cone of the quadric $N_s^n(c) \subset \mathbb{R}^{n+1}$ gives the absolute quadric $Q_s^{n-1} \subset P^n(\mathbb{R})$, which determines the projective metric of curvature c. Two vectors of $\mathbb{R}_s^{n+1}$ or $\mathbb{R}_{s+1}^{n+1}$ are orthogonal iff the corresponding points of $P^n(\mathbb{R})$ are polar with respect to Q_s^{n-1}. The m-dimensional geodesic submanifolds of $N_s^n(c)$ give the m-planes of $P^n(\mathbb{R})$; every hyperplane has its polus with respect to Q_s^{n-1}.

Let an orbit of a Lie group, acting in $N_s^n(c)$ by isometries, be pseudo-Riemannian (intrinsically). It is *symmetric* (extrinsically, i.e. with respect to reflections of $N_s^n(c)$ in its normal subspaces) iff $\overline{\nabla} h = 0$, where h is the second fundamental form and $\overline{\nabla}$ is the van der Waerden–Bortolotti connection (see [9] for $s = c = 0$, [1] for $s = 0$, $c \neq 0$ and [2] for $s > 0$; also [28]). A pseudo-Riemannian submanifold M^m of $N_s^n(c)$, satisfying $\overline{\nabla} h = 0$ (i.e having parallel h) is called *parallel* (see [34], [23] for $s = 0$ and [2], [24] for $s > 0$); it is a symmetric orbit or its open part.

A pseudo-Riemannian submanifold M^m of $N_s^n(c)$, satisfying the integrability condition $\overline{R} \circ h = 0$ of the system $\overline{\nabla} h = 0$, is called *semi-parallel* (see [4] - [8] for $s = 0$), also *semi-symmetric* (extrinsically, see [12] - [16] for $s = 0$).

It is shown [14] for the case $s = 0$ that every of them is a 2nd order envelope of the symmetric orbits (see also [23]). This result can be easily generalized to the case of semi–parallel pseudo–Riemannian M^m of $N^n_s(c)$. Intrinsically they are semi–symmetric pseudo–Riemannian manifolds (in the sense that $R \circ R = 0$; the Riemannian case see [27], Ch.II, §3., the pseudo–Riemannian (Lorentz) case see [11]).

In the last case the following *Nomizu problem* is known: by which conditions a semi–symmetric Riemannian space reduces to a symmetric one. The conjecture [25] that completeness and irreducibility are sufficient for this was refuted in [26] and [31] by using the semi–symmetric hypersurfaces (see also [33] for the pseudo–Riemannian case). The first classification of semi–symmetric Riemannian spaces is given in [29], [30].

All symmetric orbits in the case $s = 0$, $c = 0$ are classified by Ferus [9]. They are standardly imbedded irreducible symmetric R-spaces and their products. This result is extended to the case $s = 0$, $c \neq 0$ in [32]. By means of the Jordan triple systems symmetric orbits are investigated in [1] (the case $s = 0$, $c \neq 0$).

By $s > 0$ some special classes of parallel submanifolds in $\mathbb{R}^n_1$ and $\mathbb{R}^n_2$ are described by Magid [21]. The Ferus's classification of complete parallel submanifolds in $\mathbb{R}^n$ by means of symmetric R–spaces is generalized to the case of pseudo–Riemannian parallel submanifold M^m_r in $N^n_s(c)$, $s > 0$, by Blomstrom [2] and Naitoh [24].

A complete classification of semi–parallel submanifolds M^m of $N^n_s(c)$ is not known yet, even by $s = 0$. Only the following cases by $s = 0$ are investigated: surfaces ($m = 2$) if $c = 0$ [4], or if $c > 0$ [23], three–dimensional submanifolds if $c = 0$ [15], hypersurfaces (i.e. $m = n - 1$) if $c = 0$ [5] (a special class [22]), or if $c \neq 0$ [7], two–codimensional submanifolds (i.e. $m = n - 2$) if $c = 0$ [12], submanifolds M^m with flat normal connection if $c = 0$ [16], [18], or if $c \neq 0$ [8]. First surveys are given: [6] (until 1989), [16] (until 1991).

A particular problem is so called *modified Nomizu problem*: by which conditions a semi–symmetric submanifold M^m of $N^n_s(c)$, as a 2nd order envelope of symmetric orbits, reduces to a single symmetric orbit (or its part)? This problem for $s = 0$, $c = 0$ and for some special types of symmetric orbits is investigated in [17] – [20].

In the present paper two extreme classes of semi–parallel pseudo–Riemannian submanifolds M^m of are $N^n_s(c)$ are investigated by arbitrary s and c.

If $c \neq 0$, the second fundamental form h of a such M^m has values, normal to M^m in the tangent space of $N_s^n(c)$, but there exists also h^* of M^m with respect to $\mathbb{R}_s^{n+1}$ (if $c > 0$) or $\mathbb{R}_{s+1}^{n+1}$ (if $c < 0$). The vector subspace, spanned on values of h at a point $x \in M^m$, is called the principal normal of M^m at x. The same for h^* is called the outer principal normal of M^m at x. Note that if $c = 0$ it coincides with the previous one.

The two classes, investigated below, are characterized by the condition that the principal normals of M^m have extremal non–zero dimension, i.e. either (I) the minimal dimension 1 or (II) the maximal dimension $\frac{1}{2}m(m+1)$, and by some complementary conditions. One of the latter is that if $s \neq 0$ then in the case (I) the principal normals and in the case (II) the outer principal normals must be non–null vector subspaces (in the sence that every of them contains a vector with non–zero scalar square).

For the case (I) the main result is

Theorem A. *Let M^m, $m \geqslant 2$, be a pseudo–Riemannian submanifold of $N_s^n(c)$ whose principal normals are non–null 1-dimensional vector subspaces. Then either*

(1) M^m has rank 1 (i.e. its tangent m-planes constitute a 1-dimensional submanifold (a curve) of the Grassmann manifold of all m-planes), or

(2) M^m is a hypersurface, i.e. lies in a $(m+1)$-plane.

In the case (1) a such M^m is semi–parallel only if $c = 0$. Conversaly, every M^m of $N_s^n(0) = \mathbb{R}_s^n$, satisfying the assumptions and having rank 1, has flat $\overline{\nabla}$ and thus is semi–parallel.

In the case (2) and not (1), if the hypersurface M^m is semi–parallel and its second fundamental tensor h is diagonalizable simultaneously with its metric tensor g, then either

(2a) $m = 2$ and M^2 is flat, or

(2b) M^m is parallel; more exactly, a totally umbilic hypersurface or an extrinsic product of a totally umbilic submanifold (i) with a plane (if $c = 0$, i.e. a cylindric hypersurface), or (ii) with another such submanifold (if $c \neq 0$), or

(2c) a rotation hypersurface with a 1-dimensional axis, whose profile curve is (i) a straight line, intersecting the axis (if $c = 0$, i.e. M^m is a rotation hypercone),

or (ii) a helix–curve in a 2-plane, containing the axis, whose directing point is the polus of the axis (if $c \neq 0$).

Here a helix–curve with a directing point q means a such line on $N_s^2(c)$ of $\mathbb{R}_s^3$ (if $c > 0$) or of $\mathbb{R}_{s+1}^3$ (if $c < 0$), whose tangent vector t, such that $\langle t, t\rangle = \varepsilon (= 1$ or $= -1)$ satisfies $\langle t, q_0\rangle =$const, where q_0 is a constant vector, determining the point q (for the case $s = 0$ see [7]).

Remarks. 1. In the case $s = 0$ the matrices of h and g are always simultaneously diagonalizable . The part of the Theorem A, concerning hypersurfaces, is proved for this case in [5] ($c = 0$) and in [7] ($c \neq 0$). The notion of helix–curve is introduced in [7] for the case $s = 0$ to describe the profile curve; also the explicit parametric equations of a helix–curve are given on $S^2(c)$ and on $H^2(c)$ (separately for $\langle q, q\rangle > 0$ and for $\langle q, q\rangle < 0$).

2. If $s \neq 0$, the matrices h and g can be non–diagonalizable simultaneously. This case needs a separate study; the first steps for parallel submanifolds by $c = 0$ are made in [21].

3. The assumptions of the Theorem A yield the flatness of the normal connection $\nabla^\perp$ of M^m. Replacing "1-dimensional" by "2-dimensional" one obtains the same consequence: $\nabla^\perp$ is flat (see [12] for the case $s = 0$, $c = 0$; generalization is obvious; in [12] also a classification is given). Recall that by $s = 0$ all semi–parallel M^m of $N^n(c)$ with flat $\nabla^\perp$ are described in [16], [18] (case $c = 0$) and in [8] (case $c \neq 0$).

The most part of the present paper deals with the case (II). Then the normal connection of M^m is surely non–flat.

Theorem B. *Let M^m, $m \geqslant 2$, be a semiparallel pseudo–Riemannian submanifold of $N_s^n(c)$, whose*

(i) principal normals are $\frac{1}{2}m(m+1)$-dimensional vector subspaces,

(ii) outer principal normals are non–null vector subspaces,

(iii) field of principal normals is parallel with respect to the normal connection.

Then M^m is a parallel submanifold of a $\frac{1}{2}m(m+3)$-plane of $N_s^n(c)$.

From the proof of this theorem, given below in §5, some geometric and group theoretic consequences are made, which show that the considered M^m is an indefinite version of the well-known Veronese submanifold.

The essential part of the Theorem B in a special case $s = 0$, $c = 0$ is previously proved in [13].

Note that some statements follow from *(i)* and *(ii)* only, e.g. the fact, that a such M^m, $m \geqslant 3$, is intrinsically a pseudo-Riemannian manifold of constant curvature, also the inequality $\sigma(m - \sigma + 1) \leqslant s$ for its signature $(\sigma, m - \sigma)$. Note also that Theorem B gives a particular solution of the modified Nomizu problem. The most important role plays here the condition *(iii)*. If to remove this condition, the assertion does not hold, as shows the following theorem.

Theorem C. *In $N_s^n(c)$ with $n = \frac{1}{2}m(m+3)+1$ there exist semi-parallel but non-parallel pseudo-Riemannian submanifolds M^m, $m \geqslant 2$, which satisfy the conditions (i) and (ii). If $m \geqslant 3$ (or $m = 2$ and a such M^2 has constant Gaussian curvature), then each such M^m is the second order envelope of a 1-parametric family of mutually congruent parallel submanifolds of the Theorem B; characteristics are mutually congruent $(m-1)$-dimensional parallel submanifolds of the Theorem B (in particular, plane lines of constant curvature, if $m = 2$).*

Every submanifold of this family, as a m-dimensional Veronese submanifold (indefinite version, in general), lies in its outer osculating space, spanned on the tangent space and outer principal normal, and has a centre in this space. All these centres form the *central curve* of M^m.

The proof of the Theorem C, given below in §6, is complemented by a proposition showing that this central curve has a constant non-zero first curvature and non-zero second curvature. A special case is indicated, when all curvatures of the central curve are constants.

Remarks 1. In the classification of semi-parallel surfaces M^2 of $N_s^n(c)$ $(s = 0)$ given for $c = 0$ in [4] and for $c \neq 0$ in [23], the most general type is characterized as an isotropic immersion with nonflat $\nabla^\perp$ and with $||H||^2 = 3K - c$, where H is the mean curvature vector and K is the Gaussian curvature. Theorem C by $m = 2$ and $K =$const gives the existence of non-parallel among such M^2 and also its geometric construction in the case $n = 6$; these problems were left open in [4] and [23].

2. In its turn the present paper leaves open the problem of existence and geometric description of semi–parallel but non–parallel pseudo–Riemannian M^m, $m \geqslant 2$, of $N_s^n(c)$, $n > \frac{1}{2}m(m+3)+1$, more general then in the case of the Theorem C. Also the same problem by $m = 2$, non–constant K and $n = 6$ is open yet.

3. Theorem C in a special case $s = 0$, $c = 0$ for general m is previously proved in [17], as well as the proposition, concerning the central curve.

§1. Preliminaries

1.1. Standard models of space forms. Let $N_s^n(c)$ be a n-dimensional real pseudo–Riemannian manifold of constant curvature c and signature $(s, n-s)$; the last condition means, that

$$ds^2 = -\sum_{\sigma=1}^{s} \omega^\sigma \otimes \omega^\sigma + \sum_{\nu=s+1}^{n} \omega^\nu \otimes \omega^\nu \tag{1.1.1}$$

with respect to the orthonormal frame bundle $O(N_s^n(c))$. It is known (see [34], Theorem 2.4.9) that a complete connected simply connected $N_s^n(c)$ is isometric to one of the following standard models:

$$\begin{aligned}
&\mathbb{R}_s^n, \quad \text{if} \quad c = 0;\\
&S_s^n(c) = \{x \in \mathbb{R}_s^{n+1} : \langle x, x\rangle = c^{-1}\}, \quad \text{if} \quad c > 0;\\
&H_s^n(c) = \{x \in \mathbb{R}_{s+1}^{n+1} : \langle x, x\rangle = c^{-1}\}, \quad \text{if} \quad c < 0;
\end{aligned}$$

except the cases (1) $c > 0$, $s = n-1$ or $c < 0$, $s = 1$ and (2) $c > 0$, $s = n$ or $c < 0$, $s = 0$ when $S_s^n(c)$ or $H_s^n(c)$, respectively, must be replaced (1) by universal coverings $\tilde{S}_{n-1}^n(c)$ or $\tilde{H}_1^n(c)$ and (2) by the component $\tilde{S}_n^n(c)$ of $(0, \ldots, 0, 1)$ or the component $\tilde{H}_0^n(c)$ of $(1, 0, \ldots, 0)$.

Further $N_s^n(c)$ denotes one of these standard models. In particular, $\mathbb{R}_1^n$ (also $\mathbb{R}_{n-1}^n$), $S_1^n(c)$ (also $\tilde{S}_{n-1}^n(c)$) and $\tilde{H}_1^n(c)$ (also $H_{n-1}^n(c)$) are called, respectively, Minkowski spacetime, de Sitter spacetime of the first kind and de Sitter spacetime of the second kind [10]; the latter by odd n with standard complex structure is called also anti–de Sitter spacetime [33].

1.2. Frame bundles and derivation formulae. Let $\mathbb{R}^n_s$ be included into $\mathbb{R}^{n+1}_s$ as a hyperplane $x^{n+1} = 0$. Then every standard model $N^n_s(c)$ is determined by a special submanifold of $\mathbb{R}^{n+1}_{s'}$ ($s' = s$ if $c \geqslant 0$, $s' = s+1$ if $c < 0$).

Let $\{x; e_1, \ldots, e_n, e_{n+1}\}$ be a moving frame in $\mathbb{R}^{n+1}_{s'}$, i.e. an element of the frame bundle $L(\mathbb{R}^{n+1}_{s'})$. The derivation formulae

$$dx = e_A \omega^A, \quad de_A = e_B \omega^B_A \tag{1.2.1}$$

$(A, B, \ldots = 1, \ldots, n+1)$ hold and yield the structural equations

$$d\omega^A = \omega^B \wedge \omega^A_B, \quad d\omega^B_A = \omega^C_A \wedge \omega^B_C; \tag{1.2.2}$$

moreover,

$$dg_{AB} = g_{CB}\omega^C_A + g_{AC}\omega^C_B \tag{1.2.3}$$

hold, where $g_{AB} = \langle e_A, e_B \rangle$.

For standard model $N^n_s(c)$, $c \neq 0$, let $x \in N^n_s(c)$, $e_{n+1} = -\sqrt{|c|} \cdot x$, $g_{I,n+1} = 0$ $(I, J, \ldots = 1, \ldots, n)$. Then $g_{n+1,n+1} = |c|c^{-1} = \operatorname{sgn} c$,

$$\omega^{n+1} = 0, \qquad \omega^{n+1}_{n+1} = 0 \tag{1.2.4}$$

$$\omega^I_{n+1} = -\sqrt{|c|}\omega^I, \qquad \omega^{n+1}_I = \operatorname{sgn} c\sqrt{|c|} g_{IJ}\omega^J, \tag{1.2.5}$$

thus for the subbundle $L(N^n_s(c))$ of such frames

$$dx = e_I\omega^I, \qquad de_I = e_J\omega^J_I - xcg_{IJ}\omega^J, \tag{1.2.6}$$

$$d\omega^I = \omega^J \wedge \omega^I_J, \qquad d\omega^J_I = \omega^K_I \wedge \omega^J_K + cg_{IK}\omega^J \wedge \omega^K \tag{1.2.7}$$

$$dg_{IJ} = g_{KJ}\omega^K_I + g_{IK}\omega^K_J \tag{1.2.8}$$

and $\langle dx, dx \rangle = g_{IJ}\omega^J \otimes \omega^J$ has the canonical form (1.1.1).

For the standard model $N^n_s(0) = \mathbb{R}^n_s \subset \mathbb{R}^{n+1}_s$ one must take $\omega^{n+1} = 0$, $e_{n+1} =$const, $g_{I,n+1} = 0$ in (1.2.1) - (1.2.3) and this leads to the particular case $c = 0$ of (1.2.6) - (1.2.8). So the last formulae are universal for all $N^n_s(c)$.

1.3. Adapted frame bundle of a submanifold. Let M^m be a m-dimensional submanifold of $N^n_s(c)$ with tangent vector space T_xM^m at the point

$x \in M^m$ (considered as the vector subspace of $T_x N_s^n(c)$). The frame bundle $L(\mathbb{R}_{s'}^{n+1})$ can be reduced to the subbundle of frames with $x \in M^m$, $e_i \in T_x M^m$ $(i, j, \ldots = 1, \ldots, m)$. Then $\omega^\varrho = 0$ $(\varrho, \sigma, \ldots = m+1, \ldots, n+1)$ and hence

$$\omega_i^\varrho = h_{ij}^\varrho \omega^j \tag{1.3.1}$$

due to (1.2.2) and Cartan lemma, where $h_{ij}^\varrho = h_{ji}^\varrho$. In particular, for subbundle $L(N_s^n(c))$

$$\omega_i^{n+1} = \operatorname{sgn} c \sqrt{|c|} g_{ij} \omega^j \tag{1.3.2}$$

from (1.2.5).

Let M^m be pseudo–Riemannian, i.e. let the inner metric of M^m, included in this subbundle by $ds^2 = g_{ij}\omega^i \otimes \omega^j$, be regular in the sense that $\det(g_{ij}) \neq 0$. Then $T_x M^m$ has in $T_x N_s^n(c)$ the $(n-m)$-dimensional orthogonal complement $T_x^\perp M^m$. Taking $e_\alpha \in T_x^\perp M^m$ one get $g_{i\alpha} = 0$, $\det(g_{\alpha\beta}) \neq 0$, $(\alpha, \beta, \ldots = m+1, \ldots, n)$ and

$$g_{ij}\omega_\alpha^j + g_{\alpha\beta}\omega_i^\beta = 0, \quad \nabla g_{ij} = 0, \quad \nabla^\perp g_{\alpha\beta} = 0 \tag{1.3.3}$$

where ∇ is the Levi–Civita connection of M^m with $gl(m, \mathbb{R})$-valued connection form $\omega = (\omega_i^j)$, i.e. $\nabla g_{ij} = dg_{ij} - g_{kj}\omega_i^k - g_{ik}\omega_j^k$, and $\nabla^\perp$is the normal connection of M^m with $gl(n-m, \mathbb{R})$-valued connection form $\omega^\perp = (\omega_\alpha^\beta)$.

For the bundle $L_{ad}(M^m, N_s^n(c))$ of frames, adapted to pseudo–Riemannian M^m of $N_s^n(c)$ so that $x \in M^m$, $e_i \in T_x M^m$, $e_\alpha \in T_x^\perp M^m$, there hold

$$dx = e_i\omega^i, \qquad de_i = e_j\omega_i^j + h_{ij}^* \omega^j, \tag{1.3.4}$$

where

$$h_{ij}^* = h_{ij} - cxg_{ij}, \qquad h_{ij} = e_\alpha h_{ij}^\alpha. \tag{1.3.5}$$

The vector subspace span$\{h_{ij}\}$ of $T_x^\perp M^m$ is the *principal normal* of the submanifold M^m of $N_s^n(c)$, the span$\{h_{ij}^*\}$ is called the *outer* principal normal of M^m. The quadratic form $h_{ij}X^iX^j$ (or $h_{ij}^*X^iX^j$) on T_xM^m with values in this normal is the *second fundamental form* of M^m with respect to $N_s^n(c)$ (or $\mathbb{R}_{s'}^{n+1}$), denoted by h (or by h^*).

1.4. Covariant differentials and curvature 2–forms. The relations (1.3.1) yield by exterior differentiation, due to (1.3.2) and (1.3.3),

$$\overline{\nabla} h_{ij}^\alpha \wedge \omega^j = 0, \tag{1.4.1}$$

where

$$\overline{\nabla} h^{\alpha}_{ij} = dh^{\alpha}_{ij} - h^{\alpha}_{kj}\omega^{k}_{i} - h^{\alpha}_{ik}\omega^{k}_{j} + h^{\beta}_{ij}\omega^{\alpha}_{\beta} \tag{1.4.2}$$

are the components of $\overline{\nabla} h$ (and also of $\overline{\nabla} h^{*}$, because $\overline{\nabla} h^{n+1}_{ij} = 0$ due to (1.3.2) and (1.3.3)), called the covariant differential of h (and of h^{*}) with respect to the van der Waerden–Bortolotti connection $\overline{\nabla} = (\nabla, \nabla^{\perp})$ of M^m. The last is a connection in $L_{ad}(M^m, N^n_s(c))$.

Due to the Cartan lemma from (1.4.1) it follows

$$\overline{\nabla} h^{\alpha}_{ij} = h^{\alpha}_{ijk}\omega^{k}, \quad h^{\alpha}_{ijk} = h^{\alpha}_{ikj} \tag{1.4.3}$$

and the next differentiation leads to

$$\overline{\nabla} h^{\alpha}_{ijk} \wedge \omega^{k} = -h^{\alpha}_{kj}\Omega^{k}_{i} - h^{\alpha}_{ik}\Omega^{k}_{j} + h^{\beta}_{ij}\Omega^{\alpha}_{\beta}, \tag{1.4.4}$$

where

$$\Omega^{j}_{i} = d\omega^{j}_{i} - \omega^{k}_{i} \wedge \omega^{j}_{k}, \quad \Omega^{\alpha}_{\beta} = d\omega^{\alpha}_{\beta} - \omega^{\gamma}_{\beta} \wedge \omega^{\alpha}_{\gamma} \tag{1.4.5}$$

are the curvature 2-forms of $\overline{\nabla}$, the first ones of ∇ and the second ones of $\nabla^{\perp}$.

Using (1.2.7) and (1.3.1) – (1.3.3) one obtains

$$\Omega^{j}_{i} = -R^{j}_{i,pq}\omega^{p} \wedge \omega^{q}, \quad \Omega^{\alpha}_{\beta} = -R^{\alpha}_{\beta,pq}\omega^{p} \wedge \omega^{q}, \tag{1.4.6}$$

where

$$R^{j}_{i,pq} = g^{jk}\langle h^{*}_{i[p} h^{*}_{q]k}\rangle = g^{jk}(\langle h_{i[p} h_{q]k}\rangle + c g_{i[p} g_{q]k}), \tag{1.4.7}$$

$$R^{\alpha}_{\beta,pq} = g_{\beta\gamma} h^{\gamma}_{i[p} h^{\alpha}_{q]j} g^{ij} \tag{1.4.8}$$

are the components of the curvature tensors: R of ∇ and $R^{\perp}$ of $\nabla^{\perp}$.

1.5. Fundamental identities. In (1.4.3) $h^{\alpha}_{ijk} = \overline{\nabla}_k h^{\alpha}_{ij}$ are the components of the covariant derivative of h, which satisfy $\overline{\nabla}_k h^{\alpha}_{ij} = \overline{\nabla}_j h \alpha_{ik}$, the classical *Peterson–Codazzi identity.* If to substitute (1.4.6) into (1.4.4) and then to use Cartan lemma one obtains $\overline{\nabla} h^{\alpha}_{ijk} = \overline{\nabla}_l h^{\alpha}_{ijk}\omega^{l}$; thus

$$\overline{\nabla}_{[k}\overline{\nabla}_{l]} h^{\alpha}_{ij} = R^{p}_{i,kl} h^{\alpha}_{pj} + R^{p}_{j,kl} h^{\alpha}_{ip} - R^{\alpha}_{\beta,kl} h^{\beta}_{ij}. \tag{1.5.1}$$

It is the *Ricci identity* for h; its right hand side gives the components of $\overline{R} \circ h$ (see Introduction).

The relations (1.4.5) yield by exterior differentiation

$$d\Omega_i^j = \omega_j^k \wedge \Omega_k^i - \Omega_i^k \wedge \omega_k^j, \quad d\Omega_\beta^\alpha = \omega_\beta^\gamma \wedge \Omega_\gamma^\beta - \Omega_\beta^\gamma \wedge \omega_\gamma^\beta, \tag{1.5.2}$$

the *Bianchi identities*. Doing the same with $\nabla g_{ij} = 0$, $\nabla^\perp g_{\alpha\beta} = 0$ one obtains

$$g_{kj}\Omega_i^k + g_{ik}\Omega_j^k = 0, \quad g_{\gamma\beta}\Omega_\alpha^\gamma + g_{\alpha\gamma}\Omega_\beta^\gamma. \tag{1.5.3}$$

§2. Parallel (or extrinsically symmetric) submanifolds

A pseudo–Riemannian submanifold M^m of $N_s^n(c)$ is said to be *parallel* (or extrinsically *symmetric*) if its second fundamental form h is parallel with respect to $\overline{\nabla}$, i.e. if $\overline{\nabla} h = 0$ or, componentwise, if $\overline{\nabla} h_{ij}^\alpha = 0$.

Lemma. *Let M^m be a pseudo–Riemannian submanifold of $N_s^n(c)$, for $c \neq 0$ considered as the standard model in $\mathbb{R}_{s'}^{n+1}$ ($s' = s$ if $c > 0$, $s' = s + 1$ if $c < 0$).*

The following conditions are equivalent:

1) M^m is parallel,

2) M^m is parallel as a submanifold of $\mathbb{R}_{s'}^{n+1}$,

3) $(\nabla h_{ij})_x \in T_x M^m$ at every point $x \in M^m$,

4) $(\nabla h_{ij}^*)_x \in T_x M^m$ at every point $x \in M^m$,

where h_{ij} and h_{ij}^* are given by (1.3.5) *and ∇ works as in* (1.3.3).

Proof. Equivalence of 1) and 2) follows from the remark by (1.4.2) (cf. [32], Lemma 1, for the case $s = 0$). Further, (1.3.5) yield

$$\nabla h_{ij} = -e_k g^{kl} \langle h_{ij}, h_{lp} \rangle \omega^p + e_\alpha \overline{\nabla} h_{ij}^\alpha, \tag{2.1}$$

$$\nabla h_{ij}^* = -e_k g^{kl} \langle h_{ij}^*, h_{lp}^* \rangle \omega^p + e_\alpha \overline{\nabla} h_{ij}^\alpha, \tag{2.2}$$

thus 3) and 4) are equivalent to 1). ■

§3. Semi–parallel (or extrinsically semi–symmetric) submanifolds

The integrability condition of $\overline{\nabla} h^{\alpha}_{ij} = 0$ is due to (1.4.3), (1.4.4), (1.5.1)

$$h^{\alpha}_{kj}\Omega^{k}_{i} + h^{\alpha}_{ik}\Omega^{k}_{j} - h^{\beta}_{ij}\Omega^{\alpha}_{\beta} = 0 \tag{3.1}$$

or

$$R^{p}_{i,kl}h^{\alpha}_{pj} + R^{p}_{j,kl}h^{\alpha}_{pj} - R^{\alpha}_{\beta,kl}h^{\beta}_{ij} = 0; \tag{3.2}$$

shortly, $R \circ h = 0$.

A pseudo–Riemannian submanifold M^m of $N^n_s(c)$, satisfying this condition, is said to be *semi–parallel* (or extrinsically *semi–symmetric*).

They constitute a subclass of the class of intrinsically semi–symmetric submanifolds, characterized by $R \circ R = 0$ (see [4]), but this subclass, for its part, contains all M^m with flat $\overline{\nabla}$ (i.e. $\overline{R} = 0$), in particular all lines ($m = 1$), and all 2-parallel M^m (i.e. with $\overline{\nabla}^2 h = \overline{\nabla}\,\overline{\nabla} h = 0$, due to (1.5.1); see [16]).

If to substitute (1.4.7) and (1.4.8) into (3.1) one can see that semi–parallel pseudo–Riemannian submanifolds M^m of $N^n_s(c)$ are charactrized by the following system of the algebraic equations of third order on the components of h:

$$\begin{aligned}\sum g^{kl}[h^{\alpha}_{kj}\langle h_{i[p}, h_{q]l}\rangle + h^{\alpha}_{ik}\langle h_{j[p}, h_{q]l}\rangle - \langle h_{ij}, h_{k[p,}\rangle h^{\alpha}_{q]l} =\\ = c[h^{\alpha}_{p(i}g_{j)q} - h^{\alpha}_{q(i}g_{j)p}].\end{aligned} \tag{3.3}$$

Thus the semi–parallelity condition is a pointwise algebraic condition.

§4. Proof of the Theorem A

4.1 Proof of the first assertion. Let span$\{h_{ij}\}$ be a 1-dimensional non–null vector subspace of $T^{\perp}_{x}M^m$. The vector e_{m+1} with $\langle e_{m+1}, e_{m+1}\rangle = \varepsilon = \pm 1$ can be taken so that $h_{ij} = h_{ij}^{m+1} e_{m+1}$, $\langle e_\varrho, e_{m+1}\rangle = 0$ $(\varrho, \sigma, \ldots = m+2, \ldots, n)$; then $h_{ij}^\varrho = 0$. Since $h_{ij}^{m+1} = h_{ji}^{m+1}$, the frame vectors in $T_x M^m$ can be taken so that $h_{ij}^{m+1} = \lambda_{(i)} \delta_{ij}$. This yields

$$\omega_i^{m+1} = \lambda_{(i)} \omega^i, \quad \omega_i^\varrho = 0. \tag{4.1.1}$$

The last relations by exterior differentiation imply $\lambda_{(i)} \omega^i \wedge \omega_{m+1}^\varrho = 0$, thus

$$\lambda_{(i)} \omega_{m+1}^\varrho = l_{(i)}^\varrho \omega^i. \tag{4.1.2}$$

Let $\operatorname{rank}(h_{ij}^{m+1}) = 1$, i.e. among $\lambda_{(i)}$ there exists only one non–zero, say $\lambda_{(1)}$, and $\lambda_{(a)} = 0$ $(a, b, \ldots = 2, \ldots, m)$. Then $\omega_a^{m+1} = 0$ and by exterior differentiation $\omega_a^1 \wedge \lambda_{(1)} \omega^1 = 0$, hence $\omega_a^1 = \varphi_a \omega^1$. Now $d\omega^1 = \omega^1 \wedge \omega_1^1 + \omega^a \wedge \varphi_a \omega^1$ shows that $\omega^1 = 0$ determines a foliation on M^m. For its leaves (1.3.4) give

$$dx = e_a \omega^a, \quad de_a = e_b \omega_a^b - cx g_{ab} \omega^b, \tag{4.1.3}$$

$$de_1 = e_1 \omega_1^1 + e_a \omega_1^a - cx g_{1a} \omega^a, \tag{4.1.4}$$

therefore these leaves are $(m-1)$-planes and the tangent m-plane of M^m is the same at the points of every of them. This gives the case (1).

Let $\operatorname{rank}(h_{ij}^{m+1}) > 1$, i.e. there are two non–zero among $\lambda_{(i)}$, say $\lambda_{(1)} \neq 0$, $\lambda_{(2)} \neq 0$. Then from (4.1.2) it follows that every ω_{m+1}^ϱ is proportional to ω^1 and to ω^2, hence $\omega_{m+1}^\varrho = 0$. Due to (1.3.4)

$$dx = e_i \omega^i, \quad de_i = e_j \omega_i^j + e_{m+1} \lambda_{(i)} \omega^i - cx g_{ij} \omega^j,$$

$$de_{m+1} = e_i \omega_{m+1}^i + e_{m+1} \omega_{m+1}^{m+1}$$

and this yields the assertion (2). (Note that $\langle e_{m+1}, e_{m+1}\rangle$=const is not needed yet.)

4.2. Semi–parallelity in the case (1). Let M^m be as the case (1). Since among h_{ij}^{m+1} the only non–zero is $h_{11}^{m+1} = \lambda_{(1)}$, the vectors e_a, which determine the generator $(m-1)$-plane of M^m, are not restricted and can be taken so that (g_{ab}) has the canonical form: $g_{ab} = \gamma_a \delta_{ab}$. Among γ_a only one can be zero, because $\det(g_{ij}) \neq 0$, and $e_1, e_2, \ldots, e_m$ can be chosen so that either

1) $g_{ij} = \varepsilon_i \delta_{ij}$, $\varepsilon_i = \pm 1$, or

2) $g_{11} = g_{mm} = g_{1p} = g_{mp} = 0$, $g_{1m} = 1$, $g_{pq} = \varepsilon_p \delta_{pq}$, where $p, q, \ldots =$
$= 2, \ldots, m-1$ and $\varepsilon_p = \pm 1$.

In the first case (1.4.6), (1.4.7) give, in particular, that $\Omega^1_a = c\varepsilon_a \omega^a \wedge \omega^1$. If now M^m is semi–parallel then (3.1) by $i = 1$, $j = a$, $\alpha = m+1$ yields $\lambda_{(1)} c\omega^a \wedge \omega^1 = 0$, thus $c = 0$.

In the second case (1.4.6), (1.4.7) imply $\Omega^1_1 = c\omega^m \wedge \omega^1$. Also, if M^m is semi–parallel then (3.1) by $i = j = 1$, $\alpha = m+1$ yields $\lambda_{(1)} c\omega^m \wedge \omega^1 = 0$, thus $c = 0$ again.

Conversely, if $c = 0$, the light calculation shows that $\Omega^j_i = \Omega^\beta_\alpha = 0$, thus $\overline{\nabla}$ is flat and M^m is semi–parallel.

*4.3. **Semi–parallelity in the case*** (2). Let M^m be as in the case (2), i.e. a hypersurface, and let h^{m+1}_{ij} be diagonalizable simultaneously with g_{ij}. Then (4.1.1) can be obtained so that $g_{ij} = \varepsilon_i \delta_{ij}$, $\varepsilon_i = \pm 1$. Now $\Omega^j_i = -(\varepsilon\varepsilon_j \lambda_{(i)} \lambda_{(j)} + c\varepsilon_i)\omega^i \wedge \omega^j$. If M^m is semi–parallel then (3.1) yields

$$(k_j - k_i)(k_i k_j + \varepsilon c) = 0$$

where $k_i = \varepsilon_i \lambda_{(i)}$. Here three different values, say k_1, k_2 and k_3, are impossible , as is easy to see. So let

$$k_1 = \ldots = k_u = k, \quad k_{u+1} = \ldots = k_m = \tilde{k},$$

$k = \varepsilon_1 \lambda_{(1)} \neq 0$, $k \neq \tilde{k}$, $k\tilde{k} + \varepsilon c = 0$. Now $h_{ab} = kg_{ab} = \varepsilon_a k \delta_{ab}$, $h_{ap} = 0$, $h_{pq} =$
$= \tilde{k} g_{pq} = \varepsilon_p \tilde{k} \delta_{pq}$, where $a, b, \ldots = 1, \ldots, u$; $p, q, \ldots = u+1, \ldots, m$ and $h_{ij} = h^{m+1}_{ij}$.
From (1.4.3) it follows that

$$\varepsilon_a \delta_{ab} dk = h_{abd}\omega^d + h_{abp}\omega^p, \tag{4.3.1}$$

$$\varepsilon_p (k - \tilde{k})\omega^p_a = h_{apb}\omega^b + h_{apq}\omega^q, \tag{4.3.2}$$

$$\varepsilon_p \delta_{pq} d\tilde{k} = h_{pqa}\omega^a + h_{pqr}\omega^r \tag{4.3.3}$$

with symmetric h_{ijk}.

The case $m = 2$, $u = 1$ gives (2a).

The case $u = m \geqslant 2$ and the case $u \geqslant 2$, $m-u \geqslant 2$ give (2b). Indeed, if $u = m$, there is no $p, q, \ldots$ and (4.3.1) by $a \neq b$ give that only h_{aaa} can be non–zero, but by $a = b$ they give that dk must be proportional to ω^1, to ω^2 etc., i.e. $dk = 0$ and M^m is parallel (totally umbilic). If $u \geqslant 2$, $m - u \geqslant 2$, then in the same way $dk = \varkappa_p \omega^p$, $d\tilde{k} = \tilde{\varkappa}_a \omega^a$ and now $k\tilde{k} + \varepsilon c = 0$ gives $\tilde{k}\varkappa_p \omega^p + k\tilde{\varkappa}_a \omega^a = 0$, thus $\varkappa_p = \tilde{\varkappa}_a = 0$ and $\omega_a^p = 0$. Hence M^m is parallel (extrinsic product of its submanifolds, determined by $\omega^q = 0$ and $\omega^b = 0$; cf. [32]). If $c = 0$, then $\tilde{k} = 0$.

It remains to consider the cases $u = 1$, $m - u \geqslant 2$ and $u = m - 1 \geqslant 2$.

If $c = 0$ and thus $\tilde{k} = 0$, only the second case gives $\operatorname{rank}(h_{ij}) > 1$. The system (4.3.1) – (4.3.3) reduces to

$$dk = \varkappa_m \omega^m, \quad \varepsilon_m k \omega_a^m = \varepsilon_a \varkappa_m \omega^a,$$

or, equivalently,

$$\omega_m^a = \lambda \omega^a, \qquad d \ln k = -\lambda \omega^m.$$

After exterior differentiation one obtains $d\lambda = -\lambda^2 \omega^m$. Thus $d(x - \lambda^{-1} e_m) = 0$, $d\omega^m = 0$,

$$de_a = e_b \omega_a^b + (k e_{m+1} - \varepsilon_m \lambda e_m) g_{ab} \omega^b,$$
$$de_m = \lambda e_a \omega^a.$$

The lines, determined by $\omega^a = 0$ on M^m, are straight lines, going through a fixed point, and the orthogonal sections of these lines, determined by $\omega^1 = 0$, are totally umbilic in parallel m-planes. This gives (2c) for the case $c = 0$.

Let $c \neq 0$; then the both cases above are equivalent each other. Further the first is considered, when $u = 1$, $m - u \geqslant 2$ and the system (4.3.1) – (4.3.3) reduces to

$$dk = \varkappa_1 \omega^1 + \varkappa_p \omega^p,$$
$$\varepsilon_p (k - \tilde{k}) \omega_1^p = \varepsilon_1 \varkappa_p \omega^1 + \varepsilon_p \tilde{\varkappa}_1 \omega^p,$$
$$d\tilde{k} = \tilde{\varkappa}_1 \omega^1,$$

but $k\tilde{k} + \varepsilon c = 0$ gives by differentiation $\varkappa_p = 0$, $k\tilde{\varkappa}_1 + \tilde{k}\varkappa_1 = 0$. Since $0 \neq k - \tilde{k} = k^{-1}(k^2 + \varepsilon c)$, this implies

$$dk = \varkappa_1 \omega^1, \quad \omega_1^p = \frac{\varepsilon c \varkappa_1}{k(k^2 + \varepsilon c)} \omega^p. \tag{4.3.4}$$

Using here exterior differentiation and Cartan lemma one obtains

$$d\varkappa_1 = \frac{3k\varkappa_1}{k^2+\varepsilon c}\omega^1 \tag{4.3.5}$$

and now exterior differentiation gives an identity. Hence a such M^m is determined by a totally integrable system and does exist with the arbitrariness of constants.

From (1.3.3) and (4.3.4) it follows that ω_p^1 is proportional to ω^p and thus $d\omega^1 = = 0$. At least locally $\omega^1 = ds$, where s is the arc length parameter of a line determined on M^m by $\omega^2 = \ldots = \omega^m = 0$. For this line

$$dx = e_1 ds, \quad de_1 = \varepsilon_1(ke_{m+1} - cx)ds, \quad de_{m+1} = -\varepsilon k e_1 ds,$$

hence this line lies on a 2-plane of $N_s^{m+1}(c)$ and k is its curvature with respect to $N_s^{m+1}(c)$.

From (4.3.4) it is seen also that $\varkappa_1 = \dot{k} = {dk}/{ds}$ and k depends on s only.

Thus all these lines are congruent and (4.3.5) means that

$$\ddot{k} = \frac{3k\dot{k}}{k^2+\varepsilon c},$$

or, equivalently

$$\left[\frac{\dot{k}^2}{(k^2+\varepsilon c)^3}\right]^{\cdot} = 0$$

This yields

$$\dot{k}^2 = a(k^2+\varepsilon c)^3 \tag{4.3.6}$$

with $a = \text{const} \neq 0$. Here a and $k^2+\varepsilon c$ have the same sign $\alpha = |a|\cdot a^{-1}(= 1$ or $= -1)$, because $a(k^2+\varepsilon c) > 0$.

A light calculation shows that the vector

$$q = e_1 + \frac{\varepsilon_1 \alpha}{\sqrt{a(k^2+\varepsilon c)}}(e_{m+1} + \varepsilon k x)$$

is constant for a line above, because $dq = 0$ if $\omega^2 = \ldots = \omega^m = 0$, and thus determines a fixed point on the 2-plane of this line. Since $\langle q, e_1\rangle = 1$, this line is a helix–curve and this point is its directing point.

Further, for vectors

$$f_1 = e_1 - \varepsilon\alpha ck^{-1}\sqrt{a(k^2+\varepsilon c)}x, \quad f_2 = e_{m+1} - ck^{-1}x$$

one obtains, after some straightforward calculations,

$$df_1 = \left[-\varepsilon\alpha ck^{-1}\sqrt{a(k^2+\varepsilon c)}f_1 + \varepsilon_1 k f_2\right]\omega^1,$$
$$df_2 = -\varepsilon k^{-1}(k^2+\varepsilon c)f_1\omega^1,$$

and this shows that f_1, f_2 span a 2-dimensional vector subspace, fixed for M^m, which intersects $N_s^{m+1}(c)$ along a fixed geodesic. This geodesic lies on the 2-plane of every helix–curve and the directing point of the latter is the polus of this geodesic, because $\langle q, f_1\rangle = \langle q, f_2\rangle = 0$.

The orthogonal sections of these helix–curves, determined by $\omega^1 = 0$, i.e. by $s = s_0$, are totally umbilic and for every of them all points of the geodesic above are fixed. All this gives (2c) and at the same time finishes the proof of Theorem A. ■

It can be added to the remarks, given after the formulation of this theorem in the Introduction, that the first assertion, concerning (1) and (2), follows directly from the well–known Serge theorem. Its proof was given here as the base for the further deduction.

Note also that (4.3.6) can be integrated easily; so the results about helix–curves, by $s = 0$ (see [7]) can be complemented to the case $s \neq 0$.

§5. Proof of the Theorem B

5.1. Consequences from the first two conditions (i) and (ii). The following proposition shows that already the first condition *(i)*, together with the initial assumptions of Theorem B, gives a significant consequence.

Proposition 5.1. *Let M^m, $m \geqslant 2$, be a semi–parallel pseudo-Riemannian submanifold of $N_s^n(c)$, whose principal normals are $\frac{1}{2}m(m+1)$-dimensional. Then the following identity holds for the second fundamental form of M^m:*

$$\langle h_{ij}, h_{kl}\rangle = K(2g_{ij}g_{kl} + g_{ik}g_{jl} + g_{il}g_{jk}) - cg_{ij}g_{kl}, \qquad (5.1.1)$$

where K is a function on M^m.

Proof. Since (5.1.1) is a relation between tensors it is sufficient to prove it by $g_{ij} = \varepsilon_i \delta_{ij}$, $g^{ij} = \varepsilon_i \delta^{ij}$. Then (3.3) reduces to

$$\sum_k \varepsilon_k [h_{kj} \langle h_{i[p} h_{q]k} \rangle + h_{ik} \langle h_{j[p} h_{q]k} \rangle - \langle h_{ij} h_{k[p} \rangle h_{q]k}] =$$

$$= c[\varepsilon_q h_{p(i} \delta_{j)q} - \varepsilon_p \delta_{p(i} h_{j)q}].$$

Let a and b be two different values from $\{1, \ldots, m\}$. Taking here $i = j = p = a$, $q = b$ one obtains for coefficients of linearly independent vectors h_{aa}, h_{ab}, h_{bb}:

$$\begin{aligned} &\langle h_{aa}, h_{ab} \rangle = 0, \\ &\langle h_{aa}, h_{aa} \rangle + \varepsilon_a \varepsilon_b [2\langle h_{ab}, h_{ab} \rangle - 3\langle h_{aa}, h_{bb} \rangle] = 2c. \end{aligned} \tag{5.1.2}$$

Similarily by $i = p = a$, $j = q = b$

$$2\langle h_{ab}, h_{ab} \rangle - \langle h_{aa}, h_{bb} \rangle = \varepsilon_a \varepsilon_b c,$$

thus

$$\begin{aligned} 2\varepsilon_a \varepsilon_b \langle h_{aa}, h_{bb} \rangle &= \langle h_{aa}, h_{aa} \rangle - c, \\ 4\varepsilon_a \varepsilon_b \langle h_{ab}, h_{ab} \rangle &= \langle h_{aa}, h_{aa} \rangle + c. \end{aligned} \tag{5.1.3}$$

Since the left part of (5.1.3) is symmetric with respect to a and b, the right part has the same value, say $4K$, for all values of a. If $m \geqslant 3$ and $c \neq a$, the same procedure gives

$$\langle h_{aa}, h_{bc} \rangle = \langle h_{ab}, h_{ac} \rangle = 0; \tag{5.1.4}$$

if $m \geqslant 4$ and a, b, c, d are four different values, then

$$\langle h_{ab}, h_{cd} \rangle = 0. \tag{5.1.5}$$

All this can be summarized as

$$\langle h_{ij}, h_{kl} \rangle = K\,[2\varepsilon_i \varepsilon_k \delta_{ij} \delta_{kl} + \varepsilon_i \varepsilon_j (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})] - c\varepsilon_i \varepsilon_k \delta_{ij} \delta_{kl}, \tag{5.1.6}$$

equivalent to (5.1.1). ■

Corollary 1. If M^m of $N^n_s(c)$ satisfies the assumptions of Proposition 5.1 then

$$\Omega^j_i = -K g_{ik} \omega^k \wedge \omega^j \tag{5.1.7}$$

This follows directly from (1.4.6), (1.4.7), (5.1.1), and in its turn implies that a such M^m, $m \geqslant 3$, is intrinsically a manifold of constant curvature K, i.e. $K = \text{const}$. The last assertion is a consequence of Schur theorem (see [34], Sect. 2.2), but can be obtained also by substituting (5.1.7) into (1.5.2).

Corollary 2. For h^*_{ij}, given by (1.3.5), the identity (5.1.1) yields

$$\langle h^*_{ij}, h^*_{kl} \rangle = K(2g_{ij}g_{kl} + g_{ik}g_{jl} + g_{il}g_{jk}). \tag{5.1.8}$$

In particular, $\frac{1}{2}m(m-1)$ vectors h^*_{ij} $(i \neq j)$ are mutually orthogonal vectors with scalar squares $\varepsilon_i \varepsilon_j K$; all of them are orthogonal to span$\{h^*_{11}, \dots, h^*_{mm}\}$. If $K = 0$, the outer principal normal is a null–space (i.e. all its vectors have zero scalar square). If $K \neq 0$ (this is the case, when *(ii)* is satisfied), the last span has a definite metric: positively definite by $K > 0$ and negatively definite by $K < 0$. Indeed, a vector $\sum h^*_{ii} X^i$ has the scalar square $4K[\sum (X^i)^2 + \sum \varepsilon_i \varepsilon_j X^i X^j]$, where in square brackets there is a positively definite quadratic form, because its principal minor of order p is $2^{-p}(p+1) > 0$.

Hence, in the situation of the Proposition 5.1 the outer principal normal of M^m either is a null–space (if $K = 0$) or has regular metric (if $K \neq 0$).

Corollary 3. If the initial assumptions and the conditions *(i)*, *(ii)* of Theorem B are satisfied, i.e. $K \neq 0$, and if M^m of $N^n_s(c)$ has signature $(\sigma, m - \sigma)$, then its outer principal normal has signature $(\sigma(m-\sigma), m_1 - \sigma(m-\sigma))$ if $K > 0$ and signature $(m_1 - \sigma(m-\sigma), \sigma(m-\sigma))$ if $K < 0$, where $m_1 = \frac{1}{2}m(m-1)$. Thus $s' \geqslant \sigma(m - \sigma + 1)$ if $K > 0$ and $s' \geqslant m_1 - \sigma(m - \sigma - 1)$ if $K < 0$, where, recall, $s' = s$ if $c \geqslant 0$ and $s' = s + 1$ if $c < 0$.

5.2. Proof in the case $m \geqslant 3$. In this case the identity (5.1.8) yields $\nabla \langle h^*_{ij}, h^*_{kl} \rangle = 0$, since $\nabla g_{ij} = 0$ and $K = \text{const}$ (see Corollary 1). Due to (2.2) and (1.4.3) this implies

$$\langle h_{ijp}, h^*_{kl} \rangle + \langle h^*_{ij}, h_{klp} \rangle = 0,$$

where $h_{ijp} = e_\alpha h^\alpha_{ijp}$ are symmetric with respect to their indices; due to (1.3.5) symmetric are also h^*_{kl}.

Thus

$$\langle h_{ijp}, h^*_{kl} \rangle = -\langle h_{klp}, h^*_{ij} \rangle = -\langle h_{kpl}, h^*_{ij} \rangle = \langle h_{ijl}, h^*_{kp} \rangle;$$

this shows that every index of the first triplet can be exchanged by every index of the second pair. Hence

$$\langle h_{ijp}, h^*_{kl}\rangle = \langle h_{klp}, h^*_{ij}\rangle = -\langle h_{ijp}, h^*_{kl}\rangle,$$

so

$$\langle h_{ijp}, h^*_{kl}\rangle = 0; \tag{5.2.1}$$

this yields

$$\langle \overline{\nabla} h_{ij}, h^*_{kl}\rangle = 0. \tag{5.2.2}$$

The condition *(iii)* implies, due to the Erbacher reduction theorem, that every $\overline{\nabla} h_{ij}$ lies in the outer principal normal, i.e. in the span $\{h^*_{kl}\}$ (this can be deduced directly from (2.2)). Since the latter has regular metric, from (5.2.1) it can be concluded that $\overline{\nabla} h_{ij} = 0$. This means that M^m is a parallel submanifold.

5.3. Proof in the case $m = 2$. Since Corollary 1 gives $K = \text{const}$ only by $m \geqslant 3$, in the case $m = 2$ it must be deduced from *(i), (ii), (iii)* separately.

In this case (5.1.8) is as follows:

$$\langle h^*_{11}, h^*_{11}\rangle = \langle h^*_{22}, h^*_{22}\rangle = 4K, \quad \langle h^*_{11}, h^*_{22}\rangle = 2\varepsilon_1\varepsilon_2 K,$$

$$\langle h^*_{12}, h^*_{12}\rangle = \varepsilon_1\varepsilon_2 K, \quad \langle h^*_{11}, h^*_{12}\rangle = \langle h^*_{22}, h^*_{12}\rangle = 0.$$

where $K \neq 0$ due to *(ii)* and M^2 lies in a $R^5_{s'}$ due to *(iii)*; the last is a consequence of the Erbacher reduction theorem. It follows that

$$\frac{1}{2}(\varepsilon_1 h^*_{11} + \varepsilon_2 h^*_{22}), \quad \frac{1}{2}(\varepsilon_1 h^*_{11} - \varepsilon_2 h^*_{22}), \quad h^*_{12}$$

are three mutually orthogonal vectors with scalar squares $3K$, K, $\varepsilon_1\varepsilon_2 K$, respectively, orthogonal to the tangent vectors e_1, e_2 of M^2. The frame bundle $L_{ad}(M^2, \mathbb{R}^5_{s'})$ can be reduced to a subbundle, consisting of vectors e_1, e_2

$$e_3 = \frac{1}{2\sqrt{3|K|}}(\varepsilon_1 h^*_{11} + \varepsilon_2 h^*_{22}), \; e_4 = \frac{1}{2\sqrt{|K|}}(\varepsilon_1 h^*_{11} - \varepsilon_2 h^*_{22}), \; e_5 = \frac{1}{\sqrt{|K|}} h^*_{12},$$

which are mutually orthogonal and have scalar squares ε_1, ε_2, η, η, $\varepsilon_1\varepsilon_2\eta$, respectively, where $\eta = K|K|^{-1}(= 1 \text{ or } = -1)$. For this subbundle the forms ω^B_A in (1.2.1) satisfy

$$\omega^1_2 = -\varepsilon_1\varepsilon_2\omega^2_1, \; \omega^3_4 = -\omega^4_3, \; \omega^3_5 = -\varepsilon_1\varepsilon_2\omega^5_3, \; \omega^4_5 = -\varepsilon_1\varepsilon_2\omega^5_4,$$

$$\omega_3^i = -\varepsilon_i \eta \omega_i^3, \; \omega_4^i = -\varepsilon_i \eta \omega_i^4, \; \omega_5^i = -\varepsilon_j \eta \omega_i^5,$$

where $i, j \in \{1,2\}$, $i \neq j$. Since

$$h_{11}^* = \varepsilon_1 \varkappa (e_3\sqrt{3} + e_4), \; h_{22}^* = \varepsilon_2 \varkappa (e_3\sqrt{3} - e_4), \; h_{12}^* = \varkappa e_5,$$

where $\varkappa = \sqrt{|K|}$, from (1.2.1) and (1.3.4) it follows that (1.3.1) are now

$$\omega_1^3 = \varepsilon_1 \varkappa \sqrt{3}\omega^1, \; \omega_1^4 = \varepsilon_1 \varkappa \omega^1, \; \omega_1^5 = \varkappa \omega^2,$$

$$\omega_2^3 = \varepsilon_2 \varkappa \sqrt{3}\omega^2, \; \omega_2^4 = -\varepsilon_2 \varkappa \omega^2, \; \omega_2^5 = \varkappa \omega^1$$

After exterior differentiation by means of (1.2.2) this system leads to

$$\begin{aligned}
&-\left(d\ln\varkappa - \frac{1}{\sqrt{3}}\omega_3^4\right) \wedge \omega^1 + \frac{\varepsilon_2}{\sqrt{3}}\omega_3^5 \wedge \omega^2 = 0,\\
&\frac{\varepsilon_1}{\sqrt{3}}\omega_3^5 \wedge \omega^1 - \left(d\ln\varkappa + \frac{1}{\sqrt{3}}\omega_3^4\right) \wedge \omega^2 = 0,\\
&\left(d\ln\varkappa - \sqrt{3}\omega_3^4\right) \wedge \omega^1 + \varepsilon_1\varepsilon_2(2\omega_1^2 - \varepsilon_1\omega_4^5) \wedge \omega^2 = 0,\\
&(2\omega_1^2 - \varepsilon_1\omega_4^5) \wedge \omega^1 - \left(d\ln\varkappa - \sqrt{3}\omega_3^4\right) \wedge \omega^2 = 0,\\
&-\left(2\omega_1^2 - \varepsilon_1\omega_4^5 - \varepsilon_1\sqrt{3}\omega_3^5\right) \wedge \omega^1 + d\ln\varkappa \wedge \omega^2 = 0,\\
&d\ln\varkappa \wedge \omega^1 + \varepsilon_1\varepsilon_2\left(2\omega_1^2 - \varepsilon_1\omega_4^5 + \varepsilon_1\sqrt{3}\omega_3^5\right) \wedge \omega^2 = 0.
\end{aligned}$$

Each of the obtained equations gives, due to the Cartan lemma, that the 1-forms standing by the basic forms ω^1 and ω^2, can be expressed as the linear combinations of the latter with the symmetric matrix of coefficients. This yields

$$\begin{aligned}
-\frac{1}{2}d\ln\varkappa &= A\omega^1 + B\omega^2,\\
\frac{1}{\sqrt{3}}\omega_3^4 &= A\omega^1 - B\omega^2,\\
\frac{1}{5}(2\omega_1^2 - \varepsilon_1\omega_4^5) &= \varepsilon_1\varepsilon_2 B\omega^1 + A\omega^2,\\
\frac{1}{\sqrt{3}}\omega_3^5 &= \varepsilon_2 B\omega^1 + \varepsilon_1 A\omega^2.
\end{aligned}$$

The same procedure leads further, after some calculations, to

$$\begin{aligned}
dA =& B\omega_1^2 + \frac{1}{5}(14\varepsilon_1\varepsilon_2 B^2 - 11A^2)\omega^1 - 5AB\omega^2,\\
dB =& -\varepsilon_1\varepsilon_2 A\omega_1^2 - 5AB\omega^1 + \frac{1}{5}(14\varepsilon_1\varepsilon_2 A^2 - 11B^2)\omega^2.
\end{aligned}$$

After exterior differentiation each of these last two equations gives that $\omega^1 \wedge \omega^2$ with a coefficient must be zero, thus the both coefficients must be zero:

$$A\left[\varepsilon_1\eta\varkappa^2 + \frac{42}{25}(A^2 + \varepsilon_1\varepsilon_2 B^2)\right] = 0,$$

$$B\left[\varepsilon_1\eta\varkappa^2 + \frac{42}{25}(A^2 + \varepsilon_1\varepsilon_2 B^2)\right] = 0.$$

Here either $A = B = 0$ or $[\ldots] = 0$. In the last case the next differentiation gives $A\varkappa^2 = B\varkappa^2 = 0$, where $\varkappa^2 = |K| \neq 0$, thus again $A = B = 0$. Hence $K = \text{const} \neq 0$, so the deduction of 5.2 can be used and finishes the proof of Theorem B. ■

Remark. The assertion of Theorem B for the case $m = 2$ can be obtained also directly from the equations above if to replace $A = B = 0$. In the particular case $s = 0$, when $\varepsilon_1 = \varepsilon_2 = \eta = 1$, this proof is given in [13] (assuming $c = 0$).

For the same case $m = 2$, $s = 0$ in [23] a result by Asperti and Mercuri is announced (to appear with a proof in Boll. Un. Mat. Ital.), which generalizes the corresponding part of theorem of [13], namely, $c = 0$ is not assumed. The case $c < 0$ is handled there in a rather complicated way, avoiding to use the Minkowski spacetime. The same generalization is included now by Theorem B (in the case $m = 2$); note that the proof in this section 5.3 does not depend directly on c.

5.4. Geometric consequences. The parallel submanifold M^m of Theorem B has remarkable geometric properties, which follow from the formulae (2.1), (2.2), where now $\overline{\nabla} h^{\alpha}_{ij} = 0$ and (5.1.6), (5.1.8) are to be used. Hence

$$\nabla h^*_{ij} = -K(2g_{ij}e_k\omega^k + e_i g_{jk}\omega^k + e_j g_{ik}\omega^k), \tag{5.4.1}$$

$$\nabla h_{ij} = \nabla h^*_{ij} + c g_{ij} e_k \omega^k. \tag{5.4.2}$$

For the mean curvature vectors $H = \frac{1}{m} h_{ij} g^{ij}$ and $H^* = \frac{1}{m} h^*_{ij} g^{ij} = H - cx$ of M^m with respect to $N^n_s(c)$ and $\mathbb{R}^{n+1}_{s'}$, respectively, there hold

$$dH^* = -\|H^*\|^2 e_k \omega^k, \quad dH = -\|H\|^2 e_k \omega^k,$$

where $\|H^*\|^2 = \langle H^*, H^* \rangle = 2K(m+1)m^{-1} = \text{const} \neq 0$, $\|H\|^2 = \|H^*\|^2 - c$ and $n = \frac{1}{2}m(m+3)$. (Note that in inequalities of Corollary 3, connecting s' and σ, the sign "$\geqslant$" is to replace now by "=".)

The point $\tilde{z}$ of $\mathbb{R}^{n+1}_{s'}$ with radius vector $z = x + \|H^*\|^{-2}H^*$ is a fixed point since $dz = 0$, also $\|x - z\| = \|H^*\|^{-1} = \text{const}$. Thus M^m lies in a hypersphere $N^n_{s^*}(\|H^*\|^2)$ of $\mathbb{R}^{n+1}_{s'}$ with the centre $\tilde{z}$.

On the other hand M^m lies in a space from $N^n_s(c)$ which is $\mathbb{R}^n_{s'}$ if $c = 0$, or in $S^n_{s'}(c)$ if $c > 0$, or in $H^n_{s'-1}(c)$ if $c < 0$. The latter hypersurface $N^n_{s^*}(\|H^*\|^2)$ intersects this $N^n_s(c)$ along a $(n-1)$-dimensional standardly imbedded space from. In the last two cases the same intersection can be obtained when $S^n_{s'}(c)$ or $H^n_{s'-1}(c)$ to intersect by a hyperplane $\mathbb{R}^n_{s^*}$ of $\mathbb{R}^{n+1}_{s'}$. This hyperplane is orthogonal to the vector between the centre O of $S^n_{s'}(c)$ or $H^n_{s'-1}(c)$ and the centre $\tilde{z}$ of the hypersurface above, i.e. to the vector z.

A submanifold of a space form is said to be minimal, if its mean curvature vector with respect this space from is zero (in some cases by $s \neq 0$ also the term "extermal" is used).

Proposition 5.4. *A parallel submanifold M^m of the Theorem B is intrinsically a manifold of constant curvature $K \neq 0$, immersed into a $(n-1)$-dimensional space from, $n = \frac{1}{2}m(m+3)$, as a minimal submanifold.*

Proof. The first assertion follows from Corollary 1 ($m \geqslant 3$) and from the result of Sect. 5.3 ($m = 2$).

Let $c = 0$. The mean curvature vector $H^* = H$ of M^m with respect to $\mathbb{R}^n_{s'}$ is collinear to $z - x$ and its component, tangent to a hypersurface with radius $\|H\|^{-1}$ and centre $\tilde{z}$, containing M^m, is zero, as needed.

Let $c \neq 0$. The mean curvature vector H of M^m with respect to $S^n_{s'}(c)$ or $H^n_{s'-1}(c)$ is due to $H - cx = \|H^*\|^2(z - x)$, coplanar with x and $z - x$, thus orthogonal to the $(n-1)$-dimensional standardly imbedded space from, intersected from $S^n_{s'}(c)$ or $H^n_{s'-1}(c)$ by the hypersphere $N^n_{s^*}(\|H^*\|^2)$ with centre $\tilde{z}$. ∎

Remark. In the case $c = s = 0$, when $K > 0$, this Proposition 5.4 reduces to a particular (extreme) case of a general result by Ferus [9], concerning all irreducible parallel submanifolds M^m of $\mathbb{R}^n$.

5.5. Group theoretic consequences. For the case of the last Remark there is shown in [9] that every irreducible connected complete parallel submanifold M^m of $\mathbb{R}^n$ is a symmetric orbit, more exactly, a standardly immbedded symmetric R-space.

If the principal normals of a such orbit have the maximal dimension then this orbit is a Veronese orbit and n reduces to $\frac{1}{2}m(m+3)$. The following proposition shows that the last result can be generalized to the case $s \neq 0$, $c \neq 0$.

Proposition 5.5 *Let M^m, $m \geqslant 2$, be a connected complete submanifold of the Theorem B. Then M^m is a symmetric orbit of a connected Lie group G, acting by iometries in this $(n-1)$-dimensional standardly imbedded space form, $n = \frac{1}{2}m(m+$ $+3)$, which contains M^m as a minimal submanifold. This G is isomorphic to the identity component of the $\frac{1}{2}m(m-1)$-parametric group of inner isometries of M^m, considered intrinsically as a pseudo-Riemannian manifold of constant curvature.*

Proof. The formulae (1.3.4), with substitution (1.3.5), and (5.4.2), with substitution (5.4.1), can be used by $g_{ij} = \varepsilon_i \delta_{ij}$. Then (5.1.1) shows that all $\langle h_{ij}, h_{kl}\rangle$ are constants, like all $\langle e_i, e_j\rangle$. Recall that $\langle x, x\rangle = c^{-1} = \text{const}$, $\langle x, e_i\rangle = \langle x, h_{ij}\rangle =$ $= \langle e_i, h_{jk}\rangle = 0$. It follows that the frame $\{x, e_i, h_{jk}\}$, adapted so to M^m, moves in $\mathbb{R}^n_{s^*}$ as a rigid system. The results of the previous section show, that every two positions of this frame can be superposed by an isometry of $\mathbb{R}^n_{s^*}$, leaving invariant the standardly imbedded $(n-1)$-dimensional space form, containing M^m. ∎

Remarks. 1. The submanifold M^m of the Proposition 5.5 is an analogue of the classical Veronese submanifold (orbit) of $\mathbb{R}^{\frac{1}{2}m(m+3)}$ and by $s \neq 0$ is futher indicated as the indefinite Veronese orbit. A submanifold of the Theorem B is in general its open part.

2 The formulae (1.3.4) and (5.4.2) by $g_{ij} = \varepsilon_i \delta_{ij}$ show that the Lie algebra $\mathcal{L}(G)$ of G, as a subalgebra in the Lie algebra of the Lie group of isometries in $\mathbb{R}^n_{s^*}$, is determined by the system

$$\omega_i^{jk} = \omega^{(j}\delta_i^{k)}, \quad \omega_{ij}^{kl} = \omega_i^{(k}\delta_j^{l)} + \omega_j^{(k}\delta_i^{l)}.$$

$$\omega_{ij}^k = -K(2\varepsilon_i\delta_{ij}\omega^k + \varepsilon_j\delta_i^k\omega^j + \varepsilon_i\delta_j^k\omega^i) + c\varepsilon_i\omega^k,$$

where ω^j, ω_j^k are the Maurer–Cartan 1-forms of G.

This representation of $\mathcal{L}(G)$ is not a standard one by pseudo-skew-symmetric matrices, because the frame $\{x, e_i, h_{jk}\}$ is not orthonormal. The formulae of Section 5.3 give by $m = 2$, for example, the following standard representation of $\mathcal{L}(G)$: it consists of matrices

$$\begin{pmatrix} 0 & \omega_1^2 & \varepsilon_1\varkappa\sqrt{3}\omega^1 & \varepsilon_1\varkappa\omega^1 & \varkappa\omega^2 \\ -\varepsilon_1\varepsilon_2\omega_1^2 & 0 & \varepsilon_2\varkappa\sqrt{3}\omega^2 & -\varepsilon_2\varkappa\omega^1 & \varkappa\omega^1 \\ -\varepsilon\varkappa\sqrt{3}\omega^1 & -\varepsilon\varkappa\sqrt{3}\omega^2 & 0 & 0 & 0 \\ -\varepsilon\varkappa\omega^1 & \varepsilon\varkappa\omega^2 & 0 & 0 & 2\varepsilon_1\omega_1^2 \\ -\varepsilon_2\varkappa\omega^2 & -\varepsilon_1\varkappa\omega^1 & 0 & -2\varepsilon_2\omega_1^2 & 0 \end{pmatrix}.$$

Such standard representations for $m \geqslant 3$ are more complicated.

§6. Proof of the Theorem C

6.1. Deduction to the corresponding Pfaff system. In the previous Theorem B some conditions *(i) – (iii)* are given, wich guarantee that a semi-parallel pseudo-Riemannian submanifold M^m of $N_s^n(c)$ reduces to a parallel one. Now the first task is to show that there exists semi-parallel pseudo-Riemannian submanifolds M^m of $N_s^n(c)$, $n > \frac{1}{2}m(m+3)$, satisfying *(i)* and *(ii)* (with *(iii)* removed!), which are not parallel. Theorem C asserts this existence in a particular case if $n = \frac{1}{2}m(m+$ $+3)+1$.

From *(i)* and *(ii)* it follows for $n > \frac{1}{2}m(m+3)$ that a such M^m by $m \geqslant 3$ has a non-zero constant sectional curvature K. Further, in order to include also surfaces M^2, there is supposed that a such M^2 has constant Gaussian curvature K, non-zero due to *(ii)*. Then the deduction in the first part of Sect. 5.2 can be used also for such M^2. Note that the problem of existence of a semi-parallel but non-parallel pseudo-Riemannian surface M^2 of $N_s^n(c)$, $n > 5$, with non-constant Gaussian curvature K, satisfying *(i)* and *(ii)*, is left here open yet.

So let M^m $m \geqslant 2$, be a semi-parallel pseudo-Riemannian submanifold of $N_s^n(c)$, $n > \frac{1}{2}m(m+3)$, satisfying the conditions *(i)*, *(ii)*, and, if $m = 2$, let $K = \text{const}$. Its frame bundle $L_{ad}(M^m, N_s^n(c))$ can be reduced to a subbundle of frames $\{x, e_i, h_{jk}, e_\pi\}$, where in the role of e_α are the vectors h_{jk}, numbered by symmetric pair-indices ij, and the vectors e_π $(\pi, \varrho, \ldots = \frac{1}{2}m(m+3)+1, \ldots, n)$, while the complementary metric conditions

$$\langle e_i, e_j\rangle = g_{ij} = \varepsilon_i\delta_{ij}, \quad \langle h_{kj}, e_\pi\rangle = 0, \quad \langle e_\pi, e_\varrho\rangle = \varepsilon_\pi, \delta_{\pi\varrho} \tag{6.1.1}$$

are satisfied, ε_i and ε_π being 1 or -1. Then (1.3.4) compared with (1.2.6) give

$$\omega^{ij} = \omega^\pi = 0, \tag{6.1.2}$$

$$\omega_i^{ij} (= \omega_i^{ji}) = \omega^j, \quad \text{all other} \quad \omega_i^{jk} = 0,\ \omega_i^\pi = 0, \tag{6.1.3}$$

and (2.1) together with (5.1.1) and (5.2.2) yield in the same way

$$\omega_{ii}^{ij} = 2\omega_i^j,\ \omega_{ij}^{ik} = \omega_j^k,\ (i \neq j), \quad \text{all} \quad \text{other} \quad \omega_{ij}^{kl} = 0. \tag{6.1.4}$$

These relations (6.1.2) – (6.1.4) are the equations of the Pfaff system of the problem.

6.2. Compatibility of the system by Cartan criterion. First the system (6.1.2) – (6.1.4) must be closed by taking exterior differentials from both sides of every equation, using (1.2.6) and the equations of the system itself.

For this the relations are needed which follow from (1.3.3) due to (5.1.1) and (6.1.1); they are as follows:

$$\varepsilon_j \omega_i^j + \varepsilon_i \omega_j^i = 0, \quad \omega_\pi^i = 0,$$

$$\omega_{ii}^i = \varepsilon_i (c - 4K) \omega^i, \tag{6.2.1}$$

$$\omega_{ii}^j = \varepsilon_i (c - 2K) \omega^j, \quad \omega_{ij}^j = -\varepsilon_i K \omega^i \quad (i \neq j), \tag{6.2.2}$$

$$\omega_{jk}^i = 0 \quad (i, j, k \quad \text{– three different}). \tag{6.2.3}$$

$$\varepsilon_\pi \omega_{ij}^\pi = \varepsilon_i (c - 2K) \delta_{ij} \sum_k \varepsilon_k \omega_\pi^{kk} - 2\varepsilon_i \varepsilon_j K \omega_\pi^{ij}, \tag{6.2.4}$$

$$\varepsilon_\varrho \omega_\pi^\varrho + \varepsilon_\pi \omega_\varrho^\pi = 0$$

(here the equations (6.1.3) and (6.1.4) are used already; summing is indicated always only by $\sum$).

If to use now the procedure, described above, by the equations (6.1.2) and by the two first group of (6.1.3), one obtain identities. The equations $\omega_i^\pi = 0$ lead in this way to

$$\omega_{ii}^\pi \wedge \omega^i + \sum_{j \neq i} \omega_{ij}^\pi \wedge \omega^j = 0 \tag{6.2.5}$$

The equations (6.1.4) after some calculations yield

$$\sum_{\pi} \omega_{ij}^{\pi} \wedge \omega_{\pi}^{kl} = 0 \tag{6.2.6}$$

for arbitrary symmetric pairs ij and kl. Multiplaying both sides of (6.2.4) exteriorily by ω_{kl}^{π} and summing then by π one get

$$\sum_{\pi} \varepsilon_{\pi} \omega_{ij}^{\pi} \wedge \omega_{kl}^{\pi} = 0 \tag{6.2.7}$$

due to (6.2.6); the result is equivalent to (6.2.6). Thus adjunction of (6.2.5) and (6.2.7) to (6.1.1) – (6.1.3) makes the system closed.

In the case of Theorem C the index π takes only one value n (equal to $\frac{1}{2}m(m+$ $+3)+1$). Then (6.2.5) yield, due to the Cartan lemma,

$$\omega_{ii}^{n} = \varrho_i \omega^i + \sum_{j \neq i} \varrho_{ij} \omega^j,$$

$$\omega_{ij}^{n} = \varrho_{ij} \omega^i + \varrho_{ji} \omega^j + \sum_{k \neq i,j} \varrho_{ijk} \omega^k \ (i \neq j),$$

where ϱ_{ijk} are symmetric with respect to indices i, j, k, which take mutually different values. If here $\varrho_1 = \ldots = \varrho_m = 0$ then substituting the expressions of ω_{ii}^{n} into (6.2.4) one obtaines $\omega_{ii}^{n} = \omega_{ij}^{n} = 0$ and so the case of Theorem B. For the purpose of Theorem C one can assume that at last one of $\varrho_1, \ldots, \varrho_m$ is nonzero and so take $\varrho_1 > 0$ by rearranging the vectors $e_1, \ldots, e_m$, if needed. Then there exist some λ_u $(u, v, \ldots = 2, \ldots, m)$ so that $\varrho_{1u} = \lambda_u \varrho_1$ and thus

$$\omega_{11}^{n} = \varrho_1 \left(\omega^1 + \sum_{u} \lambda_u \omega^u \right). \tag{6.2.8}$$

Due to (6.2.7) now

$$\omega_{1u}^{n} = \lambda_u \omega_{11}^{n}, \tag{6.2.9}$$

$$\omega_{uv}^{n} = \lambda_u \lambda_v \omega_{11}^{n}. \tag{6.2.10}$$

These equations (6.2.8) – (6.2.10) are to be added to the system (6.1.2) – (6.1.4).

To close the new system the procedure above must be used for the added equations. After some light calculations it leads to

$$\Theta \wedge \omega_{11}^{n} + \varrho_1 \Psi_u \wedge \omega^u = 0, \tag{6.2.11}$$

$$\Psi_u \wedge \omega_{11}^n = 0, \tag{6.2.12}$$

$$(\lambda_u \Psi_v + \lambda_v \Psi_u) \wedge \omega_{11}^n = 0, \tag{6.2.13}$$

where

$$\Theta = d \ln \varrho_1 - 3\lambda_u \omega_1^u,$$

$$\Psi_u = d\lambda_u - \lambda_v \omega_u^v + \lambda_u \lambda_v \omega_1^v - \omega_u^1.$$

The 1-forms ω_{11}^n and ω^u are linearly independent and can be considered as m basic forms. The 1-forms Θ and Ψ_u are m linearly independent secondary forms. Since (6.2.13) are the consequences of (6.2.12), the first character (i.e. the rank of the polar system for (6.2.11) and (6.2.12)) is $s_1 = m$ and hence the Cartan number Q is m. From (6.2.11) and (6.2.12) it follows, due to the Cartan lemma, that

$$\Theta = \mu_1 \omega_{11}^n + \varrho_1 \mu_u \omega^u, \tag{6.2.14}$$

$$\Psi_u = \mu_u \omega_{11}^n. \tag{6.2.15}$$

Here μ_1 and μ_u are m new independent coefficients. Since the number of the latter coincides with the Cartan number Q, the Cartan criterion (see [3]) is satisfied. This proves the first assertion of the Theorem C.

6.3. Second assertion: geometric description. Let M^m be a submanifold of the Theorem C, if $m = 2$ then with $K =$ const , so that the formulae of two previous Sections 6.1 and 6.2 can be used.

Let provisionally a point $x \in M^m$ be fixed. Then $dx = 0$ and $\omega^1 = \omega^u = 0$; due to (6.2.8) – (6.2.10), (6.2.14), (6.2.15) also

$$\omega_{11}^n = \omega_{1u}^n = \omega_{uv}^n = \Theta = \Psi_u = 0.$$

Thus

$$d \ln \varrho_1 = 3\lambda_u \omega_1^u, \quad d\lambda_u = \lambda_u \omega_u^v - \lambda_u \lambda_v \omega_1^v + \omega_u^1$$

and for the non–zero vector $\varepsilon_1 e_1 + g^{uv} \lambda_u e_v$ of $T_x M^m$ it follows (by a fixed x) that

$$d(\varepsilon_1 e_1 + g^{uv} \lambda_u e_v) = \lambda_w \omega_1^w (\varepsilon_1 e_1 + g^{uv} \lambda_u e_v),$$

i.e. this vector has an invariant direction. Hence a direction field is determined on M^m. The frame vector e_1 can be taken in the direction of this field in every point $x \in M$. Then $\lambda_u = 0$ on M^m and thus

$$\omega^n_{11} = \varrho_1 \omega^1, \quad \omega^n_{1u} = \omega^n_{uv} = 0, \tag{6.3.1}$$

$$d\varrho_1 = \varrho_1(\nu_1 \omega^1 + \nu_u \omega^u), \quad \omega^1_u = -\nu_u \omega^1, \tag{6.3.2}$$

where $\nu_1 = \varrho_1 \mu_1$, $\nu_u = \varrho_1 \mu_u$. Since now $d\omega^1 = \omega^u \wedge \omega^1_u = \omega^1 \wedge \nu_u \omega^u$, the equation $\omega^1 = 0$ is completely intergrable and thus determines a 1-codimensional foliation of M^m.

For every leaf (integral $(m-1)$-dimensional submanifold $\tilde{M}^{m-1}$) of this foliation there hold $\omega^1 = \omega^1_u = \omega^n_{uv} = 0$, hence

$$dx = e_u \omega^u, \quad de_u = -cx\omega^u + e_v \omega^v_u + h_{uv}\omega^v, \tag{6.3.3}$$

$$\nabla h_{uu} = e_u \varepsilon_u (c - 4K)\omega^u + \sum_{v \neq u} e_v \varepsilon_u (c - 2K)\omega^v, \tag{6.3.4}$$

$$\nabla h_{uv} = -K(\varepsilon_u e_v \omega^u + \varepsilon_v e_u \omega^v)\ (u \neq v);$$

(here also (6.2.1) – (6.2.3) were used). Due to Lemma of §2 this submanifold $\tilde{M}^{m-1}$ is a parallel one. The result of §5 imply that it is a Veronese orbit (indefinite, in general) or its part.

The latter is a submanifold not only of M^m but also of a m-dimensional Veronese orbit (or its part) $\tilde{M}^m$, determined by the completely integrable system of equations (6.1.2) – (6.1.4), where $\pi = n$, and of

$$\omega^n_{11} = \omega^n_{1u} = \omega^n_{uv} = 0,$$

i.e. of (6.3.1) by $\varrho_1 = 0$. The tangent spaces and the components h_{ij} of h coincide for M^m and $\tilde{M}^m$ at every point $x \in \tilde{M}^{m-1} = \tilde{M}^m \cap M^m$. This means that M^m is the 2-order envelope of all these $\tilde{M}^m$ (in the sense that for every point $x \in M^m$ there exists a $\tilde{M}^m$, so that in every direction of $T_x M^m \equiv T_x \tilde{M}^m$ on M^m and $\tilde{M}^m$ through x go lines which have the 2nd order tangency; see [14]). Moreover, the leaves $\tilde{M}^{m-1}$ are the caracteristics of the 1-parameter family of $\tilde{M}^m$. Since $K = \text{const}$, the Veronese orbits, containing these $\tilde{M}^m$, are congruent each other. The same is true for characteristics $\tilde{M}^{m-1}$.

If $m = 2$ the index u takes only one value 2 and from (6.3.3), (6.3.4) it follows that characteristics $\tilde{M}^1$ are congruent plane lines of constant curvature

This finishes the proof of Theorem C.

6.4. Central curve. This curve is defined in the Introduction, after the Theorem C was formulated, and lies in $\mathbb{R}^n_s$ (if $c = 0$) or in $\mathbb{R}^{n+1}_{s'}$ (if $c \neq 0$), $n = \frac{1}{2}m(m+3)+1$. As an important geometric object connected with the submanifold M^m of this theorem it has some remarkable properties.

Proposition 6.4 *The first curvature of the central curve of a submanifold M^m, described in the second part of the Theorem C, is a constant and its second curvature is non–zero. Among such M^m there exist special submanifolds, whose central curves are helical, i.e. all their curvatures are constants.*

Proof. The centre $\tilde{z}$ of a Veronese orbit $\tilde{M}^m$ is determined by the radius vector $z = x + \|H^*\|^{-2}H^*$ (see Sect. 5.4). For the study of the central curve of M^m the formulae (1.2.6) are needed for the subbundle of $L(N^n_s(c))$ consisting of frames $\{x, e_i, h^*_{jk}, e_n\}$ whose basic vectors are introduced in Sect. 1.3. The first of these formulae are (1.3.4), the next are (2.2), in which $e_\alpha \overline{\nabla} h^\alpha_{ij}$ are now, due to (5.1.8), (5.2.1) and (6.3.1), zero, except $e_\alpha \overline{\nabla} h^\alpha_{11} = e_n \varrho_1 \omega^1$. Denoting in the decompositions of dh^*_{ij} the coefficients before h^*_{kl}, e_k by $\overset{*}{\omega}{}^{kl}_{ij}$, $\overset{*}{\omega}{}^{k}_{ij}$ one obtains for these 1-forms the same expressions as in the right sides of (6.1.4), (6.2.1) – (6.2.3) with only difference, that in the latter c must be replaced by 0. In the same way instead of (6.2.4) one obtain

$$\varepsilon_n \varrho_1 \omega^1 = -2K \left[\varepsilon_1 \left(\varepsilon_1 \overset{*}{\omega}{}^{11}_n + \sum_v \varepsilon_u \overset{*}{\omega}{}^{vv}_n \right) + \overset{*}{\omega}{}^{11}_n \right],$$

$$0 = -2\varepsilon_1 \varepsilon_u K \overset{*}{\omega}{}^{1u}_n,$$

$$0 = -2K \left[\varepsilon_u \left(\varepsilon_1 \overset{*}{\omega}{}^{11}_n + \sum_v \varepsilon_u \overset{*}{\omega}{}^{vv}_n \right) + \overset{*}{\omega}{}^{uu}_n \right],$$

$$0 = -2\varepsilon_u \varepsilon_v K \overset{*}{\omega}{}^{uv}_n \quad (u \neq v).$$

The last three give

$$0 = \overset{*}{\omega}{}^{1u}_n = (m-1)\varepsilon_1 \overset{*}{\omega}{}^{11}_n + m \sum_v \varepsilon_v \overset{*}{\omega}{}^{vv}_n = \omega^{uv}_n \qquad (u \neq v),$$

thus

$$\overset{*}{\omega}{}^{11}_n = -\frac{\varepsilon_n m \varrho_1}{2K(m+1)} \omega^1, \qquad \overset{*}{\omega}{}^{uu}_n = -\frac{\varepsilon_1 \varepsilon_u}{m} \overset{*}{\omega}{}^{11}_n.$$

Now for the central curve the following formulae hold:

$$dz = e_n d\sigma, \quad de_n = \varepsilon_n(-\varepsilon_1 m h^*_{11} + \sum_v \varepsilon_v h^*_{vv}) d\sigma,$$

where

$$d\sigma = \frac{\varepsilon_1 \varrho_1}{2K(m+1)} \omega^1. \tag{6.4.1}$$

Here the scalar square of $de_n : d\sigma$ is, due to (5.1.8), equal $2Km(m+1) = \text{const}$. On the other hand it is the square of the first curvature of the central curve.

The second curvature of this curve is zero iff differential of $de_n : d\sigma$ is collinear to e_n. But this is not the case, because

$$\begin{gathered} d\left(-\varepsilon_1 m h^*_{11} + \sum_v \varepsilon_v h^*_{vv}\right) = -2Km(m+1) e_n d\sigma + \\ + 4\varepsilon_1 K (m+1)^2 \varrho_1^{-1} \left(K e_1 + \sum_v \varepsilon_v \nu_v h^*_{1v}\right) d\sigma \end{gathered} \tag{6.4.2}$$

due to (2.2) (with $e_\alpha \nabla h^\alpha_{ij}$ indicated above). Here $K \neq 0$ and this proves the first assertion.

To prove the second assertion some new formulae are needed. By exterior differentiation and Cartan lemma the relations (6.3.2) yield

$$d\nu_1 = \tau \omega^1 + \tau_u \omega^u, \tag{6.4.3}$$

$$d\nu_u = \sum_v \nu_v (\omega^v_u + \nu_u \omega^v) + (\tau_u - \nu_1 \nu_u) \omega^1 + \varepsilon_u K \omega^u. \tag{6.4.4}$$

Now the vector $\sum_u \varepsilon_u \nu_u e_u$ is invariant at every fixed point $x \in M^m$ since $\omega^1 = = \omega^u = 0$ imply $d(\sum_u \varepsilon_u \nu_u e_u) = 0$. This vector is non-zero, because otherwise (6.4.3) gives a contradiction with $K \neq 0$, and it can be used for a further adaption of the frame so that e_2 is collinear to this vector. Then $\nu_{u'} = 0$; $u', v', \ldots = 3, \ldots, m$. Denoting $\nu_2 = k$ one obtain

$$\begin{gathered} dk = (k^2 + \varepsilon_2 K)\omega^2 + (\tau_2 - \nu_1 k)\omega^1, \\ 0 = k\omega^2_{u'} + \tau_{u'}\omega^1 + \varepsilon_{u'} K \omega^{u'} \end{gathered}$$

due to (6.4.4).

Now let a special case is considered when $\nu_1 = 0$. Then $\tau = \tau_u = 0$ due to (6.4.3) and thus

$$dln\varrho_1 = k\omega^2, \quad \omega_2^1 = -k\omega^1, \quad \omega_{u'}^1 = 0,$$

$$dk = (k^2 + \varepsilon_2 K)\omega^2, \qquad k\omega_{u'}^2 + \varepsilon_{u'} K\omega^{u'} = 0.$$

The previous Pfaff system after enlarging by these new equations gives a closed system (i.e. exterior differentiation yields identities), hence M^m in this case does exist with arbitrariness of constants.

Since now $d[\varrho_1^{-2}(k^2 + \varepsilon_2 K)] = 0$ and the vector $\varrho_1^{-1}(Ke_1 + \sum_v \varepsilon_v \nu_v h_{1v}^*)$ in (6.4.2) has the scalar square $\varepsilon_1\varepsilon_2 K\varrho_1^{-2}(k^2 + \varepsilon_2 K)$, the second curvature of the central curve is in this case a constant.

A similar direct calculation shows futher that also the next curvatures of the central curve are constants in this case (cf. [17]). ■

Remark. The condition $\nu_1 = 0$, characterizing the last case, means geometrically that the arc length parameter σ of the central curve is proportional to that of every orthogonal trajectory of the 1-parameter family of characteristic Veronese submanifolds $\tilde{M}^{m-1}$. This follows immediately from (6.4.1).

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