GLOBAL MATRIC MASSEY PRODUCTS AND THE COMPACTIFIED JACOBIAN OF THE E_6-SINGULARITY

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Abstract.
In this paper we compute the compactified Jacobian of the singularity E_6. By [6] this singularity has only a finite number of isomorphism classes of rank 1 torsionfree modules. Using the theory of Matric Massey products [10] we compute the local formal moduli with its local versal family for each local module, and we study the degeneracy of each local module. We give a proof of how the local theory connects to the global theory, i.e. we prove that the morphism from the local formal moduli of a local module to the local ring at the point corresponding to the module on the compactified Jacobian is a smooth morphism. In the case where $M = \overline{E}_6$, that is the normalization, this morphism is an isomorphism. Thus the degeneracy (stratification) diagram for the compactified Jacobian can be found from the degeneracy of the normalization in the local case.

Introduction.
Let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_X$-module over a k-scheme X, k a field. When $\dim_k \operatorname{Ext}^1_X(\mathcal{M},\mathcal{M}) \leq \infty$, Schlessingers theorem [13] tells us that there exists a hull $\hat{H}_{\mathcal{M}}$ of $\operatorname{Def}_{\mathcal{M}}$ - a complete k-algebra - which we shall call the formal moduli of $\mathcal{M}$, and a smooth morphism $\operatorname{Mor}(\hat{H}_{\mathcal{M}}, -) \longrightarrow \operatorname{Def}_{\mathcal{M}}$ such that

$$\operatorname{Mor}(\hat{H}_{\mathcal{M}}, k[x]/(x^2)) \longrightarrow \operatorname{Def}_{\mathcal{M}}(k[x]/(x^2))$$

is a bijection, and there exists a formal versal family. By Laudal [9] (4.2.4), there exists a morphism of complete local k-algebras

$$o : T^2 = Sym_k((\operatorname{Ext}^2_X(\mathcal{M},\mathcal{M})^*)^{\widehat{}} \longrightarrow Sym_k((\operatorname{Ext}^1_X(\mathcal{M},\mathcal{M})^*)^{\widehat{}}$$

determined by a system of generalized Massey products, such that

$$\hat{H}_{\mathcal{M}} \cong T^1 \otimes_{T^2} k.$$

In fact $\hat{H}_{\mathcal{M}}$ is isomorphic to $k[[u_1,...,u_d]]/(f_1,...,f_r)$ where $ext^1(\mathcal{M},\mathcal{M}) = d$, and $r \leq ext^2(\mathcal{M},\mathcal{M})$. The coefficients of these power series f_i are constructed from the Massey products $< \underline{\alpha}, \underline{n} >$ for sequences $\underline{\alpha} = \alpha_1, ..., \alpha_d$ of elements in $\operatorname{Ext}^1_A(\mathcal{M},\mathcal{M})$ and some d-tuples $\underline{n} \in \mathrm{N}^d$. Letting $\{y_1,...,y_r\}$ be a dual basis for $\operatorname{Ext}^2_A(\mathcal{M},\mathcal{M})$, $\{x_1,...,x_d\}$ a dual basis for $\operatorname{Ext}^1(\mathcal{M},\mathcal{M})$, we then have

$$\hat{H}_{\mathcal{M}} \cong k[[u_1,...,u_d]]/(f_1,...,f_r)$$

where

$$f_j = \sum y_j(< \underline{x}^\star; \underline{n} >)\underline{u}^{\underline{n}}.$$

The Massey products $<\underline{\alpha},\underline{n}>\in \text{Ext}^2_X(\mathcal{M},\mathcal{M})$ turns out to be "ordinary" Massey products in the differential graded k-algebra

$$\mathcal{D}(\mathcal{U},\mathcal{H}om^{\cdot}(\mathcal{L}.,\mathcal{L}.))$$

and the computation of the formal moduli reduces to computing these Massey products. This enable us to develop an algorithm for computing the formal moduli, and the formal versal family.

Let X be a k-scheme and $U=\text{Spec}(A)\hookrightarrow X$ an open affine subscheme. If $\mathcal{M}$ is a quasi coherent $\mathcal{O}_X$-module and $M=H^0(U,\mathcal{M})$, we have a restriction morphism of functors $\text{Def}_{\mathcal{M}}\xrightarrow{\phi}\text{Def}_M$ from the global to the local deformation functor. From the universal properties of the formal moduli there exists a k-algebra homomorphism

$$\hat{\phi}:\hat{H}_M\longrightarrow\hat{H}_{\mathcal{M}}.$$

For curves X, $\hat{\phi}$ turns out to be a smooth morphism. This result is known to be untrue when X is of higher dimension, see R.M.Roig [12], where she proves that there exists moduli spaces of invertible sheaves on projective spaces, that have singular points.

The smoothness of the morphism from the local formal moduli to the global formal moduli is then used to confirm and extend some results of Rego[11] and Cook [2]. In the paper "Compactified Jacobians and Curves with simple singularities" [2], Cook study the compactified Jacobian $\overline{P}$ for integral projective curves X over an algebraically closed field k.

$\overline{P}$ is the moduli space of degree 0, rank 1 torsion free sheaves on X, and it is proved to be a projective scheme containing $J_0(X)$, the generalised Jacobian of degree 0 invertible sheaves on X, as an open subscheme, D'Souza [3]. When X is contained in a smooth surface, $\overline{P}$ is irreducible and $\overline{J_0(X)}=\overline{P}$. Thus it makes sense to talk about boundary points of $J_0(X)$.

D'Souza describes this boundary when X has only nodes and simple cusps. This is possible because at any singular point, the stalk of any rank 1 torsion free sheaf is either free or isomorphic to the maximal ideal $\underline{m}$.

In the Gorenstein case, Greuel and Knörrer [6] proves that the simple singularities (i.e. of type ADE) has only a finite number of isomorphism classes of torsion free rank 1 modules. Thus a natural problem is to describe the compactified Jacobians of curves with simple singularities. This is the aim of Cook's paper [2].

Thus Cook gives us the stratification diagrams, implying among other things, the number of irreducible components of the boundary of the generalised Jacobian. This is Rego's result [11].

Let X be a projective curve with only one simple singularity

$$x_0\in U=\text{Spec}(A)\subseteq X.$$

When $\mathcal{M}=\overline{\mathcal{O}}_X$, the normalization of X, and $M=\overline{A}=H^0(U,\mathcal{M})$, we have an isomorphism

$$\hat{\mathcal{O}}_{\overline{P},\mathcal{M}}\cong\hat{H}_M.$$

This follows because the dimension of the fibre of the global to local morphism $\underline{\hat{H}}_{\mathcal{M}} \longrightarrow \underline{\hat{H}}_M$ is $h^1(\mathcal{E}nd_X(\overline{\mathcal{O}}_X)) = p_a(X) = 0$, because $\overline{X}$ is rational. Thus the local to global map is an isomorphism.

We stratify H_M, an algebraization of $\hat{H}_M$, using the stratum

$$S_\nu = \{\underline{t} \in \underline{H} | \mathbf{M}(t) \cong M_\nu\},$$

where $\mathbf{M}$ denotes the versal family and M_ν are the different isomorphism classes of modules over the singularity. A result of Greuel and Pfister [7], states that the only closed stratum is the stratum of the normalization. Thus the stratification diagram for H_M gives the stratification diagram for $\overline{P}$. We then prove that the boundary of the compactified Jacobian is given by the local discriminant

$$\underline{\Delta} = \{p \in \underline{H}_M | \dim \mathbf{V} < d\},$$

where $d = \dim H$ and $\mathbf{V}$ is the kernel of the Kodaira Spencer morphism.

We give an example in the case where $A = E_6$, computing the formal local moduli and the local versal family explicitly, with its degeneracy diagram, confirming and extending the results of Rego [11] and Cook [2].

1.The structure theorem

In this chapter we will use the notation and results in Schlessinger[13]. The connection between moduli-theory and deformation-theory is well known and can be found in this reference. We will give a slightly different proof of the structure theorem (4.2.4) in [9].

Let $F : \underline{\ell} \longrightarrow \underline{\text{sets}}$ be a functor with a prorepresenting hull (hull for short) $\hat{H}$. Put $A^1 = F(k[\varepsilon])$ and assume there is a finite-dimentional k-vectorspace A^2 such that for any surjection $\pi : R \longrightarrow S$ in $\underline{\ell}$ with $\ker \pi \cdot \underline{m}_R = 0$ and for $M_S \in F(S)$, there is an element $o(\pi, M_S) \in A^2 \otimes_k \ker \pi$ such that

$$o(\pi, M_S) = 0 \Leftrightarrow M_S \text{ can be lifted to } R \text{ via } \pi \; .$$

We call this o the obstruction for lifting M_S via π. Assume furthermore that o satisfy the following property : Given a commutative diagram of small morphisms

$$\begin{array}{ccc} S_2 & \xleftarrow{\pi_3'} & R_3 \\ \downarrow & & \downarrow \\ S_1 & \xleftarrow{\pi_2'} & R_2 \end{array}$$

If $M_1 \in F(S_1)$ and $M_2 \in F(S_2)$ is a lifting of M_2, then $\kappa(o(M_2, \pi_3')) = o(M_1, \pi_2')$, where

$$\kappa : A^2 \otimes_k \ker \pi_3' \longrightarrow A^2 \otimes_k \ker \pi_2'.$$

is the obvious morphism.

Assuming this, we are going to actually compute the prorepresenting hull of F.

Choose bases $\{x_1, ..., x_d\}$ for A^{1^*}, $\{y_1, ..., y_r\}$ for A^{2^*}.
Let $\{x_1^\star, ..., x_d^\star\}$ respectively $\{y_1^\star, ..., y_r^\star\}$ denote the corresponding dual bases.
From Nakayama's Lemma, we then have that

$$\hat{H}/\underline{m}^2 \cong k[x_1, ..., x_d]/\underline{m}^2$$

where $\underline{m} = (x_1, ..., x_d)$.
Put

$$S_1 = k[u_1, ..., u_d]/\underline{m}^2 = k[\underline{u}]/\underline{m}^2, \; R_2 = k[\underline{u}]/\underline{m}^3,$$

and let $X_\phi \in F(S_1)$ correspond to $\phi_1 : \hat{H} \longrightarrow \hat{H}/\underline{m}^2, \phi_1(x_i) = u_i$. If $\mathbf{a} \subseteq R_2$ is a least ideal such that X_{ϕ_1} can be lifted to $R_2/\mathbf{a} = S_2$, then $R_2/\mathbf{a} \cong \hat{H}/\underline{m}^3$.
Consider the diagram

$$\begin{array}{ccc} & & R_2 = k[\underline{u}]/\underline{m}^3 \\ & & \downarrow{\scriptstyle \pi_2'} \\ \hat{H} & \xrightarrow{\phi_1} & S_1 = k[\underline{u}]/\underline{m}^2 \end{array}$$

We now pick a monomial basis for S_1 of the form $\{\underline{u}^{\underline{n}}\}_{\underline{n} \in \overline{B}_1}$,
$\overline{B}_1 = \{\underline{n} \in \mathbf{N}^d : |\underline{n}| \leq 1\}$, and we let $\{\underline{u}^{\underline{n}}\}_{\underline{n} \in B_2'}, B_2' = \{\underline{n} \in \mathbf{N}^d : |\underline{n}| = 2\}$ be a basis for $\ker \pi_2'$. Then we have that

$$o(X_{\phi_1}, \pi_2') \in A^2 \otimes_k \ker \pi_2' \Rightarrow o(X_{\phi_1}, \pi_2') = \sum_{\underline{n} \in B_2'} a_{\underline{n}} \otimes \underline{u}^{\underline{n}} = \sum_{i=1}^{r} y_i^\star \otimes f_i^2(\underline{u}).$$

Definition 1.1.
ϕ_1 is called a defining system for the Massey products

$$< \underline{x}^\star; \underline{n} > = a_{\underline{n}}, \underline{n} \in B_2'.$$

Notice that with this notation we have

$$f_j^2(\underline{u}) = \sum_{\underline{n} \in B_2'} y_j(< \underline{x}^\star; \underline{n} >)\underline{u}^{\underline{n}}.$$

It is then clear that

$$\hat{H}/\underline{m}^3 \cong R_2/(f_1^2, ..., f_r^2) = k[\underline{u}]/(\underline{m}^3 + (f_1^2, ..., f_r^2)) = S_2.$$

Now consider the diagram

$$\begin{array}{ccc} & & R_3 = k[\underline{u}]/\underline{m}^4 + \underline{m}(f_1^2, ..., f_r^2) \\ & & \downarrow{\scriptstyle \pi_3'} \\ \hat{H} & \xrightarrow{\phi_2} & S_2 \\ & \searrow & \downarrow{\scriptstyle \pi_2} \\ & & S_1 \end{array}$$

Since $o(X_{\phi_1}, \pi_2) = \kappa o(X_{\phi_2}, \pi_2') = \sum y_i^\star \otimes f_i^2(\underline{u}) = 0$, we can lift X_{ϕ_1} to S_2. Thus we can find $\phi_2 : \hat{H} \longrightarrow S_2$ as above. And if $\mathbf{a} \subseteq R_3$ is a least ideal such that $X_{\phi_2} \in F(S_2)$ can be lifted to $R_3/\mathbf{a}$, then it is clear that $\hat{H}/\underline{m}^4 \cong R_3/\mathbf{a}$.

Pick a monomial basis for $\ker \pi_2 = \underline{m}^2/\underline{m}^3 + (f_1^2, ..., f_r^2)$ of the form

$$\{\underline{u}^{\underline{n}}\}_{\underline{n} \in B_2},$$

put

$$\overline{B}_2 = \overline{B}_1 \cup B_2.$$

Then $\overline{B}_2$ is a basis for S_2 implying that for every $\underline{n}$ with $|\underline{n}| \leq 2$ we have a unique relation in S_2:

$$\underline{u}^{\underline{n}} = \sum_{\underline{m} \in \overline{B}_2} \beta_{\underline{n},\underline{m}} \underline{u}^{\underline{m}}.$$

Because of later needs, we notice the following

Corollary 1.2.

$\sum_{\underline{n} \in B_2'} \beta_{\underline{n},\underline{m}} < \underline{x}^\star; \underline{n} >= 0$

Proof.

This is simply because $o(X_{\phi_1}, \pi_2) = 0$.

Write

$$\begin{aligned} \ker \pi_3' &= \underline{m}^3 + (f_1^2, ..., f_r^2)/\underline{m}^4 + \underline{m}(f_1^2, ..., f_r^2) = \\ & (f_1^2, ..., f_r^2)/\underline{m}(f_1^2, ..., f_r^2) \oplus \underline{m}^3/(\underline{m}^4 + \underline{m}^3 \cap \underline{m}(f_1^2, ..., f_r^2)) \\ &= \mathbf{a} \oplus I_3. \end{aligned}$$

Let $\{\underline{u}^{\underline{n}}\}_{\underline{n} \in B_3'}$ be a monomial basis for I_3 where we may assume that for $\underline{n} \in B_3'$ we have that $\underline{u}^{\underline{n}} = u_k \underline{u}^{\underline{m}}$ for some $\underline{m} \in B_2$. Put $\overline{B}_3' = \overline{B}_2 \cup \overline{B}_3'$. Note that for every $\underline{n}$ with $|\underline{n}| \leq 3$, we have a unique relation in R_3, namely

$$\underline{u}^{\underline{n}} = \sum_{\underline{n} \in \overline{B}_3'} \beta'_{\underline{n},\underline{m}} \underline{u}^{\underline{m}} + \sum_j \beta_{\underline{n},j} f_j^2.$$

With the condition on o we have that

$$o(X_{\phi_2}, \pi_3') = \sum_j y_j^\star \otimes (\sum_j \alpha_j \otimes f_j^2) + \sum_{\underline{n} \in B_3'} a'_{\underline{n}} \otimes \underline{u}^{\underline{n}},$$

with $\kappa o(X_{\phi_2}, \pi_3') = \sum y_j^\star \otimes (\sum \alpha_j \otimes f_j^2) = \sum y_j^\star \otimes f_j^2 \Rightarrow \sum \alpha_j \otimes f_j^2 \equiv f_j^2 \text{mod} \underline{m}^3$ implying that

$$o(X_{\phi_2}, \pi_3') = \sum y_j^\star \otimes f_j^2 + \sum_{\underline{n} \in B_3'} a_{\underline{n}} \otimes \underline{u}^{\underline{n}} = \sum y_j^\star \otimes f_j^3.$$

definition 1.3.
Any map ϕ_2 is called a defining system for the Massey products

$$< \underline{x}^\star ; \underline{n} >= a_{\underline{n}}, \underline{n} \in B_3'.$$

Notice that with this notation we have

$$f_j^3 = \sum_{\underline{n}\in B_2'} y_j < \underline{x}^\star ; \underline{n} > \underline{u}^{\underline{n}} + \sum_{\underline{n}\in B_3'} y_j < \underline{x}^\star ; \underline{n} > \underline{u}^{\underline{n}}.$$

Put

$$S_3 = R_3/(f_1^3, ..., f_r^3) = k[\underline{u}]/\underline{m}^4 + (f_1^3, ..., f_r^3).$$

Then $\hat{H}/\underline{m}^4 \cong S_3$. Let π_3 be the natural morphism $S_3 \longrightarrow S_2$ Pick, as before, a monomial basis $\{\underline{u}^{\underline{n}}\}_{\underline{n}\in B_3}$ for $\ker \pi_3$ such that $B_3 \subseteq B_3'$ and put $\overline{B}_3 = \overline{B}_2 \cup B_3$. For every $\underline{n}$ with $|\underline{n}| \leq 3$ we then have a unique relation in S_3,

$$\underline{u}^{\underline{n}} = \sum_{\underline{n}\in \overline{B}_3} \beta_{\underline{n},\underline{m}} \underline{u}^{\underline{n}},$$

and for later needs again , we state the following

Corollary 1.4.
$\sum_{l=2}^{3} \sum_{\underline{n}\in B_l'} \beta_{\underline{n},\underline{m}} < \underline{x}^\star ; \underline{n} >= 0.$

Proof.
The same reason as before: $o(X_{\phi_2}, \pi_3) = 0$.

Again we have a morphism $\phi_3 : \hat{H} \longrightarrow S_3$ and we continue by induction. For any $k \geq 1$, assume we have found $S_{2+k} = k[\underline{u}]/\underline{m}^{2+k+1} + (f_1^{2+k}, ..., f_r^{2+k})$ such that

$$\hat{H}/\underline{m}^{2+k+1} \cong S_{2+k},$$

and consider the diagram

$$\begin{array}{ccc} & & R_{2+k+1} = k[\underline{u}]/\underline{m}^{2+k+2} + \underline{m}(f_1^{2+k}, ..., f_r^{2+k}) \\ & & \downarrow \pi'_{2+k+1} \\ \hat{H} & \xrightarrow{\phi_{2+k}} & S_{2+k} = k[\underline{u}]/\underline{m}^{2+k+1} + (f_1^{2+k}, ..., f_r^{2+k}) \\ & \searrow & \downarrow \\ & & S_1 \end{array}$$

We write as before

$$\begin{aligned} \ker \pi'_{2+k+1} &= (f_1^{2+k}, ..., f_r^{2+k})/\underline{m}(f_1^{2+k}, ..., f_r^{2+k}) \\ &\oplus \underline{m}^{2+k+1}/(\underline{m}^{2+k+2} + \underline{m}^{2+k+1} \cap \underline{m}(f_1^{2+k}, ..., f_r^{2+k})) = \mathbf{a} \oplus I_{2+k+1}. \end{aligned}$$

We then pick a monomial basis for I_{2+k+1} of the form $\{\underline{u}^{\underline{n}}\}_{\underline{n}\in B'_{2+k+1}}$ where we may assume that for $\underline{n} \in B'_{2+k+1}$, we have $\underline{u}^{\underline{n}} = u_k \underline{u}^{\underline{n}}$ for some $\underline{m} \in B_{2+k}$. Put $\overline{B}'_{2+k+1} = \overline{B}_{2+k} \cup B'_{2+k+1}$. Then we have a unique relation in R_{2+k+1} for every $\underline{n}$ such that $|\underline{n}| \leq 2+k+1$:

$$\underline{u}^{\underline{n}} = \sum_{\underline{n}\in\overline{B}'_{2+k+1}} \beta'_{\underline{n},\underline{m}} \underline{u}^{\underline{n}} + \sum_j \beta_{\underline{n},j} f_j^{2+k}.$$

With the condition on o we have that

$$o(X_{\phi_{2+k}}, \pi'_{2+k+1}) = \sum_j y_j^\star \otimes \hat{f}_j^{2+k} + \sum_{\underline{n}\in B'_{2+k+1}} a'_{\underline{n}} \otimes \underline{u}^{\underline{n}},$$

with $\hat{f}_j^{2+k} \equiv f_j^2 \mod \underline{m}(f_1^{2+k-1}, ..., f_r^{2+k-1})$, that is

$$o(X_{\phi_{2+k}}, \pi'_{2+k+1}) = \sum_j y^\star \otimes f_j^{2+k} + \sum_{\underline{n}\in B'_{2+k+1}} a_{\underline{n}} \otimes \underline{u}^{\underline{n}} = \sum_j y_j^\star \otimes f_j^{2+k+1}.$$

Definition 1.5.
Any map ϕ_{2+k} is called a defining system for the Massey products

$$< \underline{x}^\star; \underline{n} >= a_{\underline{n}}, \underline{n} \in B'_{2+k+1}.$$

Notice that with this notation we have

$$f_j^{2+k+1} = \sum_{l=0}^{2+k+1} \sum_{\underline{n}\in B'_{2+l}} y_j < \underline{x}^\star; \underline{n} > \underline{u}^{\underline{n}}.$$

Then as before we have $\hat{H}/\underline{m}^{2+k+2} \cong S_{2+k+1} = k[\underline{u}]/\underline{m}^{2+k+2} + (f_1^{2+k+1}, ..., f_r^{2+k+1})$. We set

$$\pi_{2+k+1} : S_{2+k+1} \longrightarrow S_{2+k},$$

and we pick a monomial basis $\{\underline{u}^{\underline{n}}\}_{\underline{n}\in B_{2+k+1}}$ for $\ker \pi_{2+k+1}$ such that

$B_{2+k+1} \subseteq B'_{2+k+1}$, and then we put $\overline{B}_{2+k+1} = \overline{B}_{2+k} \cup B_{2+k+1}$. Again we get a unique relation for $|\underline{n}| \leq 2+k+1) = 0$:

$$\underline{u}^{\underline{n}} = \sum_{\underline{n}\in\overline{B}_{2+k+1}} \beta_{\underline{n},\underline{m}} \underline{u}^{\underline{m}},$$

and because $o(X_{\phi_{2+k}}, \pi_{2+k+1}) = 0$, we have the usual corollary

$$\sum_{l=2}^{2+k+1} \sum_{\underline{n}\in B'_l} \beta_{\underline{n},\underline{m}} < \underline{x}^\star; \underline{n} >= 0.$$

Now ,by induction, we have proved the following:

$$\hat{H} \cong \varprojlim_k S_{2+k} = k[[\underline{u}]]/(\bar{f}_1, ..., \bar{f}_r)$$

where $\bar{f}_j = \varprojlim f_j^{2+k}$. That is

Theorem 1.6.(Structure theorem of Laudal).

There exists a morphism of complete local k-algebras

$$o : T^2 = sym_k(A^{2^*})\hat{} \longrightarrow T^1 = sym_k(A^{1^*})\hat{}$$

such that $\hat{H} \cong T^1 \otimes_{T^2} k$. *Furthermore, for any small morphism* $\pi : R \longrightarrow S$, *in the diagram*

$$\begin{array}{ccccc} A^{2^*} \subseteq \underline{m}_{T^2} \subseteq T^2 & \xrightarrow{o} & T^1 & \xrightarrow{\bar{\phi}} & R \\ & & \downarrow & & \downarrow \pi \\ & & H & \xrightarrow{\phi} & S \end{array}$$

The obstruction for lifting X_ϕ *to* R *is the restriction of* $o \circ \bar{\phi}$ *to* A^{2^*}.

Proof.

Just put $o(y_j) = \bar{f}_j$.

Remark 1.7.

The good thing about this theorem is that the prorepresenting hull is computed using only the obstruction theory of the functor F. Very soon we are going to see that this enables us to define some "ordinary" Massey products which completely determine the hull.

Warning.

Do not believe that the Massey products are defined by the obstructions. In fact a Massey product is defined as soon as we have a defining system in a differential graded algebra, defined by the rules above. The obstruction theory shows that defining systems exists in the relevant differential graded algebras. We may therefore forget the obstruction calculus in the computation of prorepresenting hull, and just be interested in the Massey products.

In [10] Laudal writes up the representation of the Matric Massey products in $\mathrm{Ext}^1_A(M, M)$ when A is a k-algebra and M an A-module. In the next chapter we will do the same in the case where X is a k-scheme and $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_X$-module.

2. Global Matric Massey Products

2.1. Resolving functors for $\varprojlim$.

The following results and definitions are given by Laudal in [9].

Let $\underline{c}$ be any small category and denote by $\underline{\mathrm{Ab}}^{\underline{c}^o}$ the category of abelian functors on $\underline{c}^o$. We define the standard resolving complex

$$C : \underline{\mathrm{Ab}}^{\underline{c}^o} \longrightarrow \underline{\mathrm{Compl.ab.gr.}}$$

by

$$C^p(G) = \prod_{c_0 \xrightarrow[\psi_1]{} c_1 \xrightarrow[\psi_2]{} \dots \xrightarrow[\psi_p]{} c_p} G(c_0),$$

with differential $d^p : C^p(G) \longrightarrow C^{p+1}(G)$ given by:

$$(d^p(\xi))(\psi_1,...,\psi_{p+1}) = G(\psi_1)(\xi(\psi_2,...,\psi_{p+1})) + \sum_{i=1}^{p}(-1)^i\xi(\psi_1,...,\psi_i\circ\psi_{i+1},...,\psi_{p+1}) + (-1)^{p+1}\xi(\psi_1,...,\psi_p).$$

The basic properties of $C^{\cdot}$ are the following:
(i) $C^{\cdot}(\underline{c}^o,-)$ is an exact functor
(ii) $H^n(C^{\cdot}(\underline{c}^o,-)) = \varprojlim_{\underline{c}^o}{}^{(n)}$ for $n \geq 0$.

Let F be an abelian functor on the category $\underline{\mathrm{morc}}$, for which the objects are the morphisms of $\underline{c}$, and a morphism (ϕ_1,ϕ_2) between $\delta_1 : c_1 \longrightarrow c_1', \delta_2 : c_2 \longrightarrow c_2'$, is a commutative diagram

$$\begin{array}{ccc} c_1 & \xrightarrow{\delta_1} & c_1' \\ {\scriptstyle\phi_1}\uparrow & & \downarrow{\scriptstyle\phi_2} \\ c_2 & \xrightarrow[\delta_2]{} & c_2' \end{array}$$

We define the functor

$$D^{\cdot} : \underline{\mathrm{Ab}}^{\underline{\mathrm{morc}}} \longrightarrow \underline{\mathrm{Compl.ab.gr.}}$$

by

$$D^p(F) = D^p(\underline{c},F) = \prod_{c_0 \xrightarrow[\psi_1]{} c_1 \xrightarrow[\psi_2]{} \cdots \xrightarrow[\psi_p]{} c_p} F(\psi_1\circ\cdots\circ\psi_p),$$

and we let $d^p : D^p(F) \longrightarrow D^{p+1}(F)$ be defined by

$$(d^p(\xi))(\psi_1,...,\psi_{p+1}) = F(\psi_1, 1_{c_{p+1}})(\xi(\psi_2,...,\psi_{p+1})) + \sum_{i=1}^{p}(-1)^i\xi(\psi_1,...,\psi_i\circ\psi_{i+1},...,\psi_{p+1}) + (-1)^{p+1}F(1_{c_0},\psi_{p+1})(\xi(\psi_1,...,\psi_p))$$

Then the functor $D^{\cdot} = D^{\cdot}(\underline{c},-)$ has the following properties:
(i) $D^{\cdot}(\underline{c},-)$ is exact
(ii) $H^n(D^{\cdot}(\underline{c},-)) = \varprojlim_{\underline{\mathrm{morc}}}{}^{(n)}$ for $n \geq 0$.

2.2. The complex $\mathcal{D}^{\cdot}(\mathcal{U},\mathcal{H}om^{\cdot}(\mathcal{L}.,\mathcal{L}.))$ and the Ext-groups.

Let X be any scheme, and let $\mathcal{U}$ be any open affine covering. We are going to consider this covering as a category, the morphisms being inclusions. Then if $\mathcal{F}$,$\mathcal{G}$ are two sheaves of $\mathcal{O}_X$ -modules, we can consider the functor
$\mathcal{H}om(\mathcal{F},\mathcal{G}) : \underline{\mathrm{Mor}_{\mathcal{U}}} \longrightarrow \underline{\mathrm{Ab}}$ given by

$$\mathcal{H}om(\mathcal{F},\mathcal{G})(U \overset{i}{\hookrightarrow} V) = \mathrm{Hom}(\mathcal{F}\mid_V, i_*(\mathcal{G}\mid_U)).$$

We always have $\mathrm{Hom}(\mathcal{F}\mid_V, i_*(\mathcal{G}\mid_U)) \cong \mathcal{H}om(\mathcal{F},\mathcal{G})(U)$ so that

$$D^{\cdot}(\mathcal{U},\mathcal{H}om(\mathcal{F},\mathcal{G})) \cong C^{\cdot}(\mathcal{U},\mathcal{H}om(\mathcal{F},\mathcal{G})),$$

and if $\mathcal{F}$ and $\mathcal{G}$ are quasi-coherent we have

$$\mathcal{H}om(\mathcal{F},\mathcal{G})(U \hookrightarrow V) \cong \mathrm{Hom}_{\mathcal{O}_X(V)}(\mathcal{F}(V),\mathcal{G}(U))$$

as $\mathcal{O}_X(V)$-modules.

We denote by $D^{\cdot}(\mathcal{U},\mathcal{H}om^{\cdot}(\mathcal{L}.,\mathcal{L}.))$ the double complex, and by $\mathcal{D}^{\cdot}(\mathcal{U},\mathcal{H}om^{\cdot}(\mathcal{L}.,\mathcal{L}.))$ the associated total complex. An open affine cover $\mathcal{U}$ of X is called good if any intersection in $\mathcal{U}$ is covered by affines in $\mathcal{U}$, contained in the intersection. Notice that in this case, $D^{\cdot}$ is an exact functor and that $H^q(D^{\cdot}(\mathcal{H}om(\mathcal{F},\mathcal{G}))) \cong H^q(X,\mathcal{H}om(\mathcal{F},\mathcal{G}))$. This is just the fact that $D^{\cdot}$ is a resolving functor for $\varprojlim$, see Laudal[9].

Let X be a scheme (over some not necessarily algebraically closed field k). Let $\mathcal{F}$, $\mathcal{G}$ be two quasi-coherent sheaves of $\mathcal{O}_X$-modules and let $C^{\cdot}: \underline{mod_{\mathcal{O}_X}} \longrightarrow \underline{Compl.ab.gr.}$ be an exact functor such that $H^p(C^{\cdot}(\mathcal{H})) = H^p(X,\mathcal{H})$ for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{H}$. (for example the Godement-functor or the Čech-functor when we assume X noetherian and separated).

Choose injective resolutions of the $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}_{\mathcal{F}}^{\cdot},\ 0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{I}_{\mathcal{G}}^{\cdot},$$

and choose locally free resolutions of the $\mathcal{O}_X$-modules

$$\mathcal{L}^{\mathcal{F}}. \longrightarrow \mathcal{F} \longrightarrow 0,\ \mathcal{L}^{\mathcal{G}}. \longrightarrow \mathcal{G} \longrightarrow 0.$$

Consider the double complex $C^{\cdot}(\mathcal{H}om(\mathcal{F},\mathcal{I}_{\mathcal{G}}^{\cdot}))$ and let $\mathcal{K}_{\mathcal{I}}^{\cdot}$ be the associated total complex. Then we have $'E_2^{pq} = H^p(X;\mathcal{E}xt^q(\mathcal{F},\mathcal{G}))$ and $H^n(\mathcal{K}_{\mathcal{I}}^{\cdot}) = Ext^n(\mathcal{F},\mathcal{G})$ (Godement[5] ,p.264).

We have mappings between regularly filtered double complexes

$$\begin{array}{ccc} C^{\cdot}(\mathcal{H}om_{\mathcal{O}_X}^{\cdot}(\mathcal{L}^{\mathcal{F}}.,\mathcal{L}^{\mathcal{G}}.)) & \longrightarrow & C^{\cdot}(\mathcal{H}om_{\mathcal{O}_X}^{\cdot}(\mathcal{L}^{\mathcal{F}}.,\mathcal{I}_{\mathcal{G}}^{\cdot})) \\ & & \uparrow \\ & & C^{\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}_{\mathcal{G}}^{\cdot})) \end{array}$$

Thus between the associated simple filtered complexes. But these are isomorphisms on $'E_2^{pq}$ Hartshorne [8],p.234, and therefore isomorphisms in cohomology. This means

Lemma 2.1.

Let $\mathcal{F}$, $\mathcal{G}$ be two quasi-coherent sheaves on X and $C^{\cdot}: \underline{mod_{\mathcal{O}_X}} \longrightarrow \underline{compl.ab.gr}$ an exact functor such that $H^p(C^{\cdot}(\mathcal{H})) = H^p(X,\mathcal{H})$ for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{H}$. Let $\mathcal{L}^{\mathcal{F}}. \longrightarrow \mathcal{F} \longrightarrow 0, \mathcal{L}^{\mathcal{G}}. \longrightarrow \mathcal{G} \longrightarrow 0$ be locally free resolutions, and consider the associated total complex $\mathcal{C}$ of the double complex $C^{\cdot}(\mathcal{H}om^{\cdot}(\mathcal{L}^{\mathcal{F}}.,\mathcal{L}^{\mathcal{G}}.))$. Then

$$Ext_X^q(\mathcal{F},\mathcal{G}) \cong H^q(\mathcal{C}).$$

2.3. The Deformation functor $\mathrm{Def}_{\mathcal{F}}$ and liftings.

Let X be a scheme/k and $\mathcal{F}$ an $\mathcal{O}_X$-module. We are studying the moduli functor $\underline{Sch}/k \longrightarrow \underline{\text{Sets}}$, given by

$$G(Y) = \{\mathcal{F}_Y \in \underline{mod}_{X \times_k Y} | \mathcal{F}_Y \text{ is } Y-\text{flat}\}/\cong .$$

This functor is very rarely representable. Nevertheless it has many applications, as we will see later.

Recall that $\underline{\ell}$ denotes the category of local artinian k-algebras with residue field k, the morphisms being local k-algebra homomorphisms. Also recall that a surjective morphism $\phi : R \longrightarrow S$ in $\underline{\ell}$ is called small if $\ker \phi \cdot \underline{m}_R = 0$.

Let the fibre functor of G at $\mathcal{F}$ be the functor $\mathrm{Def}_{\mathcal{F}} : \underline{\ell} \longrightarrow \underline{\text{sets}}$ given by

$$\mathrm{Def}_{\mathcal{F}}(S) = \{(\mathcal{F}_S, \theta) | \mathcal{F} \in \underline{mod}_{X \times S}, \mathcal{F}_S \text{ is } S\text{-flat and } \mathcal{F}_S(*) \overset{\theta}{\cong} \mathcal{F}\}/\cong .$$

Notice that a subfunctor $G' \subseteq G$ defined by imposing conditions on the fibres of the objects such as for $\mathrm{Hilb}_{p(t)}$, constant rank, reflexivity, without torsion etc. has the same fibre functor. Thus it is interesting to compute the prorepresenting hull $\hat{H}$ of $\mathrm{Def}_{\mathcal{F}}$ when it exists. This will then be the local ring at $\mathcal{F}$ of the fine moduli-space if this last one exists.

In the section (2.5) we are going to show that this hull can be computed using generalized Massey products in the differential graded k-algebra $\mathcal{D}(\mathcal{U}, \mathcal{H}om^{\cdot}(\mathcal{L}., \mathcal{L}.))$.

Let $0 \longrightarrow I \longrightarrow R \longrightarrow S \longrightarrow 0$ be a small morphism in $\underline{\ell}$. If $\mathcal{F}_R \in \mathrm{Def}_{\mathcal{F}}(R)$ is mapped to $\mathcal{F}_S \in \mathrm{Def}_{\mathcal{F}}(S)$, we get the exact sequence $0 \longrightarrow I \otimes_R \mathcal{F}_R \longrightarrow \mathcal{F}_R \longrightarrow \mathcal{F}_S \longrightarrow 0$. But

$$I \otimes_R \mathcal{F}_R = I \otimes_k (k \otimes_R \mathcal{F}_R) = I \otimes_k \mathcal{F},$$

such that we get an exact sequence

$$0 \longrightarrow I \otimes_k \mathcal{F} \longrightarrow \mathcal{F}_R \longrightarrow \mathcal{F}_S \longrightarrow 0.$$

Conversely, suppose given an exact sequence as above. Now $\mathcal{F}_S$ is S-flat implies $\mathcal{F}_R$ is R-flat and the sequence $(I \otimes_k \mathcal{F}) \otimes_R S \longrightarrow \mathcal{F}_R \otimes_R S \longrightarrow \mathcal{F}_S \otimes_R S \longrightarrow 0$ is exact. We know that the image of

$$(I \otimes_k \mathcal{F}) \otimes_R S = (\mathcal{F} \otimes_k I) \otimes_R S$$

in $\mathcal{F}_R \otimes_R S$ is zero and that

$$\mathcal{F}_S \otimes_R S = \mathcal{F}_S \otimes_R R/I = \mathcal{F}_S,$$

so this tells us that $\mathcal{F}_R$ is a lifting of $\mathcal{F}_S$ to R. Thus we have proved the following

Lemma 2.2.
Let $0 \longrightarrow I \longrightarrow R \longrightarrow S \longrightarrow 0$ *be a small morphism in* $\underline{\ell}$, *and let* $\mathcal{F}_S \in \mathrm{Def}_{\mathcal{F}}(S)$. *Then*

$$\{ \textit{ liftings } \mathcal{F}_R \textit{ of } \mathcal{F}_S \textit{ to } R\} \cong \{0 \longrightarrow I \otimes_k \mathcal{F} \longrightarrow \mathcal{F}_R \longrightarrow \mathcal{F}_S \longrightarrow 0\}/\sim .$$

We have that $\mathcal{F} \in \mathrm{Def}_{\mathcal{F}}(k)$, and that $0 \longrightarrow (\varepsilon) \longrightarrow k[\varepsilon] \longrightarrow k \longrightarrow 0$ is a small morphism in $\underline{\ell}$. Furthermore, every element in $\mathrm{Def}_{\mathcal{F}}(k[\varepsilon])$ is a lifting of $\mathcal{F}$ such that the lemma gives

$$\mathrm{Def}_{\mathcal{F}}(k[\varepsilon]) \cong \{0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow 0\}/_{\sim} = Ext^1_X(\mathcal{F},\mathcal{F}).$$

Corollary 2.3.

$$T_{\mathrm{Def}_{\mathcal{F}}} \cong Ext^1_X(\mathcal{F},\mathcal{F}).$$

In the following we are only studying the cases where the dimension of the above vector space is finite, even if the full case can be treated. A main point in the computation of the formal moduli is then to find a basis for the tangent space. Thus the first problem is to compute the $Ext^1_X(\mathcal{F},\mathcal{F})$.

2.4. Obstruction theory for $\mathrm{Def}_{\mathcal{F}}$**.**
Let X be a scheme, $\mathcal{U} = \{U_i\}_{i\in I}$ an open affine cover such that for all $i,j \in I$ there is a subset $I_{i,j} \subseteq I$ with $\bigcup_{k\in I_{i,j}} U_k = U_i \cap U_j$. I will call this a good cover of X (when X is separated, $\mathcal{U}$ can be any open affine cover $\mathcal{U}$, including all intersections). Let $\mathcal{F} \in \underline{\mathrm{mod}}_{\mathcal{O}_X}$ and consider the deformation functor $\mathrm{Def}_{\mathcal{F}} : \underline{\ell} \longrightarrow \underline{\mathrm{sets}}$ defined by

$$\mathrm{Def}_{\mathcal{F}}(S) = \{(\mathcal{F},\theta) | \mathcal{F}_S \in \underline{\mathrm{mod}}_{\mathcal{O}_{X\times S}}, \mathcal{F}_S \text{ is } S\text{-flat}, \mathcal{F}_S(\star) \overset{\theta}{\cong} \mathcal{F}\}/\cong .$$

Suppose given $\mathcal{F}_S \in \mathrm{Def}_{\mathcal{F}}(S)$ with locally free resolution $\mathcal{L}_{.,S}$, and assume that $\mathcal{U}$ and $\mathcal{L}_{.,S}$ is choosen so that $\mathcal{L}_{.,S}|_{U_i\otimes S}$ is free for $i \in I$ (If X is quasi-projective over some algebraically closed field k, then $X \times_k S$ is quasi-projective over S, such that we have a resolution and $\mathcal{U}$ can be any good cover. In fact, for $\mathcal{F}_S$ to have such a resolution, it will be enough for $\mathcal{F}$ to have one). Then $\mathcal{L}_{.,S} \otimes_S k = \mathcal{L}_{.,S}(\star) := \mathcal{L}.$ is a locally free $\mathcal{O}_X$-resolution of $\mathcal{F}$, and $\mathcal{L}_{p,S}(U_i \otimes_k S) = \mathcal{L}_p(U_i) \otimes_k S \; \forall p$. So in this case what we have is

Lemma 2.4.
Given $\mathcal{F}_S \in \mathrm{Def}_{\mathcal{F}}(S)$. *Then there exists a locally free resolution*

$$\mathcal{L}. \longrightarrow \mathcal{F} \longrightarrow 0$$

together with morphisms:

$$\forall U_i \hookrightarrow U_j, \phi(U_i \hookrightarrow U_j) : \mathcal{L}.(U_j) \otimes_k S \longrightarrow \mathcal{L}.(U_i) \otimes_k S,$$

$$\forall U_i, d(U_i) : \mathcal{L}.(U_i) \otimes_k S \longrightarrow \mathcal{L}.(U_i)(-1) \otimes_k S,$$

such that the following conditions holds for all i, j, k:

(1) *If* $U_k \hookrightarrow U_j \hookrightarrow U_i$ *then* $\phi(U_j \hookrightarrow U_i)\phi(U_k \hookrightarrow U_j) = \phi(U_k \hookrightarrow U_i)$ *and* $\phi(U_i = U_i) = id$.

(2)

$$\begin{array}{ccc} \mathcal{L}.(U_i) \otimes_k S & \xrightarrow{d(U_i)} & \mathcal{L}.(U_i)(-1) \otimes_k S \\ {\scriptstyle \phi(U_i \hookrightarrow U_j)} \downarrow & & {\scriptstyle \phi(U_i \hookrightarrow U_j)} \downarrow \\ \mathcal{L}.(U_j) \otimes_k S & \xrightarrow{d(U_j)} & \mathcal{L}.(U_j)(-1) \otimes_k S \end{array}$$

Commutes.

(3) $(d(U_i))^2 = 0$.

Remark 2.5.

Let X be a scheme, $\mathcal{U}$ a good cover of X. Then to give a sheaf $\mathcal{F}$ on X is equivalent to give for each $U \in \mathcal{U}$ an object $\mathcal{F}(U)$ together with morphisms

$$\rho(U \hookrightarrow V) : \mathcal{F}(V) \longrightarrow \mathcal{F}(U)$$

for every $U \hookrightarrow V \in \underline{mor}_{\mathcal{U}}$, subject to conditions (1) and (2) above. For example, when we start with an open affine cover and include all possible intersections, we may look at the sheaves $\check{C}^p(\mathcal{U};\mathcal{F}) \longrightarrow \check{C}^{p+1}(\mathcal{U};\mathcal{F})$ which obviously is a complex of sheaves with the given restriction maps. But then $H^0(\check{C}(\mathcal{U};\mathcal{F}))$ is a sheaf on X with the given restriction maps. If we have any good cover of X, we have to use Laudal's $\mathcal{C}(\mathcal{U},\mathcal{F})$, the sheafyfied version of his C.

Now this last remark makes the lemma of course trivial, it is just the same as saying that $\mathcal{F}_S$ has a locally $\mathcal{O}_{X\times_k S}$-free resolution. What is not trivial, and in fact what we will need in the following, is lemma 2.7, which we will state after simplifying the proof with the following

Corollary 2.6.

Given $\mathcal{F}_S \in \mathrm{Def}_{\mathcal{F}}(S)$. Then there exists a locally free resolution $\mathcal{L}. \longrightarrow \mathcal{F} \longrightarrow 0$ and a morphism of double complexes $\mathcal{C}(\mathcal{U},\mathcal{L}. \otimes_k S) \longrightarrow \mathcal{C}(\mathcal{U},\mathcal{L}.)$ such that

$$\mathcal{F}_S =' H^0(''H_0(\mathcal{C}(\mathcal{U},\mathcal{L}. \otimes_k S)).$$

Proof.

Given $\mathcal{F}_S$ there exists mappings $\phi(U \hookrightarrow V)$ as above. In fact these are the only restriction maps used to define the complex $\mathcal{C}(\mathcal{U} \otimes_k S, \mathcal{L}. \otimes_k S)$ together with the d's. So this is indeed a double complex, lifting from $\mathcal{C}(\mathcal{U},\mathcal{L}.)$ by construction. Also by construction $''H_0(\mathcal{C}(\mathcal{U} \otimes_k S, \mathcal{L}. \otimes_k S)) = \mathcal{C}(\mathcal{U} \otimes_k S, \mathcal{F}_S)$ so that

$$'H^0(''H_0(\mathcal{C}(\mathcal{U} \otimes_k S, \mathcal{L}. \otimes_k S))) =' H^0(\mathcal{C}(\mathcal{U} \otimes_k S, \mathcal{F}_S)) = \mathcal{F}_S.$$

Lemma 2.7.

Given a good covering $\mathcal{U}$ of X and a locally free resolution $\mathcal{L}. \longrightarrow \mathcal{F} \longrightarrow 0$. Suppose given maps

$$\phi(U \hookrightarrow V) : \mathcal{L}.(V) \otimes_k S \longrightarrow \mathcal{L}.(U) \otimes_k S$$

for each $U \hookrightarrow V \in \underline{mor}_{\mathcal{U}}$ subject to condition (1) above, and maps

$$d^S(V) : \mathcal{L}.(V) \otimes_k S \longrightarrow \mathcal{L}.(V)(-1) \otimes_k S$$

for all $V \in \mathcal{U}$ subject to condition (2) and (3) above, and such that the diagram

$$\begin{array}{ccc} \mathcal{L}.(V) \otimes_k S & \xrightarrow{d^S(V)} & \mathcal{L}.(V)(-1) \otimes_k S \\ \downarrow & & \downarrow \\ \mathcal{L}.(V) & \xrightarrow{d(V)} & \mathcal{L}.(V)(-1) \end{array}$$

is commutative. Then there exists a lifting of double complexes

$$\mathcal{C}(\mathcal{U} \otimes_k S, \mathcal{L}. \otimes_k S) \longrightarrow \mathcal{C}(\mathcal{U}, \mathcal{L}.),$$

and $\mathcal{F}_S =' H^0(''H_0(\mathcal{C}(\mathcal{U} \otimes_k S, \mathcal{L}. \otimes_k S))) \in \mathrm{Def}_{\mathcal{F}}(S)$.

Proof.

By a lifting of double complexes is just meant the obvious, namely a commutative diagram

$$\begin{array}{ccccccc} \mathcal{C}(\mathcal{U} \otimes_k S, \mathcal{L}_0 \otimes_k S) & \longleftarrow & \mathcal{C}(\mathcal{U} \otimes_k S, \mathcal{L}_1 \otimes_k S) & \longleftarrow & \mathcal{C}(\mathcal{U} \otimes_k S, \mathcal{L}_2 \otimes_k S) & \longleftarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{C}(\mathcal{U}, \mathcal{L}_0) & \longleftarrow & \mathcal{C}(\mathcal{U}, \mathcal{L}_1) & \longleftarrow & \mathcal{C}(\mathcal{U}, \mathcal{L}_2) & \longleftarrow & \cdots \end{array}$$

where the mappings are understood to be mappings of complexes. But in the case with mappings respecting the usual restriction properties, it is obvious that we have such a lifting. So let's assume such a lifting is given. We are going to prove the condition of the lemma by induction on n where $\underline{m}_S^n = 0$.
if $n = 1$ then $S = k$ and $'H^0(''H_0(\mathcal{C}(\mathcal{U}, \mathcal{L}.))) = \mathcal{F}$ is indeed an element in $\mathrm{Def}_{\mathcal{F}}(k)$. So let us assume the condition true for $n-1$, $n \geq 2$. Then we have a small morphism in $\underline{\ell}$:

$$0 \longrightarrow \underline{m}_S^{n-1} \longrightarrow S \longrightarrow S/\underline{m}_S^{n-1} \longrightarrow 0$$

giving us the following liftings:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longleftarrow & \mathcal{C}^{\cdot}(\mathcal{F}_S \otimes_k \underline{m}_S^{n-1}) & \longleftarrow & \mathcal{C}^{\cdot}(\mathcal{L}_0 \otimes_k \underline{m}_S^{n-1}) & \longleftarrow & \mathcal{C}^{\cdot}(\mathcal{L}_1 \otimes_k \underline{m}_S^{n-1}) & \longleftarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longleftarrow & ''H^0(\mathcal{C}^{\cdot}(\mathcal{L}. \otimes_k S)) & \longleftarrow & \mathcal{C}^{\cdot}(\mathcal{L}_0 \otimes_k S) & \longleftarrow & \mathcal{C}^{\cdot}(\mathcal{L}_1 \otimes_k S) & \longleftarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longleftarrow & \mathcal{C}^{\cdot}(\mathcal{F}_S/\underline{m}_S^{n-1}) & \longleftarrow & \mathcal{C}^{\cdot}(\mathcal{L}_0 \otimes_k S/\underline{m}_S^{n-1}) & \longleftarrow & \mathcal{C}^{\cdot}(\mathcal{L}_1 \otimes_k S/\underline{m}_S^{n-1}) & \longleftarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Where the bottom row is exact by induction, the top row is exact by flatness of $\underline{m}_S^{n-1}$ over k, and the commutativity is given by functoriality. Now looking downwards at this diagram, we write up the long-exact sequence

$$\cdots \longrightarrow {}''H_i(\mathcal{C}(\mathcal{L}. \otimes_k \underline{m}_S^{n-1})) \longrightarrow {}''H_i(\mathcal{C}(\mathcal{L}. \otimes_k S)) \longrightarrow {}''H_i(\mathcal{C}(\mathcal{L}. \otimes_k S/\underline{m}_S^{n-1}))$$

$$\longrightarrow {}''H_{i-1}(\mathcal{C}(\mathcal{L}.\otimes_k \underline{m}_S^{n-1})) \longrightarrow {}''H_{i-1}(\mathcal{C}(\mathcal{L}.\otimes_k S)) \longrightarrow {}''H_{i-1}(\mathcal{C}(\mathcal{L}.\otimes_k S/\underline{m}_S^{n-1})) \longrightarrow$$

$$\cdots \longrightarrow {}''H_0(\mathcal{C}(\mathcal{L}.\otimes_k \underline{m}_S^{n-1})) \longrightarrow {}''H_0(\mathcal{C}(\mathcal{L}.\otimes_k S)) \longrightarrow {}''H_0(\mathcal{C}(\mathcal{L}.\otimes_k S/\underline{m}_S^{n-1})) \longrightarrow 0.$$

This gives ${}''H_i(\mathcal{C}(\mathcal{L}. \otimes_k S)) = 0$ for $i > 0$ which shows that $\mathcal{C}(\mathcal{L}. \otimes_k S)$ is a resolution of complexes in the second variable, and that we have a short exact sequence of complexes

$$0 \longrightarrow \mathcal{C}(\mathcal{F}_S \otimes_k \underline{m}_S^{n-1}) \longrightarrow {}''H_0(\mathcal{C}(\mathcal{L}. \otimes_k S)) \longrightarrow {}''H_0(\mathcal{C}(\mathcal{L}. \otimes_k S/\underline{m}_S^{n-1})) \longrightarrow 0.$$

The long exact sequence of this short exact sequence gives

$$0 \longrightarrow \mathcal{F} \otimes_k \underline{m}_S^{n-1} \longrightarrow {}' H^0({}''H_0(\mathcal{C}(\mathcal{L}. \otimes_k S)) \longrightarrow {}' H^0({}''H_0(\mathcal{C}(\mathcal{L}. \otimes_k S/\underline{m}_S^{n-1})) \longrightarrow 0,$$

with zero on the right because $\mathcal{F} \otimes_k \underline{m}_S^{n-1}$ is a sheaf and then $\mathcal{C}(\mathcal{F})$ a resolution. Then by the induction hypothesis ${}'H^0({}''H_0(\mathcal{C}(\mathcal{L}. \otimes_k S/\underline{m}_S^{n-1}))$ is a lifting of $\mathcal{F}$ to $S/\underline{m}_S^{n-1}$, and from lemma 2.1 (page 86) we have that

$${}'H^0({}''H_0(\mathcal{C}(\mathcal{L}. \otimes_k S))) \in \mathrm{Def}_{\mathcal{F}}(S)$$

which proves the lemma.

Notice that the proof works because $\mathcal{C}$ is a resolution when we have the right restriction maps.

Proposition 2.8.
For $X, \mathcal{U}, \mathcal{L}. \longrightarrow \mathcal{F} \longrightarrow 0$ as above, the following are equivalent:

(1) *To give a lifting $\mathcal{F}_S$ of $\mathcal{F}$ to S.*
(2) *To give morphisms ϕ, d subject to the conditions above.*
(3) *To give a lifting of double complexes $\mathcal{C}(\mathcal{U} \otimes_k S, \mathcal{L}. \otimes_k S) \longrightarrow \mathcal{C}(\mathcal{U}, \mathcal{L}.)$.*

Now we are going to use this proposition to make an obstruction theory for $\mathrm{Def}_{\mathcal{F}}$, taking into account that it also should be fitted to an obstruction calculus leading to generalized Massey products at the end. So on with it:

Let $0 \longrightarrow I \longrightarrow R \longrightarrow S \longrightarrow 0$ be a small morphism in $\underline{\ell}$, and assume $\mathcal{F}_S \in \mathrm{Def}_{\mathcal{F}}(S)$ is represented by morphisms ϕ, d as above. Then because (by assumption) $\mathcal{L}.(U) \otimes_k S$ is $\mathcal{O}_X(U) \otimes_k S$ -free, we may always lift the mappings ϕ, d to ϕ', d' on $\mathcal{L}. \otimes_k R$. Thus we have a mapping of doubly graded sheaves

$$\mathcal{C}(\mathcal{U} \otimes_k R, \mathcal{L}. \otimes_k R) \longrightarrow \mathcal{C}(\mathcal{U} \otimes_k S, \mathcal{L}. \otimes_k S),$$

where we do not know wether or not the first doubly graded sheaf is a double complex. Denoting the induced twisted maps as in the diagram

$$
\begin{array}{ccc}
\mathcal{C}^p(\mathcal{U}\otimes_k R, \mathcal{L}_q \otimes_k R) & \xrightarrow{\ 'd'_R\ } & \mathcal{C}^p(\mathcal{U}\otimes_k R, \mathcal{L}_q \otimes_k R) \\
{''d'_R}\big\downarrow & & {''d'_R}\big\downarrow \\
\mathcal{C}^p(\mathcal{U}\otimes_k R, \mathcal{L}_{q-1} \otimes_k R) & \xrightarrow{\ 'd'_R\ } & \mathcal{C}^p(\mathcal{U}\otimes_k R, \mathcal{L}_{q-1} \otimes_k R)
\end{array} \tag{1}
$$

We have that the three expressions $('d'_R)^2, (('d'_R)(''d'_R) - (''d'_R)('d'_R)), (''d'_R)^2$ gives an element in $Ext^2_X(\mathcal{F}, \mathcal{F}\otimes_k I) = Ext^2_X(\mathcal{F},\mathcal{F})\otimes_k I$, given by the class of the expressions in d', ϕ'. Now, if this element, say $o(\mathcal{F},\phi) = 0$, then there is an element $\xi \in \mathcal{D}^1(\mathcal{U}, \mathcal{H}om^{\cdot}(\mathcal{L}, \mathcal{L}\otimes_k I))$ such that $d\xi = -o$. Put $\xi = \xi' + \xi''$ and then

$$\phi(U \hookrightarrow V) = \phi'(U \hookrightarrow V) + \xi'(U \hookrightarrow V)$$

and

$$d(U) = d'(U) + \xi''(U)$$

and they will define a double differential on $\mathcal{C}(\mathcal{U}\otimes_k R, \mathcal{L}.\otimes_k R)$ and thus a lifting. Now, to get explicit formulas for the obstruction calculus, I will describe

$$o(\mathcal{F},\phi) \in Ext^2_X(\mathcal{F},\mathcal{F})\otimes_k I.$$

So let X be a scheme with the good covering $\mathcal{U} = \{U_i\}_{i\in I}$. Let $\mathcal{F} \in \underline{\text{mod}}_{\mathcal{O}_X}$ and $\mathcal{F}_S \in \text{Def}_{\mathcal{F}}(S)$, given by the maps

$$\phi_S(U \hookrightarrow V) : \mathcal{L}.(V)\otimes_k S \longrightarrow \mathcal{L}.(U)\otimes_k S,\ d_{i_S}(U) : \mathcal{L}_i(U)\otimes_k S \longrightarrow \mathcal{L}_{i-1}(U)\otimes_k S$$

for all $U \in \mathcal{U}$ and $U \hookrightarrow V \in \underline{\text{mor}}_{\mathcal{U}}$. Notice in the following that we are assuming $\mathcal{L}.|_U$ to be free. This may not be true in all cases, but the statements will be true for $\mathcal{L}.|_U$ projective, so we may assume it is free without loss of generality. Let

$$0 \longrightarrow I \longrightarrow R \xrightarrow{\ \phi\ } S \longrightarrow 0$$

be a small morphism in $\underline{\ell}$. Then we may always lift the maps ϕ_S and d_S to

$$\phi'_R(U \hookrightarrow V) : \mathcal{L}.(V)\otimes_k R \longrightarrow \mathcal{L}.(U)\otimes_k R,\ d'_{i_R}(U) : \mathcal{L}_i(U)\otimes_k R \longrightarrow \mathcal{L}_{i-1}(U)\otimes_k R.$$

Thus we get the obviously defined maps on $\mathcal{C}(\mathcal{U}\otimes_k R, \mathcal{L}.\otimes_k R)$, and we compute the obstructions where we, as before, name the maps as in the diagram (1). It is well known and easy to compute, is that these maps define the structure of a double complex on $\mathcal{C}(\mathcal{U}\otimes_k R, \mathcal{L}.\otimes_k R)$, i.e. a lifting, if and only if the element

$$
\begin{aligned}
&\phi'_R(U_1 \hookrightarrow U_2)\phi'_R(U_0 \hookrightarrow U_1) - \phi'_R(U_0 \hookrightarrow U_2) \\
&\quad \oplus d'_R(U_1)\phi'_R(U_0 \hookrightarrow U_1) - \phi'_R(U_0 \hookrightarrow U_1)d'_R(U_0) \oplus (d'_R(U_0))^2
\end{aligned}
$$

$\in \mathcal{D}^2(\mathcal{U}, \mathcal{H}om^{\cdot}(\mathcal{L}., \mathcal{L}. \otimes_k I))$ is zero. Anyway this element is a cocycle η whose class

$$\bar{\eta} = o(\mathcal{F}_S, \phi) \in Ext^2_X(\mathcal{F}, \mathcal{F}) \otimes_k I$$

is called the obstruction for lifting $\mathcal{F}_S$ by ϕ. If $o(\mathcal{F}_S, \phi) = 0$ then there exist a

$$\xi \in \mathcal{D}^1(\mathcal{U}, \mathcal{H}om^{\cdot}(\mathcal{L}., \mathcal{L}. \otimes_k I))$$

such that $d\xi = -\eta$. Put

$$\phi_R(U_0 \hookrightarrow U_1) = \phi'_R(U_0 \hookrightarrow U_1) + \xi(U_0 \hookrightarrow U_1),\ d_R(U_0) = d'_R(U_0) + \xi(U_0).$$

Then straight forward computation proves that we have
(i) $\phi_R(U_1 \hookrightarrow U_2)\phi_R(U_0 \hookrightarrow U_1) - \phi_R(U_0 \hookrightarrow U_2) = 0$
(ii) $d_R(U_1)\phi_R(U_0 \hookrightarrow U_1) - \phi_R(U_0 \hookrightarrow U_1)d_R(U_0) = 0$
(iii) $(d_R(U_0))^2 = 0$

What we have then is the following

Proposition 2.9.

Let $0 \longrightarrow I \longrightarrow R \xrightarrow{\phi} S \longrightarrow 0$ *be a small morphism in* $\underline{\ell}$ *.Then for each* $\mathcal{F}_S \in \mathrm{Def}_{\mathcal{F}}(S)$ *there exist an element*

$$o(\phi, \mathcal{F}_S) \in Ext^2_X(\mathcal{F}, \mathcal{F})$$

such that $\mathcal{F}_S$ *can be lifted to* R *if and only if* $o(\phi, \mathcal{F}_S) = 0$*. Furthermore, if this is true, then* $\mathrm{Def}_{\mathcal{F}}(R)$ *is a torsor (principal homogeneus space) over* $Ext^1_X(\mathcal{F}, \mathcal{F})$*.*

2.5. Massey products and formal moduli for $\mathrm{Def}_{\mathcal{F}}$.

Now we have at hand a nice obstruction theory for $\mathrm{Def}_{\mathcal{F}}$, given entirely in terms of the locally free resolution $\mathcal{L}.$ and its liftings. Using this, we are going to apply the constructions of chapter 1 and compute the Massey products $< \underline{x}^\star, \underline{n} >$ for $\underline{n} \in B'_{N+k}$. It will turn out from this that the $< \underline{x}^\star, \underline{n} >$'s of chapter 1 are some generalized "ordinary" Massey products of the differential graded algebra $\mathcal{D}(\mathcal{U}, \mathcal{H}om^{\cdot}(\mathcal{L}., \mathcal{L}.))$.

So to start, we let as before X be a scheme, $\mathcal{F}$ a coherent sheaf of $\mathcal{O}_X$-modules with a locally free resolution (locally projective)

$$\mathcal{L}. \longrightarrow \mathcal{F} \longrightarrow 0$$

which we now fix. Assume that $\mathcal{U} = \{U_i\}$ is a good covering of X and that we may choose bases

$$\{x_1^\star, ..., x_d^\star\} \in Ext^1_X(\mathcal{F}, \mathcal{F}),\ \{y_1^\star, ..., y_r^\star\} \in Ext^2_X(\mathcal{F}, \mathcal{F})$$

(take away the star for the corresponding dual bases). Now put $S_1 = k[u_1, ..., u_d]/(u_1, ..., u_d)^2 = k[\underline{u}]/\underline{m}^2$, $R_2 = k[\underline{u}]/\underline{m}^3$ and consider the diagram

$$\begin{array}{ccc} T^1 & \xrightarrow{\bar{\phi}_1} & R_2 \\ {\scriptstyle\rho}\downarrow & & {\scriptstyle\pi}\downarrow \\ H & \xrightarrow{\phi_1} & S_1 \end{array}$$

where $\rho\phi_1(x_i) = u_i(mod\underline{m}^2), \underline{m} = (u_1, ..., u_d)$. Let $\mathcal{F}_{S_1}$ correspond to ϕ_1 in

$$mor(H, S_1) \longrightarrow \mathrm{Def}_{\mathcal{F}}(S_1).$$

Then the Massey products for $|\underline{n}| = 2$ is given in terms of the obstruction

$$o(\mathcal{F}_{S_1}, \pi) \in Ext^2_X(\mathcal{F}, \mathcal{F}) \otimes_k I$$

($I = \ker\pi$ is a finite-dimentional k-vectorspace so this is true). Thus we would like to have a purely cohomological expression for this one. $\mathcal{F}_{S_1}$ corresponds to

$$\phi_{S_1}(U \hookrightarrow V) : \mathcal{L}.(V) \otimes_k S_1 \longrightarrow \mathcal{L}.(U) \otimes_k S_1, \forall (U \hookrightarrow V)$$

and

$$d_{S_1}(U) : \mathcal{L}.(U) \otimes_k S_1 \longrightarrow \mathcal{L}.(U)(-1) \otimes_k S_1, \forall U.$$

(Recall that we are assuming the restriction maps of an $\mathcal{O}_X$-module to respect linearity). Then $\phi_{S_1}(U \hookrightarrow V)$ is uniquely defined by the maps
$\alpha_{\underline{m}}(U \hookrightarrow V) : \mathcal{L}.(V) \longrightarrow \mathcal{L}.(U), |\underline{m}| < 2$ given by

(1) $$\phi_{S_1}(U \hookrightarrow V)|_{\mathcal{L}.(V)\otimes_k 1} = \sum_{|\underline{m}|<2} \alpha_{\underline{m}}(U \hookrightarrow V) \otimes \underline{u}^{\underline{m}}$$

$d_{S_1}(U)$ is uniquely defined by the maps $\alpha_{\underline{m}}(U) : \mathcal{L}.(U) \longrightarrow \mathcal{L}.(U)(-1)$ given by

(2) $$d_{S_1}(U)|_{\mathcal{L}.\otimes 1} = \sum_{|\underline{m}|<2} \alpha_{\underline{m}}(U) \otimes \underline{u}^{\underline{m}}$$

Thus $\mathcal{F}_{S_1}$, or ϕ_1, or the defining system for the Massey products $< \underline{x}^\star, \underline{n} >, |\underline{n}| \leq 2$ corresponds to a family $\{\alpha_{\underline{m}}\}_{|\underline{m}|<2}$ of cochains in $\mathcal{D}^1(\mathcal{U}, \mathcal{H}om^{\cdot}(\mathcal{L}., \mathcal{L}.))$. Writing up the fact that this family defines the lifting $\mathcal{F}_{S_1}$, we have:
(i)

$$\forall U_0 \hookrightarrow U_1 \hookrightarrow U_2 :$$
$$\phi_{S_1}(U_1 \hookrightarrow U_2)\phi_{S_1}(U_0 \hookrightarrow U_1) - \phi_{S_1}(U_0 \hookrightarrow U_2) = 0$$
$$\Updownarrow$$
$$\forall |\underline{m}| < 2; (\sum_{\underline{m}_1+\underline{m}_2=\underline{m}, |\underline{m}_i|<2} \alpha_{\underline{m}_1}(U_1 \hookrightarrow U_2)\alpha_{\underline{m}_2}(U_0 \hookrightarrow U_1)) - \alpha_{\underline{m}}(U_0 \hookrightarrow U_2)) = 0.$$

(ii)

$$\forall U_0 \hookrightarrow U_1 :$$
$$d_{S_1}(U_1)\phi_{S_1}(U_0 \hookrightarrow U_1) - \phi_{S_1}(U_0 \hookrightarrow U_1)d_{S_1}(U_0) = 0$$
$$\Updownarrow$$
$$\forall |\underline{m}| < 2; \sum_{\underline{m}_1+\underline{m}_2=\underline{m}, |\underline{m}|<2} (\alpha_{\underline{m}_1}(U_1)\alpha_{\underline{m}_2}(U_0 \hookrightarrow U_1) - \alpha_{\underline{m}_1}(U_0 \hookrightarrow U_1)\alpha_{\underline{m}_2}(U_0)) = 0.$$

(iii)

$$\begin{gathered}\forall U_0 : \\ d_{S_1}(U_0)d_{S_1}(U_0) = 0 \\ \Updownarrow \\ \forall |\underline{m}| < 2; \quad \sum_{\underline{m}_1+\underline{m}_2=\underline{m}, |\underline{m}|<2} \alpha_{\underline{m}_1}(U_0)\alpha_{\underline{m}_2}(U_0) = 0.\end{gathered}$$

Defining the liftings ϕ'_{R_2}, d'_{R_2} to R_2 by simply extending the expressions (1) and (2) to R_2, we find that the Massey products $< \underline{x}^\star; \underline{m} >$, for $|\underline{m}| = 2$ is represented by

$$\sum_{\underline{m}_1+\underline{m}_2=\underline{m}, |\underline{m}|<2} (\{\alpha_{\underline{m}_1}(U_1 \hookrightarrow U_2)\alpha_{\underline{m}_2}(U_0 \hookrightarrow U_1)\}_{U_0 \hookrightarrow U_1 \hookrightarrow U_2},$$

(2.5.1)

$$\{\alpha_{\underline{m}_1}(U_1)\alpha_{\underline{m}_2}(U_0 \hookrightarrow U_1) - \alpha_{\underline{m}_1}(U_0 \hookrightarrow U_1)\alpha_{\underline{m}_2}(U_0)\}_{U_0 \hookrightarrow U_1}, \{\alpha_{\underline{m}_1}(U_0)\alpha_{\underline{m}_2}(U_0)\}_{U_0})$$

In fact, what is just stated is that

$$< \underline{x}^\star; \underline{n} >= cl(\sum_{\underline{m}_1+\underline{m}_2=\underline{m}, |\underline{m}|<2} \alpha_{\underline{m}_1}\alpha_{\underline{m}_2})$$

with the product in $\mathcal{D}(\mathcal{U}, \mathcal{H}om^{\cdot}(\mathcal{L}., \mathcal{L}.))$ defined by the above expression, giving us some "ordinary" generalized Massey products.

Proposition 2.10.

Given a sequence of p cohomology classes $\alpha_1, ..., \alpha_p \in Ext^1_X(\mathcal{F}, \mathcal{F})$. Then a defining system for the Massey products

$$< \alpha_1, ..., \alpha_p; \underline{n} >, |\underline{n}| \leq N,$$

if it exists, correspond to a family $\{\alpha_{\underline{m}}\}_{|\underline{m}|<N}$ of 1-cochains of $\mathcal{D}(\mathcal{U}, \mathcal{H}om^{\cdot}(\mathcal{L}., \mathcal{L}.))$ such that $\alpha_0 = (\phi, d)$ where ϕ, d is the restriction map and the differential of $\mathcal{L}.$ respectively, and such that $\alpha_{e_i} = \alpha_i$ for $i = 1, ..., i = p$. Furthermore this family satisfies for each $\underline{m}$ with $|\underline{m}| < N$ the equality

$$\sum_{\underline{m}_1+\underline{m}_2=\underline{m}, |\underline{m}_i|<N} \alpha_{\underline{m}_1}\alpha_{\underline{m}_2} - \prod_{U_0 \hookrightarrow U_1 \hookrightarrow U_2} \alpha_{\underline{m}}(U_0 \hookrightarrow U_2) = 0 \tag{2.5.2}$$

Moreover, given a family $\{\alpha_{\underline{m}}\}$ as above, satisfying (2.5.2), then there exists a defining system for the Massey products $< \alpha_1, ..., \alpha_p; \underline{n} >$ for all $|\underline{n}| < N$, and in fact $< \alpha_1, ..., \alpha_p; \underline{n} >$ is represented by the cocycle

$$Y(\underline{n}) = \sum_{\underline{m}_1+\underline{m}_2=\underline{n}, |\underline{m}_i|<|\underline{n}|} \alpha_{\underline{m}_1}\alpha_{\underline{m}_2}$$

with the product defined as in (2.5.1).

Proof.

Just extend the computations for $|\underline{n}| = 2$ to the general case, noticing that a defining system $\{\alpha_{\underline{m}}\}_{|\underline{m}|<N}$ defines a lifting to $S_{N-1} = k[\underline{u}]/\underline{m}^N$, thus a morphism ϕ_{N-1} by the smoothness of the map $\text{Mor}(H, .) \longrightarrow \text{Def}_{\mathcal{F}}$.

So now we continue from the case $N = 2$. Put $f_j^2 = \sum_{|\underline{n}|=2} y_j(< \underline{x}^\star; \underline{n} >)\underline{u}^{\underline{n}}$ and look at the diagram

$$\begin{array}{ccccc}
T^2 & \xrightarrow{o} & T^1 & \xrightarrow{\bar{\phi}_2} & R_3 \\
 & & \downarrow & & \downarrow{\scriptstyle \pi_3'} \\
 & & H & \xrightarrow{\phi_2} & S_2 \\
 & & & & \downarrow{\scriptstyle \pi_2} \\
 & & & & S_1
\end{array}$$

Pick a monomial basis $\{\underline{u}^{\underline{n}}\}_{\underline{n}\in\bar{B}_1}$ for S_1. i.e. $\overline{B}_1 = \{\underline{n} \in \mathbf{N}^d : |\underline{n}| \leq 1\}$ and pick a monomial basis $\{\underline{u}^{\underline{n}}\}_{\underline{n}\in B_2}$ for $\ker \pi_2$. Put $\overline{B}_2 = \overline{B}_1 \cup B_2$. For every $\underline{n}$ with $|\underline{n}| \leq 2$ we then have a unique relation in S_2

$$\underline{u}^{\underline{n}} = \sum_{\underline{n}\in\overline{B}_2} \beta_{\underline{n},\underline{m}} \underline{u}^{\underline{m}}$$

We get the relation $\sum_{|\underline{n}|=2} \beta_{\underline{n},\underline{m}} < \underline{x}^\star; \underline{n} >= 0$ for every $\underline{m} \in \overline{B}_2$, translating into;

$$\text{for all } \underline{m} \in \overline{B}_2, \sum_{|\underline{n}|=2} \beta_{\underline{n},\underline{m}} Y(\underline{n}) \text{ is a coboundary.}$$

Thus we may find, for all $\underline{m} \in B_2$, a 1-cochain $\alpha_{\underline{m}} \in \mathcal{D}^1(\mathcal{U}, \mathcal{H}om^{\cdot}(\mathcal{L}., \mathcal{L}.))$ such that $d\alpha_{\underline{m}} = -\sum_{\underline{m}\in B_2'} \beta_{\underline{n},\underline{m}} Y(\underline{n})$. Consider the family $\{\alpha_{\underline{m}}\}_{\underline{m}\in\overline{B}_2}$ and define the maps ϕ_{S_2}, d_{S_2} the usual way. Then we have that (ϕ_{S_2}, d_{S_2}) satisfies the conditions for defining a lifting. (This is just the observation that

$$o(\mathcal{F}_1, \pi_2) = \sum_{\underline{m}\in B_2} (\sum_{\underline{m}\in B_2'} \beta_{\underline{n},\underline{m}} Y(\underline{n})) \otimes \underline{u}^{\underline{m}}),$$

so that $d\{\alpha_{\underline{m}}\}_{\underline{m}\in B_2'} = -o(\mathcal{F}, \pi_2))$. Therefore we may find maps ϕ_2 and $\bar{\phi}_2$ such that ϕ_2 corresponds to this lifting, say $\mathcal{F}_2$. Write

$$\begin{aligned}
\ker \pi_3' &= \underline{m}^3 + (f_1^2, ..., f_r^2)/\underline{m}^4 + \underline{m}(f_1^2, ..., f_r^2) \\
&= (f_1^2, ..., f_r^2)/\underline{m}(f_1^2, ..., f_r^2) \oplus \underline{m}^3/(\underline{m}^4 + \underline{m}^3 \cap \underline{m}(f_1^2, ..., f_r^2)).
\end{aligned}$$

Pick a monomial basis for $I_3 = \underline{m}^3/(\underline{m}^4 + \underline{m}^3 \cap \underline{m}(f_1^2, ..., f_r^2))$ on the form $\{\underline{u}^{\underline{n}}\}_{\underline{n}\in B_3'}$, where we may assume that for $\underline{n} \in B_3'$, $\underline{u}^{\underline{n}}$ is of the form $u_k \underline{u}^{\underline{m}}$ for some $\underline{m} \in B_2$. Put $\overline{B}_3' = \overline{B}_2 \cup B_3'$. For every $\underline{n}$ with $|\underline{n}| \leq 3$ we have a unique relation in R_3:

$$\underline{u}^{\underline{n}} = \sum_{\underline{m}\in\overline{B}_3'} \beta_{\underline{n},\underline{m}}' \underline{u}^{\underline{m}} + \sum_j \beta_{\underline{n},j}' f_j^2.$$

Define the maps ϕ'_{R_3}, d'_{R_3} by

$$\phi'_{R_3}(U_0 \hookrightarrow U_1) = \sum_{\underline{m}\in\overline{B}_2} \alpha_{\underline{m}}(U_0 \hookrightarrow U_1) \otimes \underline{u}^{\underline{m}} \text{ and}$$
$$d'_{R_3} = \sum_{\underline{m}\in\overline{B}_2} \alpha_{\underline{m}}(U_0) \otimes \underline{u}^{\underline{m}}.$$

Then, by computation, $o(\mathcal{F}_2, \pi'_3)$ is given by the following element in $Ext^2_X(\mathcal{F},\mathcal{F}) \otimes_k I$:

$$\sum_{\underline{n}\in B'_3}(\{\sum_{|\underline{m}|\leq 3}\sum_{\underline{m}_1+\underline{m}_2=\underline{m},\underline{m}_i\in\overline{B}_2} \beta'_{\underline{m},\underline{n}}\alpha_{\underline{m}_1}(U_1 \hookrightarrow U_2)\alpha_{\underline{m}_2}(U_0 \hookrightarrow U_1)\}_{U_0\hookrightarrow U_1\hookrightarrow U_2},$$
$$\{\sum_{|\underline{m}|\leq 3}\sum_{\underline{m}_1+\underline{m}_2=\underline{m},\underline{m}_i\in\overline{B}_2} \beta'_{\underline{m},\underline{n}}(\alpha_{\underline{m}_1}(U_1)\alpha_{\underline{m}_2}(U_0 \hookrightarrow U_1)-$$
$$\alpha_{\underline{m}_1}(U_0 \hookrightarrow U_1)\alpha_{\underline{m}_2}(U_0))\}_{U_0\hookrightarrow U_1},$$
$$\{\sum_{|\underline{m}|\leq 3}\sum_{\underline{m}_1+\underline{m}_2=\underline{m},\underline{m}_i\in\overline{B}_2} \beta'_{\underline{m},\underline{n}}\alpha_{\underline{m}_1}(U_0)\alpha_{\underline{m}_2}(U_0)\}_{U_0}) \otimes \underline{u}^{\underline{n}}+$$

$$\sum_{j=1}^{r}(\{\sum_{|\underline{m}|\leq 3}\sum_{\underline{m}_1+\underline{m}_2=\underline{m},\underline{m}_i\in\overline{B}_2} \beta'_{\underline{m},j}\alpha_{\underline{m}_1}(U_1 \hookrightarrow U_2)\alpha_{\underline{m}_2}(U_0 \hookrightarrow U_1)\}_{U_0\hookrightarrow U_1\hookrightarrow U_2},$$
$$\{\sum_{|\underline{m}|\leq 3}\sum_{\underline{m}_1+\underline{m}_2=\underline{m},\underline{m}_i\in\overline{B}_2} \beta'_{\underline{m},j}(\alpha_{\underline{m}_1}(U_1)\alpha_{\underline{m}_2}(U_0 \hookrightarrow U_1)$$
$$-\alpha_{\underline{m}_1}(U_0 \hookrightarrow U_1)\alpha_{\underline{m}_2}(U_0))\}_{U_0\hookrightarrow U_1},$$
$$\{\sum_{|\underline{m}|\leq 3}\sum_{\underline{m}_1+\underline{m}_2=\underline{m},\underline{m}_i\in\overline{B}_2} \beta'_{\underline{m},\underline{n}}\alpha_{\underline{m}_1}(U_0)\alpha_{\underline{m}_2}(U_0)\}_{U_0}) \otimes f_j.$$

Then we have proved the following:

Proposition 2.11.
Given a defining system $\{\alpha_{\underline{m}}\}_{\underline{m}\in\overline{B}_2}$ *for the Massey products*

$$< \underline{x}^\star; \underline{n} >, \underline{n} \in B'_3,$$

$< \underline{x}^\star; \underline{n} >$ *is represented by the cocycle*

$$Y(\underline{n}) = \sum_{|\underline{m}|\leq 3}\sum_{\underline{m}_1+\underline{m}_2=\underline{m},\underline{m}_i\in\overline{B}_2} \beta'_{\underline{m},\underline{n}}\alpha_{\underline{m}_1}\alpha_{\underline{m}_2}$$

with the product in $\mathcal{D}(\mathcal{U}, \mathcal{H}om^{\cdot}(\mathcal{L}.,\mathcal{L}.))$ *as defined in proposition 2.10.*

Next we put

$$f_j^3 = f_j^2 + \sum_{\underline{n}\in B'_3} y_j(< \underline{x}^\star; \underline{n} >)\underline{u}^{\underline{n}} = \sum_{\underline{n}\in B'_2} y_j(< \underline{x}^\star; \underline{n} >)\underline{u}^{\underline{n}} + \sum_{\underline{n}\in B'_3} y_j(< \underline{x}^\star; \underline{n} >)\underline{u}^{\underline{n}},$$

we put $S_3 = R_3/(f_1^3, ..., f_r^3) = k[\underline{u}]/(\underline{m}^4 + (f_1^3, ..., f_r^3))$,
$R_4 = k[\underline{u}]/(\underline{m}^5 + \underline{m}(f_1^3, ..., f_r^3))$ and we consider the diagram:

$$\begin{array}{ccccc} T^2 & \xrightarrow{o} & T^1 & \xrightarrow{\bar{\phi}_3} & R_4 \\ & & \downarrow & & \downarrow{\scriptstyle \pi_4'} \\ & & H & \xrightarrow{\phi_3} & S_3 \\ & & & & \downarrow{\scriptstyle \pi_3} \\ & & & & S_2 \end{array}$$

Then by construction, $o(\mathcal{F}, \pi_3) = 0$, allowing us to find maps $\phi_3, \bar{\phi}_3$ with ϕ_3 corresponding to $\mathcal{F}_3$ in the correspondence $\mathrm{Mor}(H, S_3) \longrightarrow \mathrm{Def}_{\mathcal{F}}(S_3)$. Pick a monomial basis $\{\underline{u}^{\underline{n}}\}_{\underline{n}\in B_3}$ for $\ker \pi_3$ such that $B_3 \subseteq B_3'$. Put $\overline{B}_3 = \overline{B}_2 \cup B_3$. Then $\{\underline{u}^{\underline{n}}\}_{\underline{n}\in \overline{B}_3}$ is a monomial basis for S_3, and such, for every $\underline{n}$ with $|\underline{n}| \leq N+1$ we therefore have a unique relation in S_3:

$$\underline{u}^{\underline{n}} = \sum_{\underline{n}\in \overline{B}_3} \beta_{\underline{n},\underline{m}} \underline{u}^{\underline{m}}.$$

Then $o(\mathcal{F}_2, \pi_3) = 0$ translates into: For every $\underline{m} \in B_3$, the 2-cochain

$$\beta_{\underline{m}} = \sum_{\underline{n}\in B_2'} \beta_{\underline{n},\underline{m}} Y(\underline{n}) + \sum_{\underline{n}\in B_3'} \beta_{\underline{n},\underline{m}} Y(\underline{n}) \in \mathcal{D}^2(\mathcal{U}, \mathcal{H}om^{\cdot}(\mathcal{L}., \mathcal{L}.))$$

is a coboundary. For each $\underline{m} \in B_3$, pick an $\alpha_{\underline{m}} \in \mathcal{D}^1(\mathcal{U}, \mathcal{H}om^{\cdot}(\mathcal{L}., \mathcal{L}.))$ such that

$$d\alpha_{\underline{m}} = -\beta_{\underline{m}},$$

and consider the family $\{\alpha_{\underline{m}}\}_{\underline{m}\in \overline{B}_3}$. Just as before, this family is seen to correspond to a defining system for the Massey products $< \underline{x}^{\star}; \underline{n} >, \underline{n} \in B_4'$, and we find representatives for the higher order Massey products, relations and bases, and we may copy the preceding procedure. We end up with the following:

Proposition 2.12.
Given a defining system $\{\alpha_{\underline{m}}\}_{\underline{m}\in \overline{B}_{2+k-1}}$ *for the Massey products* $< \underline{x}^{\star}; \underline{n} >, \underline{n} \in B_{2+k}'$. *Then* $< \underline{x}^{\star}; \underline{n} >$ *is represented by the 2-cocycle*

$$Y(\underline{n}) = \sum_{|\underline{m}|\leq 2+k} \sum_{\underline{m}_1+\underline{m}_2=\underline{m}, \underline{m}_i\in \overline{B}_{2+k-1}} \beta_{\underline{m},\underline{n}}' \alpha_{\underline{m}_1} \alpha_{\underline{m}_2},$$

the product in $\mathcal{D}(\mathcal{U}, \mathcal{H}om^{\cdot}(\mathcal{L}., \mathcal{L}.))$ *being defined the right way. Moreover, the polynomials*

$$f_j^{2+k} = \sum_{l=0}^{k} \sum_{\underline{n}\in B_{2+l}'} y_j(< \underline{x}^{\star}; \underline{n} >) \underline{u}^{\underline{n}},$$

$j = 1, ..., j = r$ induce the identity

$$\sum_{l=0}^{k} \sum_{\underline{n} \in B'_{2+l}} \beta_{\underline{n},\underline{m}} < \underline{x}^{\star}; \underline{n} >= 0,$$

such that if we for every $\underline{m} \in B_{2+k}$ pick a cochain $\alpha_{\underline{m}} \in \mathcal{D}^1(\mathcal{U}, \mathcal{H}om^{\cdot}(\mathcal{L}., \mathcal{L}.))$ with

$$d\alpha_{\underline{m}} = -\sum_{l=0}^{k} \sum_{\underline{n} \in B'_{2+l}} \beta_{\underline{n},\underline{m}} Y(\underline{n}),$$

then the family $\{\alpha_{\underline{m}}\}_{\underline{m} \in \overline{B}_{2+k}}$ is a defining system for the Massey products

$$< \underline{x}^{\star}; \underline{n} >, \underline{n} \in B'_{2+k+1}.$$

We sum up the content of this paragraph as follows:

Theorem 2.13.
Given $\mathcal{F} \in \underline{mod}_{\mathcal{O}_X}$, the formal moduli $\hat{H}$ of $\mathrm{Def}_{\mathcal{F}}$ is determined by the Massey products of $Ext^{\bullet}_X(\mathcal{F}, \mathcal{F})$. In fact

$$\hat{H} \cong k[[x_1, ..., x_d]]/(f_1, ..., f_r)$$

where $f_j = \sum_{l=2}^{\infty} \sum_{\underline{n} \in B'_l} y_j(< \underline{x}^{\star}; \underline{n} >)\underline{x}^{\underline{n}}$.

3. The connection between global and local Massey Products

3.1 Definition of the local to global map.

Consider a curve X containing a singularity $x_0 \in U = \operatorname{Spec} A \subseteq X$. Then for $\mathcal{M}$ a coherent $\mathcal{O}_X$-module, we denote the prorepresenting hull of $\mathrm{Def}_{\mathcal{M}}$ by $\hat{H} = \hat{H}_{\mathcal{M}}$. When restricting $\mathcal{M}$ to U, we get an A-module $M_0 = H^0(U, \mathcal{M}|_U)$. We denote the prorepresenting hull of Def_{M_0} by $\hat{H}_0 = \hat{H}_{M_0}$. By definition of prorepresenting hulls, we have smooth morphisms of functors, bijections on the tangent spaces;

$$\mathrm{Mor}(\hat{H}, -) \longrightarrow \mathrm{Def}_{\mathcal{M}} \text{ and } \mathrm{Mor}(\hat{H}_0, -) \longrightarrow \mathrm{Def}_{M_0}.$$

By restricting global modules, we have a morphism of functors

$$\mathrm{Def}_{\mathcal{M}} \longrightarrow \mathrm{Def}_{M_0}.$$

Filling in $\hat{H}$, we have the morphisms

$$\begin{array}{ccc} \mathrm{Mor}(\hat{H}, \hat{H}) & & \mathrm{Mor}(\hat{H}_0, \hat{H}) \\ \downarrow & & \downarrow \\ \mathrm{Def}_{\mathcal{M}}(\hat{H}) & \longrightarrow & \mathrm{Def}_{M_0}(\hat{H}) \end{array}$$

Where the vertical arrows are surjections by the smoothness. Thus the proversal family of $\mathrm{Def}_{\mathcal{M}}$ induces a morphism

$$\hat{H}_0 \xrightarrow{\phi} \hat{H}. \tag{3.1}$$

If $\mathrm{Def}_{\mathcal{M}}$ and Def_{M_0} are prorepresentable, and if it can be proved that $\mathrm{Def}_{\mathcal{M}} \longrightarrow \mathrm{Def}_{M_0}$ is smooth, then we understand that the diagram

$$\begin{array}{ccc} \mathrm{Mor}(\hat{H},-) & \longrightarrow & \mathrm{Mor}(\hat{H}_0,-) \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{Def}_{\mathcal{M}} & \xrightarrow[\text{smooth}]{} & \mathrm{Def}_{M_0} \end{array}$$

gives $\mathrm{Mor}(\hat{H},-) \longrightarrow \mathrm{Mor}(\hat{H}_0,-)$ smooth, which again implies the smoothness of ϕ.

Now $\mathrm{Def}_{\mathcal{M}} \longrightarrow \mathrm{Def}_{M_0}$ is smooth for curves, this is the main point to prove, but not in general for higher dimension. Nor are the functors prorepresentable. Thus there are no obvious reasons why the map ϕ is smooth.

3.2. Smoothness of the local to global map.

In this section, X is a plane projective curve over k with only one isolated singularity x_0. $\mathcal{F}$ is a torsionfree $\mathcal{O}_X$-module of finite rank with locally free resolution $\mathcal{L}. \longrightarrow \mathcal{F} \longrightarrow 0$.

The condition of $\mathcal{F}$ is crucial: A finitely generated module over a Dedekind domain is locally free. This fact will be used frequently.

In this subsection , we will assume that X can be covered by two open affines U_0 and U_1 such that $x_0 \in U_0$ but $x_0 \notin U_1$.

In the next section, the results of this section will be generalised.

We have a good covering $\mathcal{U}$ of X consisting of U_0, U_1 together with their intersection. Then $\mathcal{U}$ is a good covering of X such that $x_0 \in U_0$ but not in any other open $U \neq U_0 \in \mathcal{U}$. We fix this covering for the rest of the chapter. We also fix the notation

$$L. = H^0(U_0, \mathcal{L}.),\ M = H^0(U_0, \mathcal{F}),\ U_0 = \mathrm{Spec}(A).$$

Then M is the "affine" A-module, and $L. = H^0(U_0, \mathcal{L}.)$ is the "affine" free resolution of M (in any case, this resolution is projective and that is enough).
Consider the morphism of differential graded k-algebras

$$\sigma : \mathcal{D}(\mathcal{U}, \mathcal{H}om^{\cdot}(\mathcal{L}., \mathcal{L}.)) \longrightarrow Hom^{\cdot}(L., L.)$$

which is just the projection to the component in U_0.

Lemma 3.1.

There is a basis for $Ext^1_X(\mathcal{F},\mathcal{F})$ represented by 1-cocycles $\tilde{x^\star}_1, ..., \tilde{x^\star}_{\tilde{d}} \in \mathcal{D}^1(\mathcal{U}, \mathcal{H}om^{\cdot}(\mathcal{L}., \mathcal{L}.))$ such that $x^\star_1 = \sigma(\tilde{x^\star}_1), ..., x^\star_d = \sigma(\tilde{x^\star}_d)$ represents a k-basis for $Ext^1_A(M, M)$ and such that

$\sigma(\tilde{x^\star}_{d+1}), ..., \sigma(\tilde{x^\star}_{\tilde{d}}) = 0$

Proof.

Because X is covered by two open affines, $D^2(\mathcal{U}, \mathcal{H}om^{\cdot}(\mathcal{L}., \mathcal{L}.)) = 0$. Let $x \in Hom^1(L., L.)$ be a cocycle and include it (by the inclusion morphism) in

$$\mathcal{D}^1(\mathcal{U}, \mathcal{H}om^{\cdot}(\mathcal{L}., \mathcal{L}.)) = D^0(\mathcal{U}, \mathcal{H}om^1(\mathcal{L}., \mathcal{L}.)) \oplus D^1(\mathcal{U}, \mathcal{H}om^0(\mathcal{L}., \mathcal{L}.)),$$

let us say $x = x \oplus 0$. Then

$$d(x) =' d(x) +'' d(x) =' d(x) \in D^1(\mathcal{U}, \mathcal{H}om^1(\mathcal{L}., \mathcal{L}.)),$$

and $'d(x)$ is by definition of complexes a cocycle. But

$$H^1(D^1(\mathcal{U}, \mathcal{H}om^0(\mathcal{L}., \mathcal{L}.))) = D^1(\mathcal{U}, \mathcal{E}xt^1_X(\mathcal{F}, \mathcal{F})) = 0,$$

because $\mathcal{F}$ is free on each intersection (which is nonsingular since $x_0 \notin U \cap V$), so there is an element $\beta \in D^1(\mathcal{U}, \mathcal{H}om^0(\mathcal{L}., \mathcal{L}.))$ such that $''d(\beta) = -'d(x)$. Put $\tilde{x^\star} = x \oplus \beta \in \mathcal{D}^1(\mathcal{U}, \mathcal{H}om^{\cdot}(\mathcal{L}., \mathcal{L}.))$. Then

$$d(\tilde{x^\star}) =''d(x) \oplus' d(x) +''d(\beta) = 0.$$

So $\tilde{x^\star}$ is a cocycle with the property $\sigma(\tilde{x^\star}) = x$. This shows that the map

$$\sigma : H^1(\mathcal{D}^{\cdot}(\mathcal{U}, \mathcal{H}om^{\cdot}(\mathcal{L}., \mathcal{L}.))) \longrightarrow H^1(Hom^{\cdot}(L., L.))$$

is surjective, so that we can represent a basis for $Ext^1_X(\mathcal{F}, \mathcal{F})$ by $\tilde{x^\star}_1, ..., \tilde{x^\star}_{\tilde{d}}$ with

$$\{x^\star_i = \sigma(\tilde{x^\star}_i)\}_{i=1}^d$$

representatives for a basis for $Ext^1_A(M, M)$ and $\{\sigma(\tilde{x^\star}_i)\}_{i=d+1}^{\tilde{d}}$ all coboundaries. For any $y \in \mathcal{D}^1(\mathcal{U}, \mathcal{H}om^{\cdot}(\mathcal{L}., \mathcal{L}.))$ such that $\sigma(y)$ is a coboundary, we can write $y = \alpha \oplus \beta$ with $\alpha \in D^0(\mathcal{U}, \mathcal{H}om^1(\mathcal{L}., \mathcal{L}.))$ a coboundary. That is to say there exist a $\gamma \in D^0(\mathcal{U}, \mathcal{H}om^0(\mathcal{L}., \mathcal{L}.))$ such that $''d(\gamma) = \alpha$. Then of course

$$cl(y) = cl(\alpha \oplus \beta + (-''d(\gamma)) \oplus' d(\gamma)) = cl(0 \oplus \beta +' d(\gamma)).$$

So if $y' = 0 \oplus \beta +' d(\gamma)$ then $cl(y) = cl(y')$ and $\sigma(y') = 0$. Using this on $\tilde{x^\star}_{d+1}, ..., \tilde{x^\star}_{\tilde{d}}$ we may find representatives for a basis as proposed in the lemma.

If we try to do this for $Ext^2_X(\mathcal{F}, \mathcal{F})$ we find something even better, namely

Lemma 3.2.

There is a basis for $Ext^2_X(\mathcal{F}, \mathcal{F})$ represented by 2-cocycles $\tilde{y^\star}_1, ..., \tilde{y^\star}_r$, such that $y^\star_1 = \sigma(\tilde{y^\star}_1), ... y^\star_r = \sigma(\tilde{y^\star}_r)$ represents a k-basis for $Ext^2_A(M, M)$.

Proof.

Exactly as in the proof of lemma 3.1, we can lift any 2-cocycle of $Hom^{\cdot}(L., L.)$ to a 2-cocycle in $D^2(\mathcal{U}, \mathcal{H}om^{\cdot}(\mathcal{L}., \mathcal{L}.))$. Let $y = y_1 \oplus y_2 \in \mathcal{D}^2(\mathcal{U}, \mathcal{H}om^{\cdot}(\mathcal{L}., \mathcal{L}.))$ be a cocycle such that $\sigma(y)$ is a coboundary. Then there is an $\alpha \in D^0(\mathcal{U}, \mathcal{H}om^1(\mathcal{L}., \mathcal{L}.))$

such that $''d(\alpha) = y_1$. Now look at $'d(-\alpha) + y_2 \in D^1(\mathcal{U}, \mathcal{H}om^1(\mathcal{L}., \mathcal{L}.))$. This is a cocycle simply because $d^2 = 0$; but again

$$H^1(D^1(\mathcal{U}, \mathcal{H}om^{\cdot}(\mathcal{L}., \mathcal{L}.))) = D^1(\mathcal{U}, \mathcal{E}xt^1_X(\mathcal{F}, \mathcal{F})) = 0,$$

so there is a $\beta \in D^1(\mathcal{U}, \mathcal{H}om^0(\mathcal{L}., \mathcal{L}.))$ such that $''d(\beta) =' d(-\alpha) + y_2$. Thus we find

$$\alpha \oplus \beta \in D^0(\mathcal{U}, \mathcal{H}om^1(\mathcal{L}., \mathcal{L}.)) \oplus D^1(\mathcal{U}, \mathcal{H}om^0(\mathcal{L}., \mathcal{L}.)) = \mathcal{D}^1(\mathcal{U}, \mathcal{H}om^{\cdot}(\mathcal{L}., \mathcal{L}.)),$$

and

$$d(\alpha \oplus \beta) = (''d(\alpha)) \oplus ('d(\alpha) +' {}'d(\beta)) = y_1 \oplus' d(\alpha) +' d(-\alpha) + y_2 = y_1 \oplus y_2.$$

Thus y is a coboundary. We have proved that

$$\sigma : H^2(\mathcal{D}(\mathcal{U}, \mathcal{H}om^{\cdot}(\mathcal{L}., \mathcal{L}.))) \longrightarrow H^2(Hom^{\cdot}(L., L.))$$

is an isomorphism, and we are through.

Now we can use the explicit computation of prorepresenting hulls given by Massey products to say something about the morphism $\hat{H}_0 \xrightarrow{\phi} \hat{H}$. For the different monomial bases, we are going to denote the global case by for instance $\{\underline{u}^{\underline{n}}\}_{\underline{n} \in \tilde{\overline{B}}_2}$ and take away the tilda for the affine case. What is needed is the following

Lemma 3.3.
For any $k \geq 0$:
a) The defining systems $\{\alpha_{\underline{m}}\}_{\underline{m} \in \tilde{\overline{B}}_{2+k}}$ can be choosen to satisfy

$$\underline{m} \in \tilde{\overline{B}}_{2+k} - \overline{B}_{2+k} \Longrightarrow \sigma(\alpha_{\underline{m}}) = 0$$

b)

$$\underline{n} \in B'_{2+k} \Longrightarrow \sigma(<\tilde{\underline{x}^{\star}}; \underline{n}>) = <\underline{x}^{\star}; \underline{n}>$$

c)

$$\underline{n} \in \tilde{B}'_{2+k} - B'_{2+k} \Longrightarrow \sigma(Y(\underline{n})) = 0.$$

Proof.
The reason why this is not obvious after the lemmas, is the fact that the computation of the Massey products involves some relations $(\beta_{\underline{m},\underline{n}}, \beta'_{\underline{m},\underline{n}})$, and some choices of defining systems, constructed by induction. The lemma should therefore be proved by induction on $|\underline{n}| \geq 2$. First of all, choose bases of $Ext^i_X(\mathcal{F}, \mathcal{F})$ as in lemma 3.1 and 3.2. Then for $|\underline{n}| = 2, \underline{n} \in \tilde{B}'_2$ the Massey products $< \underline{x}^{\star}; \underline{n} >$ is represented by the cocycle

$$Y(\underline{n}) = \sum_{\substack{\underline{m}_1 + \underline{m}_2 = \underline{n} \\ \underline{m}_i \in \tilde{\overline{B}}_1}} \alpha_{\underline{m}_1} \alpha_{\underline{m}_2}.$$

By the choices of basis, b) and c) follows. a) is obvious for $|\underline{n}| = 2$.

Assume the conditions of the lemma true for $|\underline{n}| \leq 2+k-1$. Pick a basis $\tilde{\overline{B}}_{2+k-1}$, and we may put

$$\overline{B}_{2+k-1} = \tilde{\overline{B}}_{2+k-1} \cap \{\underline{n} \in \mathbb{N}^{\tilde{d}} : n_{d+1} = \cdots = n_{\tilde{d}} = 0\}$$

because there are no mixed relations. Now the defining systems are constructed as follows:

$$b_{\underline{m}} = \sum_{l=0}^{k-1} \sum_{\underline{n} \in \tilde{B}'_{2+l}} \beta_{\underline{n},\underline{m}} Y(\underline{n})$$

is a coboundary, and we pick for each $\underline{m} \in \tilde{B}_{2+l}$ an $\alpha_{\underline{m}}$ mapping to $b_{\underline{m}}$ (notice that $\tilde{B}'_{2+l}$ and B'_{2+l} can be constructed as above because there are no mixed relations). If $\underline{m} \in \tilde{\overline{B}}_{2+k-1} - \overline{B}_{2+k-1}$, then the only relations between the $\underline{u}^{\underline{m}}$ are "in there" and so the $Y(\underline{n})$'s in the sum are all with

$$\underline{n} \in \tilde{B}'_{2+k-1} - B'_{2+k-1}.$$

But in this case $\sigma(Y(\underline{n})) = 0 \Longrightarrow \sigma(b_{\underline{m}}) = 0$, say $b_{\underline{m}} = 0 \oplus b'_{\underline{m}}$. Then we may put $\alpha_{\underline{m}} = 0 \oplus \alpha'_{\underline{m}}$ where $\alpha_{\underline{m}} \in D^1(\mathcal{U}, \mathcal{H}om^0(\mathcal{L}., \mathcal{L}.))$ satisfies $''d(\alpha_{\underline{m}}) = -b'_{\underline{m}}$. Such an $\alpha_{\underline{m}}$ exists (as before) because $b_{\underline{m}}$ is a cocycle and $H^1(D^1(\mathcal{U}, \mathcal{H}om(\mathcal{L}., \mathcal{L}.)) = 0$. Then we find $d(\alpha_{\underline{m}}) = 0 \oplus -b'_{\underline{m}} = -b_{\underline{m}}$, and $\sigma(\alpha_{\underline{m}}) = 0$, which was what we wanted. Again since there are no mixed relations and because of the choices of polynomial bases and defining systems, we have conditions b) and c) satisfied.

Proposition 3.4.

Let X be a curve with only one singularity, $\mathcal{F}$ a torsionfree sheaf on X of finite rank. Let $ext^1_X(\mathcal{F},\mathcal{F}) = \tilde{d}$ and $ext^2_X(\mathcal{F},\mathcal{F}) = r$. Let $x_0 \in U = \operatorname{Spec} A$ be the singular point and $M = H^0(U,\mathcal{F})$. If $\hat{H}_0$ is the formal moduli of Def_M and $\hat{H}$ is the prorepresenting hull of $\operatorname{Def}_{\mathcal{F}}$, then there exists power series $f_1, \ldots, f_r$ such that

$$\hat{H}_0 \cong k[[x_1, \ldots, x_d]]/(f_1, \ldots, f_r).$$

$$\hat{H} \cong k[[x_1, \ldots, x_d, x_{d+1}, \ldots, x_{\tilde{d}}]]/(f_1, \ldots, f_r).$$

Proof.

$$\tilde{f}_j^n = \sum_{l=0}^{n-2} \sum_{\underline{n} \in \tilde{B}'_{2+l}} \tilde{y}_j(<\underline{x}^\star; \underline{n}>)\underline{x}^{\underline{n}} = f_j.$$

Corollary 3.5.

The natural map

$$\phi : \hat{H}_0 \longrightarrow \hat{H}$$

is (formally) smooth

Proof.
Because $\hat{H}$ is a formal power series ring over $\hat{H}_0$, in a finite number of variables.

Assume we have defined an obstruction theory for $\mathcal{O}_X$-modules $\mathcal{M}$ on a scheme X and an obstruction theory for $\mathcal{O}_{X,x}$-modules M such that for any small morphism $\pi : R \longrightarrow S$ and any lifting $\mathcal{M}_S$ of $\mathcal{M}$ to S

$$o((\mathcal{M}_S)_x;\pi) = o(\mathcal{M}_S;\pi)_x.$$

If we know that the localization morphism

$$\mathrm{Ext}^1_X(\mathcal{M},\mathcal{M}) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{X,x}}(\mathcal{M}_x,\mathcal{M}_x)$$

is surjective,

$$\mathrm{Ext}^2_X(\mathcal{M},\mathcal{M}) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}_{X,x}}(\mathcal{M}_x,\mathcal{M}_x)$$

injective, then we can prove the smoothness by abstract nonsense, without involving Massey products:

Alternative proof of corollary 3.5.
In the diagram

$$\begin{array}{ccc} \mathrm{Def}_{\mathcal{M}} & \xrightarrow{\ \rho\ } & \mathrm{Def}_{\mathcal{M}_x} \\ \text{Smooth}\uparrow & & \uparrow\text{Smooth}, \\ \mathrm{Mor}(\hat{H}_{\mathcal{M}},-) & \xrightarrow{\ \psi\ } & \mathrm{Mor}(\hat{H}_{\mathcal{M}_x},-) \end{array}$$

if we can prove that ρ is smooth, it is well known that ψ is smooth, which again gives the corollary.

The global and local obstruction theory are clearly defined such that the above assumption holds. The statements of the localization morphisms also holds for curves.

Using the fact that $\mathrm{Def}_{\mathcal{M}}(R)$ is a torsor (principal homogeneus space) over $\mathrm{Ext}^1_X(\mathcal{M},\mathcal{M})$ together with the surjectivity, respectively injectivity of $\mathrm{Ext}^i_X(\mathcal{M},\mathcal{M}) \longrightarrow \mathrm{Ext}^i_{\mathcal{O}_{X,x}}(\mathcal{M}_x,\mathcal{M}_x), i = 1,2$, putting $I = \ker\pi$, a simple diagram chasing in the diagram

$$\begin{array}{ccc} \mathrm{Ext}^1(\mathcal{M},\mathcal{M}\otimes I) & \longrightarrow & \mathrm{Ext}^1(\mathcal{M}_x,\mathcal{M}_x\otimes I) \\ \downarrow & & \downarrow \\ \mathrm{Def}_{\mathcal{M}}(R) & \longrightarrow & \mathrm{Def}_{\mathcal{M}_x}(R) \\ \downarrow & & \downarrow \\ \mathrm{Def}_{\mathcal{M}}(S) & \longrightarrow & \mathrm{Def}_{\mathcal{M}_x}(S) \end{array}$$

proves the smoothness of ρ.

This proof anyway, does not suggest anything but smoothness. The stringent proof in the next section gives the morphism explicitly, in particular the smoothness, and also suggest a generalization of the smoothness.

3.3. Generalizations by use of spectral sequences.

In this section, we use the notation

$$D^{\cdot} = \mathcal{D}(\mathcal{U}, \mathcal{H}om^{\cdot}(\mathcal{L}., \mathcal{L}.))$$

for simplicity. Also F denotes the first filtration of $D^{\cdot}$ and its cohomology. See Bredon[1] for generalities on spectral sequences.

Let X be any scheme, $\mathcal{F}$ any coherent $\mathcal{O}_X$-module. Pick a basis for

$$Ext^1_X(\mathcal{F},\mathcal{F}) = H^1(D^{\cdot}) \cong H^1(D^{\cdot})/F^1H^1(D^{\cdot}) \oplus F^1H^1(D^{\cdot}) =' E^{0,1}_{\infty} \oplus' E^{1,0}_{\infty}$$

on the form $\{x^{\star}_i\}_{i=1}^{d} \cup \{t^{\star}_i\}_{i=d+1}^{\tilde{d}}$ such that $\{e(x^{\star}_i)\}$ is a basis for $'E^{0,1}_{\infty}$ and $\{t^{\star}_i\} \subseteq F^1H^1(D^{\cdot})$. Then we have

Lemma 3.6.

Assume $H^2(F^1D^{\cdot}) = 0$. Then if $\underline{n} \in \mathbf{N}^{\tilde{d}}$ has $n_l \neq 0$ for some $l > d$, then

$$< \underline{x}^{\star}\underline{t}^{\star}; \underline{n} > \in F^1H^2(D^{\cdot}) \subseteq H^2(D^{\cdot}) = Ext^2_X(\mathcal{F},\mathcal{F}).$$

Proof.

We do the proof by induction on k, $\underline{n} \in B'_{N+k}$. First, because $t^{\star}_i \in F^1H^1(D^{\cdot}) = \mathrm{Im}(H^1(F^1D^{\cdot}) \longrightarrow H^1(D^{\cdot}))$, we may represent $t^{\star}_i = \overline{0 \oplus \beta}$. Then for $k = 0$, we have $< \underline{x}^{\star}\underline{t}^{\star}; \underline{n} >= \overline{Y(\underline{n})}$ with

$$Y(\underline{n}) = \sum_{\substack{\underline{m}_1+\underline{m}_2=\underline{n} \\ \underline{m}_i \in \overline{B}_1}} \alpha_{\underline{m}_1}\alpha_{\underline{m}_2}.$$

if $\underline{n}$ is a mixing, then every product in the sum is on the form $0 \oplus \beta' \oplus \gamma'$, and thus $Y(\underline{n})$ also. This gives

$$< \underline{x}^{\star}\underline{t}^{\star}; \underline{n} >= \overline{Y(\underline{n})} \in F^1H^2(D^{\cdot}),$$

and so proves the conditon for $k = 0$. Assume the lemma true upto k. Then the $k+1$' th order defining systems are constructed as follows: For each $\underline{m} \in B_k$, $b_{\underline{m}}$ is a coboundary. Because there are no earlier mixings, we have that when $\underline{m}$ is a mix, $b_{\underline{m}} \in F^1H^2(D^{\cdot})$. In fact $b_{\underline{m}} \in F_1D^2$, so $b_{\underline{m}}$ is a coboundary in F^1D^2, allowing us to pick an $\alpha_{\underline{m}} \in F^1D^1 \hookrightarrow D^1$ mapping to $b_{\underline{m}}$. Then again because there are no earlier mixed relations, the terms in the sum $< \underline{x}^{\star}\underline{t}^{\star}; \underline{n} > \in F^1H^2(D^{\cdot})$ when $\underline{n} \in B'_{2+k+1}$ is a mixing. This proves the lemma.

Let $U = \mathrm{Spec}\, A \in \mathcal{U}$ be any open affine in the good covering $\mathcal{U}$. Put (as before) $M = H^0(U,\mathcal{F}), L. = H^0(U,\mathcal{L}.)$ and consider the composition

$$D^{\cdot} \longrightarrow D^{\cdot}/F^1D^{\cdot} \longrightarrow Hom^{\cdot}(L., L.).$$

Let Z denote the kernel complex so that we obtain a short exact sequence

$$0 \longrightarrow Z^{\cdot} \longrightarrow D^{\cdot} \xrightarrow{\sigma} Hom^{\cdot}(L., L.) \longrightarrow 0 \qquad (2.8.2)$$

where σ factors trough $D^{\cdot}/F^1D^{\cdot}$. Now look at

$$H^1(D^{\cdot}) \xrightarrow{\sigma} H^1(D^{\cdot}/F^1D^{\cdot}) \longrightarrow H^1(Hom^{\cdot}(L., L.)).$$

This composition is a surjection by the assumption σ surjective, i.e. $H^2(F^1D^{\cdot}) = 0$, so we may assume that $\{x^{\star}_i\}_{i=1}^{d'} \subseteq H^1(D^{\cdot})$ maps to a basis for $Ext^1_A(M,M)$, $\{x^{\star}_i\}_{i=d'+1}^{d} \subseteq Im(H^1(Z^{\cdot}) \longrightarrow H^1(D^{\cdot}))$. With this notation we have the following:

Lemma 3.7.
Assume $H^2(F^1D^{\cdot}) = 0$. Then if $\underline{n} \in \mathbf{N}^d$ has $n_l \neq 0$ for some $l > d'$, then

$$< \underline{x}^\star \underline{t}^\star; \underline{n} > \in \mathrm{Im}(H^2(Z^{\cdot}) \longrightarrow H^2(D^{\cdot})) \subseteq H^2(D^{\cdot}) = Ext^2_X(\mathcal{F}, \mathcal{F}).$$

Proof.
We copy the procedure of the previous lemma: For $i > d'$,

$$x_i^\star \in Im(H^1(Z^{\cdot}) \longrightarrow H^1(D^{\cdot})),$$

so we may represent $x^\star = \overline{\alpha \oplus \beta}$ with $p_U(\alpha) = 0$ where $p_U : D^{01} \longrightarrow Hom^{\cdot}(L., L.)$ is the projection. Then because the products are taken componentwise in D^{01}, we have $Y(\underline{n}) = \alpha' \oplus \beta'$ with $p_U(\alpha') = 0$ when $\underline{n}$ has $n_l \neq 0$ for some $l > d'$. Thus

$$< x^\star; \underline{n} > \in \mathrm{Im}(H^2(Z^{\cdot}) \longrightarrow H^2(D^{\cdot}))$$

in this case. If there are no earlier mixings, $b_{\underline{m}} \in Z^2$ maps to a coboundary in D^2. Looking at the long exact sequence of the short exact sequence 2.8.2, we have

$$\cdots \longrightarrow H^1(D^{\cdot}) \longrightarrow H^1(Hom^{\cdot}(L., L.)) \longrightarrow H^2(Z^{\cdot}) \xrightarrow{\rho} H^2(D^{\cdot}) \longrightarrow \cdots$$

so that by assumption, ρ is injective. Thus $b_{\underline{m}} \in Z^2$ is a boundary, and we may choose $\alpha_{\underline{m}} \in Z^1$ when $\underline{m}$ is a mix. Again because there are no earlier mixings, $< \underline{x}^\star \underline{t}^\star; \underline{n} > \in H^2(Z^{\cdot})$ when $\underline{n} \in \mathbf{N}^d, n_l \neq 0$ for some $l > d'$. Then the lemma is proved by induction.

Of course the proof of both lemmas require that we know the exact relations in S_{2+k}, R_{2+k} at each step. To restrict the mass of the proofs of the two next propositions, I have decided to treat this separately. Thus the two following propositions both holds at each step in their respective lemmas.

Proposition 3.8.
Let X be any scheme, $\mathcal{F}$ any coherent $\mathcal{O}_X$-module. Pick any basis of

$$Ext^1_X(\mathcal{F}, \mathcal{F}) = H^1(D^{\cdot}) \cong H^1(D^{\cdot})/F^1H^1(D^{\cdot}) \oplus F^1H^1(D^{\cdot}) =' E^{01}_\infty \oplus' E^{10}_\infty$$

on the form $\{x_i^\star\}_{i=1}^d \cup \{t_i^\star\}_{i=d+1}^{\tilde{d}}$ where $\{x_i^\star\}_{i=1}^d$ maps to a basis of $H^1(D^{\cdot})/F^1H^1(D^{\cdot})$ and

$$\{t_i^\star\}_{i=d+1}^{\tilde{d}} \subseteq F^1H^1(D^{\cdot}).$$

Also pick any basis of

$$Ext^2_X(\mathcal{F}, \mathcal{F}) = H^2(D^{\cdot}) \cong H^2(D^{\cdot})/F^1H^2(D^{\cdot}) \oplus F^1H^2(D^{\cdot})/F^2H^2(D^{\cdot}) \oplus F^2H^2(D^{\cdot}) = {}'E^{0,2}_\infty \oplus' E^{1,1}_\infty \oplus' E^{2,0}_\infty$$

on the form $\{y_i^\star\}_{i=1}^r \cup \{z_i^\star\}_{i=r+1}^{r'} \cup \{u_i^\star\}_{i=r'+1}^{\tilde{r}}$ where $\{y_i^\star\}_{i=1}^r$ maps to a basis for ${}'E^{02}_\infty$, $\{z_i^\star\}_{i=r+1}^{r'}$ to a basis for ${}'E^{11}_\infty$,$\{u_i^\star\}_{i=r'+1}^{\tilde{r}}$ to a basis for ${}'E^{20}_\infty$.

Assume $H^2(F^1D^\cdot) = 0$. Then

$$\hat{H}_{\mathcal{F}} \cong k[[\underline{x},\underline{t}]]/(\underline{\tilde{y}})$$

with $\tilde{y}_i \in k[[\underline{x}]]$

Proof.
From lemma 2.9 it follows that $< \underline{x}^\star \underline{t}^\star; \underline{n} > \in F^1H^1(D^\cdot)$ when $\underline{n}$ is a mix. Thus

$$\tilde{y}_i = \sum_{k=0}^{\infty} \sum_{\underline{n}\in B'_{2+k}} y_i(< \underline{x}^\star \underline{t}^\star; \underline{n} >)(\underline{xt})^{\underline{n}} \in k[[x]].$$

Now, consider the morphism of differential graded k-algebras

$$\sigma : D^\cdot \longrightarrow Hom^\cdot(L.,L.).$$

This morphism sends defining systems to defining systems and Massey products to Massey products whenever this makes sense, giving rise to the following:

Proposition 3.9.
With the notations and conditions of proposition 2.7, let $U = \operatorname{Spec} A \in \mathcal{U}$ be any open affine in the good covering $\mathcal{U}$. Assume $\{x_i^\star\}_{i=1}^{d'} \in H^1(D^\cdot)$ maps to a basis for $Ext_A^1(M,M)$, and that $x_i^\star \in \ker(H^1(D^\cdot) \longrightarrow Ext_A^1(M,M))$ for $i > d'$.
Then the mapping from the global to the local hull is a smooth morphism followed by a closed imersion; in fact

$$\hat{H}_M \cong k[[x_1,...,x_{d'}]]/(\tilde{y}_1,...,\tilde{y}_{r'}) \longrightarrow k[[\underline{x},\underline{t}]]/(\tilde{y}_1,...,\tilde{y}_{r'},\tilde{y}_{r'+1},...,\tilde{y}_r) \cong \hat{H}_{\mathcal{F}}$$

Proof. We have

$$H^2(F^1D^\cdot) \longrightarrow H^2(D^\cdot) \longrightarrow H^2(D^\cdot)/F^1H^2(D^\cdot) \longrightarrow 0$$

exact and

$$H^2(F^1D^\cdot) \longrightarrow H^2(D^\cdot) \longrightarrow H^2(D^\cdot/F^1D^\cdot)$$

exact, so that

$$H^2(D^\cdot) \longrightarrow H^2(D^\cdot)/F^1H^2(D^\cdot) \hookrightarrow H^2(D^\cdot/F^1D^\cdot) \longrightarrow Ext_A^2(M,M)$$

with the first and the second map surjective. Thus we may assume that $\{y_i^\star\}_{i=1}^{r'}$ maps to a partial basis for $Ext_A^2(M,M)$,$\{y_i^\star\}_{i=r'+1}^{r} \subseteq H^2(Z^\cdot)$. Then from the lemmas, we have that for $i = 1,...,r'$

$$\tilde{y}_i = \sum_{k=0}^{\infty} \sum_{\underline{n}\in B'_{2+k}} y_i(< \underline{x}^\star \underline{t}^\star; \underline{n} >)(\underline{xt})^{\underline{n}} = \sum_{k=0}^{\infty} \sum_{\underline{n}\in C'_{2+k}} y_i(< \underline{x}^\star \underline{t}^\star; \underline{n} >)(\underline{xt})^{\underline{n}}$$

where $C'_{2+k} = \{\underline{n} \in B'_{2+k} : n_l = 0 \text{ for } l > d'\}$.
Thus

$$\tilde{y}_i = \sum_{k=0}^{\infty} \sum_{\underline{n}\in C'_{2+k}} y_i(< \underline{x}^\star \underline{t}^\star; \underline{n} >)(\underline{xt})^{\underline{n}} = \sum_{k=0}^{\infty} \sum_{\underline{n}\in C'_{2+k}} \sigma(y_i)(\sigma(< \underline{x}^\star \underline{t}^\star; \underline{n} >))(\underline{xt})^{\underline{n}}$$

which is the local power series. The remaining local power series will then be zero, because there are no other y ' s in the Massey products.

Theorem 3.10.

Let X be a curve with only isolated singularities $\{p_1,...,p_s\}$, $\mathcal{F}$ a coherent torsionfree $\mathcal{O}_X$- module. Then

$$\hat{H}_{\mathcal{F}} = (\otimes_{i=1}^{s} \hat{H}_{\mathcal{F}_{p_i}}) \otimes k[[t_{d+1},...,t_{\bar{d}}]].$$

Proof.

We have that $'E_2^{-2,2} \longrightarrow' E_2^{0,1} \longrightarrow' E_2^{2,0}$ and

$$(2.8.3) \qquad 'E_2^{p,q} =' H^p(''H^q(D^{\cdot})) = H^p(X, \mathcal{E}xt_X^q(\mathcal{F},\mathcal{F}))$$

so that

$$'E_2^{-2,2} =' E_2^{2,0} = 0 \Longrightarrow' E_\infty^{0,1} =' E_2^{0,1} = H^0(X, \mathcal{E}xt_X^q(\mathcal{F},\mathcal{F})) = \oplus_{i=1}^{s} Ext^1_{\mathcal{O}_{X,x_i}}(\mathcal{F}_{x_i},\mathcal{F}_{x_i}).$$

Also

$$H^1(D^{\cdot}/F^1D^{\cdot}) ='\,'H^1(D^{0\cdot}) ='\,'H^1(D^0(\mathcal{U}, \mathcal{H}om^{\cdot}(\mathcal{L}.,\mathcal{L}.))) = \oplus_{i=1}^{s} Ext^1_{\mathcal{O}_{X,x_i}}(\mathcal{F}_{x_i},\mathcal{F}_{x_i})$$

so that the diagram

$$\begin{array}{ccccc} H^1(D^{\cdot}) & \longrightarrow & H^1(D^{\cdot})/F^1H^1(D^{\cdot}) & \xrightarrow{\cong} & \oplus_{i=1}^{s} Ext^1_{\mathcal{O}_{X,x_i}}(\mathcal{F}_{x_i},\mathcal{F}_{x_i}) \\ & & \downarrow & & \| \\ & & H^1(D^{\cdot}/F^1D^{\cdot}) & \xrightarrow{\cong} & \oplus_{i=1}^{s} Ext^1_{\mathcal{O}_{X,x_i}}(\mathcal{F}_{x_i},\mathcal{F}_{x_i}) \end{array}$$

implies σ surjective, and that we may choose the basis $\{x_i^\star\}_{i=1}^{d}$ so that $\{x_i^\star\}_{i=1}^{d_1}$ maps to a basis for $Ext^1_{\mathcal{O}_{X,x_1}}(\mathcal{F}_{x_1},\mathcal{F}_{x_1})$,..., $\{x_i^\star\}_{i=d_{s-1}+1}^{d_s=d}$ maps to a basis for $Ext^1_{\mathcal{O}_{X,x_s}}(\mathcal{F}_{x_s},\mathcal{F}_{x_s})$. Furthermore, we find

$$'E_\infty^{1,1} =' E_\infty^{2,0} = 0$$

giving us that

$$H^2(D^{\cdot}) \cong' E_\infty^{0,2} =' E_2^{0,2} = \oplus_{i=1}^{s} Ext^2_{\mathcal{O}_{X,x_i}}(\mathcal{F}_{x_i},\mathcal{F}_{x_i}).$$

At last, it should be mentioned that for curves, it is easy to see that $H^2(F^1D^{\cdot}) = 0$. Indeed, the exact sequence

$$0 \longrightarrow F^1D^{\cdot} \longrightarrow D^{\cdot} \longrightarrow D^{\cdot}/F^1D^{\cdot} \longrightarrow 0$$

gives the section of the long exact sequence

$$H^1(D^{\cdot}) \xrightarrow{\phi} H^1(D^{\cdot}/F^1D^{\cdot}) \xrightarrow{\alpha} H^2(F^1D^{\cdot}) \xrightarrow{\beta} H^2(D^{\cdot}) \xrightarrow{\psi} H^2(D^{\cdot}/F^1D^{\cdot}).$$

From the above statements, follows that ϕ is surjective and ψ an isomorphism. Thus

$$H^2(F^1D^{\cdot}) = \ker\beta = \operatorname{Im}\alpha = 0.$$

Thus the corollary follows from the proposition.

The smoothness of the local to global morphism then follows as a particular case of this corollary:

Theorem 3.11.
Let X be a curve with only one isolated singularity, $\mathcal{F}$ a coherent torsionfree $\mathcal{O}_X$-module. Then

$$\hat{H}_{\mathcal{F}} \cong k[[x_1,...,x_d,x_{d+1},...,x_{\tilde{d}}]]/(f_1,...,f_r) \longleftarrow k[[x_1,...,x_d]]/(f_1,...,f_r) \cong \hat{H}_{\mathcal{F}_p},$$

i.e. the morphism from the local to the global formal moduli is smooth.

4. The local formal moduli of torsionfree modules over E_6 the affine case

Let B be the singularity E_6, that is $B = k[[x,y]]/(x^4+y^3)$. Then the torsionfree B-modules of rank 1 are given by the list of Greuel and knörrer in [6].
Eisenbud [4] tells us that every torsionfree module can be given by a matrix factorisation. Thus it is easy to write up the torsionfree modules:

Lemma 4.1.
Let $\mathcal{M}$ be the family of indecomposable, finitely generated maximal Cohen-Macaulay modules of rank 1 over B. Then $\mathcal{M}$ consists of the B-modules given by the following free resolutions (matrix factorizations):

$$0 \longleftarrow M_0 \longleftarrow B \overset{(x^4+y^3)}{\longleftarrow} B \overset{(1)}{\longleftarrow} B \longleftarrow \cdots$$

or

$$0 \longleftarrow M_0 \cong B \longleftarrow 0 \longleftarrow 0 \cdots$$

$$0 \longleftarrow M_1 \longleftarrow B^2 \overset{\begin{pmatrix} x & -y^2 \\ y & x^3 \end{pmatrix}}{\longleftarrow} B^2 \overset{\begin{pmatrix} x^3 & y^2 \\ -y & x \end{pmatrix}}{\longleftarrow} B^2 \longleftarrow \cdots$$

$$0 \longleftarrow M_2 \longleftarrow B^2 \overset{\begin{pmatrix} x^3 & -y^2 \\ y & x \end{pmatrix}}{\longleftarrow} B^2 \overset{\begin{pmatrix} x & y^2 \\ -y & x^2 \end{pmatrix}}{\longleftarrow} B^2 \longleftarrow \cdots$$

$$0 \longleftarrow M_3 \longleftarrow B^2 \overset{\begin{pmatrix} x^2 & -y^2 \\ y & x^2 \end{pmatrix}}{\longleftarrow} B^2 \overset{\begin{pmatrix} x^2 & y^2 \\ -y & x^2 \end{pmatrix}}{\longleftarrow} B^2 \longleftarrow \cdots$$

$$0 \longleftarrow M_4 \longleftarrow B^3 \overset{\begin{pmatrix} y & -x^2 & 0 \\ x & 0 & -y \\ 0 & -y & -x \end{pmatrix}}{\longleftarrow} B^3 \overset{\begin{pmatrix} y^2 & x^3 & -yx^2 \\ -x^2 & xy & -y^2 \\ xy & -y^2 & -x^3 \end{pmatrix}}{\longleftarrow} B^3 \longleftarrow \cdots$$

In this chapter I shall give the complete list of formal moduli, refering (for the case $B = E_6$) to the list of Lemma 4.1.
In [10] Laudal gives the theory. In [16], the author gives a Singular program computing the following k-algebras. Some detailed hand computations can be found in [15].

M_0.

$$H_0 = k,$$

and of course $\tilde{M_0}(0,0) = M_0$.

M_1.

$$H_1 = k[t,u]/(t^4+u^3), \tilde{M_1}(t,u) = co\ker \begin{pmatrix} x-t & -y^2+uy-u^2 \\ y+u & x^3+tx^2+t^2x+t^3 \end{pmatrix}.$$

M_2.

$$H_2 = k[t,u]/(t^4+u^3), \tilde{M_2}(t,u) \cong B, (t,u) \neq (0,0)$$

M_3.

$$H_3 = k[t_1,...,t_4]/(p,q), \tilde{M_3}(t_1,...,t_4) = \mathrm{Coker} \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}$$

where

$$p = -t_1^2 + t_2^3 + 3t_1t_2t_4^2 - \frac{3}{2}t_2^2t_3t_4 + \frac{1}{2}t_3^4 - \frac{3}{2}t_2t_3^2t_4^2 + \frac{3}{4}t_2^2t_4^4 - \frac{3}{4}t_3^3t_4^3,$$

$$q = -2t_1t_3 + 3t_2^2t_4 - t_3^3 + t_1t_4^3 + 3t_2t_3t_4^2 + t_3^2t_4^3,$$

$$\alpha = x^2 + t_1 + t_3x + t3^2 + t_4^2y - 2t_2t_4^2 - t_3t_4^3,$$

$$\beta = -y^2 + t_2y + t_4xy - t_2^2 - 2t_2t_4x - t_1t_4^2 - t_3t_4^2x,$$

$$\delta = y + t_2 + t_4x,$$

$$\gamma = x^2 - t_1 - t_3x.$$

M_4.

$$H_4 = k[t_1,...,t_6]/(f_1,f_2,f_3)$$

where

$$f_1 = 2t_1t_4t_5t_6 + t_1t_5t_6^2 - t_2t_4t_5^2 + \frac{3}{2}t_2t_5^2t_6 + 4t_3t_4t_5t_6 + 2t_3t_5t_6^2 + t_4^4 + 2t_4^3t_6$$
$$+ 3t_4^2t_6^2 + 2t_4t_6^3 + \frac{1}{2}t_6^4 - t_2^2t_5 - 3t_2t_3t_4 - \frac{3}{2}t_2t_3t_6 + t_3^3,$$

$$f_2 = -2t_1t_5t_6 - t_2t_5^2 - 4t_3t_5t_6 - 2t_4^2t_6 - 2t_4t_6^2 - t_6^3 + 2t_1t_2 + t_2t_3,$$

$$f_3 = -t_1t_5^2 - 2t_3t_5^2 - 2t_4^2t_5 - t_5t_6^2 + t_1^2 + t_1t_3 - 2t_2t_4 - t_2t_6 + t_3^2,$$

$$\tilde{M}_4(t_1,..,t_6) = \text{Coker}\begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} \end{pmatrix}$$

where

$$\alpha_{1,1} = y + t_1 + t_3,\ \alpha_{1,2} = -x^2 + xt_6 + yt_5 - t_3t_5 - t_4^2 - t_4t_6 - t_6^2,$$
$$\alpha_{1,3} = t_2,\ \alpha_{2,1} = x + t_4 + t_6,$$
$$\alpha_{2,2} = 2t_5t_6 - t_2,\ \alpha_{2,3} = -y + t_3,$$
$$\alpha_{3,1} = -t_5,\ \alpha_{3,2} = -t_5^2 - y + t_1,$$
$$\alpha_{3,3} = -x + t_4.$$

Computation of the boundary of the compactified Jacobian

5.1 Properties of the global to local morphism.

Let X be a projective curve. When X has only isolated singularities, we may assume that $\text{Sing}(X) \subseteq U = \text{Spec}(A) \subseteq X$. For simplicity, we will assume that $\text{Sing}(X) = \{x_0\}$.

Let $\mathcal{M}$ be any reflexive quasi coherent rank 1 $\mathcal{O}_X$-module. Then there is a morphism

$$\text{Def}_{\mathcal{M}}(-) \longrightarrow \text{Def}_{\mathcal{M}|_U}(-) = \text{Def}_{M_0}(-),$$

where $M_0 = H^0(U, \mathcal{M})$, inducing a morphism

$$\hat{H}_{M_0} \longrightarrow \hat{H}_{\mathcal{M}}.$$

This last morphism is proved to bee smooth (theorem 3.11.), i.e $\hat{H}_{\mathcal{M}}$ is smooth over $\hat{H}_{M_0}$.

If $\mathcal{C}$ is a coarse moduli for a family of modules, then by definition there exists a morphism $\underline{\hat{H}}_{\mathcal{M}} \longrightarrow \mathcal{C}$ in which 0 is sendt to $\mathcal{M}$. This induces a morphism

$$\hat{\mathcal{O}}_{\mathcal{C},\mathcal{M}} \longrightarrow \underline{\hat{H}}_{\mathcal{M}}.$$

From the geometric invariant theory, it follows that $\underline{\hat{H}}_{\mathcal{M}}^G = \hat{\mathcal{O}}_{\mathcal{C},\mathcal{M}}$, where G is a quotient of $\text{Aut}_X(\mathcal{M})$, implying that G is a discrete group. It can be proved, Laudal [9], that if $\underline{\hat{H}}_{\mathcal{M}}^G$ is algebraisable, then so is $\underline{\hat{H}}_{\mathcal{M}}$, the algebraizations giving open affines (up to étale isomorphism).

In our case, there is no problem because $\overline{P}$ is a fine moduli, thus $\hat{\mathcal{O}}_{\overline{P},\mathcal{M}} \cong \hat{H}_{\mathcal{M}}$. When $\hat{H}_{M_0} \longrightarrow \hat{H}_{\mathcal{M}}$ is smooth, the same is true for the algebraizations:

$H_{M_0} \hookrightarrow H_{\mathcal{M}}$ is smooth (and injective).
Now look at the morphism

$$\underline{H}_{\mathcal{M}} = \operatorname{Spec} H_{\mathcal{M}} \xrightarrow[\phi]{} \operatorname{Spec} H_{M_0} = \underline{H}_{M_0}.$$

Then $\mathcal{M} \in \underline{H}_{\mathcal{M}}$ maps to $M_0 \in \underline{H}_{M_0}$. Rego [11] proves that if two global modules, say P, Q maps to the same module locally, then $P = Q \otimes \mathcal{L}$ for some $\mathcal{L} \in \operatorname{Pic}^0 X$. Thus the fiber of ϕ over M_0 is

$$\mathcal{M} \cdot \operatorname{Pic}^0 X = o(\mathcal{M}) \subseteq \underline{H}_{\mathcal{M}}.$$

Because $\hat{H}_{M_0} \hookrightarrow \hat{H}_{\mathcal{M}}$ is smooth, $\hat{H}_{\mathcal{M}}$ is just a formal power series over $\hat{H}_{M_0}$, and the number of indeterminates is $\mathrm{h}^1(\mathcal{E}\mathrm{nd}_X(\mathcal{M}))$, which is easily seen writing up the tangent spaces. Thus the dimension of the fiber is $\mathrm{h}^1(\mathcal{E}\mathrm{nd}(\mathcal{M}))$.

Lemma 5.1.

$$\mathrm{h}^1(\mathcal{E}\mathrm{nd}_X(\overline{\mathcal{O}}_X)) = \mathrm{p_a}(\overline{X}),$$

thus $\mathrm{H}^1(\mathcal{E}\mathrm{nd}_X(\overline{\mathcal{O}}_X))$ *is minimal among the fibers .*

Proof.
The exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \overline{\mathcal{O}}_X \longrightarrow \overline{\mathcal{O}}_X/\mathcal{O}_X \longrightarrow 0$$

gives the long exact sequence

$$0 \longrightarrow \mathrm{H}^0(X, \mathcal{O}_X) \longrightarrow \mathrm{H}^0(X, \overline{\mathcal{O}}_X) \longrightarrow \mathrm{H}^0(\overline{\mathcal{O}}_X/\mathcal{O}_X) \longrightarrow \mathrm{H}^1(X, \mathcal{O}_X) \longrightarrow \mathrm{H}^1(X, \overline{\mathcal{O}}_X) \longrightarrow 0$$

implying that

$$\mathrm{p_a}(X) = \mathrm{h}^1(X, \mathcal{O}_X) = \mathrm{h}^1(X, \overline{\mathcal{O}}_X) + \mathrm{h}^0(X, \overline{\mathcal{O}_X}/\mathcal{O}_X) = \mathrm{p_a}(\overline{X}) + \delta(X).$$

On the other hand, there is also a short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E}\mathrm{nd}_X(\overline{\mathcal{O}}_X) \longrightarrow \overline{\mathcal{O}}_X/\mathcal{O}_X \longrightarrow 0$$

implying the exact sequence

$$0 \longrightarrow \mathrm{H}^0(X, \mathcal{O}_X) \longrightarrow \mathrm{H}^0(X, \mathcal{E}\mathrm{nd}_X(\overline{\mathcal{O}}_X)) \longrightarrow \mathrm{H}^0(\overline{\mathcal{O}}_X/\mathcal{O}_X) \longrightarrow \mathrm{H}^1(X, \mathcal{O}_X) \longrightarrow \mathrm{H}^1(X, \mathcal{E}\mathrm{nd}_X(\overline{\mathcal{O}}_X)) \longrightarrow 0$$

giving that

$$\mathrm{h}^1(X, \mathcal{E}\mathrm{nd}_X(\overline{\mathcal{O}}_X)) = \mathrm{p_a}(\overline{X}).$$

Looking at reflexive modules as fractional ideals, gives the minimality, see Cook [2].

Thus in this case with $p_a(\overline{X}) = 0$ (i.e. $\overline{X}$ is rational) it follows that we have an isomorphism

$$\hat{H}_{\mathcal{M}} \cong \hat{H}_{M_0}.$$

Now the local study of $\bar{P}$ can be reduced; because of the smoothness, we know that $\hat{\mathcal{O}}_{\bar{P},\mathcal{M}}$ is a formal power series ring over $\hat{H}_{\mathcal{M}_0}$, implying that $\mathcal{O}_{\bar{P},\mathcal{M}}$ is a polynomial ring over $H_{\mathcal{M}_0}$, upto some choice of étale sheet. Thus the local study on $\bar{P}$ reduces to the affine case.

5.2. Stratification of the miniversal family.

By a stratification of $\underline{H}_M = \mathrm{Spec}(H)$, we understand a family of locally closed subsets $\{S_\nu\}_{\nu\in I}$ of $\underline{H}_M$ such that $\coprod S_\nu = \underline{H}_M$. We put an ordering on the strata by defining

$$\nu < \nu' \Leftrightarrow \bar{S}_\nu \supseteq S_{\nu'} \text{ and } \nu \neq \nu'.$$

Now we put $M = \bar{A}$ and use the stratification

$$S_\nu = \{\underline{t} \in \underline{H} | M(\underline{t}) \cong M_\nu\}.$$

Theorem 5.2.(Greuel and Pfister).

For any M, the only closed stratum is S_{ν_0}, the stratum of the normalization.

Proof. Greuel and Pfister [7]

This means in paricular that S_{ν_0} must be a spezialization of all other strata $\Longrightarrow \bar{A}$ deforms to every other reflexive A-module of rank 1. Also it implies that every stratum meets $H_{\bar{A}}$.

Now put $\mathcal{M} = \bar{\mathcal{O}}_X$. Then from the theorem of Greuel and Pfister, it is enough to study the corresponding boundary points of $\underline{H}_M, M = \bar{A}$. The closure of the strata of the not locally free sheaves then gives the components of the boundary.

5.3. The Kodaira Spencer Map.

The notation now is that M is an A-module. Then the exact sequence

$$0 \longrightarrow I/I^2 \longrightarrow (A \otimes A)/I^2 \underset{\overset{\longleftarrow}{i_2}}{\overset{\overset{i_1}{\longleftarrow}}{\longrightarrow}} A \longrightarrow 0$$

gives the possibility of defining

$$c(M) = i_{1*}M - i_{2*}M \in \mathrm{Ext}^1_A(M, M \otimes_A \Omega_A).$$

Put $H = H_M$ and let $\tilde{M}$ be the miniversal family. Then we want to define

$$g : \mathrm{Der}_k(H) \longrightarrow \mathrm{Ext}^1_{H\otimes_k A}(\tilde{M}, \tilde{M}).$$

Consider the diagram

$$\begin{array}{ccccc} \mathrm{Der}_k(H) & \longrightarrow & \mathrm{Der}_k(H \otimes_k A) & \longrightarrow & \mathrm{Ext}^1_{H\otimes_k A}(\tilde{M}, \tilde{M}) \\ \downarrow{\scriptstyle \rho_t} & & & & \downarrow{\scriptstyle \rho_t} \\ \mathrm{Der}_k(H, k(\underline{t})) & & \xrightarrow{t_{\phi_t}} & & \mathrm{Ext}^1_A(M(\underline{t}), M(\underline{t})) \end{array}.$$

From the versality of $H_{M(\underline{t})}$, we know there is a surjection

$$\mathrm{Mor}(H, H_{M(\underline{t})}/\underline{m}_t^n) \twoheadrightarrow \mathrm{Def}_{M(\underline{t})}(H_{M(\underline{t})}/\underline{m}_t^n)$$

which is a bijection on the tangent space level. Thus there is a unique

$$t_{\phi_{\underline{t}}} : \mathrm{Der}_k(H, k(\underline{t})) \longrightarrow \mathrm{T}_{H_{M(\underline{t})},0} .$$

To define g such that the diagram commutes for all t, it is enough to define

$$\tilde{g} : \mathrm{Der}_k(A) \longrightarrow \mathrm{Ext}^1_A(M, M)$$

in general:
$\delta \in \mathrm{Der}_k(A) = \mathrm{Hom}_A(\Omega_A, A)$, gives a morphism

$$\delta_* : \mathrm{Ext}^1_A(M, M \otimes_A \Omega_A) \longrightarrow \mathrm{Ext}^1_A(M, M \otimes_A A) = \mathrm{Ext}^1_A(M, M),$$

and so we may define

$$\tilde{g}(\delta) = \delta_*(c(M)).$$

Consider the strata on $\underline{H} = \underline{H}_{\tilde{A}}$. Then

$$\mathrm{T}_{S_\nu, \underline{t}} \subseteq \mathrm{Der}_k(H, k(\underline{t})) = \mathrm{T}_{\underline{H}, \underline{t}} .$$

We understand that if $\mathrm{T}_{S_\nu,\underline{t}}$ is of maximal rank $\dim \underline{H}$, then the point $\underline{t}$ represents a locally free (=projective) A-module. Thus the points corresponding to the boundary of the compactified Jacobian, is the points where $\mathrm{T}_{S_\nu,\underline{t}}$ has not maximal rank.

It can be proved that

$$\mathrm{Im} = \mathbb{V},$$

where $\mathbb{V}$ is the kernel of g and Im means the image of the tangent spaces on the stratification components.

Definition 5.3.
The discriminant is defined as $\delta = \det \mathbb{V} = \bigwedge_H^d \mathbb{V},\ d = \dim H$.

Now δ is a H-module of rank 1 and gives rise to a divisor $\underline{\delta}$.

Corollary 5.4. (of the smoothness of global to local theorem).
The boundary of the compactified Jacobian is given by the local discriminant

$$\underline{\delta} = \{p \in \underline{H} | \dim \mathbb{V} < d\}.$$

5.4. The results of Cook and Rego.

In the Gorenstein case, Greuel und Knörrer [6] proves that the simple singularities (i.e. of type ADE) has only a finite number of isomorhism classes of torsion free rank 1 modules. Thus a natural problem is to describe the compactified Jacobians of curves with simple singularities. This is the aim of Cook's paper Cook[2].

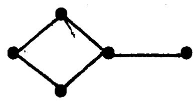
Cook stratisfies $\mathcal{M}(X)$ in the following way: Let $\underline{M}$ denote a collection of rank 1 torsionfree modules, one M_x for each singular point $x \in X$. Write

$$i(M) = \sum_x i(M_x)$$

where $i(M_x) = dim_k(\text{End}(M_x)/\mathcal{O}_x)$, and let $\mathcal{U}_{\underline{M}}$ be the subset of $\mathcal{M}(X)$ of sheaves $\mathcal{F}$ s.t. $\mathcal{F}_x \cong M_x$ for each singular point $x \in X$. Note that $\mathcal{U}_{\underline{M}}$ is not empty, Rego[11].

Cook shows that $\mathcal{U}_{\underline{M}}$ is a smooth irreducible locally closed subvariety of $\mathcal{M}(X)$ and $\dim \mathcal{U}_{\underline{M}} = g(X) - i(\underline{M})$.

He also computes the number of $\mathcal{U}_{\underline{M}}$s of given codimension. This determines the stratification diagrams for the singularities A_n , $D_{4,5}$ and E_6 (in the other cases the theory of local parabolic models is needed). For example, E_6 has the following stratification diagram:



Here each vertex corresponds to a locally closed subvariety $\mathcal{U}_M$, for some torsion free rank 1 module M. Two vertices are joined if and only if the closure of the component on the left contains the component on the right. The leftmost vertex corresponds to $\mathcal{U}_{\mathcal{O}} = J(X)$. The codimension of a given stratum is the distance from this vertex in the diagram. In each case the total length of the diagram is equal to $\delta(X) = \dim_k(\overline{\mathcal{O}}/\mathcal{O}, \overline{\mathcal{O}})$ denotes the normalization of $\mathcal{O}$.

Thus Cook give us the stratification diagrams, implying among other things, the number of irreducible components of the boundary of the generalised Jacobian. This is Rego's result.

5.5. Example.

When $X = E_6$, then we computed (see chapter 1) H_M for all reflexive M. For $M_4 = \bar{A}$ the result is

$$H_4 = k[t_1, ..., t_6]/(f_1, f_2, f_3)$$

where

$$f_1 = 2t_1t_4t_5t_6 + t_1t_5t_6^2 - t_2t_4t_5^2 + \frac{3}{2}t_2t_5^2t_6 + 4t_3t_4t_5t_6 + 2t_3t_5t_6^2 + t_4^4 + 2t_4^3t_6$$
$$+ 3t_4^2t_6^2 + 2t_4t_6^3 + \frac{1}{2}t_6^4 - t_2^2t_5 - 3t_2t_3t_4 - \frac{3}{2}t_2t_3t_6 + t_3^3$$

$$f_2 = -2t_1t_5t_6 - t_2t_5^2 - 4t_3t_5t_6 - 2t_4^2t_6 - 2t_4t_6^2 - t_6^3 + 2t_1t_2 + t_2t_3,$$

$$f_3 = -t_1t_5^2 - 2t_3t_5^2 - 2t_4^2t_5 - t_5t_6^2 + t_1^2 + t_1t_3 - 2t_2t_4 - t_2t_6 + t_3^2,$$

$$\tilde{M}_4(t_1, .., t_6) = \text{Coker}\begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} \end{pmatrix}$$

where

$$\alpha_{1,1} = y + t_1 + t_3,\ \alpha_{1,2} = -1x^2 + xt_6 + yt_5 - 1t_3t_5 - 1t_4^2 - 1t_4t_6 - 1t_6^2,$$
$$\alpha_{1,3} = t_2,\ \alpha_{2,1} = x + t_4 + t_6,$$
$$\alpha_{2,2} = 2t_5t_6 - 1t_2,\ \alpha_{2,3} = -1y + t_3,$$
$$\alpha_{3,1} = -1t_5,\ \alpha_{3,2} = -1t_5^2 - 1y + t_1,$$
$$\alpha_{3,3} = -1x + t_4.$$

To find the degeneration properties of the versal family, we would like to compute the kernel of the Kodaira-Spencer morphism. The method for doing this is easy: First compute

$$\Omega_H \cong \bigoplus_{i=1}^{d} H du_i / (df_i)_{i=1}^{r}.$$

Then compute $Der_k(H) \cong \mathrm{Hom}_H(\Omega, H)$ as $Ext^0_H(\Omega, H)$. It is sad, but true, that with todays computers, this is to big to compute, and we just have to leave it for the moment.

We are left with one possibility: Make some qualified guesses on points on H, then compute the Ext-dimention of the versal deformation in the actual points and prove that these are the different modules on E_6.

Point	Ext^1-dimention
$(0,1,0,0,0,0)$	0
$(0,0,0,0,1,0)$	2
$(1,0,0,0,1,0)$	2
$(1,0,-1,1,1,0)$	4

Now the isomorphismclass of the first point has to be the one and only free module. Checking out the two points of Ext^1-dimention 2, that is filling the point into the versal family, we find that the module corresponding to the point $(0,0,0,0,1,0)$ is the cokernel of

$$\begin{pmatrix} x^2+y^2 & xy \\ xy+x & x^2+y \end{pmatrix}$$

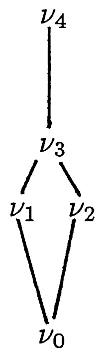
and that the module corresponding to the point $(1,0,0,0,1,0)$ is the cokernel of

$$\begin{pmatrix} x^2+y^2 & xy+x \\ xy & x^2+y \end{pmatrix}.$$

These two modules are isomorphic to $M = \begin{pmatrix} xy+x & -x^2-y \\ -x^2-y^2 & xy \end{pmatrix}$ and $N = \begin{pmatrix} -xy & -x^2-y \\ -x^2-y^2 & -xy-x \end{pmatrix}$ respectiveley, and we find that

$$MN = NM = fI.$$

This proves that the two modules are nonisomorphic (look at the given matrix factorisations of M_1 and M_2) and so they are toghether M_1 and M_2. The last point has to be M_3, and so the stratification diagram looks like:



Where each vertex corresponds to a locally closed subvariety $\mathcal{U}_M$, for some torsion free rank 1 module M.Two vertices are joined if and only if the closure of the component below contains the component above. The lowermost vertex corresponds to $\mathcal{U}_{\mathcal{O}} = J(X)$. The codimension of a given stratum is the distance from this vertex in the diagram. The boundary is given by $\bar{S}_{\nu_1} \cup \bar{S}_{\nu_2}$ which is closed and irreducible. Thus in this case we have two components.

This gives the number of components and the properties of deforming. But using the local theory like this, we can do more: We can describe the boundary (locally) completely.

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