Stochastic control problems where small intervention costs have big effects

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Abstract

We study an impulse control problem where the cost of interfering in a stochastic system with an impulse of size $\zeta \in \mathbb{R}$ is given by

$$c + \lambda |\zeta|$$

where $c$ and $\lambda$ are positive constants. We call $\lambda$ the proportional cost coefficient and $c$ the intervention cost. We find the value/cost function $V_c$ for this problem for each $c > 0$ and we show that $\lim_{c \to 0^+} V_c = W$, where $W$ is the value function for the corresponding singular stochastic control problem. Our main result is that

$$\frac{dV_c}{dc} = \infty \quad \text{at} \quad c = 0.$$

This illustrates that the introduction of an intervention cost $c > 0$, however small, into a system can have a big effect on the value function: The increase in the value function is in no proportion to the increase in $c$ (from $c = 0$).
1 Introduction

Many mathematical models make stylized assumptions which strictly speaking are not satisfied in real applications. For example, in mathematical finance it is often assumed that the transactions can be performed continuously (i.e. in continuous time) and that there are no transaction costs. In recent years one has also studied models where the transaction costs are proportional to the size $|\zeta|$ of the transaction (see e.g. [DN] and the references therein). An even more realistic model is obtained by assuming that the interventions occur at discrete times and then with a cost given by

$$c + \lambda|\zeta|$$

where $c \geq 0$, $\lambda > 0$ are constants. This leads to the impulse control model (see below). A natural question is how the result $V_{c,\lambda}$ (the value function) depends on these parameters $c, \lambda$. In this paper we fix $\lambda > 0$ and study how $V_c = V_{c,\lambda}$ depends on $c$ near $c = 0$ in a specific case. It has been shown in [MRob] and [MRof] that (under some conditions) $V_c \to V_0$ as $c \to 0^+$. The purpose of this paper is to show that $V_c$ may not be differentiable at $c = 0$.

We give an example where

$$\left[ \frac{dV_c}{dc} \right]_{c=0} = \infty. \quad \text{(Theorem 2.6)}$$

We now describe this in more detail.

Suppose that – if there are no interventions – the state $Y_t \in \mathbb{R}^n$ at time $t$ of the system we consider, satisfies an Itô stochastic differential equation of the form

$$Y_t = y + \int_0^t b(Y_r)dr + \int_0^t \sigma(Y_r)dB_r; \quad t \geq 0. \quad (1.1)$$

Here $b: \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma: \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are given Lipschitz continuous functions, $B_t = B_t(\omega)$; $t \geq 0$, $\omega \in \Omega$, is a Brownian motion in $\mathbb{R}^m$ with filtration $\mathcal{F}_t$ and probability law $P$ when starting at $0 \in \mathbb{R}^m$. The point $y \in \mathbb{R}^n$ is the starting value of the system at time 0. (See e.g. [O] for background on stochastic differential equations.)

An impulse control for this system is a (possibly finite) sequence

$$v = (\tau_1, \tau_2, \ldots, \tau_k, \ldots; \zeta_1, \zeta_2, \ldots, \zeta_k \ldots)_{k \leq N} \quad (N \leq \infty) \quad (1.2)$$

where $0 \leq \tau_1 \leq \tau_2 \leq \ldots$ are $\mathcal{F}_\tau$-stopping times and $\zeta_1, \zeta_2, \ldots$ belong to a given set $\mathcal{Z} \subset \mathbb{R}$. We interpret $\tau_1, \tau_2, \ldots$ as the intervention times, i.e. the
times when we decide to intervene. The quantities $\zeta_1, \zeta_2, \ldots$ are the impulses we give the system at the times $\tau_1, \tau_2, \ldots$ respectively. If the system is in state $x \in \mathbf{R}^n$ when it gets the impulse $\zeta \in Z$, we assume that it jumps immediately to a new state $x + g(\zeta)$, where $g: Z \to \mathbf{R}^n$ is a given function.

If the impulse control $v$ given by (1.2) is applied to the system $\{Y_t\}$, it gets the form $\{Y_t^v\}$, which is given by

\[
(1.3) \quad Y_t^v = y + \int_0^t b(Y_r^v)dr + \int_0^t \sigma(Y_r^v)dB_r + \sum_{\tau_k \leq t} g(\zeta_k); \quad 0 \leq t < T^*
\]

We still call $y = Y_0^v$ the starting point of $Y_t$ at $t = 0$, although we need not have $Y_0 = y$, because $Y_t$ could possibly jump at $t = 0$. Here $T^* = T^*(\omega)$ is the explosion time of the process $Y_t^v$, defined by

\[
(1.4) \quad T^*(\omega) = \lim_{R \to \infty} (\inf\{t > s; |Y_t^v(\omega)| \geq R\}) \leq \infty
\]

Let $Q_y = Q_y^v$ denote the law of the stochastic process $\{Y_t^v\}_{t \geq 0}$ starting at $y = Y_0^v$ at time $t = 0$.

Suppose we are given a family $\mathcal{V}$ of impulse controls, called the admissible impulse controls. We assume that all $v = (\tau_1, \tau_2, \ldots; \zeta_1, \zeta_2, \ldots) \in \mathcal{V}$ satisfy

\[
(1.5) \quad \tau_k \to T^* \quad \text{a.s.} \quad Q_y \quad \text{for all} \quad y \in \mathbf{R}^n.
\]

Suppose that the cost rate when the system is in state $x$ is $f(x)$, where $f: \mathbf{R}^n \to \mathbf{R}$ is a given function. Moreover, suppose that the impulse cost $K(x, \zeta)$ of giving the system the impulse $\zeta \in Z$ is given by

\[
(1.6) \quad K(x, \zeta) = c + \lambda|\zeta| \quad \text{(independent of} \ x) \]

where $c \geq 0$, $\lambda > 0$ are constants. Thus the impulse cost consists of a fixed, minimum cost $c$ (the intervention cost) plus a cost $\lambda|\zeta|$ proportional to the size $|\zeta|$ of the intervention $\xi$. Then if we apply $v = (\tau_1, \tau_2, \ldots; \zeta_1, \zeta_2, \ldots) \in \mathcal{V}$, we get the performance or total expected cost $J^v_c(y)$ defined by

\[
(1.7) \quad J^v_c(y) = E_y[\int_0^\infty f(Y_t^v)dt + \sum_{k=1}^N (c + \lambda|\zeta_k|)e^{-\rho \tau_k}]
\]

where $E_y$ denotes expectation with respect to $Q_y$. The corresponding impulse control problem is the following:
**PROBLEM 1.1 (General impulse control problem)** Find the value function $V_c = V_c(y)$ defined by

$$V_c(y) = \inf_{v \in \mathcal{Y}} J^v_c(y); \quad y \in \mathbb{R}^n$$

and find an optimal $v^* \in \mathcal{Y}$ such that

$$V_c(y) = J^{v^*}_c(y); \quad y \in \mathbb{R}^n.$$ 

We refer the reader to [BL] for more information about impulse control and its relation to quasi-variational inequalities (see Theorem 2.1).

A study of impulse control problems for diffusions with jumps can be found in [M].

Note that Problem 1.1 makes sense also if $c = 0$. The infimum is still taken over impulse controls with finitely many jumps in compact time intervals in $[0, T^*)$. However, if $c = 0$ it is conceivable that no optimal impulse control $v^* \in \mathcal{Y}$ exists: Since the number of interventions do not matter, only the sizes of them, it is natural to guess that the infimum is obtained by letting the number of interventions go to infinity and at the same time the sizes of the interventions go to zero. This would in the limit bring us to a control outside $\mathcal{Y}$. We will show that this is indeed the case in the problem we consider.

From now on let us assume that we only allow non-negative impulses, i.e. that

$$Z = [0, \infty),$$

and that the impulse $\zeta$ acts linearly on just one of the components of $Y_t$, say the last one:

$$g(\zeta) = (0, \ldots, b\zeta) \in \mathbb{R}^n \quad \text{for} \quad \zeta \in Z,$$

where $b \in \mathbb{R}$ is a constant.

Then if $c = 0$ it is also natural to model the problem above as a *singular stochastic control* problem, as follows:

Let our space $\Gamma$ of admissible controls consist of all $\mathcal{F}_t$-adapted, $t$-right continuous processes $\gamma_t(\omega)$ such that the function $t \mapsto \gamma_t(\omega)$ is non-decreasing for a.a. $\omega \in \Omega$. We can associate to $\gamma_t(\omega)$ the measure $d\gamma_t(\omega)$ with the property that

$$\int_{s^+}^{t} d\gamma_t(\omega) = \int_{(s,t]} d\gamma_t(\omega) = \gamma_t(\omega) - \gamma_s(\omega); \quad t > s.$$
Note that with this notation we have

\[(1.13) \quad \int_s^t d\gamma_r(\omega) = \Delta \gamma_s(\omega) + \gamma_t(\omega) - \gamma_s(\omega)\]

where in general

\[(1.14) \quad \Delta \gamma_t(\omega) = \gamma_t(\omega) - \gamma_{t-}(\omega)\]

is the jump of \(\gamma_t(\omega)\) at \(t = t_i\).

In analogy with (1.3) we now assume that if the admissible control \(\gamma_t \in \Gamma\) is applied to the system \(\{Y_t\}_{t \geq 0}\), it gets the form \(\{Y^\gamma_t\}_{t \geq 0}\) given by the equation (using (1.11))

\[(1.15) \quad Y^\gamma_t = y + \int_0^t b(Y^\gamma_r)dr + \int_0^t \sigma(Y^\gamma_r)dB_r + (0, \ldots, b \int_0^t d\gamma_r); \quad t \geq 0.\]

Note that this coincides with \(Y^v_t\) given by (1.3) if \(\gamma_t\) only increases at \(t = \tau_k\) and then performs jumps of size \(\zeta_k\), \(k = 1, 2, \ldots\). Furthermore, in analogy with (1.7) for \(c = 0\), we assume that the cost of applying the control \(\gamma \in \Gamma\) is given by

\[(1.16) \quad J^\gamma(y) = E_y[\int_0^\infty f(Y^\gamma_t)dt + \lambda \int_0^\infty e^{-\lambda t}d\gamma_t].\]

Hence the singular stochastic control problem corresponding to \(c = 0\) is the following:

**PROBLEM 1.2 (General singular stochastic control problem)** Find the value function \(W(y)\) defined by

\[(1.17) \quad W(y) = \inf_{\gamma \in \Gamma} J^\gamma(y); \quad y \in \mathbb{R}^n\]

and find an optimal \(\gamma^* \in \Gamma\) such that

\[(1.18) \quad W(y) = J^{\gamma^*}(y); \quad y \in \mathbb{R}^n\]

Since we can regard the set of impulse controls \(v \in \mathcal{V}\) as a subset of the set of singular stochastic controls \(\gamma \in \Gamma\) (by identifying an impulse \(\zeta > 0\) with the jump \(\Delta \gamma = \zeta\)), we have

\[(1.19) \quad W(y) \leq V_c(y) \quad \text{for all } c \geq 0.\]
It was proved in [MRob, Theorem 2.1 and Theorem 2.5] and [MRof] (if \( \lambda = 0 \))
that
\[
W(y) = V_0(y) = \lim_{c \to 0^+} V_c(y).
\]

In particular, the function \( c \to V_c(y) \) is continuous at \( c = 0 \).

In this paper we study a specific impulse control problem of the form
(1.8)–(1.9) in detail (see Problem 2.2) and we investigate how \( V_c(y) \) and
other quantities of the problem depends on \( c \). Our main result is that \( V_c(y) \)
is not differentiable at \( c = 0 \). More precisely, we show that
\[
\lim_{c \to 0^+} \frac{d}{dc} V_c(y) = \left[ \frac{dV_c}{dc} \right]_{c=0} = \infty \quad \text{for all } y.
\]
(see Theorem 2.6).

This means that going from \( c = 0 \) to a small positive \( c \) causes an increase
in the value function which is in no proportion to the increase in \( c \).

We also solve explicitly the corresponding singular stochastic control
problem when \( c = 0 \) (see Problem 3.1) and we show that the reflecting
barrier \( \bar{x} \) in the solution of this problem is the common limit of the impulse
barriers \( x_0(c) \) and \( x_1(c) \) of the impulse problem as \( c \to 0 \). (See Theorem 3.3.)

This work was motivated by a discussion of a combined stochastic control
and impulse control of problem in [MÖ], where a property similar to (1.21)
was deduced under certain conditions. (See (4.47) and (4.48) in [MÖ].)

This work is also related to [JS], where it is proved that the limit of the
impulse barriers as \( c \to 0 \) (of a different impulse control problem) is equal to
the reflecting barrier of a corresponding singular control problem. However,
the property (1.21) is not studied there.

It is natural to conjecture that the statement (1.21) holds not just for
the case we consider, but more generally for a large class of impulse control
problems where the cost of interfering has the form given by (1.6). This will
be discussed in a forthcoming paper by Ubøe, Zhang and the author ([OUZ]).

2 The impulse control problem

We begin by stating a verification theorem of quasi-variational type for im-
pulse control problems. The result is a special case of Theorem 3.1 in [BO2],
which again is an elaboration of results in [BL]. First we introduce some
notation.

Define the intervention operator \( \mathcal{N} \) on the space of functions \( h: \mathbb{R}^n \to \mathbb{R} \)
by
(2.1) \[ N h(x) = \inf_{\zeta \in Z} \{ h(x + g(\zeta)) + c + \lambda |\zeta| \} \]

Suppose that for each \( x \in \mathbb{R}^n \) there exists at least one \( \zeta = \hat{\zeta}(x) \in Z \) such that the infimum in (2.1) is attained and that a measurable selection \( \hat{\zeta} = R_h(x) \) of such minimum points \( \zeta \) exists. Then we have

(2.2) \[ N h(x) = h(x + R_h(s, x)) + c + \lambda |R_h(s, x)| \]

Note that if we don’t have any interventions then \( Y_t \) is an Itô diffusion with a generator \( A \) which coincides on \( C^2_0(\mathbb{R}^n) \) with the partial differential operator

(2.3) \[ L = \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} (\sigma \sigma^T)_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} \]

As is customary, we let \( C^1(\mathbb{R}^n) \), resp. \( C^2(\mathbb{R}^n) \) denote the set of all once, resp. twice, continuously differentiable functions on \( \mathbb{R}^n \). And \( C^2_0(\mathbb{R}^n) \) denotes the set of functions in \( C^2(\mathbb{R}^n) \) with compact support.

The Green measure of \( Y_t \) is the “total expected occupation measure” \( G = G_Y(\cdot, y) \) defined for each \( y \in \mathbb{R}^n \) by

(2.4) \[ G_Y(F, y) = E_y[\int_{0}^{\infty} \chi_F(Y_t) dt]; \quad F \subset \mathbb{R}^n \quad \text{(Borel)}. \]

We say that a continuous function \( f: \mathbb{R}^n \to \mathbb{R} \) is stochastically \( C^2 \) (with respect to \( Y \)) if \( Lf(x) \) is defined (pointwise) for a.a. \( x \) with respect to the Green measure \( G_Y(\cdot, y) \) and the generalized Dynkin formula holds for \( f \):

(2.5) \[ E_y[f(Y_T)] = f(y) + E_y[\int_{0}^{T} Lf(Y_t) dt]; \quad y \in \mathbb{R}^n \]

for all bounded stopping times \( T \) which are less than the exit time for \( Y_t \) from some compact set. By the classical Dynkin formula all \( C^2(\mathbb{R}^n) \) functions are stochastically \( C^2 \). More generally, functions which are \( C^2 \) except on a “small” set (for \( Y_t \)) are stochastically \( C^2 \). See [BÔ1] for more details.

**Theorem 2.1** [BÔ2] (Sufficient quasi-variational inequalities for impulse control)

Let \( c \geq 0 \).
a) Suppose we can find a continuous function \( \varphi : \mathbb{R}^n \to \mathbb{R}, \varphi \geq 0 \) such that
\[
(2.6) \quad \varphi \text{ is stochastically } C^2 \text{ with respect to } Y.
\]
\[
(2.7) \quad L\varphi(y) + f(y) \geq 0 \quad \text{for a.a. } y \text{ with respect to } G_Y(\cdot, z) \text{ for all } z \in \mathbb{R}^n.
\]
\[
(2.8) \quad \varphi(y) \leq \mathcal{N}\varphi(y) \quad \text{for all } y \in \mathbb{R}^n
\]
\[
(2.9) \quad \lim_{t \to \infty} \varphi(Y^v_t) = 0 \quad \text{a.s. } Q^v_y \text{ for all } v \in \mathcal{V} \text{ and all } y \in \mathbb{R}^n
\]
\[
(2.10) \quad \text{The family } \{\varphi(X^u_{\tau}); \tau \text{ stopping time}\} \text{ is } Q_y\text{-uniformly integrable}
\]
for all \( y \in \mathbb{R}^n \) and all \( v \in \mathcal{V} \).

Then
\[
\varphi(y) \leq J^v(y) \quad \text{for all } v \in \mathcal{V}, y \in \mathbb{R}^n,
\]
where \( J^v(y) \) is defined by (1.7).

b) Define the continuation region \( D \) by
\[
(2.11) \quad D = \{y; \varphi(y) < \mathcal{N}\varphi(y)\}
\]
Suppose that, in addition to (2.6)–(2.10) above, we have
\[
(2.12) \quad L\varphi(y) + f(y) = 0 \quad \text{for all } y \in D
\]
Define the impulse control
\[
\hat{v} = (\hat{\tau}_1, \hat{\tau}_2, \ldots; \hat{\zeta}_1, \hat{\zeta}_2, \ldots)
\]
inductively as follows:

Put \( \hat{\tau}_0 = 0 \) and
\[
(2.13) \quad \hat{\tau}_{k+1} = \inf\{t > \hat{\tau}_k; Y^{(k)}_t \notin D\},
\]
\[
(2.14) \quad \hat{\zeta}_{k+1} = \mathcal{R}\varphi(Y^{(k)}_{\hat{\tau}_{k+1}}),
\]
where \( Y^{(k)}_t \) is the result of applying the impulse control \( \hat{v}: = (\hat{\tau}_1, \ldots, \hat{\tau}_k; \hat{\zeta}_1, \ldots, \hat{\zeta}_k), k = 1, 2, \ldots \) to the process \( Y_t \). Suppose
\[
(2.15) \quad \hat{\tau} \in \mathcal{V}
\]
Then
\[
(2.16) \quad \varphi(y) = J^\hat{v}(y) \quad \text{for all } y.
\]
Hence
\[
(2.17) \quad \varphi(y) = V_c(y) \quad \text{(defined by (1.8))}
\]
and therefore
\[
(2.18) \quad v^* = \hat{v} \quad \text{is optimal (i.e. satisfies (1.9))}.
\]
A special case

We now apply this to a special case. The following impulse control problem is related to the problems studied in [HST] and [V]. When $\lambda = 0$ this problem was solved in [MR, Section 1].

If there are no interventions, we assume that the system $Y_t$ is given by

$$Y_t = (s + t, x + B_t); \quad t \geq 0$$

where $B_t$ is a 1-dimensional Brownian motion starting at 0 (so $Y_t$ starts at $y = (s, x)$). If the impulse control $v = (\tau_1, \tau_2, \ldots; \zeta_1, \zeta_2, \ldots) \in \mathcal{V}$ is applied to $\{Y_t\}$, it gets the form

$$Y^v_t = (s + t, X^v_t)$$

where

$$X^v_t = x + B_t - \sum_{\tau_k \leq t} \zeta_k; \quad 0 \leq t < T^* = \infty.$$  

We assume that the cost of applying the impulse $\zeta \geq 0$ is given by

$$K(\zeta) = c + \lambda \zeta$$

where $c \geq 0$, $\lambda \geq 0$ are constants. Assume that the cost rate is

$$f(r, \xi) = e^{-r^2 \xi^2}.$$  

This leads to a total expected cost $J^v_c(s, x)$ given by

$$J^v_c(s, x) = E_{s,x} \left[ \int_0^\infty e^{-r(s+t)} (X^v_t)^2 dt + \sum_k (c + \lambda |\zeta_k|) e^{-r(s+\tau_k)} \right]$$  

Note that it is not optimal to move $X_t$ downwards if $X_t$ is already below 0. Hence it is enough to consider impulse controls $v = (\tau_1, \tau_2, \ldots; \zeta_1, \zeta_2, \ldots)$ such that $\zeta_k \leq x + B_{\tau_k} - \sum_{j=1}^{k-1} \zeta_j$ i.e. $\sum_{j=1}^k \zeta_j \leq x + B_{\tau_k}$. We let the family of such $v$ satisfying (1.5) be our admissible family $\mathcal{V}$.

**PROBLEM 2.2 (Special impulse control problem)**

Let $J^v_c(s, x)$ be as in (2.24), $X^v_t$ as in (2.21).

For all $c > 0$ find the value function $V_c(s, x)$ and the optimal impulse control $v^*_c = v^*_c \in \mathcal{V}$ such that

$$V_c(s, x) = \inf_{v \in \mathcal{V}} J^v_c(s, x) = J^{v^*_c}(s, x).$$
In order to solve this problem we make some guesses about the value function $V_c(s, x)$ and the continuation region $D$ given by (2.11). Then we verify that this is indeed the solution by applying Theorem 2.1.

First, as a candidate $\varphi(s, x) = \varphi_c(s, x)$ for $V_c(s, x)$ let us try to put

\begin{equation}
\varphi(s, x) = e^{-\rho s} \psi(x)
\end{equation}

and as a candidate for the continuation region $D$ let us try

\begin{equation}
D = \{(s, x); x < x_1\}
\end{equation}

where $\psi(x)$ and $x_1 \in \mathbb{R}$ remain to be determined. With this choice of $\varphi$ we get

\[ L\varphi(s, x) + f(s, x) = \frac{\partial \varphi}{\partial s} + \frac{1}{2} \frac{\partial^2 \varphi}{\partial x^2} + e^{-\rho s} x^2 = e^{-\rho s} \left( \frac{1}{2} \psi''(x) - \rho \psi(x) + x^2 \right), \]

so (2.12) indicates that we choose $\psi(x)$ in $(-\infty, x_1)$ as a solution $h(x)$ of the equation

\begin{equation}
\frac{1}{2} h''(x) - \rho h(x) + x^2 = 0
\end{equation}

The general solution of (2.28) is

\begin{equation}
h(x) = C_1 e^{\sqrt{2}\rho x} + C_2 e^{-\sqrt{2}\rho x} + \frac{1}{\rho} x^2 + \frac{1}{\rho^2}
\end{equation}

where $C_1, C_2$ are arbitrary constants.

If we make no interventions, then the expected total cost is

\[ J^0(s, x) = E_{s,x} \left[ \int_0^\infty e^{-\rho(s+t)} (B_t^2)^2 dt \right] = e^{-\rho s} \int_0^\infty e^{-\rho t} (x^2 + t) dt = e^{-\rho s} \left( \frac{1}{\rho} x^2 + \frac{1}{\rho^2} \right). \]

Hence we must have

\[ 0 \leq \psi(x) \leq \frac{1}{\rho} x^2 + \frac{1}{\rho^2} \quad \text{for all } x \leq x_1 \]

and comparing with (2.29) we see that the last inequality is only possible if $C_2 \leq 0$. The first inequality rules out that $C_2 < 0$. Hence $C_2 = 0$. Then $C_1 \leq 0$ and we put

\begin{equation}
\psi(x) = \frac{1}{\rho} x^2 + \frac{1}{\rho^2} - ae^{\sqrt{2}\rho x} \quad \text{for } x \leq x_1
\end{equation}
where $a = -C_1 \geq 0$. Let us guess that $a > 0$.

For $x \geq x_1$ we have $\varphi(x) = N \varphi(x)$ and hence

$$\psi(x) = \inf_{\zeta > 0} \{\psi(x - \zeta) + c + \lambda \zeta\}$$

(2.31)

To find the minimum of $h(\zeta) = \psi(x - \zeta) + c + \lambda \zeta$ we look for $\hat{\zeta} = \hat{\zeta}(x)$ such that $h'(\hat{\zeta}) = -\psi'(x - \hat{\zeta}) + \lambda = 0$ i.e.

$$\psi'(x_0) = \lambda,$$

(2.32)

where

$$x_0 = x - \hat{\zeta}(x).$$

(2.33)

From (2.31) we get that $\psi(x) = \psi(x - \hat{\zeta}(x)) + c + \lambda \hat{\zeta}(x)$, i.e.

$$\psi(x) = \psi(x_0) + c + \lambda(x - x_0) \quad \text{for } x \geq x_1$$

(2.34)

In particular,

$$\psi(x_1) = \psi(x_0) + c + \lambda(x_1 - x_0)$$

(2.35)

and

$$\psi(x_1) = \psi(x_0) + c + \lambda(x_1 - x_0)$$

(2.36)

To summarize we put

$$\psi(x) = \begin{cases} \frac{1}{\rho^2} x^2 + \frac{1}{\rho^2} - ae^{\sqrt{2\rho} \frac{x}{\rho}} & \text{for } x \leq x_1 \\ \psi(x_0) + c + \lambda(x - x_0) & \text{for } x \geq x_1 \end{cases}$$

(2.37)
where \( x_0, x_1 \) and \( a \) are determined by the 3 equations (2.32), (2.35), (2.36), i.e.

\[
(2.38) \quad a \sqrt{2 \rho} e^{\sqrt{2 \rho} x_0} = \frac{2}{\rho} x_0 - \lambda \\
(2.39) \quad a \sqrt{2 \rho} e^{\sqrt{2 \rho} x_1} = \frac{2}{\rho} x_1 - \lambda \\
(2.40)' \quad a e^{\sqrt{2 \rho} x_1} - ae^{\sqrt{2 \rho} x_0} = \frac{1}{\rho} (x_1^2 - x_0^2) - c - \lambda (x_1 - x_0)
\]

where \( 0 \leq x_0 < x_1 \) and \( a > 0 \).

Note that by subtracting (2.38) from (2.39) and rearranging we get

\[
(2.40) \quad (x_1 - x_0)(x_1 + x_0 - 2 \hat{x}) = c \rho
\]

where

\[
(2.41) \quad \hat{x} = \frac{\rho x}{2} + \frac{1}{\sqrt{2 \rho}}.
\]

To study the solutions of (2.38)–(2.40), we first consider the function

\[
(2.42) \quad g(x) = g_a(x) = a \sqrt{2 \rho} e^{\sqrt{2 \rho} x} - \frac{2}{\rho} x + \lambda \quad \text{for fixed } a > 0
\]

Note that

\[
g'(x) = 2 \rho a e^{\sqrt{2 \rho} x} - \frac{2}{\rho}
\]

so that

\[
(2.43) \quad g'(x) = 0 \iff 2 \rho a e^{\sqrt{2 \rho} x} = \frac{2}{\rho}
\]

which has a unique solution \( x = \overline{x} = \overline{x}(a) \).

Since

\[
(2.44) \quad g''(x) = \sqrt{2 \rho} 2 \rho a e^{\sqrt{2 \rho} x} > 0 \quad \text{for all } x,
\]

we see that \( x = \overline{x} \) is a minimum point for \( g(x) \). Then note that from (2.43) we have

\[
g(\overline{x}) < 0 \iff a \sqrt{2 \rho} e^{\sqrt{2 \rho} \overline{x}} - \frac{2}{\rho} \overline{x} + \lambda < 0
\]

\[
(2.45) \quad \iff \frac{2}{\rho \sqrt{2 \rho}} - \frac{2}{\rho} \overline{x} + \lambda < 0 \iff \overline{x} > \frac{\rho \lambda}{2} + \frac{1}{\sqrt{2 \rho}} = \hat{x}.
\]
Equivalently, since by (2.43)

\[ a = \frac{1}{\rho^2} e^{-\sqrt{2} \rho x} \]

we get that \( g(\bar{x}) \) < 0 if and only if \( a < A \), where

\[ A = \frac{1}{\rho^2} \exp \left( -1 - \frac{\lambda \rho \sqrt{2 \rho}}{2} \right) = \frac{1}{\rho^2} \exp(-\sqrt{2 \rho} \hat{x}) \].

From this we conclude that equations (2.38), (2.39) have exactly two solutions \( x_0 = x_0(a) \), \( x_1 = x_1(a) \) such that

\[ 0 < x_0(a) < \bar{x}(a) < x_1(a) \]

if and only if \( 0 < a < A \).

From now on assume that \( 0 < a < A \). To study how \( x_0 = x_0(a) \) depends on \( a \) we differentiate (2.38) with respect to \( a \) and get

\[ \sqrt{2 \rho} e^{\sqrt{2} \rho x_0(a)} + a e^{\sqrt{2} \rho x_0(a)} x_0'(a) = \frac{2}{\rho} x_0'(a) \]

or

\[ x_0'(a) = \frac{\sqrt{2 \rho} e^{\sqrt{2} \rho x_0(a)}}{\frac{3}{\rho} - 2 \rho a e^{\sqrt{2} \rho x_0(a)}} = \frac{\sqrt{2 \rho} e^{\sqrt{2} \rho x_0(a)}}{-g'(x_0)} > 0. \]

Similarly

\[ x_1'(a) = \frac{\sqrt{2 \rho} e^{\sqrt{2} \rho x_1(a)}}{\frac{3}{\rho} - 2 \rho a e^{\sqrt{2} \rho x_1(a)}} = \frac{\sqrt{2 \rho} e^{\sqrt{2} \rho x_1(a)}}{-g'(x_1)} < 0. \]

Hence \( x_0(a) \) decreases as \( a \to 0 \) and from (2.38) it follows that

\[ \lim_{a \to 0} x_0(a) = \frac{\lambda \rho}{2}. \]
On the other hand, \( x_1(a) \) increases as \( a \to 0 \) and from (2.39) it follows that

\[
\lim_{a \to 0} x_1(a) = \lim_{a \to 0} a e^{\sqrt{2} \rho x_1(a)} = \infty .
\]

If \( a \to A \) then \( \overline{x}(a) \to \frac{\rho \lambda}{2} + \frac{1}{\sqrt{2} \rho} \) by (2.45) and (2.47) and hence

\[
g_0(\overline{x}(a)) = \frac{\sqrt{2} \rho}{\rho^2} - \frac{2}{\rho} \overline{x}(a) + \lambda \to 0 ,
\]

while \( g_0''(\overline{x}(a)) \) is bounded away from 0. Therefore

\[
\lim_{a \to A} x_0(a) = \lim_{a \to A} x_1(a) = \lim_{a \to A} \overline{x}(a) = \frac{\rho \lambda}{2} + \frac{1}{\sqrt{2} \rho} = \hat{x} .
\]

Moreover, we claim that

\[
x_0'(a) + x_1'(a) < 0 \quad \text{for all} \quad a \in (0, A).
\]

To prove this we use (2.48), (2.49) and rewrite (2.53) as (with \( x_0 = x_0(a), x_1 = x_1(a) \))

\[
\frac{e^{\sqrt{2} \rho x_0}}{\frac{\rho}{2} - 2 \rho a e^{\sqrt{2} \rho x_0}} < \frac{e^{\sqrt{2} \rho x_1}}{2 \rho a e^{\sqrt{2} \rho x_1} - \frac{\rho}{2}}
\]

By (2.46) this inequality is equivalent to

\[e^{\sqrt{2} \rho x_0} (e^{\sqrt{2} \rho (x_1 - \overline{x})} - 1) < e^{\sqrt{2} \rho x_1} (1 - e^{\sqrt{2} \rho (x_0 - \overline{x})})\]

or, by multiplying with \( e^{\sqrt{2} \rho (\overline{x} - x_0)} \),

\[
e^{\sqrt{2} \rho \overline{x}} (e^{\sqrt{2} \rho (x_1 - \overline{x})} - 1) < e^{\sqrt{2} \rho x_1} (e^{\sqrt{2} \rho (\overline{x} - x_0)} - 1).
\]

Since \( g'''(x) = 4 \rho^2 a e^{\sqrt{2} \rho x} > 0 \) for all \( x \) we must have

\[x_1 - \overline{x} < \overline{x} - x_0\]

and since we also have \( \overline{x} < x_1 \), we conclude that (2.54) – and hence (2.53) – holds.

We have now proved the following:
LEMMA 2.3 For all \( a \in (0, A) \) there exists a unique solution \( x_0 = x_0(a) < x_1 = x_1(a) \) of equations (2.38), (2.39). Moreover,

\[
(2.55) \quad x_0'(a) > 0, \quad x_1'(a) < 0, \quad x_0(a) + x_1'(a) < 0
\]

\[
(2.56) \quad \lim_{a \to 0} x_0(a) = \frac{\lambda \rho}{2}, \quad \lim_{a \to 0} x_1(a) = \infty
\]

\[
(2.57) \quad \lim_{a \to A} x_0(a) = \lim_{a \to A} x_1(a) = \hat{x} = \frac{\lambda \rho}{2} + \frac{1}{\sqrt{2\rho}}.
\]

Accordingly,

\[
(2.58) \quad \frac{\lambda \rho}{2} < x_0(a) < \hat{x} < x_1(a)
\]

and

\[
(2.59) \quad x_0(a) + x_1(a) > 2\hat{x} \quad \text{for all} \quad a \in (0, A).
\]

Using this result we get the existence of a solution \( a, x_0, x_1 \) of equations (2.38), (2.39), (2.40):

LEMMA 2.4 For all \( c > 0 \) there exists \( a = a^*(c) \in (0, A) \) such that (2.40) holds. With this choice of \( a = a^* \) the triple \( a = a^*, x_0 = x_0(a^*), x_1 = x_1(a^*) \) is a solution of the system (2.38), (2.39), (2.40).

Proof Let \( L(a) = (x_1(a) - x_0(a))(x_1(a) + x_0(a) - 2\hat{x}) \) denote the left hand side of (2.40). Then by Lemma 2.3 we have

\[
\lim_{a \to 0} L(a) = \infty, \quad \lim_{a \to A} L(a) = 0.
\]

Hence, for all \( c > 0 \) there exists \( a = a^*(c) \) such that \( L(a) = cp \). Now use this value of \( a \) in Lemma 2.3.

We can summarize this as follows:

THEOREM 2.5 For \( c > 0 \) let \( a^* = a^*(c), x_0 = x_0(a^*) \) and \( x_1 = x_1(a^*) \) be as in Lemma 2.4. Define

\[
(2.60) \quad \varphi_c(s, x) = e^{-\rho s} \psi_c(x)
\]

with

\[
(2.61) \quad \psi_c(x) = \begin{cases} 
\frac{1}{\rho} x^2 + \frac{1}{\rho^2} - a^* e^{\sqrt{2\rho x}} & \text{for} \quad x \leq x_1(a^*) \\
\psi(x_0(a^*)) + c + \lambda (x - x_0(a^*)) & \text{for} \quad x > x_1(a^*)
\end{cases}
\]
Then \( \varphi_c \) satisfies all the conditions of Theorem 2.1. Hence \( \varphi_c \) solves Problem 2.2, i.e.

\[
(2.62) \quad \varphi_c(s, x) = V_c(s, x) \quad \text{defined in (2.25)}
\]

and the following impulse control \( v^* = (\tau_1^*, \tau_2^*, \ldots; \zeta_1^*, \zeta_2^*, \ldots) \) is optimal:

Set \( \tau_0^* = 0 \) and define inductively, as in (2.13),

\[
(2.63) \quad \tau_{k+1}^* = \inf \{ t > \tau_k^*; X_t^{(k)} \geq x_1(a^*) \}
\]

and (from (2.33))

\[
(2.64) \quad \zeta_{k+1}^* = \zeta(x_1(a^*)) = x_1(a^*) - x_0(a^*); \quad k = 0, 1, 2, \ldots
\]

**REMARK**

a) Note that since \( \zeta_{k+1}^* \) does not depend on \( k \), we clearly have that

\[
\tau_k^* \to \infty \quad \text{as} \quad k \to \infty
\]

and therefore \( v^* \in \mathcal{V} \).

b) It follows from Theorem 2.5 that \( a^* \) must be unique (and hence \( x_0(a^*) \) and \( x_1(a^*) \)), because the corresponding \( \varphi_c(y) \) is the (unique) value function \( V_c(y) \).

We proceed to study how the solution depends on \( c > 0 \). Let \( a = a(c) \), \( x_0 = x_0(c) \) and \( x_1 = x_1(c) \) denote the unique solution we have found for the 3 equations (2.38)–(2.40). Differentiating (2.38) and (2.39) with respect to \( c \) we get, with \( a' = a'(c) \), \( x_0' = x_0'(c) \) and \( x_1' = x_1'(c) \):

\[
(2.65) \quad a' \sqrt{2 \rho e^{\sqrt{2 \rho} x_0}} + a 2 \rho e^{\sqrt{2 \rho} x_0} \cdot x_0' = \frac{2}{\rho} x_0'
\]

\[
(2.66) \quad a' \sqrt{2 \rho e^{\sqrt{2 \rho} x_1}} + a 2 \rho e^{\sqrt{2 \rho} x_1} \cdot x_1' = \frac{2}{\rho} x_1'
\]

Differentiating (2.40) with respect to \( c \) gives

\[
(x_1' - x_0')[x_1 + x_0 - 2 \hat{a}] + (x_1 - x_0)[x_1' + x_0'] = \rho
\]

or

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(2.67) \[(2x_1 - \lambda \rho)x'_0 = (2x_0 - \lambda \rho)x'_0 + \frac{2}{\sqrt{2\rho}}(x'_1 - x'_0) + \rho\]  

Now (2.65), (2.66) can be written, using (2.38), (2.39),

(2.68) \[a'\sqrt{2\rho}e^{\sqrt{2\rho}x_0} + \sqrt{2\rho}\left(\frac{2}{\rho}x_0 - \lambda\right)x'_0 = \frac{2}{\rho}x'_0\]  
(2.69) \[a'\sqrt{2\rho}e^{\sqrt{2\rho}x_1} + \sqrt{2\rho}\left(\frac{2}{\rho}x_1 - \lambda\right)x'_1 = \frac{2}{\rho}x'_1\]

Subtracting (2.68) from (2.69) and using (2.67) we get

\[a'\sqrt{2\rho}(e^{\sqrt{2\rho}x_1 - e^{\sqrt{2\rho}x_0}}) + \frac{\sqrt{2\rho}}{\rho}\left(\frac{2}{\sqrt{2\rho}}(x'_1 - x'_0) + \rho\right) = \frac{2}{\rho}(x'_1 - x'_0)\]

or

(2.70) \[a'(c) = -(e^{\sqrt{2\rho}x_1(c)} - e^{\sqrt{2\rho}x_0(c)})^{-1} < 0\]

Combining (2.48) with (2.70) we get

(2.71) \[x'_0(c) = \frac{dx_0}{dc} = \frac{dx_0}{da} \cdot \frac{da}{dc} < 0\]

and

(2.72) \[x'_1(c) = \frac{dx_1}{dc} = \frac{dx_1}{da} \cdot \frac{da}{dc} > 0\]

We conclude from (2.70), (2.71) and (2.72) that

(2.73) \[\hat{a} := \lim_{c \to 0} a(c), \quad \hat{x}_0 := \lim_{c \to 0} x_0(c) \quad \text{and} \quad \hat{x}_1 := \lim_{c \to 0} x_1(c)\]

exist. To find these limits, we note that they must solve the system (2.38)--(2.40) with \(c = 0\), i.e.

(2.74) \[\hat{a}\sqrt{2\rho}e^{\sqrt{2\rho}\hat{x}_0} = \frac{2}{\rho}\hat{x}_0 - \lambda\]

(2.75) \[\hat{a}\sqrt{2\rho}e^{\sqrt{2\rho}\hat{x}_1} = \frac{2}{\rho}\hat{x}_1 - \lambda\]

(2.76) \[(\hat{x}_1 - \hat{x}_0)(\hat{x}_1 + \hat{x}_0 - 2\xi) = 0\]

Clearly \(\check{a} \leq A\). Assume \(\check{a} < A\). Then by Lemma 2.3 we must have \(\hat{x}_0 < \hat{x}_1\) and \(\hat{x}_0 + \hat{x}_1 > 2\hat{a}\). But this contradicts (2.76). Hence \(\hat{a} = A\) and therefore

\[\hat{x}_0 = \lim_{a \to A} x_0(a) = \hat{x}\quad \text{and} \quad \hat{x}_1 = \lim_{a \to A} x_1(a) = \hat{x}\]

From (2.70) it follows that \(a'(c) \to -\infty\) as \(c \to 0^+\). Then by (2.71), (2.71) combined with (2.48), (2.49) we see that \(x'_0(c) \to -\infty\) and \(x'_1(c) \to \infty\) as \(c \to 0^+\).

We summarize this as follows:
THEOREM 2.6 Let \( a(c), a_0(c) \) and \( x_1(c) \) be the solution given in Theorem 2.5 of the special impulse control problem (Problem 2.2) for \( c > 0 \). Then

\[
(2.77) \quad \lim_{c \to 0^+} x_0(c) = \lim_{c \to 0^+} x_1(c) = \frac{\lambda \rho}{2} + \frac{1}{\sqrt{2\rho}} =: \hat{x}
\]

and

\[
(2.78) \quad \lim_{c \to 0^+} a(c) = \frac{1}{\rho^2} \cdot \exp(-\sqrt{2\rho} \hat{x}) =: A
\]

Moreover

\[
(2.79) \quad a'(c) = -(e^{\sqrt{2\rho}x_1(c)} - e^{\sqrt{2\rho}x_0(c)})^{-1} \to -\infty \quad \text{as} \quad c \to 0^+.
\]

and

\[
(2.80) \quad x_0'(c) \to -\infty, \quad x_1'(c) \to \infty \quad \text{as} \quad c \to 0^+.
\]

Hence

\[
(2.81) \quad \lim_{c \to 0^+} V_c(s, x) = e^{-\rho s} \psi_0(x),
\]

where

\[
(2.82) \quad \psi_0(x) = \begin{cases} 
\frac{1}{\rho} x^2 + \frac{1}{\rho^2} - Ae^{\sqrt{2\rho}x} & \text{for} \ x \leq \hat{x} \\
\psi_0(\hat{x}) + \lambda(x - \hat{x}) & \text{for} \ x > \hat{x}
\end{cases}
\]

Moreover,

\[
(2.83) \quad \lim_{c \to 0^+} dV_c(s, x) = \infty \quad \text{for all} \ s, x.
\]

3 The singular stochastic control problem

We now consider the case \( c = 0 \) only. As explained in the introduction (see Problem 1.2) there is a natural singular control problem interpretation of Problem 2.2, as follows:

For \( \gamma \in \Gamma \) let the state \( Y_t^\gamma \) of our system be given by

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\[(3.1) \quad Y_t^\gamma = (s + t, x + B_t - \int_0^t d\gamma_t); \quad t \geq 0\]

Assume that the cost of applying the control $\gamma \in \Gamma$ is given by

\[(3.2) \quad J^\gamma(y) = J^\gamma(s, x) = E_{s,x}[\int_0^\infty e^{-\rho(s+t)}(X_t^\gamma)^2 dt + \lambda \int_0^\infty e^{-\rho(s+t)} d\gamma_t],\]

where

\[(3.3) \quad X_t^\gamma = x + B_t - \int_0^t d\gamma_t. \quad \text{(See (2.20)-(2.24))}\]

This leads to the following problem:

**PROBLEM 3.1 (Special singular stochastic control problem when $c = 0$)** Let $J^\gamma(s, x)$ be as in (3.2). Find the value function $W(s, x)$ and the optimal control $\gamma^* \in \Gamma$ such that

\[(3.4) \quad W(s, x) = \inf_{\gamma \in \Gamma} J^\gamma(s, x) = J^{\gamma^*}(s, x).\]

We briefly recall the concept of a reflected Itô diffusion:

Let $Y_t$ be an Itô diffusion in $\mathbb{R}^n$ given by

\[dY_t = b(Y_t)dt + \sigma(Y_t)dB_t; \quad t \geq 0, \quad Y_0 = y \in \mathbb{R}^n.\]

Let $D$ be a domain in $\mathbb{R}^n$ whose boundary has a tangent at every point. For each $x \in \partial D$ choose a vector $\eta(x) \in \mathbb{R}^n$ pointing into $D$. Choose $y \in \overline{D}$, the closure of $D$. Then consider the problem of finding a pair $(\tilde{Y}_t, \xi_t)$ of continuous, $\mathcal{F}_t$-adapted stochastic processes with the following properties:

\[(3.5) \quad \tilde{Y}_t \in \overline{D} \quad \text{for all } t \geq 0\]

\[(3.6) \quad \xi_t \text{ is a nondecreasing process, increasing only when } t \in \Lambda := \{t; \tilde{Y}_t \in \partial D\}\]

\[(3.7) \quad \Lambda = \Lambda(\omega) \subset [0, \infty) \quad \text{has Lebesgue measure 0 for a.a. } \omega\]

and

\[(3.8) \quad d\tilde{Y}_t = b(\tilde{Y}_t)dt + \sigma(\tilde{Y}_t)dB_t + \eta(\tilde{Y}_t)d\xi_t; \quad t \geq 0\]

\[(3.9) \quad \tilde{Y}_0 = y \in \overline{D} \quad \text{and} \quad \xi_0 = 0.\]
The equations (3.8)–(3.9) (with the conditions (3.5)–(3.7)) is called a Skorohod stochastic differential equation. If it has a unique solution then \( \tilde{Y}_t \) is called the reflection of \( Y_t \) at \( \partial D \) (in the direction of the vector fields \( \eta(x), x \in \partial D \)) and \( \xi_t \) is called the local time of \( Y_t \) at \( \partial D \). For more information see [F] and the references there.

In the special case when \( Y_t = Z_t \) is a constant \( \sigma \) times a Brownian motion plus a constant drift in \( \mathbb{R} \) and \( D \) has the form \( D = (-\infty, x^*) \), then there is a simple construction of the reflected process \( \tilde{Z}_t \), which in this case is called the downward reflection of \( Z_t \) at \( x^* \). In fact, \( \tilde{Z}_t \) is given by

\[
\tilde{Z}_t = Z_t - \xi_t,
\]

where

\[
\xi_t = \sup_{s \leq t} (Z_s - x^*)^+,
\]

is the local time of \( \{Z_t\} \) at \( x^* \). This is due to Skorohod (see e.g. [RY, p. 222]). Here we have used the standard notation \( y^+ = \max(y, 0) \).

The pair \( (\tilde{Z}_t, \xi_t) \) satisfies all the requirements above, except possibly (3.7). If \( \sigma \) non-zero, it also satisfies (3.7). However, (3.7) is not needed in the result below.

Analogous to the quasi-variational inequality verification result for impulse control (Theorem 2.1) there is a variational inequality verification result for singular stochastic control. The following formulation is sufficient for our purposes.

Note that it is not optimal to move \( X_t \) downwards if \( X_t \) is already below 0. Hence we have

\[
W(s, x) = \inf_{\gamma \in \Gamma} J^\gamma(s, x) = \inf_{\gamma \in \Gamma_0} J^\gamma(s, x),
\]

where

\[
\Gamma_0 = \{ \gamma \in \Gamma; \gamma_t \text{ increases only when } X_t \geq 0 \}.
\]

**THEOREM 3.2** [LØ] (Sufficient variational inequalities for the singular stochastic control problem)

Suppose we can find \( \varphi \in C^2(\mathbb{R}^2), \varphi \geq 0 \) such that

\[
\frac{\partial \varphi}{\partial x} \leq \lambda e^{-\rho t} \quad \text{everywhere,}
\]

\[
\frac{\partial \varphi}{\partial t} + \frac{1}{2} \frac{\partial^2 \varphi}{\partial x^2} + e^{-\rho t} x^2 \geq 0 \quad \text{everywhere}
\]

and

\[
\lim_{R \to \infty} E_{s, x}[\varphi(T_R, X^\gamma_{T_R})] = 0 \quad \text{for all } \gamma \in \Gamma_0,
\]

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where

\[ T_R = R \wedge \inf\{t > 0; \frac{|X_t^0|}{t} \geq \frac{1}{R}\} \]

Then

\[ \varphi(t, x) \leq W(t, x). \tag{3.13} \]

Now define the non-intervention region

\[ D = \{(t, x); \frac{\partial \varphi}{\partial x} < \lambda e^{-\rho t}\} \tag{3.14} \]

Assume that

\[ D = \{(t, x); x < x^*\} \text{ for some } x^* \in \mathbb{R} \tag{3.15} \]

and that

\[ \frac{\partial \varphi}{\partial t} + \frac{1}{2} \frac{\partial^2 \varphi}{\partial x^2} + e^{-\rho t}x^2 = 0 \text{ in } D. \tag{3.16} \]

Let \( X_t^0 = x + B_t \) (corresponding to \( \gamma = 0 \) in (3.3)). Define the following control \( \tilde{\gamma} \in \Gamma_0 \):

\[ \tilde{\gamma}_t = \begin{cases} (x - x^*)^+ & \text{for } t = 0 \\ \xi_t & \text{for } t > 0 \end{cases} \tag{3.17} \]

where

\[ \xi_t = \sup_{s \leq t} (X_s^0 - x^*)^+. \tag{3.18} \]

Then

\[ \varphi(t, x) = W(t, x) \tag{3.19} \]

and

\[ \gamma^* = \tilde{\gamma} \text{ is optimal.} \tag{3.20} \]

The proof of Theorem 3.2 is similar to the proof of Theorem 3.3 in [LO] and is omitted.

Note that the first statement of the following Theorem is just a special case of a result in [MRob] (see (1.20)).
THEOREM 3.3 With $V_t(s,x)$ as in Theorem 2.6 for $c > 0$ and $W$ as in Problem 3.1 we have
\[
\lim_{c \to 0^+} V_t(s,x) = W(s,x).
\]
Moreover, the optimal singular stochastic control $\gamma^* \in \Gamma$ for Problem 3.1 coincides with the local time $\xi_t$ at $x = \hat{x}$ where $\xi$ is as in Theorem 2.6, i.e.
\[
\dot{x} = \frac{\lambda \rho}{2} + \frac{1}{\sqrt{2} \rho} = \lim_{c \to 0^+} x_0(c) = \lim_{c \to 0^+} x_1(c),
\]
corresponding to an optimal state process $Y_t^\gamma^*$ being the downward reflection of the process $Y_t^0 = (s + t, X_t^0)$ at $\partial D = \{(t, x); x = \hat{x}\}$.

Proof We verify that $\varphi(s, x) = e^{-\rho s} \psi_0(x)$ with $\psi_0$ given by (2.82) satisfies the requirements of Theorem 3.2. By (2.82) it suffices to verify that $\psi_0(x) \in C^2(R)$ and that the following holds:
\[
\psi_0'(x) \leq \lambda \quad \text{everywhere}
\]
\[
-\rho \psi_0(x) + \frac{1}{2} \psi_0''(x) + x^2 \geq 0 \quad \text{everywhere}
\]
\[
D_\gamma = \{(t, x); \psi_0(x) < \lambda\} = \{(t, x); x < \hat{x}\}
\]
\[
-\rho \psi_0(x) + \frac{1}{2} \psi_0''(x) + x^2 = 0 \quad \text{for } x < \hat{x}
\]
Moreover, we must check that (3.12) holds, i.e.
\[
\lim_{R \to \infty} E_{s,x}[e^{-\rho t_R} \psi_0(X_t^\gamma)] = 0 \quad \text{for all } \gamma \in \Gamma_0.
\]
To check that $\psi_0 \in C^2(R)$ we first note that
\[
\frac{d}{dx} \left( \frac{1}{\rho} x^2 + \frac{1}{\rho^2} - A e^{2p x} \right) = \frac{2}{\rho} x - A \sqrt{2} e^{2p x} = \lambda
\]
for $x = \hat{x}$ by (2.74) and (2.77). Since $\frac{d}{dx} (\psi_0(\hat{x}) + \lambda (x - \hat{x})) = \lambda$ also, we have $\psi_0 \in C^2(R)$. Next, since
\[
\frac{d^2}{dx^2} \left( \frac{1}{\rho} x^2 + \frac{1}{\rho^2} - A e^{2p x} \right) = \frac{2}{\rho} - 2 \rho A e^{2p x}
\]
\[
= \frac{2}{\rho} - 2 \rho \cdot \frac{1}{\rho^2} \exp(\sqrt{2} p (x - \hat{x})) = 0 \quad \text{for } x = \hat{x},
\]
we conclude that $\psi_0 \in C^2(R)$.
For $x \geq \hat{x}$ we have $\psi_0'(x) = \lambda$, while for $x < \hat{x}$ we have

$$\psi_0'(x) = \frac{2}{\rho} x - A \sqrt{2\rho} e^{\sqrt{2\rho} x} < \lambda$$

because $\psi_0''(x) > 0$ for $x < \hat{x}$ by (3.27). Hence (3.22) holds. Moreover,

$$D = \{ x ; \psi_0'(x) < \lambda \} = \{ x ; x < \hat{x} \}.$$

To verify (3.23) we first show that the function $h(x)$ defined by

$$h(x) = -\rho(\psi_0(\hat{x}) + \lambda(x - \hat{x})) + \frac{1}{2}(\psi_0(\hat{x}) + \lambda(x - \hat{x}))'' + x^2$$

$$= -\rho\psi_0(\hat{x}) - \rho\lambda(x - \hat{x}) + x^2$$

satisfies

(3.28) $h(x) \geq 0$ for $x \geq \hat{x}$

Since $h'(x) = -\rho\lambda + 2x > 0$ for $x > \frac{\rho A}{2}$ and $\hat{x} = \frac{\rho A}{2} + \frac{1}{\sqrt{2\rho}} > \frac{\rho}{2}$, it suffices to verify (3.27) for $x = \hat{x}$, i.e. to verify that

(3.29) $-\rho\psi_0(\hat{x}) + \hat{x}^2 \geq 0$

By (2.82) we have

$$-\rho\psi_0(\hat{x}) + \hat{x}^2 = \rho A e^{\sqrt{2\rho} \hat{x}} - \frac{1}{\rho} = 0.$$

To verify (3.25), and thereby also completing the verification of (3.23), we note that for $x < \hat{x}$ we have

$$-\rho\psi_0(x) + \frac{1}{2}\psi_0''(x) + x^2 = 0$$

by (2.82) and (2.28).

Finally we note that (3.26) clearly holds, because if $\gamma \in \Gamma_0$ then

$$\psi_0(X^\gamma) \leq \frac{1}{\rho} B_t^2 + C,$$

for some constant $C$ not depending on $t$. That completes the proof of Theorem 3.3.
SUMMARY

We have studied an impulse control problem and found its value function $V_c$ for all positive intervention costs $c$. Then we have shown that

$$\lim_{c\to 0^+} \frac{d}{dc} V_c = \infty,$$

which implies that increasing the intervention cost from $c = 0$ to a positive $c$, albeit small, can have a big effect on the value function for the problem.

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References


