On the Limit Distributions of Random Matrices with independent or free entries

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Abstract

We state limit distribution results for random matrices with independent or free entries, also addressing when we get freeness in the limit and semicircular and circular limits. The results generalize some already known results about asymptotic freeness of large random matrices, but our goal is to get a more optimal flavour on these results. When having matrices with identically distributed entries, we show that freeness in the limit is typical when we have free entries, but nontypical when the entries are independent, and restricted to the case of circular limits.

1 Introduction

The theory of limit distributions of random matrices has recently (see [13], [1]) found applications in the form of free group factors in Von Neumann algebra theory. The clue to this application is the multimatrix version of Wigners semicircle law ([15], [16]), discovered by Voiculescu ([13]), which says that independent Gaussian random matrices in the limit become free semicircular random variables, free from sets of constant block diagonal matrices.

Although Gaussian random matrices have been sufficient in obtaining the applications to free group factors, one can ask the question of how general random matrices one can use in order to obtain the same limit distribution results. This question is for instance addressed in [2], where an optimal result was obtained for when one gets a semicircular limit distribution for the eigenvalues of certain random matrices. The task of this paper is to work towards a similar optimal-flavoured result for convergence in distribution, i.e. that of convergence of moments of the distributions. We will also consider the multimatrix problem, and we will be especially interested in how the limit distributions relate: When do we have asymptotic freeness in the limit, as in the application to free group factors?

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We will start by looking at selfadjoint random matrices with (up to symmetry) independent entries, and prove a generalized version (theorem 1) of the asymptotic freeness results for random matrices and constant block diagonal matrices appearing in [13], [1]. The proof is rather different from that in [13], [1], and hinges on a characterization of freeness due to Nica and Speicher ([6], [7]). This new proof and the combinatorics appearing there is an essential part of the paper. In the proof, we keep at the same time track of how the moments of the entries can be allowed to grow as $n \to \infty$ in order to get a limit distribution.

There is a similar result, theorem 2, without the symmetry condition on the matrices, i.e. *all* entries are assumed independent. This result is surely also known. In this direction we also find an 'optimal' result, theorem 3, for what growth conditions we can put on the moments of the entries in order to obtain convergence in distribution for random matrices with identically distributed entries (for what optimal should mean in this case, see the comments preceding theorem 3).

It will also be clear from this when one obtains free distributions in the limit, and that the circular limit distributions are what usually appear when this is the case. This shows that the results in [13], [1] and the matrices considered there are in a certain way representative for the possibilities in obtaining free limit distributions. When working towards the 'optimal' result in theorem 3, we get a whole class of limit distribution laws, and we will show that the structure of this class is governed by a set of partitions, called clickable partitions, in the same way free random variables are governed by the set of noncrossing partitions. We characterize these partitions at the end of the paper. There is an obstruction for attaining free random variables in the limit, this is roughly that the set of clickable partitions is somewhat larger than the set of noncrossing partitions. We also make some comments regarding the 'selfadjoint version' of theorem 3 and the possible limit distributions that can be constructed by using as entries random variables which have infinitely divisible distribution.

If one replaces the word independent with $*$-free, the situation gets to be different. There is a similar result, theorem 4, as theorem 3 in this case, but, vaguely speaking, the obstruction mentioned above that the set of clickable partitions is larger than the set of noncrossing partitions has disappeared in the calculations. Freeness in the limit will be a consequence of this. Also, the R-transform of the joint limit $*$-distribution of the matrices can be described in terms of properties of the entries in a nice way, and one can see from this that in the limit we typically get (free) random variables giving rise to R-diagonal pairs (see definition 6). This is perhaps the most striking result of the paper. R-diagonal pairs have been studied already in the litterature [7]. For the limit distribution we obtain a nice interpretation of the cumulants and the noncrossing partitions in terms of the properties of the entries and matrix multiplication, when writing out the moments of the matrices as sums of products of the entries. We also show that we in the selfadjoint version of theorem 4 can get *all* even, infinitely divisible probability measures with compact support in the limit.

Random matrices with free entries have also been studied by Shlyakhtenko [9].

The conclusion is that asymptotic freeness of random matrices (at least when one has identically distributed entries) becomes natural when the entries are $*$-free, but is unnatural when the entries are independent and is then in a certain way restricted to the free circular limit distributions.

2 Combinatorial preliminaries

We will occupy ourselves with certain noncommutative probability spaces. A noncommutative probability space is a pair (A, ϕ) where A is a unital $*$-algebra and ϕ is a normalized (i.e. $\phi(1) = 1$) linear functional on A. The elements of A are called random variables.

An important particular case is the case where $A = L = \cap_{1 \leq p < \infty} L^p(\sigma)$ with σ some probability measure on a measure space, i.e. the algebra of complex valued random variables, having bounded moments of all orders. The state on L, given by integration with respect to σ, is denoted E (serving as ϕ).

Definition 1. *A family of unital $*$-subalgebras $(A_i)_{i \in I}$ will be called a free family if*

$$\left\{ \begin{array}{c} a_j \in A_{i_j} \\ i_1 \neq i_2, i_2 \neq i_3, ..., i_{n-1} \neq i_n \\ \phi(a_1) = \phi(a_2) = \cdots = \phi(a_n) = 0 \end{array} \right\} \Rightarrow \phi(a_1 \cdots a_n) = 0. \tag{1}$$

The family $(\{a_{11}, ..., a_{1k_1}\}, ..., \{a_{n1}, ..., a_{nk_n}\})$ will be called a $*$-free family if the $*$-algebras $A_i = * - alg(a_{i1}, ..., a_{ik_i})$ form a free family (we sometimes write free for $*$-free if the sets $\{a_{i1}, ..., a_{ik_i}\}$ are selfadjoint).

$C\langle X_1, ..., X_n \rangle$ will be the unital algebra of complex polynomials in n noncommuting variables. Unital complex linear functionals on $C\langle X_1, ..., X_n \rangle$ will be called distributions. The set of all distributions will be denoted Σ_n (or simply Σ_I for a general index set I). If $a_1, ..., a_n$ are elements in some noncommutative probability space (A, ϕ), their joint distribution $\mu_{a_1,...,a_n} \in \Sigma_n$ is defined by having mixed moments

$$\mu_{a_1,...,a_n}(X_{i_1} \cdots X_{i_m}) = \phi(a_{i_1} \cdots a_{i_m}).$$

Definition 2. *We will say that random variables $(a_n(1), a_n(2), ...) \subset (A_n, \phi_n)$ converge in $*$-distribution (to random variables $(a(1), a(2), ...) \subset (A, \phi)$) if*

$$\lim_{n \to \infty} \phi_n(a_n(i_1)^{g(1)} \cdots a_n(i_k)^{g(k)}) \textit{ exists (is equal to } \phi(a(i_1)^{g(1)} \cdots a(i_k)^{g(k)})) \tag{2}$$

for all choices of $k, i_1, ..., i_k$ and functions $g : \{1, ..., k\} \to \{\cdot, *\}$. If this is the case, and the $(a(1), a(2), ...)$ are $*$-free in (A, ϕ), we will say that $(a_n(1), a_n(2), ...)_n$ is an asymptotically $*$-free family.

If $(a_n(1), a_n(2), ...)$ converge in $*$- distribution as above and there is no mention of (A, ϕ) and $(a(1), a(2), ...)$ in the limit, we will think of $a(i), a(i)^*$ as the random variables X_i, X_i^* in $(C\langle X_1, X_1^*, X_2, X_2^*, ...\rangle, \mu)$ with μ the unital linear functional defined by $\mu(X_{i_1}^{g(1)} \cdots X_{i_k}^{g(k)}) = \lim_{n \to \infty} \phi_n(a_n(i_1)^{g(1)} \cdots a_n(i_k)^{g(k)})$ (i.e. μ is the limit distribution).

Given a sequence $\{\{a_n(i, j; k)\}_{1 \leq i,j \leq n}\}_n$ of random variables from (A, ϕ), (possibly subject to the symmetry condition $a_n(i, j; k) = a_n(j, i; k)^*$), we will consider the matrices $A_n(k) = \sum_{1\leq i,j\leq n} a_n(i,j;k)e_n(i,j)$ in $(M_n(A),\phi_n)$, where $e_n(i,j)$ is the canonical system of matrix units, and ϕ_n is ϕ tensored with the normalized trace on the $n\times n$-matrices. We will occupy ourselves with the limit distribution of the $A_n(k)$, together with sets of constant block diagonal matrices (this is to be specified). Mostly we will try to conclude the mere existence of a joint limit $*$-distribution or asymptotic $*$-freeness of such random matrices subject to a freeness or independence condition on the entries. It turns out that all limit distributions we encounter naturally can be expressed in terms of cumulants, either the free cumulants or some cumulants coming from a different setting.

To be able to express the joint limit $*$-distributions of our matrices, we will need the following definitions and results:

2.1 Preliminaries on noncrossing partitions and the R-transform

The set of all partitions of $\{1,...,m\}$ will be denoted $\mathcal{P}(m)$. A partition π will have block structure $\{B_1,...,B_k\}$, $|\pi| = k$ will be the number of blocks and $|B_i|$ will denote the number of elements in each block. Also, $|\pi|_k$ will be the number of blocks of cardinality k. We will also write $B_i = \{v_{i1},...,v_{i|B_i|}\}$, with the v's written in increasing order, and write $i\sim j$ when i and j are in the same block (or $i\sim_\pi j$ when we need specify the partition). A partition will be called even if all blocks have even cardinality, and $\mathcal{P}(m)_{even}$ denotes the set of even partitions. The following class of partitions will be important:

Definition 3. *A partition π is called noncrossing if whenever we have $i<j<k<l$ with $i\sim k, j\sim l$ we also have $i\sim j\sim k\sim l$ (i.e. i,j,k,l are* all *in the same block). The set of all noncrossing partitions is denoted $NC(n)$. We will also write $NC(n)_2$ for the noncrossing partitions with all blocks of cardinality two.*

The fact that $i<j<k<l$ could actually be taken in the general sense that i,j,k and l lie in clockwise order (or more precisely, $i<j<k<l<i$) on the circle when one identifies $\{1,...,n\}$ with points on the circle as one does in the circular representation of a partition (see section 2.2 for the definition of the circular representation). This also gives the notion of successors in blocks meaning, by addressing the next element of the block in the clockwise direction.

$NC(n)$ becomes a lattice with the refinement order on the set of partitions, i.e. the partial ordering given by refinement of partitions (the maximal and minimal elements are denoted 1_n and 0_n, which are the partitions with 1 block and n blocks, respectively). We will have use for the complementation map of Kreweras, a lattice anti-isomorphism $NC(n)\to NC(n)$. We denote it by K. It is usually defined in terms of a circular representation of the partitition ([3], [6]). In this paper we will not have use for K defined on the set of all noncrossing partitions, but rather on a smaller set of partitions, for which we will define it through a different circular representation, see section 2.2.

The multidimensional R-transform, also defined in [4], is an important transform $\Sigma_n\to\Theta_n$, where Θ_n denotes the set of all power series with vanishing constant term in n noncommuting variables. In referring to the coefficients of a power series $f = \sum a_{i_1,\ldots,i_m} z_{i_1} \cdots z_{i_m}$ we will write

$$[\text{ coef } (i_1, \ldots, i_m)](f) = a_{i_1,\ldots,i_m},$$

and if $\pi = \{B_1, \ldots, B_k\} \in \mathcal{P}(m)$,

$$[\text{ coef } (i_1, \ldots, i_m)|B_i](f) = a_{(i_j)_{j\in B_i}},$$

$$[\text{ coef } (i_1, \ldots, i_m); \pi](f) = \prod_i [\text{ coef } (i_1, \ldots, i_m)|B_i](f).$$

We will define the R-transform in the following way, which is not the way it was defined first in the litterature (the characterization below can be derived from the *real* definition):

Definition 4. *If $\mu \in \Sigma_n$ then $R(\mu) \in \Theta_n$ is the unique power series such that*

$$\mu(X_{i_1} \cdots X_{i_m}) = \sum_{\pi \in NC(m)} [\text{ coef } (i_1, \ldots, i_m); \pi](R(\mu)). \tag{3}$$

for all monomials $X_{i_1} \cdots X_{i_m}$.

One can show (by induction on m in (3), also called the moment-cumulant formula, the R-transform coefficients are sometimes referred to as cumulants) that (3) provides a bijection from Σ_n to Θ_n.

Note that the odd moments of μ_a are all zero if and only if all the odd R-transform coefficients are zero. This is easy to show by induction from the formula (3). Such random variables a are called even.

Having the R-transform, one can define semicircular and circular random variables (again, this is not the way these concepts were defined first, but they can be derived from the real definition):

Definition 5. *A random variable a is called*

1. *(centered) semicircular (of radius $r > 0$) if it is selfadjoint and its R-transform is given by $R(\mu_a)(z) = \frac{r^2}{4} z^2$. A semicircular family is a family of free semicircular random variables*
2. *(centered) circular (of radius $r > 0$) if the R-transform is given by $R(\mu_{a,a^*})(z, z^*) = \frac{r^2}{4} zz^* + \frac{r^2}{4} z^* z$.*
3. *a creation operator (on the full Fock space) if its R-transform is given by $R(\mu_{a,a^*})(z, z^*) = z^* z$ (creation operators should really be defined through the vacuum expectation on the full Fock space, but we will not have use for this characterization).*

The quantity $\alpha = \frac{r^2}{4}$ in the above could also be called variance, since we assume centeredness (i.e. the first moment is zero). We will also need the following definition related to the R-transform:

Definition 6. *([7]) $\{a, b\}$ is called an R-diagonal pair if*

$$R(\mu_{a,b})(z_1, z_2) = \sum_{k=1}^{\infty} \Big(b_k (z_1 z_2)^k + b_k (z_2 z_1)^k \Big) \tag{4}$$

for some sequence of complex numbers $\{b_k\}$. We will say that a random variable a gives rise to an R-diagonal pair if $\{a, a^*\}$ is an R-diagonal pair. The sequence $\{b_n\}_n$ is called the defining sequence (or determining series) of the R-diagonal pair.

The simplest example of a random variable giving rise to an R-diagonal pair is the circular random variable, as can easily be seen from 2 of definition 5.

The R-series is sometimes used in connection with the moment series of a distribution:

Definition 7. *The moment series of the distribution $\mu \in \Sigma_n$ is the power series $M(\mu)$ in Θ_n given by*

$$M(\mu)(z_1, ..., z_n) = \sum_{m \geq 1} \sum_{i_1, ..., i_m} \mu(X_{i_1} \cdots X_{i_m}) z_{i_1} \cdots z_{i_m}.$$

The Kreweras complementation map enters the picture due to a convolution product which is an important tool in recognizing freeness of random variables:

Definition 8. *The boxed convolution $f \boxed{\star} g$ (see [5], [6], [7]) of two power series f and g is the power series defined by*

$$[\ coef\,(i_1, ..., i_m)](f \boxed{\star} g) = \sum_{\pi \in NC(m)} [\ coef\,(i_1, ..., i_m); \pi](f) [\ coef\,(i_1, ..., i_m); K(\pi)](g), \tag{5}$$

With this definition at hand, we can state the characterization of freeness which we will use ([6]):

Lemma 9. *If ϕ is a trace, the following are equivalent*

1. *$\{a_1, ..., a_n\}$ and the unital algebra D are free in (A, ϕ)*
2. *$\phi(a_{i_1} d_1 \cdots a_{i_k} d_k) = [\ coef\,(1, ..., k)](R(\mu_{a_{i_1}, ..., a_{i_k}}) \boxed{\star} M(\mu_{d_1, ..., d_k}))$ for all choices of $k, 1 \leq i_1, ..., i_k \leq n$ and $d_1, ..., d_k \in D$.*

This characterization of freeness can be formulated also without referring to the coefficients of the series (see [7], [8]), but it is the coefficients *themselves* which will appear naturally in our calculations.

This says nothing about mutual freeness of the a_i. For this we will use the following 'no mixed terms' characterization of freeness([4]):

Lemma 10. *The following are equivalent:*

1. *$(\{a_{1,1}, ..., a_{1,m_1}\}, ..., \{a_{n,1}, ..., a_{n,m_n}\})$ is a free family in (A, ϕ)*

2. the coefficient of $z_{i_1,j_1}\cdots z_{i_k,j_k}$ *in*

$$R(\mu_{a_{1,1},\ldots,a_{1,m_1},\ldots,a_{n,1},\ldots,a_{n,m_n}})(z_{1,1},\ldots,z_{1,m_1},\ldots,z_{n,1},\ldots,z_{n,m_n}) \tag{6}$$

vanishes whenever we don't have $i_1 = i_2 = \cdots = i_k$.

When referring to monomials in $C\langle X_1,\ldots,X_n\rangle$ we will use the following terminology:

Definition 11. *The signed partition* σ *of a monomial* $X_{i_1}\cdots X_{i_m}$ *is the partition obtained by saying that* j *and* k *are in the same block, say* σ_l*, if and only if* $i_j = i_k = l$. σ *gives rise to the sign map, defined by* $\sigma(k) = i_k$ *for* $k \in \{1,\ldots,m\}$. *If* $\pi = \{A_1,\ldots,A_h\} \leq \sigma$ *we will also write* $\sigma(A_i) = r$ *if* $\sigma(k) = r$ *for all* $k \in A_i$.

Note from lemma 10 that $*$-freeness of $(a_1, a_2, \ldots,)$, which is the same as freeness of $(\{a_1, a_1^*\}, \{a_2, a_2^*\}, \ldots)$, is the same as having only to sum over $\pi \leq \sigma$ in (3), i.e.

$$\phi(a_{i_1}^{g(1)}\cdots a_{i_m}^{g(m)}) = \sum_{\pi\in NC(m)\leq\sigma} [\text{ coef } ((i_1, g(1)),\ldots,(i_m, g(m)));\pi] R(\mu_{a_1,a_1^*,a_2,a_2^*,\ldots}) \tag{7}$$

for all $m, i_1,\ldots,i_m, g(1),\ldots,g(m)$ with σ the signed partition of the monomial $a_{i_1}\cdots a_{i_m}$ as in definition 11.

Using (7), we will need the following combinatorial descriptions of the distributions appearing in definitions 5 and 6, as this is the form they will appear in the limit distributions of our matrices. Roughly speaking, the partitions π appear as ways to identify dependent (or non-free) entries from the matrices when they are multiplied, while the cumulants appear as (scaled) moments of the individual entries.

1. The R-transform of a semicircular family $(a_i)_i$ is (due to lemma 10) $\sum_k \alpha_k z_k^2$ with α_k their variances. This means that due to (7), with σ meaning the same,

$$\phi(a_{i_1}\cdots a_{i_m}) = \sum_{\pi\in NC(m)_2\leq\sigma} [\text{ coef } (i_1,\ldots,i_m);\pi] R(\mu_{a_1,a_2,\ldots}) = \sum_{\substack{\pi\in NC(m)_2\leq\sigma\\ \pi=\{A_1,\ldots,A_h\}}} \prod_{i=1}^{h} \alpha_{\sigma(A_i)}. \tag{8}$$

2. If $(a_i)_i$ is a circular family with variances α_k, the R-transform is $\sum_k \alpha_k(z_k z_k^* + z_k^* z_k)$, and we have

$$\phi(a_{i_1}^{g(1)}\cdots a_{i_m}^{g(m)}) = \sum_{\pi\in NC(m)\leq\sigma} [\text{ coef } ((i_1, g(1)),\ldots,(i_m, g(m)));\pi] R(\mu_{a_1,a_1^*,a_2,a_2^*,\ldots}) =$$

$$\sum_{\substack{\pi\in NC(m)_2\leq\sigma\\ \pi=\{A_1,\ldots,A_h\}=\{\{v_{i1},v_{i2}\}\}_i\\ (g(v_{i1}),g(v_{i2}))=(\cdot,*)\text{ or }(*,\cdot)}} \prod_i \alpha_{\sigma(A_i)}. \tag{9}$$

3. To make the picture complete for definition 5, for $*$-free creation operators $(a_i)_i$ we have

$$\phi(a_{i_1}^{g(1)} \cdots a_{i_m}^{g(m)}) = \sum_{\substack{\pi \in NC(m)_2 \leq \sigma \\ \pi = \{\{v_{i1}, v_{i2}\}\}_i \\ (g(v_{i1}), g(v_{i2})) = (*, \cdot)}} 1, \tag{10}$$

and it is not difficult to see that only one π can appear in the sum. This formula was used by Shlyakhtenko in his paper [9].

Finally, if $\{a_k, a_k^*\}$ are free R-diagonal pairs with defining sequences $\{\alpha_{k,2m}\}_{m \geq 1}$, then we have due to the alternation of z and z^* in their R-series,

$$\phi(a_{i_1}^{g(1)} \cdots a_{i_m}^{g(m)}) = \sum_{\substack{\pi \in NC(m)_{even} \leq \sigma \\ \pi = \{A_1, \dots, A_h\} = \{\{v_{ij}\}_j\}_i \\ (g(v_{i1}), g(v_{i2}), \dots) = \\ (\cdot, *, \cdot, *, \dots) \text{ or } (*, \cdot, *, \cdot, \dots)}} \prod_i \alpha_{\sigma(A_i), |A_i|}. \tag{11}$$

The expression on the right will also come out of our calculations.

2.2 Preliminaries on oriented partitions

We will use the following circular representation of a partition, which is suited for what we will call oriented partitions. All elements i of $\{1, \dots, n\}$ will be represented as edges in the inscribed n-gon of the circle, the labelling of the edges being done clockwise. Each block of the partition will be the corresponding assemble of edges in the n-gon, we indicate this by drawing connecting lines between the midpoints of successive edges in the blocks (the notion of successors in a block giving meaning as before), see figure 1.

Definition 12. *A partition $\pi \in \mathcal{P}(n)$ where each block of π has an equivalence relation (the orientation) with at most two equivalence classes is called an oriented partition. In the circular representation of the partition, the orientation on each block will be decribed by a direction for each edge, clockwise or anti-clockwise. The set of all oriented partitions will be denoted $\mathcal{OP}(n)$.*

This concept will be important to the combinatorial calculations in our matrix multiplications.

We view the set of oriented partitions as a partially ordered set by saying that $\pi_1 \leq \pi_2$ if $\pi_1 \leq \pi_2$ as ordinary partitions, and any orientation class of π_1 is contained in some orientation class of π_2. The cardinalities of the orientation classes B^+, B^- of a block B will be denoted $|B^+|, |B^-|$.

Note that the signed partition $\sigma = \{\sigma_1, \dots, \sigma_r\}$ of the $*$-monomial $X_{i_1}^{g(1)} \cdots X_{i_m}^{g(m)}$ can be viewed as an oriented partition by saying that $k \in \sigma_j^+$ (the positive orientation class of σ_j) if and only if $i_k = j$ and $g(k) = \cdot$ (σ_j^- defined similarly with $g(k) = *$ instead). We will follow this convention.

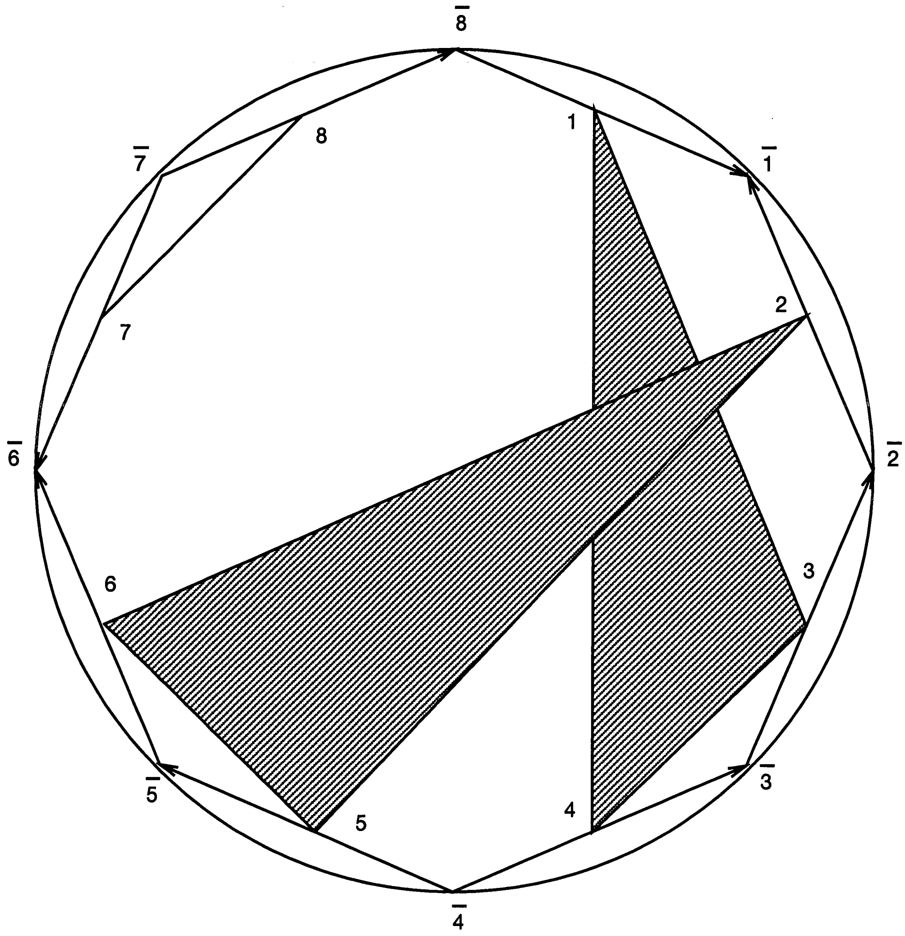

Figure 1: *The circular representation of an oriented partition.*

Definition 13. *The quotient graph $\overline{\pi}$ of an oriented partition $\pi \in \mathcal{OP}(n)$ is the graph that appears when the set of edges $\{1, ..., n\}$ in the circular representation are identified with the other edges in the same block, with directions preserved if and only if they have the same orientation. The block structure of $\{\overline{1}, ..., \overline{n}\}$, the connecting vertices between the edges in the circular representation, that appear when we do the identifications of the edges will produce a partition, denoted $K'(\pi)$, its blocks consisting of vertices that are identified (we do not give $K'(\pi)$ an orientation).*

Remark: This representation of a partition was used (implicitly) by Dykema in his paper [1]. The map K' is not to be confused with the complementation map of Kreweras, denoted K. They are denoted by the same letter since they coincide in the important special case of partitions from $NC(n)_2$ (given a certain orientation), which is the class of partitions we will encounter most often. To be more precise, if such a π is given with orientation so that the two members of a block *always* are given opposite orientation, then $K = K'$. This is not too hard to inferr from the circular definition of K (in which $\{1, ..., n\}$ are vertices on the circle, and $\{\overline{1}, ..., \overline{n}\}$ are the midpoints on the circular arcs connecting these vertices, connecting lines drawn as before). Actually, we will throughout mostly concentrate on (oriented) noncrossing partitions which have alternating orientations within the blocks (meaning that the successor of an element in a block always has opposite orientation), and one can also inferr from the circular representation that $K'(\pi) > K(\pi)$ with *strict* inequality if $\pi \notin NC(n)_2$ has such an orientation. Roughly, the explanation for this is that the mapping K' produce many more identifications of vertices than the mapping K does. This follows really from the connection between the two circular representations; When we identify two successive edges which have opposite orientation, we usually get *two* identifications of vertices (for instance identifying the edges 1 and 3 in figure 1 leads to an identification of the vertices $\overline{1}$ and $\overline{2}$, and also the vertices $\overline{3}$ and $\overline{8}$), while we in the definition of K only would get *one* identification ($\overline{1}$ and $\overline{2}$), this explains why K' in general produce more identifications, hence $K'(\pi) > K(\pi)$.

Corollary 14 (of definition). *The partition $K'(\pi)$ is the equivalence relation on $\{\overline{1}, ..., \overline{n}\}$ generated by the relations (running through all pairs of edges (i, j) in the same block):*

$$i \sim_\pi j \text{ with opposite orientation } \Rightarrow \overline{i} \sim_{K'(\pi)} \overline{j-1}, \overline{i-1} \sim_{K'(\pi)} \overline{j}, \tag{12}$$

$$i \sim_\pi j \text{ with same orientation } \Rightarrow \overline{i-1} \sim_{K'(\pi)} \overline{j-1}, \overline{i} \sim_{K'(\pi)} \overline{j}$$

(here numbers are taken mod n).

The following set of partitions will appear in the combinatorics when we describe the limit distributions of our random matrices. Note that the number of vertices in the quotient graph, $|K'(\pi)|$, is at most $|\pi| + 1$ (as $|\pi|$ is the number of edges in the graph), and that this can be the case only if the quotient graph is a tree (see also lemma 21).

Definition 15. *An oriented partition $\pi \in \mathcal{OP}(n)$, is said to be clickable if the number of vertices in the quotient graph $\overline{\pi}$ (which is $|K'(\pi)|$) is $|\pi| + 1$. The set of all such is denoted $C(n)$.*

The name clickable stems from [1], where a click was defined as a certain identification of edges, namely two edges lying next to each other being identified with opposite orientation. The importance of the clickable partitions here is due to the fact that we will run into calculations where a degree of freedom is assigned to each vertex in the quotient graph of the partition, and it is only partitions with enough such degrees of freedom (i.e. the clickable ones) which can give contribution in the limit in our calculations.

The clickable partitions do not become a lattice under the refinement order. We will describe the structure of these partitions at the end of the paper. There we will also prove a fact we will use, namely that $C(n)_2 = NC(n)_2$ as partitions. The corresponding orientation for a block of $\pi \in NC(n)_2$ is given by one in each orientation class. Actually, we will show that *any* clickable π has alternating orientation within it's blocks (as discussed above), and that *all* even noncrossing partitions become clickable with this choice of orientation.

Note that, if we in the index of the summand in (9) (or in (11)) assign the orientation to $\pi \in NC(m)_2$ (or $\pi \in NC(m)_{even}$) by letting v_{ij} have positive orientation if $g(v_{ij}) = \cdot$ and negative orientation if $g(v_{ij}) = *$, we get that we sum over all partitions $\pi \in C(m)_2 \leq \sigma$ (or $\pi \in C(m) \leq \sigma$).

3 Random matrices with independent entries

We shall need the terminology about oriented partitions when we look at limit distributions of the random matrices $A_n(1), ..., A_n(k)$ as in the introduction. We will first do the computations for independent selfadjoint random matrices with entries from L satisfying the following criteria:

$$\begin{aligned} &1.\ \text{The entries } a_n(i,j;k) \text{ satisfy } E(|a_n(i,j;k)|^2) = \tfrac{\alpha_k}{n} \\ &2.\ \sup_{i,j} E(|a_n(i,j;k)|^m) = o(n^{-1}) \text{ for } m \neq 2 \text{ and every } k \\ &3.\ \text{all } \{a_n(i,j;k)\}_{k,1\leq i\leq j\leq n} \text{ are independent,} \end{aligned} \tag{13}$$

where the condition 3 sometimes is replaced by

3'. all $\{a_n(i,j;k)\}_{k,1\leq i,j\leq n}$ are independent,

i.e. the symmetry condition is removed. Here $o(n^\alpha)$ denotes any sequence $\gamma = \{\gamma_n\}_n$ such that $\lim_{n\to\infty} \gamma_n n^{-\alpha} = 0$. Of course, we have that $o(n^\alpha)o(n^\beta) = o(n^{\alpha+\beta})$, and also that γ is $o(n^\alpha) \Rightarrow \gamma$ is $o(n^\beta)$ whenever $\alpha < \beta$. A sequence converges to 0 if and only if it is $o(n^0) = o(1)$. The reasons for the conditions in (13) will become clearer during the proof of theorem 1 and 3, as $\alpha = -1$ turns out to be a critical value for the existence of a limit distribution, and is therefore related to our 'optimal' results.

We have that $n\phi_n(A_n(i_1)\cdots A_n(i_m))$ (or more generally $n\phi_n(A_n(i_1)^{g(1)}\cdots A_n(i_m)^{g(m)})$) is a sum of terms on the form

$$E\left(a_n(j_m,j_1;i_1)a_n(j_1,j_2;i_2)\cdots a_n(j_{m-2},j_{m-1};i_{m-1})a_n(j_{m-1},j_m;i_m)\right), \tag{14}$$

with $j_1, ..., j_m \in \{1, ..., n\}, i_k \in I$.

Definition 16. *In a term as (14) we will consider the oriented partition π of $\{1, ..., m\}$ given by dependence of the random variables involved, i.e. $k \sim_\pi l$ if and only if the k'th and l'th factor in the above are not independent due to the conditions imposed (in particular we must have $i_k = i_l$ due to 3 of (13), i.e. $\pi \leq \sigma$ with σ the signed partition of $X_{i_1} \cdot X_{i_m}$).*

The orientation of π is defined by letting k and l have the same orientation if and only if the corresponding random variables are placed in the same position in the matrices. Opposite orientation is the case if the two entries are placed symmetrically about the the diagonal.

One sees from this definition that opposite orientation for k and l in some block B with 1,2,3' holding (instead of 1,2,3) is possible only if one of them comes from some $A_n(i)$ with the other coming from the opposite side of the diagonal of $A_n(i)^*$. We give positive orientation to the one coming from $A_n(i)$. We then have $\pi \leq \sigma$ as oriented partitions for any term giving π, σ the oriented signed partition of the $*$-monomial $X_{i_1}^{g(1)} \cdots X_{i_m}^{g(m)}$.

In definition 16 we say that the j_i give (rise to) π. Note that the number of vertices in the quotient graph of π is related to the number of choices of j_i giving π (same order as $n^{|K'(\pi)|}$) as the j_i can be identified with the vertices in the quotient graph (the random variables are the edges) due to our definition.

In addition to random matrices as above, consider constant block diagonal matrices (with n a multiple of N)

$$D_n(t) = \sum_{\substack{0 \leq b \leq \frac{n}{N}-1 \\ 1 \leq i,j \leq N}} d_n(Nb+i, Nb+j; t) e(Nb+i, Nb+j; n),$$

for t in some index set T:

Definition 17. *We call the set $\{D_n(t)\}_{t \in T}$ as above (with n running through multiples of N) a set of constant block diagonal matrices if*

1. *$D_n(t)$ has a limit distribution as $n \to \infty$ for any $t \in T$*
2. *$\sup_{i,j,n} |d_n(i, j; t)| < \infty$ for any $t \in T$*
3. *For any $t_1, t_2 \in T$ there exists $t_3 \in T$ such that $D_n(t_1)D_n(t_2) = D_n(t_3) \forall n$.*

Let us formulate the first theorem. We will go through its proof in detail, and then discuss the generalizations (theorem 2, 3, 4) which follow.

Theorem 1. *Under the conditions of (13) and definition 17 we get that $(A_n(1), A_n(2), ..., \{D_n(t)\}_t)$ is an asymptotically free family as $n \to \infty$ (through multiples of N). Moreover, the $A_n(k)$ converge in distribution to centered semicircular random variables of variance α_k.*

Proof: Let σ be the signed partition of the monomial $X_{i_1} \cdots X_{i_m}$. We write

$$\phi_n(A_n(i_1)D_n(t_1) \cdots A_n(i_m)D_n(t_m)) = \tag{15}$$

$$\sum_{\pi\in\mathcal{OP}(m)\leq\sigma}\ \sum_{\substack{j_1,\dots,j_m,\\ k_1,\dots,k_m\\ \text{giving } \pi}} \frac{1}{n}E(a_n(j_1,k_1;i_1)d_n(k_1,j_2;t_1)\cdots a_n(j_m,k_m;i_m)d_n(k_m,j_1;t_m))$$

where the oriented partition π is defined by the term involved as in definition 16 (with the obvious meaning when additional matrix entries $d_n(k,j;t)$ are put in between). If $\pi = \{A_1,...,A_h\} = \{\{w_{ij}\}_j\}_i$ and $K'(\pi) = \{B_1,...,B_k\} = \{\{v_{ij}\}_j\}_i$ we get from independence that this equals

$$\sum_{\pi\in\mathcal{OP}(m)\leq\sigma}\ \sum_{\substack{j_1,\dots,j_m,\\ k_1,\dots,k_m\\ \text{giving } \pi}} \frac{1}{n}\left(\prod_{i=1}^{h}E\left(\prod_{r=1}^{|A_i|}a_n(j_{w_{ir}},k_{w_{ir}};i_{w_{ir}})\right)\right)d_n(k_1,j_2;t_1)\cdots d_n(k_m,j_1;t_m). \tag{16}$$

Note that the number of choices of $j_1,...,j_m,k_1,...,k_m$ giving π is at most $n^{|K'(\pi)|}N^{2|\pi|-|K'(\pi)|}$, since the j's and the k's are attached to the vertices in the quotient graph: Recall that the d's are block diagonal, this is where the powers of N come from; We need not have $k_r = j_{r+1}$ to get a nonzero term, but rather $|k_r - j_{r+1}| \leq N$ as $D_n(t)$ is zero outside block diagonals. Using this one sees that the exponent $2|\pi| - |K'(\pi)|$ comes out by counting in the quotient graph, which is a graph with $|K'(\pi)|$ vertices and $|\pi|$ edges.

Set $d := sup_{k\in\{1,...,m\},n,i,j}|d_n(i,j;t_k)|^m$, which is $< \infty$ from condition 2 of definition 17. If there are blocks of cardinality $\neq 2$, we get from the conditions on the $a_n(i,j;k)$ and the fact that $|E(f)| \leq E(|f|)$, that the π-term in (16) is dominated by

$$\left| o(n^{-|\pi|-1}) \sum_{\substack{j_r,k_r\\ \text{giving } \pi}} d_n(k_1,j_2;t_1)\cdots d_n(k_m,j_1;t_m)\right|,$$

since some factor in (16) in this case is dominated by $E(|a_n(i,j;k)|^r)$, with $r \neq 2$, which is $o(n^{-1})$ by our assumptions. The π-term is then also dominated by

$$o(n^{-|\pi|-1})n^{|K'(\pi)|}N^{2|\pi|-|K'(\pi)|}d = ro(n^{|K'(\pi)|-|\pi|-1}) \tag{17}$$

(with $r = N^{2|\pi|-|K'(\pi)|}d$), which is $o(1)$ since $|K'(\pi)| - |\pi| - 1 \leq 0$. Therefore, we may assume that all blocks have cardinality two, i.e. $|\pi| = |\pi|_2$, in order to get contribution in the limit. The estimate (17) shows that for such π the quantity is dominated by (a constant times) $n^{|K'(\pi)|-|\pi|-1}$, this means that π must be clickable (i.e. $|K'(\pi)|-|\pi|-1 = 0$) in order to get contribution in the limit, so that $\pi \in NC(m)_2$ with alternating orientation by our analysis of $C(m)$ in the last section. For such partitions K and K' coincide, so that we can simply write K for them from now on. We add to (16) for each π

$$\sum_{\substack{j_1,\dots,j_m,\\ k_1,\dots,k_m\\ \text{giving any } \pi'>\pi}} \frac{1}{n}\left(\prod_{i=1}^{h}E\left(\prod_{r=1}^{|A_i|}a_n(j_{w_{ir}},k_{w_{ir}};i_{w_{ir}})\right)\right)d_n(k_1,j_2;t_1)\cdots d_n(k_m,j_1;t_m), \tag{18}$$

which is $o(1)$: This follows since at most $n^{|K'(\pi')|}N^{2|\pi'|-|K'(\pi')|}$ choices of j's and k's can give π' as quotient graph exactly as above, so that estimating as in (17) we get (for some constant r')

$$r'n^{|K'(\pi')|}n^{-|\pi|-1} \leq r'n^{|\pi'|+1}n^{-|\pi|-1} = r'n^{|\pi'|-|\pi|} \leq r'n^{-1} = o(1)$$

since $|\pi'| < |\pi|$. Modulo terms that are $o(1)$, (16) thus gets to be

$$\sum_{\pi \in NC(m)_2 \leq \sigma} \sum_{\substack{j_1,\ldots,j_m, \\ k_1,\ldots,k_m \\ \text{giving any } \pi' \geq \pi}} \frac{1}{n} \left(\prod_{i=1}^{h} E \left(\prod_{r=1}^{|A_i|} a_n(j_{w_{ir}}, k_{w_{ir}}; i_{w_{ir}}) \right) \right) d_n(k_1, j_2; t_1) \cdots d_n(k_m, j_1; t_m). \tag{19}$$

Noting that the fact that the j's and the k's give rise to any $\pi' \geq \pi$ is equivalent to $j_r = k_s, k_r = j_s$ whenever $r \sim s$ in π, and replacing the expectations in (19) by $\frac{\alpha_{i w_{ir}}}{n}$, we see that this is

$$\sum_{\pi \in NC(m)_2 \leq \sigma} n^{-|\pi|-1} \left(\prod_{i=1}^{h} \alpha_{\sigma(A_i)} \right) \left(\sum_{\substack{j_r, k_r | \\ j_r = k_s, k_r = j_s \text{ if } r \sim s \text{ in } \pi}} \prod_{i=1}^{k} \prod_{t=1}^{|B_i|} d_n(k_{v_{it}}, j_{(v_{it}+1)}; t_{v_{it}}) \right) \tag{20}$$

where we have split up the product as dictated by the partition $K(\pi)$. Consider the term corresponding to some π. I claim that

$$j_{(v_{it}+1)} = k_{v_{i(t+1)}}, i = 1, \ldots, k, t = 1, \ldots, |B_i| (\mod |B_i|) \tag{21}$$

and that these relations run through the same relations as

$$j_r = k_s, k_r = j_s \text{ with } r \sim s \text{ in } \pi. \tag{22}$$

The number of relations is m for both sets of relations as is easily checked. If $r \sim s$ in π, then $r - 1 = v_{it}, s = v_{i(t+1)}$ and $r = v_{j(s+1)}, s - 1 = v_{js}$ for suitable choices of i, j, s and t as can easily be seen from the circular representation. $j_r = k_s$ then says that $j_{(v_{it}+1)} = k_{v_{i(t+1)}}$, while $k_r = j_s$ says that $k_{v_{j(s+1)}} = j_{(v_{js}+1)}$, which all are relations from (21), so that (22)$\subset$(21). As all relations from (22) are distinct and the numbers of relations are the same, we also have equality here so that the relations are the same.

All this means that the product $\prod_{t=1}^{|B_i|}$ inside the summand of (20) can be written

$$\prod_{t=1}^{|B_i|} d_n(k_{v_{it}}, k_{v_{i(t+1)}}; t_{v_{it}}),$$

and summing over all $k_{v_{it}}$ gives that this equals

$$n\phi_n(D_n(t_{v_{i1}}) \cdots D_n(t_{v_{i|B_i|}})) \tag{23}$$

so that the entire product $\prod_{i=1}^{k}\prod_{t=1}^{|B_i|}$ in (20) equals

$$n^{|K(\pi)|}\prod_{i=1}^{k}\phi_n\left(\prod_{t=1}^{|B_i|}D_n(t_{v_{it}})\right).$$

Since $|K(\pi)|-|\pi|-1=0$ and since all $\prod_t D_n(t_{v_{it}})=D_n(t_i')$ have limit distributions as $n\to\infty$ due to 1 of definition 17 (denote the limit variables by $D(t_i)$), we get that the limit contribution for π in (20) exists and is

$$\left(\prod_{i=1}^{h}\alpha_{\sigma(A_i)}\right)\left(\prod_{j=1}^{k}\phi_n\left(\prod_{r\in B_j}D(t_r)\right)\right). \tag{24}$$

If we first choose all the $D_n(t_k)$'s to be the identity (the assumption $I\in\{D_n(t)\}_{t\in T}$ is irrelevant), we get by summing over all $\pi\in NC(m)_2$ that the limit of (15) is

$$\lim_{n\to\infty}\phi(A_n(i_1)\cdots A_n(i_m))=\sum_{\substack{\pi\in NC(m)_2\leq\sigma\\ \pi=\{A_1,\ldots,A_h\}}}\prod_{i=1}^{h}\alpha_{\sigma(A_i)}, \tag{25}$$

from which we see from definition 2 and equation (8) that $(A_n(1),A_n(2),\ldots)$ converge in distribution to a semicircular family $(X_1,X_2,\ldots)$ of variances α_i.

As [coef $(i_1,\ldots,i_m);\pi]R(\mu_{a_1,a_2,\ldots})$ equals [coef $(1,\ldots,m);\pi]R(\mu_{a_{i_1},\ldots a_{i_m}})$ (this is shown in [8]), the limit quantity (24) is seen to be

$$[coef(1,\ldots,m);\pi]\left(R(\mu_{X_{i_1},\ldots,X_{i_m}})\right)[coef(1,\ldots,m);K(\pi)]\left(M(\mu_{D(t_1),\ldots,D(t_m)})\right). \tag{26}$$

By the definition of boxed convolution, for general D's the limit quantity is (by summing over $\pi\in NC(m)_2$)

$$[coef(1,\ldots,m)]\left(R(\mu_{X_{i_1},\ldots,X_{i_m}})\boxed{\star}M(\mu_{D(t_1),\ldots,D(t_m)})\right) \tag{27}$$

which implies asymptotic freeness with $(\{D_n(t)\}_{t\in T})$ by Nica and Speichers characterization of freeness in lemma 9. ■

This reproves Voiculescus results on limit distributions of random matrices ([11], [14]), and simplifies Dykemas proof of the same statement in [1] (they had stronger moment estimates on the entries). Dykema used the trace-0 definition (definition 1) of freeness directly in order to show asymptotic freeness. This meant that for every power of a random matrix or constant block diagonal matrix he had to subtract its trace (times the identity) in order to get centered random variables (before he multiplied them together to find that the product has trace 0), and many combinatorial sides had to be resolved in this direction. The boxed convolution characterization of freeness in lemma 9 is nicer with respect to this since the assumption of zero trace on the random variables involved is irrelevant; One need not modify (i.e. subtract the trace times the identity) the random variables to work with them, and this leads to a more direct proof of the asymptotic freeness result. One can say that freeness from constant block diagonal matrices comes out when one factors out the trace of the block diagonal matrices in (24).

Note the following version of theorem 1:

Theorem 2. *If the matrices $(A_n(1), A_n(2), ...)$ have entries satisfying 1 and 2 and 3' of (13), then the statement of theorem 1 holds with semicircular replaced by circular.*

Proof: To see this one goes through the proof of theorem 1 again, starting by replacing (15) by $\phi_n(A_n(i_1)^{g(1)}D_n(t_1)\cdots A_n(i_m)^{g(m)}D_n(t_m))$, and $a_n(j_r, k_r; i_r)$ by the (j_r, k_r)-entry of $A_n(i_r)^{g(r)}$. σ is as before, but now it can be thought of as an oriented partition, as we are dealing with a $*$-monomial. Summing over $\pi \leq \sigma$ (in the sense of oriented partitions) in (16), we get because of the same estimates as before that clickable π with blocks only of cardinality two are the only ones giving contribution in the limit. This means that $\pi \in NC(m)_2$ with alternating orientation in the blocks due to lemma 21. This says that if $\pi = \{A_1, ..., A_h\} = \{\{v_{i1}, v_{i2}\}\}_i$, then v_{i1} and v_{i2} have opposite orientation, i.e. $(g(v_{i1}), g(v_{i2})) = (\cdot, *)$ or $(*, \cdot)$. All the calculations go exactly as before, and we get by adding up for different π in (24) (modulo terms that are $o(1)$)

$$\sum_{\substack{\pi \in NC(m)_2 \leq \sigma \\ \pi=\{A_1,...,A_h\}=\{\{v_{i1},v_{i2}\}\}_i \\ (g(v_{i1}),g(v_{i2}))=(\cdot,*) \text{ or } (*,\cdot)}} \left(\prod_{i=1}^{h} \alpha_{\sigma(A_i)}\right)\left(\prod_{j=1}^{k} \phi_n\left(\prod_{r\in B_j} D(t_r)\right)\right). \tag{28}$$

Choosing the $D_n(t)$'s to be the identity first we arrive at exactly the same expression as in (9) for the joint limit $*$-distribution, and this is the same as saying that $(A_n(1), A_n(2), ...)$ converge in distribution to $*$-free circular random variables. Exactly as in theorem 1 we then also get asymptotic freeness with constant block diagonal matrices. ∎

The proof above is actually not far from giving an 'optimal' result for when one can hope for a limit distribution in the case of each random matrix consisting of identically distributed random variables. The 'optimal' result is not stated in terms of convergence in distribution itself, for which such a nice characterization may not exist, but rather in terms of convergence in distribution with the sums (15) kept absolutely bounded (i.e. keeping $\sum |a_n|$ bounded with a_n the terms appearing in (15), i.e. the terms coming from writing the moments as sums of powers of the entries).

With this kind of convergence in distribution we will show that we obtain an entire class of limit distribution laws, and that the clickable partitions play an important role in the description of these distributions. Freeness of the limit distributions we get will be very rare. The reason for this is roughly that the clickable and noncrossing partitions do not coincide: First of all only even partitions arise in the combinatorics of our calculations, and the set of clickable partitions consists of all the even noncrossing partitions, plus a large class of partitions having crossings. Only in the case of partitions with all blocks of cardinality two there is a correspondence between the clickable and the even noncrossing partitions, and this suggests why the circular limit distribution should be the only one appearing in the case of freeness in the limit: Circular random variables have cumulants *only* of order two.

Theorem 3. *Assume the matrices $A_n(k)$ consist of independent, identically distributed entries from L. Then we have convergence in distribution with the sums (15) for the joint $*$-distribution kept absolutely bounded if and only if the following conditions hold:*

1. $\lim_{n\to\infty} nE(|a_n(i,j;k)|^{2m})$ *exists for all integers* k *and* $m \geq 1$

2. $\lim_{n\to\infty} n^\alpha E(a_n(i,j;k)^p \overline{a_n(i,j;k)}^q) = 0$ *for all* $\alpha < 1$ *and all integers* k, p, q *(i.e. the expectations are* $o(n^\alpha)$ *for all* $\alpha > -1$*).*

Moreover, the joint limit $*$-distributions we then get are in one to one correspondence with the sequences of limits

$$(\lim_{n\to\infty} nE(|a_n(i,j;k)|^2), \lim_{n\to\infty} nE(|a_n(i,j;k)|^4), \lim_{n\to\infty} nE(|a_n(i,j;k)|^6), \ldots)_k$$

in such a way that, if $(\alpha_{k,2}, \alpha_{k,4}, \ldots)$ *is the sequence of limits for the random matrices* $A_n(k)$*, the joint limit $*$-distribution of the matrices* $A_n(k)$ *is given by*

$$\lim_{n\to\infty} \phi_n(A_n(k_1)^{g(1)} \cdots A_n(k_m)^{g(m)}) = \sum_{\pi \in C(m)} [\ coef\ ((k_1, g(1)), \ldots, (k_m, g(m))); \pi](\alpha), \tag{29}$$

where α *is the power series without mixed terms in the variables* $(z_i, z_i^*)_i$*,*

$$\alpha(z_1, z_1^*, z_2, z_2^*, \ldots,) = \sum_k \sum_m \alpha_{k,2m}((z_k z_k^*)^m + (z_k^* z_k)^m) \tag{30}$$

with $[\ coef\ ((k_1, g(1)), \ldots, (k_m, g(m)))](\alpha)$ *the coefficient of* $z_{k_1}^{g(1)} \cdots z_{k_m}^{g(m)}$ *in* α*.*

In particular, we get $*$-free circular limit distributions if and only if $\alpha_{k,2m} = 0$ *for* $m \neq 1$*. In this case we get also asymptotic freeness with constant block diagonal matrices.*

Proof: We assume first that we have convergence in distribution with absolute boundedness for the sums (15) in the limit distribution. This means that the quantity for each π in (15) stays bounded as $n \to \infty$. Put $\pi = 1_{2m}$ with orientation of π chosen so that π is clickable. The corresponding quantity in (15) coming from π for the mixed moment $\phi_n(A_n(k)A_n(k)^* \cdots A_n(k)A_n(k)^*)$ is easily seen to be

$$n^{|K'(\pi)|-1} E(|a_n(i,j;k)|^{2m}) = nE(|a_n(i,j;k)|^{2m})$$

as $|K'(\pi)| = 2$ (it is easy to calculate the exact number of j's giving π in this case), thus the quantities in 1 stay bounded as $n \to \infty$ (convergence of these quantities will be proved later).

The fact that the quantities in 2 stay bounded as $n \to \infty$ is a bit harder. We will need the following lemma for this.

Lemma 18. *For all* $s > 0$*,* $p \neq q$ *there exists an oriented partition* $\pi = \{B_1, \ldots, B_s\}$ *such that*

1. *all* $|B_i|^+ = p$*,* $|B_i|^- = q$ *(so that* $|B_i| = p + q$*)*

2. $|K'(\pi)| = |\pi|$ *(i.e. the number of vertices is one from being maximal).*

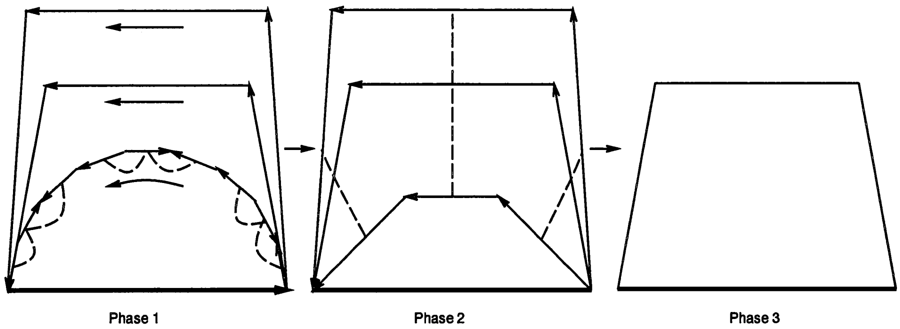

Figure 2: *Doing the identifications of edges in obtaining the quotient graph of* π.

Proof: Assume $p > q$. The π we want to construct is in $\mathcal{OP}(s(p+q))$. We will construct π so that the orientation of edges in each block is given by (in increasing order, $\cdot$ meaning positive orientation, $*$ meaning negative orientation)

$$\cdot, *, \cdot, *, ..., \cdot, *, \cdot, \cdot, ..., \cdot, \tag{31}$$

i.e. the first $2q$ elements are given by alternating $\cdot$'s and $*$'s till the q $*$'s (i.e. the edges having negative orientation) are used up, the rest are $\cdot$'s.

We will construct the circular representation of π by adding edges with orientation, so that the end product is an oriented partition with quotient graph having one loop (it is easy to see that this implies $|K'(\pi)| = |\pi|$). First place $m = p + q$ edges on the circle, they are to make out the first block B_1 of π, and let their orientation be determined by (31). The first $2q + 1$ of the edges should be connected (this we call the largest segment of B_1), the rest ($p - q - 1$ edges) should not be connected, as we will place the remaining $(p + q)(s - 1)$ edges in the $p - q$ intervals we now have. To see how the rest of the edges (with orientation) should be placed, do first the identifications in B_1 so that we obtain the loops in phase 1 of figure 2, each loop corresponding to one of the $p - q$ mentioned intervals (with clockwise direction in the circular representation indicated in the figure, the innermost loops appearing first in the circular representation, the B_1-block indicated in bold). It is not hard to see from this how one can add the remaining edges so that the loops actually collapse to one loop when one does the remaining identifications: The inner loop in the above could be made up of $s - 1$ consecutive copies of the largest segment of B_1 (so that doing identifications here we arrive at phase two of figure 2, dotted lines drawn to indicate what edges are identified), the other loops $s - 1$ consecutive copies of a single edge with positive orientation, i.e. copies of the other segments of B_1. When one does the identifications from phase two to phase three of figure 2, we obtain in the end one loop. Obviously, this partition also satisfies the conditions in 1.

For $p < q$ we interchange the roles of $\cdot$ and $*$ in the above argument to come to the desired conclusion. ■

Say that the order $\cdot$'s and $*$'s appeared in the above is given by the function $g : \{1, ..., (p + q)s\} \to \{\cdot, *\}$. Then the term coming from the above π in (15) is, for the mixed moment $\phi_n(A_n(k)^{g(1)} \cdots A_n(k)^{g((p+q)s)})$ (we need only look at the case with one matrix), equal to $n^{s-1}E(a_n(i,j;k)^p\overline{a_n(i,j;k)}^q)^s$ (as $|\pi| = |K'(\pi)|$) + terms negligible compared to this (coming from terms added as in (18)). This must stay bounded as $n \to \infty$, so that $n^{\frac{s-1}{s}}E(a_n(i,j;k)^p\overline{a_n(i,j;k)}^q)$ stays bounded as $n \to \infty$. As s was arbitrary we obtain that $\lim_{n\to\infty} n^{\alpha}E(a_n(i,j;k)^p\overline{a_n(i,j;k)}^q) = 0$ for all p, q and $\alpha < 1$.

Knowing this, we see that no nonclickable π can give contribution in (15), as for any such $\pi = \{B_1, ..., B_s\}$ ($|K'(\pi)| \leq |\pi|$ for such π) its contribution would be dominated by (a constant times)

$$n^{|\pi|-1}\prod_{l=1}^{s} E(a_n(i,j;k)^{|B_l^+|}\overline{a_n(i,j;k)}^{|B_l^-|}),$$

which is, after distributing the powers of n among the factors

$$\prod_{l=1}^{s} n^{\frac{|\pi|-1}{|\pi|}} E(a_n(i,j;k)^{|B_l^+|}\overline{a_n(i,j;k)}^{|B_l^-|}), \tag{32}$$

which converges to zero from what we have shown.

Convergence of the quantities $nE(|a_n(i,j;k)|^{2m})$ follows by induction: If this is shown for $m' < m$, convergence of the mixed moments of order $2m$ is equivalent to convergence of $nE(|a_n(i,j;k)|^{2m})$ (as $\pi \neq 1_{2m}$ gives convergent quantities in (15) from induction), and we are through showing that 1 and 2 are fulfilled when we assume convergence in distribution with such absolute boundedness. The expressions for the limit distribution is obtained in the same way as in the proof of theorem 1, but we now have to sum over *all* clickable partitions. This means that we in (24) sum over all $\pi \in C(m) \leq \sigma$ with $\pi = \{A_1, ..., A_h\} = \{\{v_{ij}\}_j\}_i$ so that all $(g(v_{i1}), g(v_{i2}), ...)$ are alternating sequences of $\cdot$'s and $*$'s. Choosing all $D_n(t) = I$ we arrive at (after replacing expectations)

$$\sum_{\substack{\pi\in C(m)\leq\sigma \\ \pi=\{A_1,...,A_h\}=\{\{v_{ij}\}_j\}_i \\ (g(v_{i1}),g(v_{i2}),...)= \\ (\cdot,*,\cdot,*,...) \text{ or } (*,\cdot,*,\cdot,...)}} \prod_i \alpha_{\sigma(A_i),|A_i|}. \tag{33}$$

It is easy to see that the summand is the same as

$$[\text{ coef } ((i_1, g(1)), ..., (i_m, g(m))); \pi](\sum_k \sum_m \alpha_{k,2m}((z_k z_k^*)^m + (z_k^* z_k)^m)),$$

where the power series is recognized as the α in (30). Since such a coefficient is zero unless $\pi \leq \sigma$ and all $(g(v_{i1}), g(v_{i2}), ...)$ give alternating sequences, we see that the sum for the limit distribution is the same if we sum over *all* $\pi \in C(m)$, i.e.

$$\sum_{\pi\in C(m)} [\text{ coef } ((i_1, g(1)), ..., (i_m, g(m))); \pi](\alpha),$$

which is what we wanted to show.

The other way, if 1 and 2 are fulfilled, we see from our arguments above that all terms from any π converge, so that we have convergence in distribution with absolute boundedness of the sums we are looking at.

The sequences of limits are now easily seen to be in bijection with the possible limit distributions, just as one shows that the R-transform is a bijection, namely by determining the cumulants recursively in terms of the moments. Therefore we get a circular limit distribution if and only if $\alpha_{k,2m} = 0$ for $m \neq 1$. These distributions are free and we get freeness with constant block diagonal matrices since the situation is the same as the one we have seen before in theorem 2. ■

We will not go into special limit distributions we get in theorem 3 (except for the circular one) (we *will* go into special limit distributions in the case of *free* entries), but make a remark here on how one can use 'nice' distributions (more precisely, the infinitely divisible ones) to obtain a large class of limit distributions. We will consider the 'selfadjoint' version of theorem 3, in which 1 and 2 are assumed to hold only for $i \geq j$, and where we have a symmetry condition on the entries instad $(a_n(i,j;k) = a_n(j,i;k)^*)$. One can then show, similarly to the proof of theorem 1, that the power series α of (30) for describing the limit distribution as in (29) instead takes the form $\sum_k \sum_{m\geq 1} \alpha_{k,2m} z^{2m}$.

More precisely, if ν is an even, (classically) infinitely divisible probability measure with compact support, I claim that we can get as a limit distribution (using entries with infinitely divisible distribution) in this matrix-like fashion, any distribution with sequence of limits $\{\alpha_{k,2}, \alpha_{k,4}, ...\}$ (these could for instance be called the clickable cumulants) equal to the even (classical) cumulants of ν. To see this, note that if ν has (classical) cumulants $0, \alpha_2, 0, \alpha_4, ...$ and $\nu = \nu_{\frac{1}{n}} * \cdots * \nu_{\frac{1}{n}}$, then the cumulants of $\nu_{\frac{1}{n}}$ are $0, \frac{\alpha_2}{n}, 0, \frac{\alpha_4}{n}, ...$ If we in the matrices A_n put independent (up to symmetry) entries a_n (we can assume them to be real valued, for the sake of simplicity) all with the same distribution $\nu_{\frac{1}{n}}$, then

$$\lim_{n\to\infty} n\phi(a_n^{2m}) = \lim_{n\to\infty} n \sum_{\pi \in \mathcal{P}(2m)} [\text{ coef } (2m); \pi](\frac{\alpha_2}{n} z^2 + \frac{\alpha_4}{n} z^4 + \cdots) = \alpha_{2m}$$

from the moment-cumulant formula in classical probability ($\mathcal{P}(n)$ takes the role of $NC(n)$ in the free setting), as only $\pi = 1_{2m}$ can give contribution in the limit, due to the fact that higher powers of n enter for other π. We see in this way that all sequences of limits equal to an even cumulant sequence from a $*$-infinitely divisible measure appear in the limit.

At present, we can't say if we for instance get the infinitely divisible distributions *themselves* in the limit.

Remark: One can in fact show that the only possibility for free limit distributions is if all except possibly *one* of the matrices give circular limit distributions. This is shown in appendix 1 of the author's PhD-thesis. The proof is rather combinatorical. It is only in the case of constant diagonal matrices ($N = 1$) we can expect a limit distribution with our matrices, as we are not able to 'factor out traces' as in (23) in the general case. Even with $N = 1$ we can't expect freeness in the general case, as summation in (26) goes over all clickable partitions, so that lemma 9 for proving freeness does not apply. These things are discussed more precisely in appendix 3 of the authors PhD-thesis.

Remark: Given two sequences $A_n(1)$ and $A_n(2)$ of independent random matrices as in theorem 3, one can find the limit $*$-distribution of the sum $A_n(1) + A_n(2)$ of these two random matrices by

1. adding the two corresponding limit sequences $(\lim_{n\to\infty} nE(|a_n(i,j;k)|^{2m}))_m$ componentwise,
2. putting this new sequence of limits into the power series α in (30),
3. determine the mixed moments of the sum from the formula (29).

This follows because of the no-mixed-terms nature of the power series in (29) for determining the moments in terms of the sequences of limits: Vaguely speaking, any transform from distributions to power series defined in a multiplicative-like fashion (as the R-transform with noncrossing partitions) over some set of partitions which attains a no-mixed-term power series for the joint distribution μ_{A_1,A_2}, necessarily must have the linearizing property which the R-transform has, i.e. $R(\mu_{A_1+A_2})(z) = R(\mu_{A_1})(z) + R(\mu_{A_2})(z)$. This is shown in appendix 2 of the authors PhD-thesis.

Remark: Note that, for a sequence k_n, the fact that $\lim_{n\to\infty} n^\alpha k_n = 0$ for all $\alpha < 1$ needs not imply that $\lim_{n\to\infty} nk_n$ exists (the converse is of course true). The sequence $k_n = \frac{\ln(n)}{n}$ provides an example of this.

4 Random matrices with free entries

Versions of theorem 1, 2 and 3 can be stated also for matrices with $*$-*free* assembles of random variables. We will see that the freeness assumption on the entries implies a certain dominance for the even noncrossing partitions inside the clickable partitions in our calculations, so that the even noncrossing partitions, instead of the clickable partitions, govern the structure of the joint limit $*$-distributions here. This means that we must get even, free random variables in the limit. We also get that these give rise to R-diagonal pairs, this is due to the alternating property for the blocks of a clickable partition, see 3b) of lemma 21.

Theorem 4. *Let the $\{A_n(k)\}_k$ be random matrices with entries in each matrix being identically distributed and $*$-free (entries in separate matrices also being free) in (A, ϕ) (for each n). If*

1. $\alpha_{k,2m} = \lim_{n\to\infty} n\phi((a_n(i,j;k)^* a_n(i,j;k))^m)$ *and*
   $\beta_{k,2m} = \lim_{n\to\infty} n\phi((a_n(i,j;k) a_n(i,j;k)^*)^m)$ *exist for* $m \geq 1$,
2. $\lim_{n\to\infty} n^\alpha \phi(\prod_{r=1}^m a_n(i,j;k)^{g(r)}) = 0$ *for all* $\alpha < 1, m$ *and* g,

then the joint limit $*$-distribution of the matrices $A_n(k)$ exists. Then we also have that asymptotic freeness of the random matrices is automatic, and the R-transform coefficients of the limit distribution of the $A_n(k)$ are related to the limits in 1, i.e.

$$R(\mu_{X_k,X_k^*})(z,z^*) = \sum_{m=1}^{\infty} \alpha_{k,2m}(z^*z)^m + \beta_{k,2m}(zz^*)^m. \tag{34}$$

In particular, if ϕ is a trace ($\Rightarrow \alpha_{k,2m} = \beta_{k,2m}$. If the entries of $A_n(k)$ are normal, ϕ is automatically a trace on the $*$-algebra the entries generate), we get in the limit random variables giving rise to R-diagonal pairs with the sequences of limits in 1 as defining sequences.

Freeness holds with sets of constant block diagonal matrices if the limit $*$-distributions in the above have cumulants only of order two.

Proof: Look at the situation without constant block diagonal matrices, i.e. the terms in (15) are instead $\frac{1}{n}\phi(a_n(j_m, j_1; i_1)\cdots a_n(j_{m-1}, j_m; i_m))$ (for convenience we drop the adjoints of the matrices in the first part of the calculations). The oriented partition π appearing in (15) should now be defined by replacing independent with $*$-free in definition 16. We need only sum over $\pi \leq \sigma$ due to the freeness condition for separate matrices (σ as in theorem 1). The calculations go as in theorem 1, but we can't replace the term above with

$$\frac{1}{n}\prod_{i=1}^{h}\phi(\prod_{r=1}^{|A_i|} a_n(j_{w_{ir}-1}, j_{w_{ir}}; i_{w_{ir}}))$$

(with notation for π and its blocks as in theorem 1) as in (16) anymore since independence has been replaced by freeness, and the expectation E has been replaced by the unital linear functional ϕ. Instead we have to split the mixed moment $\phi(a_n(j_m, j_1; i_1)\cdots a_n(j_{m-1}, j_m; i_m))$ into sums of products of the individual moments using definition 1. For any (oriented) $\sigma' = \{\sigma'_1, ..., \sigma'_r\} \leq \pi$ we get a 'submoment' $m_{\sigma'} = \prod_i \phi(\prod_{r\in\sigma'_i} a_n(j_{r-1}, j_r; i_r))$, and we can write

$$\phi(a_n(j_m, j_1; i_1)\cdots a_n(j_{m-1}, j_m; i_m)) = \sum_{\sigma'\leq\pi} t(\pi; \sigma')m_{\sigma'} \quad (35)$$

for some constants $t(\pi; \sigma')$ (depending only on the partitions π, σ', not on the particular random variables involved. In particular the constants are the same if some of the random variables are replaced by their adjoints), following notation in [10]. A result in [10] says that $t(\pi, \pi) \neq 0$ if and only if π is noncrossing and that $t(\pi; \pi) = 1$ for such π. Any $\sigma' < \pi$ has more than $|\pi|$ blocks, so that putting the summands $t(\pi; \sigma')m_{\sigma'}$ into (16) we get terms that are negligible for large n, after distributing powers of n as in (32) and using condition 2 (the maximum possible power of n is $n^{\frac{|\pi|}{|\sigma'|}}$ for each factor there, this happens if π is clickable), and for π which are crossing we get no terms which contribute in the limit since $t(\pi; \pi) = 0$ for crossing π. Therefore, in the limit we get only contribution from $t(\pi; \pi)$ with π noncrossing, this means that we instead of (33) get

$$\sum_{\substack{\pi\in NC(m)_{even}\leq\sigma \\ \pi=\{A_1,...,A_h\}=\{\{v_{ij}\}_j\}_i \\ (g(v_{i1}),g(v_{i2}),...)= \\ (\cdot,*,\cdot,*,...) \text{ or } (*,\cdot,*,\cdot,...)}} \prod_i (\alpha \text{ or } \beta)_{\sigma(A_i),|A_i|} \quad (36)$$

(where we choose α or β for i depending on whether $(g(v_{i1}), g(v_{i2}), ...) = (*, \cdot, *, \cdot, ...)$ or $(\cdot, *, \cdot, *, ...)$, respectively) since only the clickable partitions which are noncrossing give contribution. If $\alpha_{k,2m} = \beta_{k,2m}$, we get from comparison with (11), in the limit (free) R-diagonal pairs with the sequences of limits as defining sequences (If the α's and the β's are different, it is not too hard to conclude that the R-transform is as in (34)).

Freeness with constant block diagonal matrices holds in the case of circular limit distributions because we end up with the same situation as in theorem 2, as $C(m)_2 = NC(m)_2$ are the only partitions appearing in this case. If $\alpha_{k,2m} \neq \beta_{k,2m}$ is is not too hard to convince oneself that one *also* gets freeness with constant block diagonal matrices, even if we do not have circular limits in this case. ■

We could also have obtained a selfadjoint version of theorem 4, either by replacing the matrices $A_n(k)$ with $A_n(k) + A_n(k)^*$ (the limit R-transform series of (34) then instead takes the form $\sum_{m=1}^{\infty}(\alpha_{k,2m} + \beta_{k,2m})z^{2m}$), or replacing 1 and 2 above with the corresponding conditions suited to a symmetry condition on the entries. The identically distributed condition on the entries is not really needed in the proof above, and could be replaced by saying that the sup of the mixed moments of the entries appearing in condition 2 should be of order $o(n^{\alpha})$ for any $\alpha > -1$, with exact values for the moments in condition 1.

As in theorem 3 it is only in the case of constant diagonal matrices we can say that limit distributions exist with all the matrices in theorem 4. We cannot conclude freeness with these even if we now have reduced summation to π noncrossing, as $K'(\pi) \neq K(\pi)$ for noncrossing π if $\pi \notin NC(m)_2$, so that (26) is still different from an application of lemma 9 (see also the comments following theorem 5).

In the general situation above, it is only in the limit we retrieve freeness, there is no reason why we should have a similar result for the finitedimensional matrices. It is not difficult to construct matrices (with free identically distributed entries) that are *not* free. Examples with freeness for the finitedimensional matrices seem to be limited, but there is an important special case if we choose the $a_n(i,j;k)$ to give rise to R-diagonal pairs. More precisely:

Theorem 5. *If the $a_n(i,j;k)$ above give rise to R-diagonal pairs with defining sequence $\{\frac{\alpha_{k,2m}}{n}\}_m$, then the matrices $A_n(k)$ give rise to R-diagonal pairs with defining sequences $\{\alpha_{k,2m}\}_m$. Moreover, $(A_n(1), A_n(2), ...)$ is a $*$-free family. In particular, there is no need to take the limit as in theorem 4.*

Proof: For an arbitrary product of the matrix entries, we use the moment-cumulant formula (3) in (35) instead, the result is that we obtain for a mixed moment with oriented signed partition σ

$$\frac{1}{n}\sum_{\pi\leq\sigma}\ \sum_{\substack{\sigma'\leq\pi\\ \sigma'\in NC(m)}} \#(j_i \text{ giving rise to } \pi)\times$$
$$[\,\mathrm{coef}\,((i_1,g(1)),...,(i_m,g(m)));\sigma'] R(\mu_{a_n(i,j;1),a_n(i,j;1)^*,a_n(i,j;2),a_n(i,j;2)^*,...}) \tag{37}$$

(when we also bring the adjoints into the picture). Summing over σ' first and noting that (exactly) $n^{|K'(\sigma')|}$ choices of j's give rise to any $\pi \geq \sigma'$ we obtain

$$\sum_{\sigma' \in NC(m) \leq \sigma} n^{|K'(\sigma')|-1} \times$$
$$[\text{ coef } ((i_1, g(1)), ..., (i_m, g(m))); \sigma'] R(\mu_{a_n(i,j;1), a_n(i,j;1)^*, a_n(i,j;2), a_n(i,j;2)^*, ...}). \tag{38}$$

From R-diagonality of the a_n's we see that the R-transform coefficient above is $n^{-|\sigma'|}$ times the same coefficient of $R(\mu_{a(1),a(1)^*,a(2),a(2)^*,...})$ with $a(k)$ giving rise to (free) R-diagonal pairs with defining sequence $\alpha_{k,2m}$ (the powers of n enter since the defining sequences of the a_n's where scaled by $\frac{1}{n}$). As R-diagonality implies that only σ' with alternating orientations within the blocks enter in the sum, the σ' we work with are clickable (as σ' is noncrossing) so that $|K'(\sigma')| - |\sigma'| - 1 = 0$ and the powers of n cancel. The result is that the matrices $A_n(k)$ have the same distribution as the $a(k)$ above since we have exhibited the cumulants of our distribution. The result follows.

■

In particular, all R-diagonal limit distributions arise and in such a way that we need not take the limit. If we in the above also wanted distributions with sets of diagonal matrices $D_n(t_k)$, then we would obtain the quantity $\phi_n(A_n(i_1)^{g(1)} D_n(t_1) \cdots A_n(i_m)^{g(m)} D_n(t_m))$ simply by replacing $n^{|K'(\sigma')|}$ in (38) by $d_n(j_1, j_1; t_1) \cdots d_n(j_m, j_m; t_m)$ and in addition add over j's giving rise to some $\pi \geq \sigma'$. Bringing the powers of n from the R-transform coefficient into play we can add up for j's and factor out traces as in (23) to see that our mixed moment equals

$$\sum_{\sigma' \leq \sigma} [\text{ coef } ((i_1, g(1)), ..., (i_m, g(m))); \sigma'] R(\mu_{a_n(i,j;1), a_n(i,j;1)^*, a_n(i,j;2), a_n(i,j;2)^*, ...})$$
$$[\text{ coef } (1, ..., m); K'(\sigma')] M(\mu_{D_n(t_1), ..., D_n(t_m)}). \tag{39}$$

Although we get an exact expression for the joint distribution with diagonal matrices, we do not obtain freeness with these in general (except in the case with only second order cumulants), as the partition $K'(\sigma')$ appears instead of $K(\sigma')$, so that we can't use lemma 9 to conclude freeness.

If some of the entries $a_n(i, j; k)$ are non-R-diagonal we see that the proof above breaks down as we can have σ' without alternating orientation in the blocks then, hence nonclickable σ'. But then we can have formula (38) with no possibility of cancelling powers of n as above for all σ'. It seems then to be difficult to at all produce matrices which possesses freeness except in the limit.

4.1 Random matrices with the same kind of entries in the matrices for all n

Theorem 4 shows how to model many free families by using free assembles of identically distributed random variables. We were allowed to choose different types of entries for each n. One can ask what one can model if the entries in *all* the matrices have the same $*$-distribution (subject to some normalization condition).

An example of such matrices is given by Shlyakhtenko [9] who used free creation operators within the matrices, and obtained also free creation operators for each n and so also in the limit. Roughly speaking, he showed this by recognizing formula (10) in the calculations (Shlyakhtenkos example is related to Voiculescus proposition 2.8 of [12] for obtaining the semicircular distribution by putting circular and semicircular entries into the matrices). The combinatorics in this case is nicer than in the general case with an arbitrary entry in all the matrices.

We will show that this example is actually close to being exhaustive for the possible limit distributions in this setting. We will also show that it is close to being the only example of when the matrices produced give the same distributions for all n.

More precisely, given a random variable a we will consider for each n the corresponding matrices A_n with (i, j) entry equal to $\frac{a_{ij}}{\sqrt{n}}$, with $(\{a_{ij}\}_{1\leq i,j\leq n})$ a $*$-free family of random variables, all having the same $*$-distribution as a. We use the normalization factor $\frac{1}{\sqrt{n}}$ for the matrices, the effect of this is amongst other things that the Hilbert Schmidt norms of the matrices stay bounded.

Corollary 19. *Let $A_n(i)$ be random matrices constructed from some random variable a_i as above, the entries of the $A_n(i)$ assumed $*$-free for separate i. Then the matrices converge in $*$-distribution if and only if the a_i are centered (i.e. $\phi(a_i) = \phi(a_i^*) = 0$). In this case the R-transform of the limit $*$-distribution is given by $\sum_i \phi(a_i^* a_i) z_i^* z_i + \phi(a_i a_i^*) z_i z_i^*$. In particular the higher moments of a_i have no influence.*

Proof: If the limit distribution exists, a_i must be centered, as $\phi(A_n(i)^2) = \phi(a_i^2) + (n^2-n)\frac{\phi(a_i)^2}{n}$ (here it is easy to calculate the exact number of choices giving $\pi = 1_2$ and 0_2 in (16), these give rise to the first and second term, respectively), and this diverges if $\phi(a_i) \neq 0$. When a_i is centered, it is easy to calculate the $*$-moments of the entries $\frac{a_i}{\sqrt{n}}$ in 1 and 2 of theorem 4. For instance we get $\lim_{n\to\infty} n\phi((\frac{a_i}{\sqrt{n}})^*(\frac{a_i}{\sqrt{n}})) = \phi(a_i^* a_i)$ and

$$\lim_{n\to\infty} n\phi\left(\left(\frac{a_i^*}{\sqrt{n}}\frac{a_i}{\sqrt{n}}\right)^m\right) = \lim_{n\to\infty} n^{1-m}\phi((a_i^* a_i)^m) = 0$$

for $m > 1$ as a_i has moments of all orders. Thus 1 and 2 are fulfilled, so that the limit distribution exists, and as all limits in the sequence of limits are zero except the first one, we see that the R-transform of the limit $*$-distribution is $\phi(a_i^* a_i) z_i^* z_i + \phi(a_i a_i^*) z_i z_i^*$ (this is circular if and only if $\phi(a_i^* a_i) = \phi(a_i a_i^*)$). ■

We see from this that the only way to reproduce the same distribution in the limit (as what we started with) is to let $R(\mu_{a_i, a_i^*})(z, z^*) = \phi(a_i^* a_i) z^* z + \phi(a_i a_i^*) z z^*$. If this is the case, one can show, just as Shlyakhtenko did when a_i was a creation operator, that there is no need to take the limit, that is one also obtains the same distribution (and freeness) for all the finitedimensional matrices.

This follows from theorem 5, because if a_i is as above then $\{\frac{a_i}{\sqrt{n}}, \frac{a_i^*}{\sqrt{n}}\}$ has R-transform $\frac{1}{n}\phi(a_i^* a_i) z^* z + \frac{1}{n}\phi(a_i a_i^*) z z^*$ from properties of the R-transform of dilations of random variables. The entries of the matrices thus have R-transforms with coefficients scaled by $\frac{1}{n}$ which makes theorem 5 apply.

With choices of other a_i one can actually show that there *always* is a need to take the limit, i.e. we have convergence to the limit which doesn't terminate at a finite number of steps. Choices of other a_i are also likely, vaguely speaking, to never produce freeness for all the finitedimensional matrices.

4.2 Infinitely divisible limit distributions

The class of limit distributions we can get for matrices with free entries as above contains, in the case of selfadjoint matrices, amongst other distributions the class of even infinitely divisible measures with compact support. To see this, we will for each such measure ν decompose it as $\nu_{\frac{1}{n}} \boxplus \cdots \boxplus \nu_{\frac{1}{n}}$ with $\nu_{\frac{1}{n}}$ even and compactly supported, where we will realize the $\nu_{\frac{1}{n}}$ as distributions of (free) selfadjoint $n \times n$-matrices whose sum is a matrix with distribution ν of the form we have been considering already.

More precisely, let a ν as above be realized as the distribution of some random variable $A + A^*$ with $\{A, A^*\}$ being an R-diagonal pair (this decomposition is possible for any even selfadjoint random variable). Let us say that $R(\mu_{A,A^*})(z, z^*) = \sum_{m\geq 1} \alpha_{2m}((z^*z)^m + (zz^*)^m)$. By theorem 5 we can realize the $*$-distribution of A as the $*$-distribution of a matrix A_n having as entries free identically distributed random variables $a_n(i,j)$ with $R(\mu_{a_n(\cdot,\cdot),a_n(\cdot,\cdot)^*})(z, z^*) = \sum_{m\geq 1} \frac{\alpha_{2m}}{n}((z^*z)^m + (zz^*)^m)$. The diagonal entries of $A_n + A_n^*$ are $a_n(i,i) + a_n(i,i)^*$, and these are selfadjoint, even random variables with R-transform $\sum_{m\geq 1} \frac{2\alpha_{2m}}{n} z^{2m}$, while the off-diagonal entries $a_n(i,j) + a_n(j,i)^*$ have R-transforms

$$R(\mu_{a_n(i,j)+a_n(j,i)^*,a_n(j,i)+a_n(i,j)^*})(z, z^*) =$$

$$R(\mu_{a_n(i,j),a_n(i,j)^*})(z, z^*) + R(\mu_{a_n(j,i)^*,a_n(j,i)})(z, z^*) =$$

$$\sum_{m\geq 1} \frac{2\alpha_{2m}}{n}((z^*z)^m + (zz^*)^m),$$

i.e. the diagonal entries are even and selfadjoint while the off-diagonal entries give rise to free R-diagonal pairs. Let $B_n = A_n + A_n^*$ have entries $b_n(i,j)$, and let $B_n(l), 1 \leq l \leq n$ be defined componentwise by $B_n(i,j;l) = b_n(i,j)$ if $i+j = l mod n$, and 0 otherwise. The $B_n(l)$ are thus selfadjoint permutation matrices with $\sum_{l=1}^{n} B_n(l) = B_n$. I claim that the $B_n(l)$ are free and have all the same distribution. This distribution must then be $\nu_{\frac{1}{n}}$, and we have our realization of ν in terms of matrices.

The following argument for showing that the $B_n(l)$ are free and all have the same distribution is due to Alexandru Nica, but has not been published before (he dealt with more general assumptions on the entries). I am indebted to Roland Speicher for communicating it to me. An odd power of the $B_n(l)$ consists of a matrix with n nonzero entries (due to the permutation nature of our matrices) with an odd power of the $b_n(i,j)$ in each entry, and these have all expectation zero from the assumptions on even distributions and freeness of the entries, so that odd moments of $B_n(l)$ are also 0. Also, $B_n(l)^2$ is diagonal with entries of the form $b_n(i,j)b_n(i,j)^*$ (or possibly $b_n(i,i)^2$) on the diagonal, and these have all the same moments. Thus we see that the even moments of the matrices all coincide with the moments of $b_n(i,j)b_n(i,j)^*$, and this shows that the $B_n(l)$ all have the same distribution.

To show that the $B_n(l)$ are free, we need only take polynomials $C_i = B_n(l_i)^{k_i} - \beta_i I_n$ from alternating subalgebras (i.e. $l_1 \neq l_2, l_2 \neq l_3, \ldots$) all with trace zero, and show that $C_1 \cdots C_n$ has expectation zero. But it is easily seen that all entries of the C_i have expectation zero from identical distribution of its entries, and that the nonzero terms in products of the entries of the C_i have expectation zero due to the freeness assumption on the entries. Adding things up ends the proof.

5 The structure of the clickable partitions

To be able to describe the clickable partitions, we will need a result on the spotting of loops in the quotient graph. Let π be an oriented partition, and let $\overline{\pi}$ be its quotient graph.

Lemma 20. *If we, after having done some identifications of edges in obtaining the quotient graph have obtained a loop where one edge in the loop is not in the same block as any of the other edges in the loop, then this really gives rise to a loop in the quotient graph when we do the rest of the identifiactions also.*

Proof: First do the rest of the identifications inside the loop. This may give us shorter loops, but the existence of the edge not being identified with any other assures us that we must end up with at least *one* loop. Doing the identifications outside the loop may also give shorter loops, but there is no way to 'break up' the loop we already have so that all loops disappear in the end. All in all, the quotient graph must at least contain *one* loop. ■

This lemma is crucial in obtaining the following properties and the recursive characterization of the clickable partitions:

Lemma 21. *The following hold:*

1. $C(n) = \emptyset$ *if* n *is odd*
2. *An oriented partition is clickable if and only if its quotient graph has no loops, i.e. it is a tree*
3. *If* $\pi = \{B_1, ..., B_k\} = \{\{v_{ij}\}_j\}_i \in C(n)$ *with* v*'s in increasing order, then:*
   (a) *All* $|B_i|$ *are even, that is* π *is an even partition*
   (b) *(Alternating property of clickable partitions) The orientation classes of each block* B_i *are* $\{v_{i1}, v_{i3}, ..., v_{i|B_i|-1}\}$ *and* $\{v_{i2}, v_{i4}, ..., v_{i|B_i|}\}$ *(i.e. we have alternating orientation within the blocks)*
4. *An oriented partition* π *with alternating orientation within its blocks as in 3b) is clickable if and only if, for any* i, j *we have that* $\pi|\{v_{ij}+1, ..., v_{i(j+1)}-1\}$ *and* $\pi| \cup_r \{v_{i(j+2r)}+1, ..., v_{i(j+2r+1)}-1\}$ *are clickable partitions.*

Proof: 2): We will not prove this, as it is a well known fact about graphs and trees (the number of vertices in a graph is at most the number of edges $+1$, and is so only if it is a tree).

3): Assume that π is clickable. We first show b): If v_{ik} and $v_{i(k+1)}$ ($k+1$ taken mod $|B_i|$) have the same orientation, we get by identifying these edges a loop. Since none of the v, $v_{ik} < v < v_{ik+1}$ are in B_i, this gives us, by lemma 20, a real loop after having done *all* the identifications. So the partition is not clickable by part 2, contra assumption. Therefore, the v_{ik} must have alternating orientations.

a): If one of the $|B_i|$ is odd, then there must be consecutive edges in B_i with the same orientation, which again is contra assumption due to 3b). Therefore a) also follows. This also implies part 1.

4): Let π be oriented with alternating orientation within its blocks. Assume first that π is clickable. First do the identifications within some block B_i. This gives us (many) separate loops, one loop from each collection $\{v_{ij}+1, ..., v_{i(j+1)}-1\}$ of edges, and each of these loops has to be clickable in order to get no loops in the end, so that the $\pi|\{v_{ij}+1, ..., v_{i(j+1)}-1\}$ are clickable. The identifications within B_i have lead to two connected collections of edges, namely $\cup_r\{v_{i(j+2r)}+1, ..., v_{i(j+2r+1)}-1\}$ and $\cup_r\{v_{i(j-1+2r)}+1, ..., v_{i(j-1+2r+1)}-1\}$. Each of these collections have to be clickable in order to obtain no loops as is easily inferred from the quotient graph, so that the two other restrictions of π as in the statement *also* have to be clickable . The other way, it is not hard to convince oneself by looking at the quotient graph that clickability of all these segments is enough to secure a quotient graph without loops, i.e. the partition is clickable. ∎

The condition 3b) indicates a canonical orientation for any even partition. A partition from $NC(m)_{even}$ will *automatically* be given this orientation.

Note the similarity between the recursive characterization 4 of the clickable partitions and the following recursive characterization of noncrossing partitions:

Lemma 22. *A partition π is noncrossing if and only if all $\pi|\{v_{ij}+1, ..., v_{i(j+1)}-1\}$ are noncrossing and there are no interidentifications of edges in different such segments.*

An easy corollary of this characterization is the following:

Corollary 23. *The following hold:*

1. *$NC(n)_2$ and $C(n)_2$ coincide*
2. *$NC(n)_{even} \subseteq C(n)$ (this inclusion is strict for n large enough)*

Proof: 1 follows from the recursive characterizations of noncrossing and clickable partitions, as we can have no interidentifications between different segments as in the proof of 4 above when we have all blocks of cardinality two as all identifications get used up within each loop.

2 is also easily seen from these characterizations. It is not too hard to see that the inclusion is strict for $n \geq 8$. ∎

From our characterizations it also follows that clickable partitions with all blocks, except possibly one, of cardinality two, are noncrossing (do the identifications within the block of cardinality $\neq 2$ first to see that this is the case). In the other direction, we see that we can find clickable and crossing partitions whenever at least two of the blocks have cardinality > 2: The partition $\{\{1,2,5,6\},\{3,4,7,8\}\}$ (with alternating orientation within the two blocks) is the easiest example here (do the identifications of the edges to convince yourself that it is clickable).

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