On Følner nets, Szegö's theorem and other eigenvalue distribution theorems

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Abstract: We give a unified C*-algebraic approach to some eigenvalue distribution theorems having the same flavour as Szegö's classical theorem. To this aim, we introduce the notion of a Szegö-pair for a unital concrete C*-algebra $\mathcal{A}$ and shows how it is related to the concept (due to Connes) of a Følner net for $\mathcal{A}$. Examples are given when $\mathcal{A}$ is the twisted group von Neumann algebra of an amenable discrete group and when $\mathcal{A}$ is a C*-algebra naturally associated with a discretized Schrödinger operator. A streamlined proof of Szegö's theorem is also presented.

Primary: 46L05, 47A58
Secondary: 46L, 46, 47A, 47, 22D25, 43A07, 81Q10.

Work partially supported by the Norwegian Research Council.
Introduction

Let $\mathcal{H}$ be a complex Hilbert space, $B(\mathcal{H})$ denote the bounded linear operators on $\mathcal{H}$ and $A$ be a self-adjoint operator in $B(\mathcal{H})$. Let $\{P_\alpha\}$ be a net of non-zero finite dimensional orthogonal projections in $B(\mathcal{H})$. Then $\mathcal{H}_\alpha = P_\alpha \mathcal{H}$ is a finite dimensional Hilbert space and $A_\alpha = P_\alpha A|_{\mathcal{H}_\alpha}$ is a self-adjoint operator in $B(\mathcal{H}_\alpha)$ for each $\alpha$. A fundamental question is how the spectrum $\text{sp}(A)$ of $A$ is related to $\text{sp}(A_\alpha)$ as $\alpha$ grows. This question has been dealt with recently in [Arv 1, Arv 2, Arv 3, Bed 1], but some special cases have been studied earlier in [e.g. Bel 1, Bel 2, BLT, CFKS, Gui, PF, Shu, Wid 2].

One possible approach to this problem is to consider for each $\alpha$ the spectral measure $\mu^\alpha_\lambda$ of $A_\alpha$ with respect to the normalized trace on $B(\mathcal{H}_\alpha)$, i.e. $\mu^\alpha_\lambda$ is the Borel probability measure on $\mathbb{R}$ supported on $\text{sp}(A_\alpha)$ given by

$$\mu^\alpha_\lambda(S) = \frac{1}{\dim \mathcal{H}_\alpha} \nu^\alpha_\lambda(S) \quad (S \subseteq \mathbb{R}, \quad S \text{ Borel}),$$

where $\nu^\alpha_\lambda(S)$ is the number of eigenvalues of $A_\alpha$ contained in $S$, multiplicities counted. Then one may try to find some conditions ensuring that the net $\{\mu^\alpha_\lambda\}$ converges weakly to some Borel probability measure $\mu$ on $\mathbb{R}$ whose (closed) support $\text{supp}(\mu)$ is related in some way to $\text{sp}(A)$. We recall that

$$\text{supp}(\mu) = \{\lambda \in \mathbb{R} | \mu(U) > 0 \text{ for all open sets } U \text{ in } \mathbb{R} \text{ containing } \lambda\}$$

and that if $F_\mu(\lambda) = \mu((\infty, \lambda])$, $\lambda \in \mathbb{R}$, denotes the cumulative distribution function of $\mu$, then $\text{supp}(\mu)$ consists of the points of non-constancy of $F_\mu$.

Mathematical physicists are especially interested in the so-called integrated density of states (IDS) of $A$ (w.r.t. of $\{P_\alpha\}$) which is defined by

$$N_A(\lambda) = \lim_\alpha \mu^\alpha_\lambda((-\infty, \lambda]]), \quad \text{i.e.} \quad N_A(\lambda) = \lim_\alpha F^\alpha_\mu(\lambda), \quad \lambda \in \mathbb{R},$$

whenever the limit exists. If the net $\{\mu^\alpha_\lambda\}$ converges weakly to a $\mu$ as above,
i.e. $\lim_{\alpha} \int g(\lambda) d\mu^{\alpha}_{A}(\lambda) = \int g(\lambda) d\mu(\lambda)$ for all $g \in C_{0}(\mathbb{R})$,

then it follows from classical arguments in probability theory that $N_{A}(\lambda)$ exists whenever $\mu(\{\lambda\}) = 0$ and is then given by $N_{A}(\lambda) = F_{\mu}(\lambda)$, the condition $\mu(\{\lambda\}) = 0$ corresponding to the fact that $F_{\mu}$ is continuous at $\lambda$.

Instead of just considering the problem as a single operator-theoretic one, we shall formulate it in a $C^{*}$-algebraic frame. So we assume without loss of generality that $A$ lies in a $C^{*}$-subalgebra $\mathcal{A}$ of $B(\mathcal{H})$ which contains the identity operator $I$ on $\mathcal{H}$. Let $\varphi$ be a state on $\mathcal{A}$. Via the functional calculus and the Riesz representation theorem, we obtain the spectral measure of $A$ w.r.t. $\varphi$, $\mu^{\varphi}_{A}$, which is the unique Borel probability measure on $\mathbb{R}$ satisfying

$$\int_{\mathbb{R}} g(\lambda) d\mu^{\varphi}_{A}(\lambda) = \varphi(g(A)) \quad \text{for all} \quad g \in C_{0}(\mathbb{R}).$$

If $(\pi_{\varphi}, \mathcal{H}_{\varphi}, \xi_{\varphi})$ denotes the GNS-triple associated to $(\mathcal{A}, \varphi)$ and $E$ denotes the projection-valued measure associated to $\pi_{\varphi}(A)$ in $B(\mathcal{H}_{\varphi})$, then $\mu^{\varphi}_{A}$ may be described by

$$\mu^{\varphi}_{A}(S) = \langle E(S) \xi_{\varphi}, \xi_{\varphi} \rangle, \quad S \subseteq \mathbb{R}, \ S \text{ Borel.}$$

Hence we have $\text{supp}(\mu^{\varphi}_{A}) \subseteq \text{sp}(\pi_{\varphi}(A)) \subseteq \text{sp}(A)$. When $\varphi$ is faithful, then $\pi_{\varphi}$ is faithful and we get $\text{sp}(\pi_{\varphi}(A)) = \text{sp}(A)$. On the other hand, when $\varphi$ is tracial, then it is well-known that $\xi_{\varphi}$ is a separating vector for $\pi_{\varphi}(\mathcal{A})''$ (cf. [KR]) and we get $\text{supp}(\mu^{\varphi}_{A}) = \text{sp}(\pi_{\varphi}(A))$. Especially, when $\varphi$ is faithful and tracial, we have $\text{supp}(\mu^{\varphi}_{A}) = \text{sp}(A)$.

As we shall review in this paper, in many interesting cases, the $C^{*}$-algebra in consideration possess a natural faithful tracial state $\varphi$ and one may choose a net $\{P_{\alpha}\}$ (usually a sequence) as above such that the $\{\mu^{\alpha}_{A}\}$ converges weakly to $\mu^{\varphi}_{A}$ for all self-adjoint $A \in \mathcal{A}$. In such cases, a numerical approximation of the cumulative distribution function of $\mu^{\varphi}_{A}$, and thereby of $\text{sp}(A) = \text{supp}(\mu^{\varphi}_{A})$, may then presumably be obtained with the help of a computer. Let us mention here that in another direction initiated by Bellissard and his co-workers, one may try to label the gaps in the spectrum of $A$ by utilizing K-theoretic
methods. We shall concentrate our attention on the numerical approach and refer the interested reader to [Bel 2] and the references given there for an excellent recent account on this other approach.

To ease our exposition, we now introduce the following terminology: Let \( \varphi \) be a state on \( \mathcal{A} \) and \( \{P_\alpha\} \) be a net in \( \mathcal{B}(\mathcal{H}) \) as before. The pair \( \{\{P_\alpha\}, \varphi\} \) is called a Szegö-pair for \( \mathcal{A} \) whenever we have

\[
\mu_A^\varphi \to \mu_A^\varphi \quad \text{(weakly) for all self-adjoint } A \in \mathcal{A}.
\]

In other words, whenever \( A \in \mathcal{A} \) is self-adjoint with eigenvalue list \( \lambda_1^\alpha, \ldots, \lambda_n^\alpha \) (repeated according to multiplicity, so \( n_\alpha = \dim \mathcal{H}_\alpha \)) for each \( A_\alpha = P_\alpha A |_{\mathcal{H}_\alpha} \), we have

\[
\lim_\alpha \left[ \frac{1}{n_\alpha} \sum_{i=1}^{n_\alpha} g(\lambda_i^\alpha) \right] = \varphi(g(A)) \quad \text{for all } g \in C_0(\mathbb{R}).
\]

As we shall see in this paper, if \( \{\{P_\alpha\}, \varphi\} \) is a Szegö-pair for \( \mathcal{A} \) then \( \varphi \) is necessarily tracial. Now, to motivate our choice of terminology, we recall the classical Szegö-theorem ([Sze, GS, Wid 1]), which was one major source of inspiration in [Arv 1] and [Wid 2], formulated in an operator-theoretic way:

Let \( \mathbb{T} \) denote the circle group with normalized Haar measure \( \bar{m} \), \( P_n \) the orthogonal projection from \( L^2(\mathbb{T}) \) onto the linear span of \( \{\varepsilon_k; k = 0, 1, \ldots, n\} \) in \( L^2(\mathbb{T}) \), where \( \varepsilon_k(z) = z^k (z \in \mathbb{T}) \) for each \( k \in \mathbb{N} \), and \( M : L^\infty(\mathbb{T}) \to \mathcal{B}(L^2(\mathbb{T})) \) is the representation of \( L^\infty(\mathbb{T}) \) as multiplication operators on \( L^2(\mathbb{T}) \). Then Szegö’s theorem asserts that \( \{\{P_n\}, \bar{m}\} \) is a Szegö-pair for \( M (L^\infty(\mathbb{T})) \), \( \bar{m} \) denoting the (faithful tracial) state on \( M (L^\infty(\mathbb{T})) \) obtained by transporting \( m \) via \( M \).

Our main result in [Bed 1] (see also [Arv 1]) assumes that the \( \text{C}^* \)-algebra \( \mathcal{A} \) has a unique tracial state \( \varphi \) and gives then a sufficient condition on a sequence \( \{P_n\} \) for \( \{\{P_n\}, \varphi\} \) to be a Szegö-pair for \( \mathcal{A} \). Of course, such an assumption on \( \mathcal{A} \) makes it impossible to deduce Szegö’s theorem from this result. On the other hand, Szegö’s theorem is, as far as
we know, the only result in the literature which exhibits a Szegö-pair for a von Neumann algebra (namely \( M(L^\infty(\mathbb{T})) \)).

As we shall show in these notes, one can in fact construct Szegö-pairs for any twisted group von Neumann algebras associated to an amenable discrete group. As a corollary, we easily obtain Szegö-type theorems for any abelian compact group \( G \), and thereby reprove Szegö’s classical theorem by taking \( G = \mathbb{T} \). An essential ingredient in our proof is the concept of a Følner net for a \( \text{C}^* \)-algebra. This concept, which is implicitly introduced by Connes in [Con 1, Con 2], is the analogue of the concept of a Følner net for a group. As pointed out in [Bed 2], it may be used to reformulate our main result in [Bed 1]. In fact, one may use it to characterize Szegö-pairs (cf. Theorem 6): the pair \( \{\{P_\alpha\}, \varphi\} \) is a Szegö-pair for \( \mathcal{A} \) if and only if \( \varphi \) is the trace per unit volume w.r.t. \( \{P_\alpha\} \) and \( \{P_\alpha\} \) is a Følner net for \( \mathcal{A} \).

The paper is organized as follows. The first section is devoted to Følner nets. The second deals with Szegö-pairs and Szegö’s theorem. The third and final section contains some applications to discretized Schrödinger operators which are nearly related to the known results for such operators (cf. [Bel 1, Bel 2, BLT, CFKS, PF, Shu]). We shall freely use standard notation and terminology in \( \text{C}^* \)-algebras: the reader may consult e.g. [KR] and/or [Ped].

Acknowledgements: It is a pleasure to thank Trond Digernes for many fruitful discussions when working on this paper.

1. Følner nets

Let \( \Gamma \) be a discrete group. We recall that a Følner net for \( \Gamma \) is a net \( \{\Gamma_\alpha\} \) of non-empty finite subsets of \( \Gamma \) satisfying

\[
\lim_{\alpha} \frac{\#(\gamma \Gamma_\alpha \Delta \Gamma_\alpha)}{\#(\Gamma_\alpha)} = 0 \quad \text{for all} \quad \gamma \in \Gamma,
\]

where \( \Delta \) denotes the symmetric difference (of sets) and \( \# \) the cardinality (of sets). Følner’s theorem asserts that \( \Gamma \) has a Følner net if and only if \( \Gamma \) is amenable (see [Pat] or [Pie]). Note that one may work with sequences instead of nets whenever \( \Gamma \) is countable and that if \( \Gamma \) is amenable then one can find a Følner net \( \{\Gamma'_\alpha\} \) for \( \Gamma \) which also satisfies
i) \( \alpha \leq \beta \Rightarrow \Gamma'_\alpha \subseteq \Gamma'_\beta \) and

ii) \( \bigcup_\alpha \Gamma'_\alpha = \Gamma \).

We stress that we don’t assume (as some authors do) that these two conditions are fullfilled in our definition of a Følner net. As we shall see later, Szegö’s classical theorem is related to the Følner net for \( \mathbb{Z} \) associated to \( \Gamma_n = \{0, 1, \ldots, n\} \) (which doesn’t satisfy ii)) and one may obtain other Szegö-type theorems by choosing other Følner nets for \( \mathbb{Z} \), e.g. \( \Gamma'_n = \{-n, \ldots, -1, 0, 1, \ldots, n\} \) (which satisfies i) and ii)) or \( \Gamma''_n = \{0, 1, \ldots, n-1\} \cup \{n+1, \ldots, 2n\} \) (which neither satisfies i) nor ii)).

The concept of Følner nets for C*-algebras was introduced in [Bed 2], but is essentially due to Connes ([Con 1, Con 2]). The definition is as follows.

Let \( \mathcal{H} \) be a complex Hilbert space, \( \{P_\alpha\} \) be a net of non-zero finite dimensional projections in \( \mathcal{B}(\mathcal{H}) \) and \( \mathcal{A} \) be a C*-subalgebra of \( \mathcal{B}(\mathcal{H}) \) containing the identity operator \( \mathbf{1} \) on \( \mathcal{H} \). We denote the canonical trace on \( \mathcal{B}(\mathcal{H}) \) by \( \text{Tr} \) and we set \( \|S\|_1 = \text{Tr}(|S|) \) and \( \|S\|_2 = (\text{Tr}(S^*S))^{1/2} \) for \( S \in \mathcal{B}(\mathcal{H}) \).

The following lemma is essential well-known, but for the convenience of the reader, we sketch the proof.

**Lemma 1:** The following four conditions on \( \{P_\alpha\} \) are equivalent:

\[
(F_p) \quad \lim_{\alpha} \frac{\|AP_\alpha - AP_\alpha\|_p}{\|P_\alpha\|_p} = 0 \quad \text{for all } A \in \mathcal{A} \quad (p = 1, 2)
\]

\[
(F'_p) \quad \lim_{\alpha} \frac{\|(I - P_\alpha)AP_\alpha\|_p}{\|P_\alpha\|_p} = 0 \quad \text{for all } A \in \mathcal{A} \quad (p = 1, 2)
\]

**Proof:** Let \( \mathcal{U} \) denote the unitary group of \( \mathcal{A} \). It is nearly obvious that \((F_p)\) is equivalent to

\[
(F''_p) \quad \lim_{\alpha} \frac{\|UP_\alpha U^* - P_\alpha\|_p}{\|P_\alpha\|_p} = 0 \quad \text{for all } U \in \mathcal{U}
\]
for each $p = 1, 2$. 
Suppose that $(F_1')$ is satisfied. The Powers-Størmer inequality ([PS]) says that for all positive trace-class operators $S$ and $T$ on $\mathcal{H}$ we have

$$\|S^{1/2} - T^{1/2}\|_2^2 \leq \|S - T\|_1.$$ 

Therefore we have

$$\left( \frac{\|UP\alpha U^* - P\alpha\|_2}{\|P\alpha\|_2} \right)^2 = \frac{\|(UP\alpha U^*)^{1/2} - P\alpha^{1/2}\|_2^2}{\|P\alpha\|_1} \leq \frac{\|UP\alpha U^* - P\alpha\|_1}{\|P\alpha\|_1}$$

for all $U \in \mathcal{U}$ and all $\alpha$, from which it follows clearly that $(F_2''')$ is satisfied. 

Conversely, suppose that $(F_2''')$ is satisfied. Then we have

$$\|UP\alpha U^* - P\alpha\|_1 = \|UP\alpha U^* - UP\alpha U^*P\alpha + UP\alpha U^*P\alpha - P\alpha\|_1 \leq \|UP\alpha U^*(UP\alpha U^* - P\alpha)\|_1 + \|(UP\alpha U^* - P\alpha)P\alpha\|_1 \leq \|UP\alpha U^*\|_2 \|UP\alpha U^* - P\alpha\|_2 + \|UP\alpha U^* - P\alpha\|_2 \|P\alpha\|_2 = 2\|P\alpha\|_2 \|UP\alpha U^* - P\alpha\|_2,$$

hence

$$\frac{\|UP\alpha U^* - P\alpha\|_1}{\|P\alpha\|_1} \leq 2\|P\alpha\|_2 \|UP\alpha U^* - P\alpha\|_2 = 2\|UP\alpha U^* - P\alpha\|_2 \|P\alpha\|_1$$

for all $U \in \mathcal{U}$ and all $\alpha$, from which it follows that $(F_1')$ is satisfied. 

Let now $p \in \{1, 2\}$. Suppose that $(F_p)$ is satisfied. Since we have

$$\|(I - P\alpha)AP\alpha\|_p = \|(I - P\alpha)(AP\alpha - P\alpha A)\|_p \leq \|I - P\alpha\| \|AP\alpha - P\alpha A\|_p \leq \|AP\alpha - P\alpha A\|_p$$
for all $A \in \mathcal{A}$, it follows that $(F'_p)$ is satisfied. Conversely, suppose that $(F'_p)$ is satisfied. Let $A \in \mathcal{A}$. Then we have

$$\lim_{\alpha \to 0} \frac{\|(I - P_{\alpha})AP_{\alpha}\|_p}{\|P_{\alpha}\|_p} = 0 \quad \text{and} \quad \lim_{\alpha \to 0} \frac{\|P_{\alpha}A(I - P_{\alpha})\|_p}{\|P_{\alpha}\|_p} = \lim_{\alpha \to 0} \frac{\|(I - P_{\alpha})A^*P_{\alpha}\|_p}{\|P_{\alpha}\|_p} = 0.$$ 

Now, as

$$\|AP_{\alpha} - P_{\alpha}A\|_p = \|AP_{\alpha} - P_{\alpha}AP_{\alpha} + P_{\alpha}AP_{\alpha} - P_{\alpha}A\|_p \leq \|(I - P_{\alpha})AP_{\alpha}\|_p + \|P_{\alpha}A(I - P_{\alpha})\|_p,$$

it follows that

$$\lim_{\alpha \to 0} \frac{\|AP_{\alpha} - P_{\alpha}A\|_p}{\|P_{\alpha}\|_p} = 0,$$

hence $(F_p)$ is satisfied. \hfill \Box

We say that the net $\{P_{\alpha}\}$ is a Følner net for $\mathcal{A}$ whenever it satisfies one of the equivalent conditions in Lemma 1.

**Remarks**

a) It follows from an elementary $\varepsilon/3$-argument that in order to show that $\{P_{\alpha}\}$ is a Følner net for $\mathcal{A}$, it is enough to check that $(F_1)$ or $(F_2)$ (resp. $(F'_1)$ or $(F'_2)$) holds for a generator set (resp. self-adjoint generator set) for $\mathcal{A}$.

b) Suppose that $\mathcal{H}$ is separable with dim $\mathcal{H} = +\infty$ and $A \in B(\mathcal{H})$ is self-adjoint. Let $\{P_n\}$ be a sequence of non-zero finite dimensional orthogonal projections in $B(\mathcal{H})$ satisfying $P_n \leq P_{n+1}$ for all $n$ and $\lim_{n \to +\infty} P_n = I$ (in the strong operator topology). Assume that the following condition holds:

$$(*) \quad \sup_{n \in \mathbb{N}} \|\|(I - P_n)AP_n\|_2 < +\infty.$$
Examples of when such a situation occurs are provided by self-adjoint operators possessing a band-limited matrix representation (cf. [Arv 1]). Arveson shows in [Arv 1; proof of Theorem 3.8] that the essential spectrum of $A$ is described by

$$\text{ess.sp}(A) = \{\lambda \in \mathbb{R} | \lim_{n \to +\infty} N_n(U) = +\infty \text{ for every open set } U \text{ in } \mathbb{R} \text{ containing } \lambda\}$$

where $N_n(U)$ denotes the number of eigenvalues of $P_n A|_{P_n(H)}$ belonging to $U$ (multiplicities counted). Now, it is clear that $(*)$ implies that

$$\lim_{n \to +\infty} \frac{\| (I - P_n) A P_n \|_2}{\| P_n \|_2} = 0,$$

so \{P_n\} is a Følner sequence for the C*-algebra generated by $A$ and $I$ (using a)).

We shall study the relationship between $\text{sp}(A)$, $\text{ess.sp}(A)$ and $\text{sp}(P_n A|_{P_n(H)})$ in the next section in the more general context when $A$ is a self-adjoint element in a C*-algebra $\mathcal{A}$ and \{\{P_n\}\} is a Følner net for $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$.

Examples of Følner nets are easily provided with the help of the next proposition.

**Proposition 2:** Assume that net \{\{P_n\}\} satisfies

$$(\text{QD}) \quad \lim_{\alpha} \| AP_{\alpha} - P_{\alpha} A \| = 0 \quad \text{for all } A \in \mathcal{A},$$

(in other words that \{\{P_n\}\} quasidiagonalizes $\mathcal{A}$).

Then \{\{P_n\}\} is a Følner net for $\mathcal{A}$.

**Proof:** Obvious from the fact that

$$\| AP_{\alpha} - P_{\alpha} A \|_1 \leq \text{rank}(AP_{\alpha} - P_{\alpha} A) \| AP_{\alpha} - P_{\alpha} A \| \leq 2 \| P_{\alpha} \|_1 \| AP_{\alpha} - P_{\alpha} A \|$$

for all $A \in \mathcal{A}$ and all $\alpha$. \qed

Suppose for example that $\mathcal{A} = \bigcup_{n \geq 1} \mathcal{A}_n \subseteq \mathcal{B}(\mathcal{H})$ is an AF-algebra, where $I \in \mathcal{A}_1 \subseteq \cdots \subseteq A_n \subseteq \mathcal{A}_{n+1} \subseteq \cdots \subseteq \mathcal{A}$ and all $\mathcal{A}_n$ are finite dimensional C*-algebras. Choose $\zeta \in \mathcal{H} \setminus \{0\}$
and let $P_n$ be the orthogonal projection from $\mathcal{H}$ onto $[A_n, \mathcal{C}]$. Then it is straightforward to check that $\{P_n\}$ satisfies (QD), hence $P_n$ is a Følner net for $\mathcal{A}$ by Proposition 2.

Going back to our general setting, suppose that $\{P_\alpha\}$ is a Følner net for $\mathcal{A}$ and consider the net $\{\Psi_\alpha\}$ of states on $\mathcal{B}(\mathcal{H})$ defined by

$$\Psi_\alpha(S) = \frac{1}{\|P_\alpha\|_1} Tr(P_\alpha S), \quad S \in \mathcal{B}(\mathcal{H}).$$

Let then $\Psi$ any weak* cluster point of $\{\Psi_\alpha\}$. Then a simple calculation using $(F_1)$ gives that $\Psi$ is a hypertrace on $\mathcal{A}$, i.e. $\Psi$ is a state on $\mathcal{B}(\mathcal{H})$ satisfying $\Psi(AS) = \Psi(SA)$ for all $A \in \mathcal{A}, S \in \mathcal{B}(\mathcal{H})$. Conversely, as explained in [Bed 2], Connes shows in [Con 1, Con 2] that one may produce (non-constructively) a Følner net for $\mathcal{A}$ whenever given a hypertrace on $\mathcal{A}$. Hence we have

**Theorem 3:** $\mathcal{A}$ has a Følner net if and only if $\mathcal{A}$ has a hypertrace.

Let now $\Gamma$ denote a (discrete) group and $\mathfrak{vN}(\Gamma)$ its (left) group von Neumann algebra, which acts on $\ell^2(\Gamma)$. Then $\mathfrak{vN}(\Gamma)$ is a finite von Neumann algebra and it is well known from Connes' work that $\Gamma$ is amenable if and only if $\mathfrak{vN}(\Gamma)$ has a hypertrace. Especially, this implies that if $\Gamma$ is amenable, then $\mathfrak{vN}(\Gamma)$ has a Følner net. However it is not obvious that this Følner net is related to some Følner net for $\Gamma$. As we have not seen a proof of this fact in the literature, we shall show that one may naturally associate a Følner net for $\mathfrak{vN}(\Gamma)$ to each Følner net for $\Gamma$. To cover a wider class of examples, we shall do this for twisted group von Neumann algebras as well.

Let us first recall some elementary facts about these algebras (the reader may consult e.g. [ZM] for more information).

Let $\sigma \in Z^2(\Gamma, \mathbb{T})$ denote a normalized 2-cocycle on $\Gamma$, i.e.

$$\sigma : \Gamma \times \Gamma \to \mathbb{T} = (\text{the circle group})$$

satisfies

$$\sigma(a, b)\sigma(ab, c) = \sigma(b, c)\sigma(a, bc) \quad \text{and}$$

$$\sigma(a, 1) = \sigma(1, a) = 1 \quad \text{for all } a, b, c \in \Gamma.$$
Let $\lambda_\sigma(a)$ denote the unitary operator on $\ell^2(\Gamma)$ given by

$$(\lambda_\sigma(a)\zeta)(b) = \sigma(b^{-1}, a)\zeta(a^{-1}b),$$

where $a \in \Gamma, \zeta \in \ell^2(\Gamma), b \in \Gamma$.

For all $a, b \in \Gamma$ we have then

$$\lambda_\sigma(a)\lambda_\sigma(b) = \sigma(a, b)\lambda_\sigma(ab),$$

and $a \rightarrow \lambda_\sigma(a)$ is usually called the (left) projective regular representation of $\Gamma$ on $\ell^2(\Gamma)$ associated with $\sigma$. We denote by $C^*_r(\Gamma, \sigma)$ (resp. $vN(\Gamma, \sigma)$) the C*-algebra (resp. the von Neumann algebra) generated by $\lambda_\sigma(\Gamma)$. The case $\sigma = 1$ just gives the reduced C*-algebra (resp. the group von Neumann algebra) of $\Gamma$ denoted by $C^*_r(\Gamma)$ (resp. $vN(\Gamma)$). The twisted group von Neumann algebra $vN(\Gamma, \sigma)$ (resp. $C^*_r(\Gamma, \sigma)$) has a canonical faithful tracial state $\tau$ given by

$$\tau(S) = \langle S\delta_1, \delta_1 \rangle, \quad S \in vN(\Gamma, \sigma) \quad (\text{resp. } C^*_r(\Gamma, r)),$$

where $\delta = \delta_1$ denotes the delta-function at 1.

If $\{\delta_a\}_{a \in \Gamma}$ denotes the canonical basis of $\ell^2(\Gamma)$ consisting of the delta-functions and $S \in vN(\Gamma, \sigma)$, an expression for $\langle S\delta_a, \delta_b \rangle$, i.e. for a matrix-element in the matrix of $S$ w.r.t. this basis, is as follows. Set $f_S = S\delta \in \ell^2(\Gamma)$ and define $\tilde{f}_S \in \ell^2(\Gamma)$ by

$$\tilde{f}_S(a) = \overline{\sigma(a, a^{-1})f_S(a)}, \quad a \in \Gamma.$$

(One checks that $\tilde{f}_S(a) = \tau(S\lambda_\sigma(a)^*)$, so $\tilde{f}_S(a)$ is the "Fourier coefficient" of $S$ at $a$).

A tedious, but elementary, computation gives that

$$\langle S\delta_a, \delta_b \rangle = \sigma(b^{-1}, ba^{-1})\tilde{f}_S(ba^{-1}), \quad a, b \in \Gamma.$$

We are now ready to prove:
**Proposition 4:** Assume that $\Gamma$ is amenable and let $\{\Gamma_\alpha\}$ be a Følner net for $\Gamma$. For each $\alpha$ identify $\ell^2(\Gamma_\alpha)$ as a subspace of $\ell^2(\Gamma)$ as usual and let $P_\alpha$ denote the orthogonal projection from $\ell^2(\Gamma)$ onto $\ell^2(\Gamma_\alpha)$. Then $\{P_\alpha\}$ is a Følner net for $\nu_N(\Gamma, \sigma)$.

**Proof:** Let $F$ be an non-empty finite subset of $\Gamma$ and let $P$ denote the orthogonal projection from $\ell^2(\Gamma)$ onto $\ell^2(F)$ (identified as a subspace of $\ell^2(\Gamma)$). Let $a, b \in \Gamma$. An easy computation gives that

$$[\lambda_\sigma(a)P - P\lambda_\sigma(a)]\delta_b = \begin{cases} 
\sigma(b^{-1}a^{-1}, a)\delta_{ab}, & b \in F \setminus a^{-1}F \\
-\sigma(b^{-1}a^{-1}, a)\delta_{ab}, & b \in a^{-1}F \setminus F, \\
0, & b \notin a^{-1}F \Delta F
\end{cases}$$

Hence, the range of $\lambda_\sigma(a)P - P\lambda_\sigma(a)$ is linearly spanned by $\{\delta_{ab} | b \in a^{-1}F \Delta F\}$, so

$$\text{rank} (\lambda_\sigma(a)P - P\lambda_\sigma(a)) = \#(a^{-1}F \Delta F).$$

Therefore we get

$$||\lambda_\sigma(a)P_\alpha - P_\alpha\lambda_\sigma(a)||_1 \leq ||\lambda_\sigma(a)P_\alpha - P_\alpha\lambda_\sigma(a)|| \cdot \text{rank} (\lambda_\sigma(a)P_\alpha - P_\alpha\lambda_\sigma(a)) \leq 2 \cdot \#(a^{-1}\Gamma_\alpha \Delta \Gamma_\alpha),$$

so

$$\frac{||\lambda_\sigma(a)P_\alpha - P_\alpha\lambda_\sigma(a)||_1}{||P_\alpha||_1} \leq 2 \frac{\#(a^{-1}\Gamma_\alpha \Delta \Gamma_\alpha)}{\#(\Gamma_\alpha)}.$$

Since $\{\Gamma_\alpha\}$ is assumed to be a Følner net for $\Gamma$, this implies that

$$\lim_\alpha \frac{||\lambda_\sigma(a)P_\alpha - P_\alpha\lambda_\sigma(a)||_1}{||P_\alpha||_1} = 0.$$

As $\{\lambda_\sigma(a)|a \in \Gamma\}$ generates $C^*_\tau(\Gamma, \sigma)$ as a C*-algebra, this shows that $\{P_\alpha\}$ is a Følner net for $C^*_\tau(\Gamma, \sigma)$ (using $(F_1)$ for $A = C^*_\tau(\Gamma, \sigma)$ and a previous remark).

To show that $\{P_\alpha\}$ is a Følner net for $\nu_N(\Gamma, \sigma)$, we shall check that $\{P_\alpha\}$ satisfies $(F'_2)$ for $A = \nu_N(\Gamma, \sigma)$. First, let $S \in \nu_N(\Gamma, \sigma), F$ and $P$ be as above. Then we have

$$\|(I - P)SP\|_2^2 = \sum_{a \in F} (\sum_{b \in \Gamma \setminus F} |\tilde{f}_S(ba^{-1})|^2) \quad \text{(using (1)).}$$

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Now, let $A \in vN(\Gamma, \sigma)$ and $\epsilon > 0$ be given. Then choose $g \in \ell^2(\Gamma)$ with finite support such that $\|\tilde{f}_A - g\|_2 < \epsilon/2$ and define $B \in C^*_r(\Gamma, \sigma)$ by $B = \sum_{a \in \Gamma} g(a)\lambda_\sigma(a)$ (finite sum). As $\tilde{f}_B = g$, we have $\|\tilde{f}_A - \tilde{f}_B\|_2 < \epsilon/2$. By the above formula (with $S = A - B$ and $P = P_\alpha$) we get

$$
\left( \frac{\|(I - P_\alpha)(A - B)P_\alpha\|_2}{\|P_\alpha\|_2} \right)^2 = \frac{1}{\#\Gamma_\alpha} \sum_{a \in \Gamma_\alpha} \left( \sum_{b \in \Gamma \setminus \Gamma_\alpha} |\tilde{f}_{(A - B)}(ba^{-1})|^2 \right)
$$

$$\leq \frac{1}{\#\Gamma_\alpha} \sum_{a \in \Gamma_\alpha} \left( \sum_{b \in \Gamma} |\tilde{f}_{(A - B)}(ba^{-1})|^2 \right)
$$

$$= \frac{1}{\#\Gamma_\alpha} \sum_{a \in \Gamma_\alpha} \|\tilde{f}_{A - B}\|_2^2
$$

$$= \|\tilde{f}_{A - B}\|_2^2 = \|\tilde{f}_A - \tilde{f}_B\|_2^2 < \left( \frac{\epsilon}{2} \right)^2$$

for all $\alpha$. Now, as we know that $\{P_\alpha\}$ is a Følner net for $C^*_r(\Gamma, \sigma)$, we may choose $\alpha_0$ such that

$$\frac{\|(I - P_\alpha)BP_\alpha\|_2}{\|P_\alpha\|_2} < \frac{\epsilon}{2} \quad \text{whenever} \quad \alpha \succ \alpha_0,$$

and we then get

$$\frac{\|(I - P_\alpha)AP_\alpha\|_2}{\|P_\alpha\|_2} \leq \frac{\|(I - P_\alpha)(A - B)P_\alpha\|_2}{\|P_\alpha\|_2} + \frac{\|(I - P_\alpha)BP_\alpha\|_2}{\|P_\alpha\|_2}
$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{whenever} \quad \alpha \succ \alpha_0.$$

This shows that $\{P_\alpha\}$ satisfies $(F_2')$ for $A = vN(\Gamma, \sigma)$ as desired. $\square$

It is usually not true that a Følner net for a C*-algebra $A \subseteq B(\mathcal{H})$ is also a Følner net for its weak closure $A''$, as the above proof could indicate. For if $A''$ has a Følner net, then $A''$ must have a hypertrace by Theorem 3, and this is impossible whenever $A''$ has

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no tracial state. As an example of when this happens, one may consider the AF-algebra 
\( A = \mathcal{K}(\mathcal{H}) + CI \), where \( \mathcal{K}(\mathcal{H}) \) denoted the compact operators on an infinite dimensional 
\( \mathcal{H} \).

We refer to [Bed 2] for other results concerning the existence/non-existence of Følner 
nets/hypertraces on C*-algebras.

To conclude this section, let \( M \) be a compact Riemannian manifold, \( \{P_n\} \) be a sequence of 
spectral projections for the Laplace-Beltrami operator on \( L^2(M) \) and let \( A \) be a self-adjoint 
order zero pseudo-differential operator with smooth symbol. Under the assumption that 
\( M \) is a rank one symmetric space, Widom ([Wid 2]) gives conditions on either \( \{P_n\} \) or \( A \) 
ensuring that \( \{P_n\} \) is a Følner net for the C*-algebra generated by \( A \) and \( I \) (expressed in 
our terminology) and he then shows a Szegő-type theorem for \( A \) (see [Gui] for a generalization of this result). He points out that, as a corollary, one reobtains Szegő’s theorem 
for smooth real-valued functions on \( \mathbb{T} \) where the sequence \( \{P_n\} \) is now the one associated 
to the subspaces spanned by \( \{e_k : k = 0, \pm 1, \ldots, \pm n\} \) of \( L^2(\mathbb{T}) \) using the same notation as 
in the introduction. We shall show in the next section that one may in fact obtain Szegő’s 
theorem in its full generality following a nearly related line of proof as his.

2. Szegő-pairs.

Let \( H \) be a Hilbert space, \( \{P_\alpha\} \) a net of non-zero finite dimensional orthogonal projections 
in \( B(\mathcal{H}) \) and set \( \mathcal{H}_\alpha = P_\alpha(\mathcal{H}), \ n_\alpha = \dim \mathcal{H}_\alpha = \|P_\alpha\|_1 \). Let \( A \) be a C*-subalgebra of \( B(\mathcal{H}) \) 
containing the identity operator \( I \) on \( \mathcal{H} \) and \( \varphi \) be a state on \( A \).

We recall from the introduction that the pair \( \{\{P_\alpha\}, \varphi\} \) is called a Szegő-pair for \( A \) 
whenever the net \( \{\mu_{\alpha}^A\} \) converges weakly to \( \mu_{\alpha}^\varphi \) for all self-adjoint \( A \in A \), where \( \mu_{\alpha}^A \) 
denotes the spectral measure of \( A_\alpha = P_\alpha A|_{\mathcal{H}_\alpha} \) w.r.t. the normalized trace on \( B(\mathcal{H}_\alpha) \) 
and \( \mu_{\alpha}^\varphi \) denoted the spectral measure of \( A \) w.r.t. \( \varphi \).

A conceptual characterization of Szegő-pairs is given in the next theorem. Its proof, which 
unifies arguments appearing in [Arv 1], [Bed 1], [Bed 2], [Bel 1] and [Wid 2], relies on the 
following lemma.
Lemma 5: Let $P$ be a finite dimensional orthogonal projection in $B(H)$, $A \in B(H)$ and $n \in \mathbb{N}$. Then we have

$$\left\| PA^n P - (PAP)^n \right\|_1 \leq n \left\| A \right\|^{n-1} \left\| (I - P)AP \right\|_1$$

Proof: We use induction on $n$. The inequality is trivially true when $n = 1$. So assume now it is true for some $n \in \mathbb{N}$. Then we have

$$\left\| PA^{n+1} P - (PAP)^{n+1} \right\|_1 = \left\| PA^n (I - P)AP + (PA^n P - (PAP)^n)AP \right\|_1$$

$$\leq \left\| PA^n \right\| \left\| (I - P)AP \right\|_1 + \left\| PA^n P - (PAP)^n \right\|_1 \left\| AP \right\|$$

$$\leq \left\| A \right\|^n \left\| (I - P)AP \right\|_1 + n \left\| A \right\|^{n-1} \left\| (I - P)AP \right\|_1 \left\| A \right\|$$

$$= (n + 1) \left\| A \right\|^n \left\| (I - P)AP \right\|_1$$

i.e. the inequality holds for $n + 1$ as desired. \qed

Theorem 6

i) The pair $\{\{P_\alpha\}, \varphi\}$ is a Szegö-pair for $\mathcal{A}$ if and only if the following two conditions hold:
   a) $\varphi$ is “the trace per unit volume w.r.t. $\{P_\alpha\}$”, i.e. $\varphi(A) = \lim_{\alpha} \frac{1}{\alpha} \text{Tr}(P_\alpha A)$ for all $A \in \mathcal{A}$.
   b) $\{P_\alpha\}$ is a Følner net for $\mathcal{A}$.

ii) If $\{\{P_\alpha\}, \varphi\}$ is a Szegö-pair for $\mathcal{A}$, then $\varphi$ is tracial. Moreover, there exists a hypertrace $\psi$ on $\mathcal{A}$ such that $\psi|\mathcal{A} = \varphi$. Conversely, if $\mathcal{A}$ has a hypertrace, then $\mathcal{A}$ has a Szegö-pair.

iii) Assume that $\mathcal{A}$ has a unique tracial state $\varphi$. Then $\{\{P_\alpha\}, \varphi\}$ is a Szegö-pair for $\mathcal{A}$ if and only if $\{P_\alpha\}$ is a Følner net for $\mathcal{A}$. 

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**Proof:** i) Suppose first that a) and b) hold. Let $A \in \mathcal{A}$ be self-adjoint. Since $\text{sp}(A)$ and $\text{sp}(A_\alpha)$ are contained in $[-\|A\|, \|A\|]$ for all $\alpha$, we have to show that

$$
\lim_{\alpha}\frac{1}{n_\alpha} Tr_\alpha(g(A_\alpha)) = \varphi(g(A)) \quad \text{for all} \quad g \in C([-\|A\|, \|A\|]),
$$

where $Tr_\alpha$ denotes the (unnormalized) trace on $\mathcal{B}(\mathcal{H}_\alpha)$ for each $\alpha$.

An easy application of the Stone-Weierstrass theorem gives that it is enough to check this only for $g(t) = t^n, n = 0, 1, 2, \ldots$, i.e. that

$$
\lim_{\alpha}\frac{1}{n_\alpha} Tr_\alpha((A_\alpha)^n) = \varphi(A^n)
$$

hold for $n = 0, 1, 2, \ldots$

The case $n = 0$ is trivial, and $n = 1$ is just a rephrasement of a).

So let $n \geq 2$. As we have

$$
\varphi(A^n) = \lim_{\alpha}\frac{1}{n_\alpha} Tr(P_\alpha A^n P_\alpha) \quad \text{(by a))} \quad \text{and}
$$

$$
Tr_\alpha((A_\alpha)^n) = Tr((P_\alpha A P_\alpha)^n),
$$

it is enough to check the following equality:

$$
\lim_{\alpha}\frac{1}{n_\alpha} Tr(P_\alpha A^n P_\alpha - (P_\alpha A P_\alpha)^n) = 0.
$$

Now we have

$$
|Tr(P_\alpha A^n P_\alpha - (P_\alpha A P_\alpha)^n)| \leq \|P_\alpha A^n P_\alpha - (P_\alpha A P_\alpha)^n\|_1 \leq n\|A\|^{n-1}\|(I - P_\alpha)A P_\alpha\|_1
$$

for all $\alpha$, using Lemma 5 to obtain the second inequality. Further, by b), we have
\[
\lim_{\alpha} \frac{1}{n_\alpha} \|(I - P_\alpha)AP_\alpha\|_1 = 0,
\]
and the desired equality clearly follows.

Next, suppose that \(\{P_\alpha, \varphi\}\) is a Szegő-pair for \(\mathcal{A}\). Then a) holds for all self-adjoint elements in \(\mathcal{A}\). To check b), let \(A \in \mathcal{A}\) be self-adjoint. Then we have

\[
\lim_{\alpha} \frac{1}{n_\alpha} Tr((P_\alpha AP_\alpha)^2) = \varphi(A^2) \quad \text{ (using the assumption) and}
\]

\[
\lim_{\alpha} \frac{1}{n_\alpha} Tr(P_\alpha A^2 P_\alpha) = \varphi(A^2) \quad \text{ (since a) holds),}
\]

hence

\[
\lim_{\alpha} \frac{1}{n_\alpha} Tr(P_\alpha A^2 P_\alpha - (P_\alpha AP_\alpha)^2) = 0.
\]

But

\[
\|(I - P_\alpha)AP_\alpha\|_2^2 = Tr(P_\alpha A(I - P_\alpha)AP_\alpha)
\]

\[
= Tr(P_\alpha A^2 P_\alpha - (P_\alpha AP_\alpha)^2),
\]

so we get

\[
\lim_{\alpha} \frac{1}{n_\alpha} \|(I - P_\alpha)AP_\alpha\|_2^2 = 0, \text{ hence } \lim_{\alpha} \frac{1}{\|P_\alpha\|_2} \|(I - P_\alpha)AP_\alpha\|_2 = 0.
\]

It follows easily that this holds for all \(A \in \mathcal{A}\), i.e. \(\{P_\alpha\}\) is a Følner net for \(\mathcal{A}\), (using \((F'_2)\)).

ii) Suppose that \(\{P_\alpha, \varphi\}\) is a Szegő-pair for \(\mathcal{A}\). As shown above, we then know that \(\{P_\alpha\}\) is a Følner net for \(\mathcal{A}\), so there exists a hypertrace \(\psi\) on \(\mathcal{A}\) given by any weak* cluster point of the net \(\{\psi_\alpha\}\) of states on \(B(\mathcal{H})\) given by

\[
\psi_\alpha(S) = \frac{1}{n_\alpha} Tr(P_\alpha S), \quad S \in B(\mathcal{H}),
\]

as explained before Theorem 3. From i), it is clear that \(\psi|_{\mathcal{A}} = \varphi\). Especially, \(\varphi\) is tracial. Conversely, suppose that \(\mathcal{A}\) has a hypertrace. Then, by Connes' result (Theorem 3), there exists a Følner net \(\{Q_\beta\}\) for \(\mathcal{A}\) and we can consider the net \(\{\Phi_\beta\}\) of states on \(\mathcal{A}\) given by

\[
\Phi_\beta(A) = \lim_{\beta} \frac{1}{\|Q_\beta\|_1} Tr(Q_\beta A), \quad A \in \mathcal{A}.
\]

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By passing if necessary to a subnet, we may assume that \( \{\Phi_\beta\} \) converges *-weakly to a (tracial) state \( \Phi \) on \( \mathcal{A} \). The pair \( \{\{Q_\beta\}, \Phi\} \) is then a Szegö-pair for \( \mathcal{A} \) by i).

iii) Assume that \( \mathcal{A} \) has a unique tracial state \( \varphi \). If \( \{P_\alpha\} \) is a Følner net for \( \mathcal{A} \) and \( \{\psi_\alpha\} \) is defined as in ii), then we have \( \psi_{\mathcal{A}} = \varphi \) for any weak* cluster point \( \psi \) of \( \{\psi_\alpha\} \). By the weak*-compactness of the state space of \( \mathcal{A} \), this implies that the weak*-limit of \( \{\psi_\alpha, \mathcal{A}\} \) exists and is equal to \( \varphi \), i.e. a) holds, and \( \{\{P_\alpha\}, \varphi\} \) is then a Szegö-pair for \( \mathcal{A} \) by i). The converse implication is a trivial consequence of i). \( \square \)

**Remarks**

1) As mentioned earlier, the question of existence of hypertraces, i.e. of Szegö-pairs by Theorem 6 ii), is studied in [Bed 2]. Let us point out here one consequence. Let \( S \in \mathcal{B}(\mathcal{H}) \) be self-adjoint. Then the C*-algebra \( \mathcal{C}^*(S,I) \) generated by \( S \) and \( I \) is abelian and therefore has a hypertrace ([Bed 2; Proposition 1.8]). Hence we know that \( \mathcal{C}^*(S,I) \) must have a Szegö-pair, but in such a general setting we have no constructive method for describing such a pair. One way out of this, proeminent in this article, is to try to find a larger C*-algebra \( \mathcal{A} \) containing \( \mathcal{C}^*(S,I) \) where Theorem 6 hopefully can be applied to verify that some natural candidate is in fact a Szegö-pair for \( \mathcal{A} \). Indeed, Szegö's theorem should not be considered as a statement concerning a single essentially bounded real-valued function on \( T \), but as a statement yielding for any self-adjoint element in \( L^\infty(T) \).

2) In the case of a unital AF-algebra \( \mathcal{A} \) acting on \( \mathcal{H} \), we have described in Section 1 some Følner nets for \( \mathcal{A} \). By (the proof of) Theorem 6, we then know that \( \mathcal{A} \) has Szegö-pairs. When \( \mathcal{A} \) has more than one tracial state, it seems difficult to give some explicit description of these pairs. On the other hand, when \( \mathcal{A} \) has a unique tracial state (e.g. \( \mathcal{A} \) is UHF), Theorem 6 iii) applies.

As pointed out in the introduction, it is desirable from the point of view of approximation of spectra to find Szegö-pairs \( \{\{P_\alpha\}, \varphi\} \) for \( \mathcal{A} \) where \( \varphi \) is *faithful*. Of course, this is automatic when \( \mathcal{A} \) is simple (e.g. \( \mathcal{A} \) is UHF). We shall see that this can also be achieved in several other interesting cases. In this connection, let us mention the following proposition (mostly folklore) which is related to our final remark in Section 1.

**Proposition 7:** Assume that \( \{\{P_\alpha\}, \varphi\} \) is a Szegö-pair for \( \mathcal{A} \) satisfying \( \lim_{\alpha} [\dim \mathcal{H}_\alpha] = +\infty \) (i.e. given \( M > 0 \), there exists \( \alpha_0 \) such that \( \dim \mathcal{H}_\alpha > M \) for all \( \alpha > \alpha_0 \)) and let
$A \in \mathcal{A}$ be self-adjoint. Then

$$\text{supp}(\mu^\varphi_A) \subseteq \text{ess.sp}(A).$$

Moreover, if $\varphi$ is faithful, then we have

$$\text{supp}(\mu^\varphi_A) = \text{ess.sp}(A) = \text{sp}(A).$$

**Proof:** By Theorem 6 we know that $\varphi$ is tracial and is the trace per unit volume w.r.t. $\{P_\alpha\}$. To show the first assertion, let $\lambda \in \mathbb{R} \setminus \text{ess.sp}(A)$. Then either $\lambda \not\in \text{sp}(A)$ or $\lambda \in \text{sp}(A)$ is an isolated eigenvalue of $A$ with finite multiplicity. Hence, the characteristic function $\chi_{\{\lambda\}}$ is continuous on $\text{sp}(A)$, so $E = \chi_{\{\lambda\}}(A)$ is an orthogonal projection in $\mathcal{A}$ satisfying $0 \leq d = \dim E(\mathcal{H}) < \infty$. Let $U$ be an open interval in $\mathbb{R}$ satisfying $U \cap \text{sp}(A) = \{\lambda\}$, and set $n_\alpha = \dim \mathcal{H}_\alpha \in (0, +\infty)$. Then we have

$$\mu^\varphi_A(U) = \int_{\text{sp}(A)} \chi_U(t) d\mu^\varphi_A(t) = \int_{\text{sp}(A)} \chi_{\{\lambda\}}(t) d\mu^\varphi_A(t)$$

$$= \varphi(E) = \lim_{\alpha} \frac{1}{n_\alpha} \text{Tr}(P_\alpha E)].$$

But

$$\frac{1}{n_\alpha} \text{Tr}(P_\alpha E)] \leq \frac{1}{n_\alpha} \|P_\alpha\| \|E\|_1 \leq \frac{d}{n_\alpha} \to 0$$

as $\alpha$ grows since $n_\alpha \to +\infty$. Hence we get $\mu^\varphi_A(U) = 0$, i.e. $\lambda \not\in \text{supp}(\mu^\varphi_A)$. This shows the first assertion. If now $\varphi$ is faithful, we have (cf. our discussion in the introduction for the first equality):

$$\text{sp}(A) = \text{supp}(\mu^\varphi_A) \subseteq \text{ess.sp}(A) \subseteq \text{sp}(A)$$

which proves the second assertion. \qed
Unless in the case where $\mathcal{A}$ has a non-zero finite dimensional invariant subspace, it seems reasonable to expect that a Szegö-pair for $\mathcal{A}$ will satisfy the assumption in Proposition 7.

We now turn our attention to twisted group von Neumann algebras. In the rest of this section we let $\Gamma$ denote a discrete group, $\sigma \in Z^2(\Gamma, \Gamma)$ and $\tau$ denote the canonical tracial state of $vN(\Gamma, \sigma)$.

**Proposition 8:** Let $\{\Gamma_\alpha\}$ be any net of non-empty finite subsets of $\Gamma$ and $\{P_\alpha\}$ be the net of non-zero finite dimensional projections in $\mathcal{B}(\ell^2(\Gamma))$ naturally associated to $\{\Gamma_\alpha\}$. Then we have

$$\tau(A) = \lim_{\alpha} \frac{1}{\|P_\alpha\|_1} Tr(P_\alpha A) \quad \text{for all } A \in vN(\Gamma, \sigma),$$

i.e. $\tau$ is the trace per unit volume w.r.t. $\{P_\alpha\}$.

**Proof:** We use the notation introduced in the proof of Proposition 5. Let $A \in vN(\Gamma, \sigma)$ and $a \in \Gamma$. Then

$$< A\delta_\alpha, \delta_\alpha \sigma = (a^{-1}, aa^{-1}) \tilde{f}_A(aa^{-1}) = \tilde{f}_A(1) = f_A(1) = < A\delta, \delta \sigma = \tau(A),$$

hence

$$Tr(P_\alpha A) = \sum_{a \in \Gamma_\alpha} < A\delta_\alpha, \delta_\alpha = \#(\Gamma_\alpha) \cdot \tau(A) = \|P_\alpha\|_1 \cdot \tau(A)$$

for all $\alpha$, and the assertion follows. \(\square\)

**Remark:** Let $A \in vN(\Gamma, \sigma)$ be self-adjoint and $E_A$ denote its projection-valued spectral measure. Then we have

$$\mu_\tau^*(S) = \tau(E_A(S)), \quad S \text{ Borel in } \mathbb{R}.$$  

As $\tau$ is faithful, we have $\sp(A) = \supp(\mu_\tau^*)$ and $\mu_\tau^*(\{\lambda\}) \neq 0$ if and only if $\lambda$ is an eigenvalue of $A$. When $\Gamma$ is infinite, it follows easily from Proposition 8 that $A$ has no eigenvalue of finite multiplicity (proceeding essentially as in the proof of Proposition 7). Of course,
eigenvalues of infinite multiplicity can not be excluded in general.

In light of Proposition 8, the next proposition shows that the condition "\{P_\alpha\} is a Følner net for \(\mathcal{A}\)" is not redundant in our characterization of a Szegö-pair for \(\mathcal{A}\) given in Theorem 6 i).

**Proposition 9:** Assume that \(\Gamma\) is non-amenable (e.g. \(\Gamma\) is a non-abelian free group). Then \(C^*_\tau(\Gamma, \sigma)\) and \(\nu\mathcal{N}(\Gamma, \sigma)\) have no Szegö-pairs.

**Proof:** When \(\Gamma\) is non-amenable, \(C^*_\tau(\Gamma, \sigma)\)(resp. \(\nu\mathcal{N}(\Gamma, \sigma)\)) has no hypertraces ([Bed 2; Corollary 1.7]), hence \(C^*_\tau(\Gamma, \sigma)\)(resp.\(\nu\mathcal{N}(\Gamma, \sigma)\)) has no Szegö-pairs by Theorem 6 ii). \(\square\)

On the other hand, we have the following result, as promised in the introduction.

**Theorem 10:** Assume that \(\Gamma\) is amenable. Let \(\{\Gamma_\alpha\}\) be any Følner net for \(\Gamma\) and \(\{P_\alpha\}\) be the net of non-zero finite dimensional projections in \(\mathcal{B}(\ell^2(\Gamma))\) naturally associated to \(\{\Gamma_\alpha\}\). Then \(\{P_\alpha, \tau\}\) is a Szegö-pair for \(\nu\mathcal{N}(\Gamma, \sigma)\) and \(\{P_\alpha, \gamma_{C^*_\tau(\Gamma, \sigma)}\}\) is a Szegö-pair for \(C^*_\tau(\Gamma, \sigma)\).

**Proof:** Combine Proposition 4 and Proposition 8 with Theorem 6 i). \(\square\)

**Remarks:** 1) When the assumptions of Theorem 10 are satisfied and \(A \in \nu\mathcal{N}(\Gamma, \sigma)\) is self-adjoint, we obtain that the IDS of \(A\) (w.r.t. \(\{P_\alpha\}\)) at \(\lambda\), \(N_A(\lambda)\), exists whenever \(\mu^*_A(\{\lambda\}) = 0\), i.e. \(\lambda\) is not an eigenvalue of \(A\), and is given by

\[
N_A(\lambda) = \mu^*_A((-\infty, \lambda]) = \tau(E_A(-\infty, \lambda]), \quad \lambda \in \mathbb{R},
\]

which is a generalization of the so-called Shubin's trace formula to our setting ([Shu], [Bel 2]).

2) When \(\Gamma \neq \{1\}\) is an ICC-group, or more generally when each \(\sigma\)-regular conjugacy class in \(\Gamma\) different from \(\{1\}\) is infinite, it is known that \(\nu\mathcal{N}(\Gamma, \sigma)\) is a II\(_1\)-factor ([Pac]), which
is the hyperfinite II$_1$-factor $\mathcal{R}$ if $\Gamma$ is moreover countable and amenable ([Con 1]). Hence Theorem 10 may be used to exhibit concrete Szegő-pairs for $\mathcal{R}$. As an example let us take $\Gamma = \mathbb{Z}^n$, $n \in \mathbb{N}$, and $\sigma \in \mathcal{L}^2(\mathbb{Z}^n, \mathbb{T})$ be given by

$$\sigma(\vec{x}, \vec{y}) = \exp(\pi i \vec{x}^T \mathcal{O} \vec{y}), \quad \vec{x}, \vec{y} \in \mathbb{Z}^n,$$

where $\mathcal{O} = [\theta_{jk}]$ is a skew-symmetric $n \times n$ real matrix. Set $U_j = \lambda_\sigma(\vec{e}_j)$, where $\vec{e}_j$, $j = 1, \ldots, n$, denote the standard generators of $\mathbb{Z}^n$. Then we have

$$U_j U_k = \exp(2\pi i \theta_{jk}) U_k U_j \quad (1 \leq j, k \leq n)$$

and $vN(\mathbb{Z}^n, \sigma)$ is the von Neumann subalgebra of $B(l^2(\mathbb{Z}^n))$ generated by $U_1, U_2, \ldots, U_n$. Further $vN(\mathbb{Z}^n, \sigma)$ is the hyperfinite II$_1$-factor whenever the skew form $(\vec{x}, \vec{y}) \rightarrow \vec{x}^T \mathcal{O} \vec{y}$ on $\mathbb{Z}^n$ is non-degenerate ([Sla], [Pac]). When $n = 2$ and $\mathcal{O} = \begin{bmatrix} 0 & \theta \\ -\theta & 0 \end{bmatrix}$ this happens precisely when $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and of special interest in this case is the so-called discrete magnetic Laplacian $\Delta = \Delta_{\theta, c}$ given by $\Delta = U_1 + U_1^* + c(U_2 + U_2^*)$, $c \in \mathbb{R} \setminus \{0\}$ being some coupling constant. We refer to [Shu] for a recent account on the known spectral properties of $\Delta$ and on the relation between $\Delta$ and the almost Mathieu operator acting on $l^2(\mathbb{Z})$. We intend to study higher-dimensional analogues of $\Delta$ in a forthcoming paper.

We conclude this section with the announced generalization of Szegő's theorem, formulated in a classical manner.

**Theorem 11:** Let $G$ denote a compact abelian group with normalized Haar measure $m$ and let $\Gamma = \hat{G}$ be its dual group (which is abelian and therefore amenable). Let $\{\Gamma_\alpha\}$ be a Følner net for $\Gamma, \Gamma_\alpha = \{\gamma_1^\alpha, \gamma_2^\alpha, \ldots, \gamma_{n_\alpha}^\alpha\}$ where $n_\alpha = \#(\Gamma_\alpha)$. Let $f \in L^\infty(G)$ be real-valued, $\hat{f} \in l^2(\Gamma)$ denote its Fourier-transform and $M_f^\alpha$ be the self-adjoint matrix given by

$$M_f^\alpha = [\hat{f}(\gamma_i^\alpha \gamma_j^\alpha)]_{1 \leq i, j \leq n_\alpha} \quad \text{for each } \alpha.$$

If $\{\lambda_1^\alpha, \lambda_2^\alpha, \ldots, \lambda_{n_\alpha}^\alpha\}$ denote the eigenvalue-list of $M_f^\alpha$, then we have
\[
\lim_{\alpha} \left[ \frac{1}{n_\alpha} \sum_{i=1}^{n_\alpha} g(\lambda_i^\alpha) \right] = \int_G g(f(t)) \, dm(t)
\]

for all \( g \in C_0(\mathbb{R}) \).

**Proof:** It is well known from harmonic analysis that \( \Gamma \) is an orthonormal basis for \( L^2(G) \) and that if \( U : L^2(G) \to \ell^2(\Gamma) \) denotes the Fourier-transform, then \( U \) is a unitary operator satisfying \( UM(L^\infty(G))U^* = vN(\Gamma) \), where \( M : L^\infty(G) \to B(L^2(G)) \) denotes the representation of \( L^\infty(G) \) as multiplication operators on \( L^2(G) \). Further, if \( \overline{m} \) is the state on \( M(L^\infty(G)) \) given by

\[
\overline{m}(M(h)) = \int_G h \, dm \quad , h \in L^\infty(G) , \quad \text{and}
\]

\( \tau \) is the canonical trace on \( vN(\Gamma) \), then we have

\[
\overline{m}(M(h)) = \tau(UM(h)U^*) \quad \text{and}
\]

\[
\overline{m}(g(M(h))) = \overline{m}(g \circ h) = \int_G g(h(t)) \, dm(t) \quad \text{for all } h \in L^\infty(G) , \ g \in C_0(\mathbb{R}) .
\]

For each \( \alpha \), let \( \mathcal{H}_\alpha \) denote the finite dimensional subspace of \( L^2(G) \) spanned by \( \Gamma_\alpha \) and \( Q_\alpha \) be the orthogonal projection from \( L^\infty(G) \) onto \( \mathcal{H}_\alpha \). If \( f \in L^\infty(G) \), then it is easily seen that the matrix of the operator \( Q_\alpha M(f)|_{\mathcal{H}_\alpha} \) in \( B(\mathcal{H}_\alpha) \) is given by

\[
M^\alpha_f = \left[ \hat{f}(\gamma_i^\alpha \gamma_j^\alpha) \right]_{1 \leq i, j \leq n_\alpha} .
\]

Thus, to show the theorem is equivalent to show that the pair \( \{Q_\alpha, \overline{m}\} \) is a Szegö-pair for \( M(L^\infty(G)) \). Now, if \( \{P_\alpha\} \) denotes the net of finite dimensional projections in \( B(\ell^2(\Gamma)) \) associated to \( \{\Gamma_\alpha\} \), then we know from Theorem 10 that \( \{\{P_\alpha\}, \tau\} \) is a Szegö-pair for \( vN(\Gamma) \). As we have \( Q_\alpha = U^* P_\alpha U \) for each \( \alpha \) and \( \overline{m}(\cdot) = \tau(U \cdot U^*) \), this clearly implies that \( \{\{Q_\alpha\}, \overline{m}\} \) is a Szegö-pair for \( M(L^\infty(G)) \) as desired. \( \square \)
Remark: Based on an idea due to T. Digernes, one can give an elementary more direct proof of Szegö's classical theorem. Indeed, let $\varepsilon_k(z) = z^k, z \in T, k = 0, 1, \ldots$, and let $Q_n$ be the orthogonal projection from $L^2(T)$ onto $\text{lin} \{\varepsilon_k; k = 0, 1, \ldots, n\}$. Denote by $\overline{m}$ the state of $M(L^\infty(T))$ obtained from the normalized Haar measure on $T$. To show that $\{\{Q_n\}, \overline{m}\}$ is a Szegö-pair for $M(L^\infty(T))$, it is enough by Theorem 6 to show that

\begin{align*}
\text{a) } \lim_{n \to +\infty} \frac{1}{n+1} \text{Tr}(Q_n M(f) Q_n) = \overline{m}(M(f)) = \int f \, dm \text{ for all } f \in L^\infty(T) \text{ and}
\end{align*}

\begin{align*}
\text{b) } \{Q_n\} \text{ is a Følner net for } M(L^\infty(T)).
\end{align*}

Now, $\text{Tr}(Q_n M(f) Q_n) = (n+1) \int f dm$ for all $f \in L^\infty(T), n \in \mathbb{N}$, so a) follows.

Further, let $f \in L^\infty(T)$. Then we have

\begin{align*}
\left( \frac{\| (I - Q_n) M(f) Q_n \|_2}{\| Q_n \|_2} \right)^2 &= \frac{1}{n+1} \sum_{i=0}^{n} \left( \sum_{j \in \mathbb{Z} \setminus \{0, 1, \ldots, n\}} |\hat{f}(j-i)|^2 \right) \\
&= \frac{1}{n+1} \sum_{i=0}^{n} \left( \sum_{j \in \mathbb{Z}} |\hat{f}(j-i)|^2 \right) - \frac{1}{n+1} \sum_{i=0}^{n} \left( \sum_{j=0}^{n} |\hat{f}(j-i)|^2 \right) \\
&= \| \hat{f} \|_2^2 - \frac{1}{n+1} \left( \sum_{k=0}^{n} (n+1+k)|\hat{f}(k)|^2 + \sum_{\ell=0}^{n} (n+1-\ell)|\hat{f}(-\ell)|^2 \right) + |\hat{f}(0)|^2.
\end{align*}

As $\hat{f} \in \ell^2(\mathbb{Z})$, the series $\sum_{k=0}^{\infty} |\hat{f}(k)|^2$ and $\sum_{\ell=0}^{\infty} |\hat{f}(-\ell)|^2$ are convergent, and therefore also Cesaro-convergent, i.e.

\begin{align*}
\lim_{n \to +\infty} \left( \sum_{k=0}^{n} \left( 1 - \frac{k}{n+1} \right) |\hat{f}(k)|^2 \right) = \sum_{k=0}^{\infty} |\hat{f}(k)|^2 \quad \text{and}
\end{align*}

\begin{align*}
\lim_{n \to +\infty} \left( \sum_{\ell=0}^{n} \left( 1 - \frac{\ell}{n+1} \right) |\hat{f}(-\ell)|^2 \right) = \sum_{\ell=0}^{\infty} |\hat{f}(-\ell)|^2.
\end{align*}

Hence, we get $\lim_{n \to +\infty} \frac{\| (I - Q_n) M(f) Q_n \|_2}{\| Q_n \|_2} = 0$. So b) follows (using $F_2'$).
3. Applications to C*-algebras associated with discretized Schrödinger operators.

Let \( \lambda \) denote the (left) regular representation of \( \mathbb{Z}^n(n \geq 1) \) on \( \ell^2(\mathbb{Z}^n) \) and \( \{ \vec{e}_i \}_{i=1,...,n} \) the standard generators of \( \mathbb{Z}^n \).

A (bounded)(self-adjoint) discretized Schrödinger operator on \( \ell^2(\mathbb{Z}^n) \) is an operator of the form

\[
H = -L + V
\]

where \( L = \sum_{i=1}^{n} [\lambda(\vec{e}_i) + \lambda(\vec{e}_i)^*) - 2nI] \) is the Laplacian on \( \ell^2(\mathbb{Z}^n) \) and \( V \) is a multiplication operator by a bounded real-valued function on \( \mathbb{Z}^n \) playing the role of a potential. It is common to consider \(+L\) instead of \(-L\), and to subsume the diagonal terms of \( L \) into the potential (cf.[CFKS]), i.e. one considers operators of the form

\[
H = L_o + V, \text{ where } L_o = \sum_{i=1}^{n} [\lambda(\vec{e}_i) + \lambda(\vec{e}_i)^*) \text{ and } V \text{ is as above.}
\]

We shall look at \( H \) as being an element of the C*-algebra \( A_V \) generated by \( V \) and \( \{ \lambda(\vec{e}_i) \}_{i=1,...,n} \) which not only contains \( H \) but also each

\[
H_{\vec{m}} = L_o + V_{\vec{m}}, \quad \vec{m} \in \mathbb{Z}^n,
\]

where \( V_{\vec{m}} = \lambda(\vec{m})V\lambda(\vec{m})^* \), which also ought to be observables of the system modelled by \( H \) in many situations (cf. [Bel 1, Bel 2]). If we denote by \( B_V \) the abelian C*-algebra generated by \( \{ V_{\vec{m}}, I | \vec{m} \in \mathbb{Z}^n \} \) then it is clear that \( A_V = C^*\{ W, \lambda(\vec{m}) | W \in B_V, \vec{m} \in \mathbb{Z}^n \} \).

To gain generality, we shall work in the setup where \( \Gamma \) is a discrete group and \( \sigma \in Z^2(\Gamma, \mathbb{T}) \). We then denote by \( \tau : \Gamma \to \ell^\infty(\Gamma) \) the action given by (left) translation, i.e.

\[
\tau_\gamma(f) = f_\gamma, \text{ where } f_\gamma(\omega) = f(\gamma^{-1}\omega), \quad f \in \ell^\infty(\Gamma) \quad \gamma, \omega \in \Gamma.
\]

When identifying \( \ell^\infty(\Gamma) \) with its natural copy in \( \mathcal{B}(\ell^2(\Gamma)) \), as we do in the sequel, one easily checks the following covariance relation:
\((*)\) \[\tau_\gamma(f) = \lambda_\sigma(\gamma) f \lambda_\sigma(\gamma)^* \quad , \gamma \in \Gamma, f \in \ell^\infty(\Gamma),\]

where \(\lambda_\sigma\) denotes the associated (left) projective regular representation of \(\Gamma\) on \(\ell^2(\Gamma)\) as in Section 2.

Let now \(\mathcal{B}\) denote a \(\tau\)-invariant \(C^*\)-subalgebra of \(\ell^\infty(\Gamma)\) containing \(I\) (e.g. \(\mathcal{B} = \mathcal{B}_\nu\) as defined earlier when \(\Gamma = \mathbb{Z}^n\)). We then define

\[\mathcal{A} = C^*(\mathcal{B}, \lambda_\sigma(\Gamma)) = C^*(\{f, \lambda_\sigma(\gamma)|f \in \mathcal{B}, \gamma \in \Gamma\}) \subseteq \mathcal{B}(\ell^2(\Gamma)).\]

When \(\mathcal{B} = \mathbb{C} \cdot I\) we just get \(\mathcal{A} = C^*_\tau(\Gamma, \sigma)\), which has been dealt with in Section 2. Our purpose in this section is to describe how Szegö-pairs for \(\mathcal{A}\) (when such exist) may be produced in other cases of interest.

**Remark:** This remark is intended for readers with some knowledge of the theory of \(C^*\)-dynamical systems. To keep our exposition as elementary as possible, we shall avoid to refer in an essential way to this theory, unless in the last result of this paper.

By identifying \(\sigma(a, b)\) with \(\sigma(a, b) I\) for all \(a, b \in \Gamma\), we may regard the pair \((\tau, \sigma)\) as a cocycle-crossed action of \(\Gamma\) on \(\mathcal{B}\). Hence \((\mathcal{B}, \Gamma, \tau, \sigma)\) is a (twisted) discrete \(C^*\)-dynamical system and we may form the associated full (resp. reduced) \(C^*\)-crossed product \(C^*(\mathcal{B}, \Gamma, \tau, \sigma)\) (resp. \(C^*_\tau(\mathcal{B}, \Gamma, \tau, \sigma)\)), cf. [ZM] or [PR]. If \(i\) denotes the "identity" representation of \(\mathcal{B}\) on \(\ell^2(\Gamma)\) (as multiplication operators), then the covariance relation says that the pair \((i, \lambda_\sigma)\) is a covariant representation of \((\mathcal{B}, \Gamma, \tau, \sigma)\) on \(\ell^2(\Gamma)\). From Proposition 12i) and [ZM; Theorem 4.22], one obtains that

\[\mathcal{A} = C^*(\mathcal{B}, \lambda_\sigma(\Gamma)) \simeq C^*_\tau(\mathcal{B}, \Gamma, \tau, \sigma).\]

Further, when \(\Gamma\) is amenable, we also have

\[C^*_\tau(\mathcal{B}, \Gamma, \tau, \sigma) \simeq C^*(\mathcal{B}, \Gamma, \tau, \sigma).\]

([ZM; Theorem 5.1]).

The following useful proposition is surely well-known, so we just sketch the proof.
Proposition 12: i) There exists a faithful conditional expectation $E$ from $\mathcal{A}$ onto $\mathcal{B}$ satisfying

$$
E(\lambda_\sigma(\gamma)) = \begin{cases} 
1, & \gamma = 1 \\
0, & \gamma \neq 1
\end{cases}
$$

ii) If $\varphi$ is a $\tau$-invariant state on $\mathcal{B}$, then $\tilde{\varphi} = \varphi \circ E$ is a tracial state on $\mathcal{A}$, which is faithful whenever $\varphi$ is faithful.

Proof: i) Let $E'$ denote the canonical conditional expectation from $\mathcal{B}(\ell^2(\Gamma))$ onto $\ell^\infty(\Gamma)$ (cf. [Str]), which is given by

$$
[E'(x)](\gamma) = \langle x\delta_\gamma, \delta_\gamma \rangle \quad x \in \mathcal{B}(\ell^2(\Gamma)), \gamma \in \Gamma.
$$

Then one checks easily that $E = E'_{\mathcal{A}}$ has the desired properties.

ii) Let $\mathcal{A}_0$ be the $*$-algebra generated by $\mathcal{B}$ and $\lambda_\sigma(\Gamma)$. By exploiting the covariance relation (*), one gets that $\mathcal{A}_0 = \{ \sum_{i=1}^n f_i \lambda_\sigma(\gamma_i) | n \in \mathbb{N}, f_1, \ldots, f_n \in \mathcal{B}, \gamma_1, \ldots, \gamma_n \in \Gamma \}$ and further that $\tilde{\varphi}$ is tracial on $\mathcal{A}_0$, hence on $\mathcal{A}$ by density. \qed

As in the previous section, we let $\{\Gamma_\alpha\}$ denote a net of non-empty finite subsets of $\Gamma$ and $\{P_\alpha\}$ the associated net of non-zero finite dimensional orthogonal projections in $\mathcal{B}(\ell^2(\Gamma))$.

Lemma 13: Assume that $\{\Gamma_\alpha\}$ is a Følner net for $\Gamma$. Then $\{P_\alpha\}$ is a Følner net for $\mathcal{A} = C^*(\mathcal{B}, \lambda_\sigma(\Gamma))$.

Proof: Trivially, we have $P_\alpha f = f P_\alpha$ for all $\alpha$ and all $f \in \mathcal{B}$.

Hence $\{P_\alpha\}$ is a Følner net for $\mathcal{B}$. Further, $\{P_\alpha\}$ is a Følner net for $C^*_r(\Gamma, \sigma) = C^*(\lambda_\sigma(\Gamma))$ by Proposition 4. This implies that $\{P_\alpha\}$ is a Følner net for $\mathcal{A}$. \qed

Proposition 14: $\mathcal{A} = C^*(\mathcal{B}, \lambda_\sigma(\Gamma))$ has a Szegö-pair if and only if $\Gamma$ is amenable.
Proof: The if part follows from the combination of Lemma 13, Theorem 3 and Theorem 6 ii). Assume now that $\mathcal{A}$ has a Szegö-pair. Then $C^*_r(\Gamma, \sigma)$ has a Szegö-pair too, hence has a hypertrace by Theorem 6 ii), which implies that $\Gamma$ is amenable by [Bed 2; Corollary 1.7].

In light of Proposition 14, we assume from now on that $\Gamma$ is amenable and $\{\Gamma_\alpha\}$ is a Følner net for $\Gamma$ with associated net $\{P_\alpha\}$ of projections in $B(B(\ell^2(\Gamma)))$.

Theorem 15: Let $\varphi$ be a $\tau$-invariant state on $B$ (such a $\varphi$ exists by the amenability of $\Gamma$) and let $\tilde{\varphi} = \varphi \circ E$ be the associated tracial state on $\mathcal{A} = C^*(B, \lambda_\sigma(\Gamma))$. Assume that $\varphi$ satisfies

$$
\varphi(f) = \lim_\alpha \left[ \frac{1}{\#\Gamma_\alpha} \sum_{\gamma \in \Gamma_\alpha} f(\gamma) \right] \quad \text{for all } f \in B.
$$

Then $\{\{P_\alpha\}, \tilde{\varphi}\}$ is a Szegö-pair for $B$.

Proof: By Theorem 6 i) and Lemma 13, it is enough to show that

$$
\lim_\alpha \left[ \frac{1}{\#\Gamma_\alpha} Tr(P_\alpha A) \right] = \tilde{\varphi}(A) = \varphi(E(A)) \quad \text{for all } A \in \mathcal{A}.
$$

From our assumption on $\varphi$, it is then clearly enough to show that

$$
Tr(P_\alpha A) = \sum_{\gamma \in \Gamma_\alpha} [E(A)](\gamma) \quad \text{for all } A \in \mathcal{A},
$$

which is satisfied since both sides are equal to

$$
\sum_{\gamma \in \Gamma_\alpha} <A \delta_{\gamma}, \delta_{\gamma}> \quad \text{for all } A \in \mathcal{A}.
$$
To apply this theorem, we shall need the following argument, which is essentially the usual one used to show that the existence of a Følner net for $\Gamma$ implies the amenability of $\Gamma$ (when choosing $B = \ell^\infty(\Gamma)$ below): For each $\alpha$, we define

$$
\psi_\alpha(f) = \left[ \frac{1}{\# \Gamma_\alpha} \sum_{\gamma \in \Gamma_\alpha} f(\gamma) \right], \quad f \in B.
$$

Then $\{\psi_\alpha\}$ is a net in the state space $S(B)$ of $B$ and there exists therefore a subnet $\{\psi_\beta\} = \{\psi_{\alpha_\beta}\}$ of $\{\psi_\alpha\}$ which converges $\ast$-weakly to a $\psi \in S(B)$. If we set $\Gamma'_\beta = \Gamma_{\alpha_\beta}$ for each $\beta$, then we get

$$
\psi(f) = \lim_\beta \left[ \frac{1}{\# \Gamma'_\beta} \sum_{\gamma \in \Gamma'_\beta} f(\gamma) \right] \quad \text{for all } f \in B,
$$

and, as $\{\Gamma'_\beta\}$ is a Følner net for $\Gamma$, one verifies without difficulty that $\psi$ is $\tau$-invariant. Note further that if $B$ is assumed to have a unique $\tau$-invariant state $\varphi$, then we must have $\psi = \varphi$ and it follows that the net $\{\psi_\alpha\}$ converges $\ast$-weakly to $\varphi$ (as all its convergent subnets do), i.e. we have

$$
\varphi(f) = \lim_\alpha \left[ \frac{1}{\# \Gamma_\alpha} \sum_{\gamma \in \Gamma_\alpha} f(\gamma) \right] \quad \text{for all } f \in B.
$$

From this discussion, we obtain the following two corollaries to Theorem 15:

**Corollary 16:** There exists a subnet $\{\Gamma'_\beta\}$ of $\{\Gamma_\alpha\}$ and a $\tau$-invariant state $\psi$ on $B$ such that $\{Q_\beta, \tilde{\psi}\}$ is a Szegö-pair for $A = C^*(B, \lambda_\sigma(\Gamma))$, where $Q_\beta$ denotes the projection in $B(\ell^2(\Gamma))$ associated to $\Gamma'_\beta$.

**Corollary 17:** Assume that $B$ has a unique $\tau$-invariant state $\varphi$. Then $\{P_\alpha, \tilde{\varphi}\}$ is a Szegö-pair for $A = C^*(B, \lambda_\sigma(\Gamma))$. 

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Corollary 16 is obviously not very satisfactory. On the other hand, Corollary 17 may be applied in many cases of interest where \( \mathcal{B} \) has a faithful unique \( \tau \)-invariant state \( \varphi \), hence \( \varphi \) is faithful on \( \mathcal{A} \) (cf. Proposition 12 ii)). One important example of such a situation is given by taking \( \mathcal{B} \) to consist of all almost periodic functions on \( \Gamma \), i.e. \( \mathcal{B} = AP(\Gamma) \). By a well-known theorem of von Neumann, \( AP(\Gamma) \) has indeed a faithful unique \( \tau \)-invariant state (see e.g. [Loo]). Especially, we get that the spectrum of a discretized Schrödinger operator \( H_0 = L_0 + V \) on \( \mathbb{Z}^n \) with an almost periodic real-valued potential function \( V \) can be approximated by the spectrum of \( P_m H_0 P_m \), where \( \{ P_m \} \) is any sequence of projections in \( B(\ell^2(\mathbb{Z}^n)) \) associated with a Følner sequence \( \{ \Gamma_m \} \) for \( \mathbb{Z}^n \), e.g. \( \Gamma_m = \{0, 1, \ldots, m \}^n \), or \( \Gamma_m = \{-m, \ldots, 0, \ldots, m\}^n \) (and this is still true if \( L_0 \) denotes a discrete magnetic Laplacian on \( \mathbb{Z}^n \), i.e.

\[
L_0 = \sum_{i=1}^n [\lambda_\sigma(\vec{e}_i) + \lambda_\sigma(\vec{e}_i)^*] \quad \text{where} \quad \sigma \in Z^2(\mathbb{Z}^n, \mathbb{T}).
\]

In many physical applications, the system under study is not modelled by a single Hamiltonian, but by a random family of operators acting on \( \ell^2(\Gamma) \) (cf. [CFKS], [PF]) indexed by a compact Hausdorff space \( X \) on which the group \( \Gamma \) acts by homeomorphisms:

if \( \Gamma \) acts on \( X \) and for each \( x \in X, \pi_x : C(X) \to \ell^\infty(\Gamma) \) is given by

\[
[\pi_x(F)](\gamma) = F(\gamma \cdot x), \quad F \in C(X), \quad \gamma \in \Gamma,
\]

then \( \{\pi_x(F)\}_{x \in X} \) gives a family of potential functions on \( \Gamma \) to which one may associate a random family \( \{H^F_x\}_{x \in X} \) of discretized Schrödinger operators e.g. when \( \Gamma = \mathbb{Z}^n \).

*In the rest of this section, we assume that \( \Gamma \) acts on \( X \) as above and keep our standing assumptions on \( \Gamma, P, \) and \( \sigma \).*

In such a setting, one may form the family \( \{\mathcal{A}_x\}_{x \in X} \) of \( \mathcal{C}^* \)-subalgebras of \( B(\ell^2(\Gamma)) \) given by

\[
\mathcal{A}_x = C^*(B_x, \lambda_\sigma(\Gamma)), \quad \text{where}
\]

\[
B_x = \pi_x(C(X)), \quad x \in X,
\]

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and we have then \( H_{x}^{F} \in A_{x} \) whenever \( H_{x}^{F} \) is defined.

If we denote by \( \alpha : \Gamma \to \text{Aut}(C(X)) \) the associated action of \( \Gamma \) on \( C(X) \), then one verifies that

\[
\pi_{x}(\alpha_{\gamma}(F)) = \tau_{\gamma}(\pi_{x}(F)) = \lambda_{\sigma}(\gamma)\pi_{x}(F)\lambda_{\sigma}(\gamma)^{*}
\]

for all \( x \in X, \gamma \in \Gamma, F \in C(X) \).

Especially, it follows that each \( B_{x} \) is \( \tau \)-invariant. i.e. \( A_{x} \) fits in our earlier setting for each \( x \in X \).

**Remark:** One may regard \((**)*\) as expressing that the pair \((\pi_{x}, \lambda_{\sigma})\) is a covariant representation of the twisted dynamical system \((C(X), \Gamma, \alpha, \sigma)\) (cf. [ZM] of [PR]). Hence there exists an associated representation \( \pi_{x} \times \lambda_{\sigma} \) of the twisted crossed product \( C^{*}(C(X), \Gamma, \alpha, \sigma) \) such that \( A_{x} = (\pi_{x} \times \lambda_{\sigma})(C^{*}(C(X), \Gamma, \alpha, \sigma)) \). Further, from Proposition 12 i), [ZM; Theorem 4.22] and the amenability of \( \Gamma \), it follows that \( \pi_{x} \times \lambda_{\sigma} \) is faithful whenever \( \pi_{x} \) is faithful, i.e. whenever the orbit \( O_{x} = \{ \gamma \cdot x | \gamma \in \gamma \} \) is dense in \( X \).

Corollary 17 may be applied to give the following:

**Corollary 18:** Assume that the action of \( \Gamma \) on \( X \) is uniquely ergodic, i.e. \( C(X) \) has a unique \( \alpha \)-invariant state \( \psi \) and let \( x \in X \). Denote by \( E_{x} \) the faithful conditional expectation of \( A_{x} \) onto \( B_{x} \) provided by Proposition 12 i). Then \( B_{x} \) has a unique \( \tau \)-invariant state \( \varphi_{x} \) determined by \( \varphi_{x} \circ \pi_{x} = \psi \), and \( \{ P_{y} \}, \varphi_{x} \) is a Szegö-pair for \( A_{x} \), where \( \varphi_{x} = \varphi_{x} \circ E_{x} \). If moreover the action of \( \Gamma \) on \( X \) is minimal, then \( \varphi_{x} \) and \( \varphi_{x} \) are faithful.

**Proof:** As \( \Gamma \) is amenable, \( B_{x} \) has at least one \( \tau \)-invariant state, say \( \varphi \). Then it follows from \((**)\) that \( \varphi \circ \pi_{x} \) is an \( \alpha \)-invariant state on \( C(X) \) hence we have \( \varphi \circ \pi_{x} = \psi \) by the uniqueness assumption. As this equation clearly determines \( \varphi \), \( \varphi \) is the only \( \tau \)-invariant state on \( B_{x} \) and we may set \( \varphi_{x} = \varphi \). The second assertion follows then from Corollary 17. Finally, suppose moreover that the action of \( \Gamma \) on \( X \) is minimal. Then we have
supp(μψ) = X, i.e. ψ is faithful (where μψ is the Borel measure on X associated with ψ), and the faithfulness of φx and φx easily follows.

When the action of Γ on X is not uniquely ergodic, a weaker result can be obtained in certain cases. Let us first remark that there always exists at least one ergodic α-invariant state ψ on C(X) (this well-known fact follows from the amenability of Γ and the Krein-Milman theorem). We shall denote by μ = μψ the Borel probability measure on X associated with ψ. Secondly, consider x ∈ Xo, where

\[ X_o = \{ x \in X | O_x \text{ is dense in } X \} = \{ x \in X | \pi_x \text{ is faithful} \} . \]

Then we may clearly define a state φx on Bx by φx(πx(F)) = ψ(F), F ∈ C(X), which is τ-invariant (by (**) ). With these notations we have:

**Corollary 19:** Assume that X is metrizable, Γ is countable and there exists a Følner sequence \{Γ_k\} for Γ which also satisfies:

i) \( Γ_k \subseteq Γ_{k+1} \), ii) \( Γ = \bigcup_{k \in \mathbb{N}} Γ_k \) and

iii) there exists a \( M > 0 \) such that \( \#(Γ_k^{-1}Γ_k)/\#(Γ_k) \leq M \) for all \( k \in \mathbb{N} \).

Then \( X_o \) is Borel and there exists a Borel subset \( Y_o \) of \( X_o \) satisfying \( μ(Y_o) = μ(X_o) \), such that for all \( x \in Y_o \) the pair \( \{P_k, φ_x\} \) is a Szegö-pair for \( A_x \), where \( \{P_k\} \) is the sequence of projections in \( B(ℓ^2(Γ)) \) associated with \( \{Γ_k\} \) and \( φ_x = φ_x \circ E_x \).

**Proof:** From the pointwise ergodic theorem for amenable groups ([E; Theorem 4'], [Pat; Theorem 5.21]) and the ergodicity of μ, we obtain that for each \( F \in C(X) \), there exists a Borel set \( X_F \subseteq X \) such that \( μ(X_F) = 1 \) and

\[ \psi(F) = \lim_{k \to +\infty} \left[ \frac{1}{\#Γ_k} \sum_{γ \in Γ_k} F(γ \cdot x) \right] \quad \text{for all } x \in X_F \]
\[ \psi(F) = \lim_{k \to +\infty} \left[ \frac{1}{\#\Gamma_k} \sum_{\gamma \in \Gamma_k} \pi(x)(F)(\gamma) \right] \quad \text{for all } x \in X_F. \]

Now, as \( X \) is metrizable, \( C(X) \) is norm separable and we may pick a dense subset \( \{F_k\}_{k \in \mathbb{N}} \) in \( C(X) \). Let \( Y = \bigcap_{k=1}^{\infty} X_{F_k} \). Then \( Y \) is a Borel subset of \( X \) such that \( \mu(Y) = 1 \) and a density argument gives:

\[ \psi(F) = \lim_{k \to +\infty} \left[ \frac{1}{\#\Gamma_k} \sum_{\gamma \in \Gamma_k} \pi(x)(F)(\gamma) \right] \quad \text{for all } x \in Y, F \in C(X). \]

Now, as \( X_\circ = \bigcap_{n=1}^{\infty} \left( \bigcup_{\gamma \in \Gamma} V_\gamma \right) \) where \( V_1, V_2, \ldots \) denote a countable base for the topology of \( X \), \( X_\circ \) is Borel. Further, if we set \( Y_\circ = Y \cap X_\circ \), then \( Y_\circ \) is a Borel subset of \( X_\circ \), \( \mu(Y_\circ) = \mu(X_\circ) \) and, as \( \varphi \circ \pi = \psi \) for all \( x \in X_\circ \), we get

\[ \varphi(x)(\pi_x(F)) = \lim_{k \to +\infty} \left[ \frac{1}{\#\Gamma_k} \sum_{\gamma \in \Gamma_k} \pi_x(F)(\gamma) \right] \quad \text{for all } x \in Y_\circ, F \in C(X). \]

The final assertion follows then from Theorem 15. \( \square \)

**Remarks:**
1) \( \Gamma_k = \{-k, \ldots, 0, \ldots, k\}^n \) gives a Følner sequence for \( \mathbb{Z}^n \) satisfying conditions i) ii) and iii). More generally, any nilpotent countable group has a Følner sequence satisfying conditions i) ii) and iii) (cf. [Pat; Problem 6.15]).

2) If moreover the action of \( \Gamma \) on \( X \) is assumed to be minimal in Corollary 19, i.e. \( X_\circ = X \), then \( \mu(Y_\circ) = 1 \). More generally, if \( \psi \) is known to be faithful, then as \( \mu \) is ergodic, we have \( \mu(X_\circ) = 1 \) (copy the proof of [Wal; Theorem 5.15]), hence again \( \mu(Y_\circ) = 1 \).

Corollary 19 is nearly connected to the result of Bellissard proved in the appendix of [Bel 1] (see also [BLT; Theorem 4.5] and [Bel 2; Theorem 9]), which we restate below in a slightly different version (as its proof refers to "the Birkhoff ergodic theorem" for a general action of a countable amenable group on a compact Hausdorff space, but should refer e.g. to [E; Theorem 4']).

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Its formulation involves the (twisted) crossed product $\mathcal{A} = C^*(C(X), \Gamma, \alpha, \sigma)$. There exists a canonical conditional expectation $E$ from $\mathcal{A}$ onto $C(X)$ and $\tilde{\psi} = \psi \circ E$ is then a tracial state on $\mathcal{A}$ (cf. [ZM]). Then we have:

**Theorem 20:** Let $X$, $\Gamma$, $\{\Gamma_k\}$ and $\{P_k\}$ be as in Corollary 19. Let $A$ be a self-adjoint element in $\mathcal{A}$ and for each $x \in X$, set $A_x = (\pi_x \times \lambda_\sigma) (A)$ (which is a self-adjoint element in $\mathcal{A}_x \subseteq B(\ell^2(\Gamma))$). Denote by $\mu_{A_x}^k$ the spectral measure of $P_k A_{x, \ell^2(\Gamma_k)}$ w.r.t. the normalized trace on $B(\ell^2(\Gamma_k))$ and by $\mu_{\mathcal{A}}$ the spectral measure of $A$ w.r.t. $\tilde{\psi}$. Then we have

$$\mu_{\mathcal{A}}^{\tilde{\psi}} = \lim_{k \to \infty} \mu_{A_x}^k \text{ (weakly)} \quad \text{for } \mu\text{-almost all } x \in X.$$ 

**Proof:** For completeness, we sketch the proof. As in the proof of Corollary 19, one obtains from [E; Theorem 4'] that there exists a Borel set $Y \subseteq X$ such that $\mu(Y) = 1$ and

$$\tilde{\psi}(A) = \psi(E(A)) = \lim_{k \to \infty} \frac{1}{\# \Gamma_k} \sum_{\gamma \in \Gamma_k} [\pi_x(E(A))] (\gamma)$$

for all $A \in \mathcal{A}, x \in Y$.

Now, as $[\pi_x(E(A))](\gamma) = [E_x(A_x)](\gamma) = \langle A_x \delta_\gamma, \delta_\gamma \rangle$ for all $x \in X, A \in \mathcal{A}, \gamma \in \Gamma$ (this follows from the definition of $E$ and $E_x$), we get

$$\tilde{\psi}(A) = \lim_{k \to +\infty} \frac{1}{\# \Gamma_k} Tr(P_k A_x) \quad \text{for all } A \in \mathcal{A}, x \in Y$$

So to show that

$$\lim_{k \to \infty} \left[ \frac{1}{\# \Gamma_k} Tr_k(g(P_k A_{x, \ell^2(\Gamma_k)}) \right] = \tilde{\psi}(g(A))$$

for all self-adjoint $A \in \mathcal{A}, g \in C_0(\mathbb{R})$ and $x \in Y$ (which proves the result), it is enough, proceeding as in the proof of Theorem 6 i), to check that
lim_{k \to +\infty} \left[ \frac{1}{\# \Gamma_k} Tr(P_k A_x^n P_k - (P_k A_x P_k)^n) \right] = 0

for all \( A \in \mathcal{A}, n \in \mathbb{N}, n \geq 2 \) and \( x \in Y \). But, as we know that \( \{P_k\} \) is a Følner sequence for \( \mathcal{A}_x \) for each \( x \in X \), this follows in the same way as in the proof of Theorem 6 i). □

Readers acquainted with crossed products will easily realize that Corollary 19 may be deduced from Theorem 20 (which is often called Shubin’s formula). This theorem generalizes earlier results on random Schrödinger operators and illustrates the link between this class of results and the Szegő-type theorems described in this paper. We refer to [Bel 1], [Bel 2], [CFKS] and [PF] for many other results and references on this subject, included many nice applications.

References


