Generators and comparison of entropies of automorphisms of finite von Neumann algebras

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Abstract

We modify slightly Voiculescu's definition of approximation entropy of automorphisms of finite von Neumann algebras and compare it with the entropy of Connes and Størmer. For this the notion of a generator is relevant, as its existence implies that the entropies coincide. Special emphasis is put on binary shifts. Examples of automorphisms without generators are also considered.

1 Introduction

At the present time there are several approaches to the study of entropy of C^*-dynamical systems, and in particular of finite von Neumann algebras, see e.g. [CS, CNT, ST, AF, V]. We shall in the present paper study the latter case, where we are given an automorphism α of a von Neumann algebra R with a faithful normal invariant tracial state τ, and we shall mainly consider the relationship between the entropy $H(\alpha)$ from [CS] with the approximation entropy $ha_\tau(\alpha)$ from [V]. These entropies have some basic differences, namely the one of Connes and Størmer in closely related to relative entropy of states and is quite abelian in its nature, c.f. the definition in [CNT], while the one of Voiculescu is a mean entropy.

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We shall impose a slight modification of Voiculescu's approximation entropy $ha_\tau(a)$ by replacing in its definition log rank A of a finite dimensional algebra A by its entropy, and denote the modified version by $Ha_\tau(\alpha)$. It should be noted that a similar change has been made by Choda [Ch2]. It is immediate from the definition that $Ha_\tau(\alpha) \leq ha_\tau(\alpha)$ and that equality represents a weak form of a Shannon, Breiman, McMillan Theorem. Voiculescu also introduced a "lower approximation entropy" $lha_\tau(\alpha)$, in which a limsup in the definition of $ha_\tau(\alpha)$ is replaced by lim inf. If we make the same modification of lha_τ as for ha_τ we get an entropy lHa_τ, and we have the inequalities $H(\alpha) \leq lHa_\tau(\alpha) \leq Ha_\tau(\alpha)$. In section 2 we give necessary and sufficient conditions for equalities in these inequalities.

Having done this, and keeping in mind the related results in [HS] it is natural to introduce the concept of a generator. In analogy with the classical abelian situation a generator as defined in section 3 is a finite dimensional von Neumann subalgebra N of R such that (1), $R = \bigvee_{-\infty}^{\infty} \alpha^i(N)$ and such that N satisfies two additional requirements, namely (2), if $m \leq n$ then $\bigvee_{i=m}^{n} \alpha^i(N)$ is finite dimensional, and (3), which will take different forms, that $H(\alpha)$ or $H(N, \alpha)$ in the notation of [CS] equals the mean entropy $\limsup \frac{1}{n} H(\bigvee_{i=0}^{n-1} \alpha^i(N))$ or lim inf. We then obtain results like $H(\alpha) = Ha_\tau(\alpha)$ or $H(\alpha) = lHa_\tau(\alpha)$. After preliminary studies of generators we consider specific cases, namely shifts on Temperley Lieb algebras and noncommutative Bernoulli shifts as defined in [CS].

In section 4 we consider binary shifts arising from nonperiodic bitstreams. It turns out that $lHa_\tau(\alpha) = \frac{1}{2}\log 2$, and if $H(\alpha) = \frac{1}{2}\log 2$ then we have a generator in the sense of lim inf above, and the generator is in the sense of lim sup if moreover the center sequence (c_n) grows like $0(n)$. We also have generators if the set X corresponding to the set of 1's in the bitstream is either contained in the even or in the odd integers. If (c_n) grows faster than $0(n)$ then the mean entropy can take any value in $(\frac{1}{2}\log 2, \log 2]$, (see Remark 4.13).

Finally, in section 5 we consider dynamical systems without generators. The first example was exhibited in [NST] as a binary shift with entropy $H(\alpha) = 0$. The existence of an uncountable number of nonconjugate examples was noted in [GS, Remark 6.4]. We shall present systems for which $0 < H(\alpha) < lHa_\tau(\alpha)$.

2 Voiculescu's approximation entropies

In [V] Voiculescu introduced several candidates for dynamical entropy of automorphisms. Technically they may be viewed as refined versions of mean entropy. The values are greater than those of the entropy $H(\alpha)$ defined in [CS]. We use the notation of [V]. Let R be a separable, hyperfinite von Neumann algebra with a faithful normal tracial state τ, and let $\|x\|_2 = \tau(x^*x)^{1/2}$ be the associated 2-norm. Let $Pf(R)$ denote the finite subsets of R. If $\omega \in Pf(R)$ and $X \subset R$ we write $\omega \subset^\delta X$ if for each $a \in \omega$ there is $x \in X$ with $\|x - a\|_2 < \delta$. Let $F(R)$ denote the set of finite dimensional C^*-subalgebras of R containing the identity 1 of R. If $A \in F(M)$, $\dim A$ is the dimension of A and $\operatorname{rank} A$ its rank, i.e. the dimension of a maximal abelian C^*-subalgebra of A. Crucial in Voiculescu's defnition is the δ-rank of ω defined by

$$r_\tau(\omega; \delta) = \inf\{\operatorname{rank} A \colon A \in F(R), \omega \subset^\delta A\}.$$

For our purposes we find it more natural to replace rank by entropy. We therefore put

$$e_\tau(\omega; \delta) = \inf\{\exp H(A) \colon A \in F(R), \omega \subset^\delta A\}.$$

Since τ will be fixed throughout our discussion, we shall from now on drop the subscript τ, and we imitate Voiculescu's definition of the approximation entropy $ha(\alpha)$ $(= ha_\tau(\alpha))$ for a τ-invariant automorphism α as follows:

$$\begin{aligned} Ha(\alpha, \omega, \delta) &= \limsup \frac{1}{n} \log e\Big(\bigcup_{j=0}^{n-1} \alpha^j(\omega); \delta\Big) \\ &= \limsup \frac{1}{n} \inf\Big\{H(A) \colon A \in F(R), \bigcup_{j=0}^{n-1} \alpha^j(\omega) \subset^\delta A\Big\}. \\ Ha(\alpha, \omega) &= \sup_{\delta > 0} Ha(\alpha, \omega, \delta) \\ Ha(\alpha) &= \sup\{Ha(\alpha, \omega) \colon \omega \in Pf(R)\}. \end{aligned}$$

Since $H(A) \leq \log \operatorname{rank} A$ it is clear that

$$Ha(\alpha) \leq ha(\alpha)\ .$$

We remark that Choda [Ch2] has also made a similar modification of $ha(\alpha)$.

Voiculescu also introduced, [V, Remark 1.6] the "lower approximation entropy" $lha(\alpha)$ $(= lha_\tau(\alpha))$ of α by replacing lim sup in the definition of

$ha(\alpha,\omega,\delta)$ by lim inf. We shall do the same and define

$$\begin{aligned} lHa(\alpha,\omega,\delta) &= \liminf \frac{1}{n} e\Big(\bigcup_{j=0}^{n-1}\alpha^j(\omega);\delta\Big) \\ lHa(\alpha,\omega) &= \sup_{\delta>0} lHa(\alpha,\omega,\delta) \\ lHa(\alpha) &= \sup\{lHa(\alpha,\omega): \omega\in Pf(F)\} \end{aligned}$$

An inspection of the proofs in section 1 in [V] shows that most of them go through for Ha and lHa. More specifically we have

2.1. If $k\in\mathbb{Z}$ then $Ha(\alpha^k)=|k|Ha(\alpha)$, and similarly for lHa.

2.2. If $\omega_j\in Pf(R)$, $j\in\mathbb{N}$, $\omega_1\subset\omega_2\subset\cdots$ are such that $\bigcup_{j\in\mathbb{N}}\bigcup_{n\in\mathbb{Z}}\alpha^n(\omega_j)$ generates R as a von Neumann algebra, then

$$Ha(\alpha)=\sup_{j\in\mathbb{N}} Ha(\alpha,\omega_j),$$

and similarly for lHa.

2.3. $H(\alpha)\le lHa(\alpha)\le Ha(\alpha)$

2.4. Let $R=R_1\otimes R_2$, $\tau=\tau_1\otimes\tau_2$, $\alpha_1\otimes\alpha_2$, then

$$Ha(\alpha_1\otimes\alpha_2)\le Ha(\alpha_1)+Ha(\alpha_2).$$

For lHa we can only prove the following.

2.5. $lHa(\alpha\otimes\alpha)\le 2lHa(\alpha)$.

Proof of 2.5. Let $\omega\in Pf(R)$ with $\|x\|\le 1$ for $x\in\omega$. Assume also that $1\in\omega$. Let $\delta>0$. Then we have:

$$\begin{aligned} &\inf\{H(A): A\in F(R\otimes R), \bigcup_0^{n-1}(\alpha\otimes\alpha)^j(\omega\otimes\omega)\subset^\delta A\} \\ &\le \inf\{H(A): A\in F(R\otimes R), \Big(\bigcup_0^{n-1}\alpha^j(\omega)\Big)\otimes\Big(\bigcup_0^{n-1}\alpha^j(\omega)\Big)\subset^\delta A\} \\ &\le \inf\{H(A_1\otimes A_2): A_i\in F(R), \bigcup_0^{n-1}\alpha^j(\omega)\subset^{\delta/2} A_i, i=1,2\} \\ &= \inf\{H(A_1)+H(A_2): A_i\in F(R), \bigcup_0^{n-1}\alpha^j(\omega)\subset^{\delta/2} A_i, i=1,2\} \\ &= 2\inf\{H(A): A\in F(R), \bigcup_0^{n-1}\alpha^j(\omega)\subset^{\delta/2} A\}, \end{aligned}$$

where the last equality follows since the inf over A_1 and A_2 is obtained for the same A. It follows from the above that

$$lHa(\alpha \otimes \alpha, \omega \otimes \omega, \delta) \leq 2lHa(\alpha, \omega, \delta/2),$$

which proves the assertion by 2.2, since sets of the form $\alpha^j \otimes \alpha^j(\omega \otimes \omega)$ with $1 \in \omega$ generate R as a von Neumann algebra.

By [SV, Lemma 3.4] the entropy H satisfies the inequality $H(\alpha_1 \otimes \alpha_2) \geq H(\alpha_1) + H(\alpha_2)$, thus the following proposition is immediate from 2.3, 2.4 and 2.5.

Proposition 2.6 (i) With the above notation, if $H(\alpha) = lHa(\alpha)$ then $H(\alpha \otimes \alpha) = 2H(\alpha)$.

(ii) If $R = R_1 \otimes R_2$, $\tau = \tau_1 \otimes \tau_2$, $\alpha = \alpha_1 \otimes \alpha_2$ and furthermore $H(\alpha_i) = Ha(\alpha_i)$, $i = 1, 2$, then $H(\alpha_1 \otimes \alpha_2) = H(\alpha_1) + H(\alpha_2)$.

For the rest of the section we shall discuss the situation when we have equality in 2.3. For this purpose we introduce two concepts which measure the deviation of $H(\alpha)$ from being a mean entropy.

Definition 2.7 Let $A \in F(R)$, α be a τ-invariant automorphism of R, ω_A be the set of matrix units in A. If $\delta > 0$ let

$$d(\alpha, A, \delta)$$
$$= \limsup \frac{1}{n} \inf\{|H(A, \ldots, \alpha^{n-1}(A)) - H(B)|: B \in F(R), \bigcup_{j=0}^{n-1} \alpha^j(\omega_A) \subset^{\delta} B\}.$$

Note that $0 < \delta' < \delta$ implies $d(\alpha, A, \delta) \leq d(\alpha, A, \delta')$. Put

$$d(\alpha, A) = \sup_{\delta > 0} d(\alpha, A, \delta) ,$$

$$\mathcal{S} = \{(A_i)_{i \in \mathbb{N}}: A_i \in F(R), A_1 \subset A_2 \subset \cdots, \left(\bigcup_{i=1}^{\infty} A_i\right)'' = R\},$$

and put

$$d(\alpha) = \inf_{(A_i) \in \mathcal{S}} \liminf_{i \to \infty} d(\alpha, A_i)$$

We put

$$ld(\alpha, A, \delta)$$
$$= \liminf \frac{1}{n} \inf\{|H(A, \ldots, \alpha^{n-1}(A)) - H(B)|: B \in F(R)) \bigcup_{j=0}^{n-1} \alpha^j(\omega_A) \subset^{\delta} B\},$$
$$ld(\alpha, A) = \sup_{\delta > 0} ld(\alpha, A, \delta), ld(\alpha) = \inf_{(A_i) \in \mathcal{S}} \liminf_{i \to \infty} d(\alpha, A_i).$$

Theorem 2.8 With α and R as before we have
(i) $d(\alpha) = 0$ if and only if $H(\alpha) = Ha(\alpha)$.
(ii) $ld(\alpha) = 0$ if and only if $H(\alpha) = lHa(\alpha)$.

Proof. We first show $d(\alpha) = 0$ implies $H(\alpha) = Ha(\alpha)$.
Let $\varepsilon > 0$ and choose $(A_i) \in \mathcal{S}$ such that

(1) $$\liminf_{i\to\infty} d(\alpha, A_i) < \varepsilon \,.$$

By the Kolmogoroff-Sinai Theorem [CS, Thm. 2] there exists $j_0 \in \mathbb{N}$ such that $j \geq j_0$ implies

(2) $$H(A_j, \alpha) - H(\alpha)| < \varepsilon \,,$$

where $H(A_j, \alpha) = \lim_n \frac{1}{n} H(A_j, \alpha(A_j), \ldots, \alpha^{n-1}(A_j))$, see [CS]. For each j let $n_j \in \mathbb{N}$ be such that $n \geq n_j$ implies

(3) $$\left|\frac{1}{n} H(A_j \ldots, \alpha^{n-1}(A_j)) - H(A_j, \alpha)\right| < \varepsilon \,.$$

Choose by (1) $j \geq j_0$ such that

$$d(\alpha, A_j) < 2\varepsilon \,.$$

Let $\delta > 0$. Then for the above j,

(4) $$d(\alpha, A_j, \delta) < 2\varepsilon \,.$$

Thus by definition of $d(\alpha, A_j, \delta)$ there is $m_j \geq n_j$ such that if $n \geq m_j$ then there exists $B_n \in F(R)$ with $\bigcup_0^{n-1} \alpha^k(\omega_{A_j}) \subset^\delta B_n$ such that

(5) $$\frac{1}{n} |H(A_j, \ldots, \alpha^{n-1}(A_j)) - H(B_n)| < 3\varepsilon$$

Choose $n \geq m_j$ such that

$$Ha(\alpha, \omega_{A_j}, \delta) < \varepsilon - \frac{1}{n} \inf\{H(B) \colon B \in F(R), \bigcup_{k=0}^{n-1} \alpha^k(A_j) \subset^\delta B\}$$

Then in particular this holds for B_n, so we have from (5), (3) and (2)

(6) $$\begin{aligned} Ha(\alpha, \omega_{A_j}, \delta) &< \varepsilon + \frac{1}{n} H(B_n) \\ &< \frac{1}{n} H(A_j, \ldots, \alpha^{n-1}(A_j)) + 4\varepsilon \\ &< H(A_j, \alpha) + 5\varepsilon \\ &< H(\alpha) + 6\varepsilon \end{aligned}$$

Since this holds for all δ

$$Ha(\alpha, \omega_{A_j}) \le H(\alpha) + 6\varepsilon .$$

Since this holds for all A_j, $j \ge j_0$, by 2.2

$$Ha(\alpha) \le H(\alpha) + 6\varepsilon .$$

Since ε is arbitrary $Ha(\alpha) \le H(\alpha)$, so they are equal by 2.3.

The proof that $ld(\alpha) = 0$ implies $H(\alpha) = lHa(\alpha)$ is similar but simpler than the proof above, and is omitted.

We next show $H(\alpha) = Ha(\alpha)$ implies $d(\alpha) = 0$.

Let $\varepsilon > 0$ and $(A_i) \in \mathcal{S}$. Choose by 2.2 and [CS, Thm. 2] j_0 such that $j > j_0$ implies

$$\begin{aligned} Ha(\alpha) &< Ha(\alpha, \omega_{A_j}) + \varepsilon \\ H(\alpha) &< H(A_j, \alpha) + \varepsilon . \end{aligned}$$

Fix $j \ge j_0$ and let by [CS, Thm. 1] $\delta > 0$ be such that if $P, N \in F(R)$, $\dim P = \dim A_j$ and the unit ball P_1 of P satisfies $P_1 \subset^\delta N_1$, then $H(P|N) < \varepsilon$, where the latter is the relative entropy as defined in [CS, Property F]. Let $0 < \eta < \delta$ be so small that $\omega_{A_j} \subset^\eta N$ implies $A_{j1} \subset^\delta N_1$, and therefore also that $\alpha^k(\omega_{A_j}) \subset^\eta N$ implies $\alpha^k(A_j)_1 \subset^\delta N_1$.

Let $n_0 \in \mathbb{N}$ be so large that $n \ge n_0$ implies

$$\frac{1}{n} H(A_j, \ldots, \alpha^{n-1}(A_j)) > H(A_j, \alpha) - \varepsilon .$$

Now we have

$$Ha(\alpha, \omega_{A_j}, \eta) = \limsup_n \frac{1}{n} \inf\{H(B) \colon B \in F(R), \bigcup_{k=0}^{n-1} \alpha^k(\omega_{A_j}) \subset^\eta B\} .$$

Therefore there exists $n_2 \ge n_1$ such that $n \ge n_2$ implies the existence of $B_n \in F(R)$ such that $\bigcup_{k=0}^{n-1} \alpha^k(\omega_{A_j}) \subset^\eta B_n$ and

$$\frac{1}{n} H(B_n) < Ha(\alpha, \omega_{A_j}, \eta) + \varepsilon .$$

For $n \ge n_2$ we have, using [CS, Property F],

$$\begin{aligned} H(\alpha) &< H(A_j, \alpha) + \varepsilon \\ &< \frac{1}{n} H(A_j, \ldots, \alpha^{n-1}(A_j)) + 2\varepsilon \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{n}H(B_n) + \frac{1}{n}\sum_{k=0}^{n-1} H(\alpha^k(A_j)|B_n) + 2\varepsilon \\
&< \frac{1}{n}H(B_n) + 3\varepsilon \\
&< Ha(\alpha, \omega_{A_j}, \eta) + 4\varepsilon \\
&\leq Ha(\alpha) + 4\varepsilon \\
&= H(\alpha) + 4\varepsilon
\end{aligned}
$$

Thus $\frac{1}{n}|H(A_j, \ldots, \alpha^{n-1}(A_j)) - H(B_n)| < 4\varepsilon$, proving that $d(\alpha, A_j, \delta) < 4\varepsilon$, and therefore that $d(\alpha) = 0$.

The proof that $H(\alpha) = lHa(\alpha)$ implies $ld(\alpha) = 0$ is similar but easier than the one above and is omitted. □

Remark 2.9 If $H(\alpha) = Ha(\alpha)$ then $d(\alpha) = \liminf d(\alpha, A_j) = 0$ for every sequence $(A_j) \in \mathcal{S}$ and similarly if $H(\alpha) = lHa(\alpha)$. This is immediate, since we started with an arbitrary sequence $(A_j) \in \mathcal{S}$ to show $d(\alpha) = 0$.

3 Generators

In classical ergodic theory a generator is a partition which together with all its translates by the ergodic transformation generates the σ-algebra in question. We shall extend this concept to the noncommutative case by replacing the partition by a finite dimensional C^*-subalgebra of R which together with its translates under the automorphism generates R and for which the entropy function $H(N, \ldots, \alpha^n(N))$, see [CS], behaves almost like mean entropy.

Definition 3.1 Let R, τ, α be as before and let $N \in F(R)$. We say N is a *generator* (resp. *lower generator*) for α, if

(i) $\bigvee_{i\in\mathbb{Z}} \alpha^i(N) = R$.

(ii) $\bigvee_{i=m}^{n} \alpha^i(N) \in F(R)$ whenever $m < n$, $m, n \in \mathbb{Z}$.

(iii) $H(N, \alpha) = \limsup_n \frac{1}{n} H(\bigvee_{i=0}^{n-1} \alpha^i(N))$

(resp. $H(N, \alpha) = \liminf_n \frac{1}{n} H(\bigvee_{i=0}^{n-1} \alpha^i(N))$).

N is called a *mean generator* (resp. *lower mean generator*) if (i) and (ii) hold and

(iv) $H(\alpha) = \limsup_n \frac{1}{n} H(\bigvee_{i=0}^{n-1} \alpha^i(N))$

(resp. $H(\alpha) = \liminf_n \frac{1}{n} H(\bigvee_{i=0}^{n-1} \alpha^i(N))$).

Note that since $H(N, \ldots, \alpha^{n-1}(N)) \leq H(\bigvee_{i=0}^{n-1} \alpha^i(N))$ we have

$$H(N, \alpha) \leq H(\alpha) \leq \liminf_n \frac{1}{n} H(\bigvee_{i=0}^{n-1} \alpha^i(N)) \leq \limsup_n \frac{1}{n} H(\bigvee_{i=0}^{n-1} \alpha^i(N))$$

for all $N \in F(R)$, satisfying (i) and (ii), see Lemma 3.2 below, hence in particular every generator is a lower generator, and similarly for mean generators. Note also that if N is a generator then

$$H(N, \alpha) = \lim_n \frac{1}{n} H(\bigvee_0^{n-1} \alpha^i(N)),$$

and similarly for mean generators. Furthermore, if N is generator (resp. a lower generator) then $H(N, \alpha) = H(\alpha)$, so in particular N is a mean generator (resp. lower mean generator).

Lemma 3.2 Suppose $N \in F(R)$ satisfies (i) and (ii) of Definition 3.1. Then

$$lHa(\alpha) \leq \liminf_n \frac{1}{n} H(\bigvee_{i=0}^{n-1} \alpha^i(N)).$$

Proof. Let $\omega \in Pf(R)$ and $\delta > 0$. Choose $j < k$ in $\mathbb{Z}$ such that $\omega \subset^\delta \bigvee_{i=j}^{k} \alpha^i(N)$. Put $M = \bigvee_j^k \alpha^i(N)$ and let $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies

$$\begin{aligned} lHa(\alpha, \omega, \delta) &< \frac{1}{n} H(\bigvee_0^{n-1} \alpha^i(M)) + \varepsilon \\ &= \frac{1}{n} H(\bigvee_0^{k+n-1-j} \alpha^i(N)) + \varepsilon \\ &= \frac{k+n-1-j}{n} \frac{1}{k+n-1-j} H(\bigvee_0^{k+n-1-j} \alpha^i(N)) + \varepsilon \,. \end{aligned}$$

This holds for all $n \geq n_0$, hence

$$lHa(\alpha, \omega, \delta) \leq \liminf_n \frac{1}{n} H(\bigvee_0^{n-1} \alpha^i(N)) + \varepsilon,$$

from which the conclusion of the lemma follows. □

Proposition 3.3 Let $N \in F(R)$. Then we have:

(i) If N is a mean generator (resp. lower mean generator) then $H(\alpha) = Ha(\alpha)$ (resp. $H(\alpha) = lHa(\alpha)$).

(ii) If N is a generator (resp. lower generator) and $M \in F(R)$ satisfies $N \subset M$ and $\bigvee_{i=m}^{n} \alpha^i(M) \in F(R)$ for $m < n$ in $\mathbb{Z}$ then M is a generator (resp. lower generator).

Proof. (i) If N is a mean generator $\frac{1}{n}H(\bigvee_0^{n-1} \alpha^i(N))$ converges, thus in the notation of the proof of Lemma 3.2 $\limsup_n \frac{1}{k+n-1-j}H(\bigvee_0^{k+n-1-j} \alpha^i(N) = \lim_n \frac{1}{n}H(\bigvee_0^{n-1} \alpha^i(N))$, hence it follows as in the proof of the lemma that

$$Ha(\alpha) \leq \lim_n \frac{1}{n}H(\bigvee_0^{n-1} \alpha^i(N)) = H(\alpha) .$$

By 2.3 $H(\alpha) \leq Ha(\alpha)$, hence they are equal. Similarly, if N is a lower mean generator then by Lemma 3.2 $lHa(\alpha) \leq H(\alpha)$, and again $lHa(\alpha) = H(\alpha)$.

(ii) To show (ii) note that the assumption implies that M satisfies (i) and (ii) of Definition 3.1. (iii) follows since $N \subset M$ implies $H(N, \alpha) \leq H(M, \alpha) \leq H(\alpha)$. □

Remark 3.4 It follows from Proposition 3.3 and 2.4 that if α_1 and α_2 are automorphisms of R_1 and R_2 respectively with mean generators then $H(\alpha_1 \otimes \alpha_2) = H(\alpha_1) + H(\alpha_2)$. If $\alpha \in \operatorname{Aut} R$ has a lower mean generator then by 2.5 $H(\alpha \otimes \alpha) = 2H(\alpha)$.

Remark 3.5 In order to get the tensor product formula for two automorphisms as in Remark 3.4 one can weaken the definition of generators to the case when (i) of Definition 3.1 does not hold, as follows. We say an increasing sequence $(N_k)_{k\in\mathbb{N}}$ in $F(R)$ is a *family of generators* for α if

(i) $\bigcup_k N_k$ is weakly dense in R.

(ii) $\bigvee_{i=m}^{n} \alpha^i(N_k) \in F(R)$ whenever $m < n$ in $\mathbb{Z}$.

(iii) $H(N_k, \alpha) = \limsup_n \frac{1}{n}H(\bigvee_{i=0}^{n-1} \alpha^i(N_k))$, $k \in \mathbb{N}$.

As before the following relations hold.

$$\begin{aligned} Ha(\alpha) &= H(\alpha) \\ H(\alpha_1 \otimes \alpha_2) &= H(\alpha_1) + H(\alpha_2) \\ H(\alpha) &= \lim_k H(N_k, \alpha) , \end{aligned}$$

where the last statement is the Kolmogoroff-Sinai Theorem [CV, Th,. 2].

Lemma 3.6 Let D be an abelian von Neumann subalgebra of R. Suppose $N \in F(R)$ satisfies (i) and (ii) of Definition 3.1. Suppose there is a sequence $(n_j)_{j\in\mathbb{N}}$ in $\mathbb{N}$ such that $D \cap \bigvee_{i=0}^{n_j} \alpha^i(N)$ is maximal abelian in $\bigvee_0^{n_j} \alpha^i(N)$ for all $j \in \mathbb{N}$, and such that

$$D \cap \bigvee_0^{n_j} \alpha^i(N) = \bigvee_0^{n_j} D \cap \alpha^i(N), \qquad j \in \mathbb{N} .$$

Then N is a lower generator.

Proof. The assumptions on D imply that

$$H(N, \ldots, \alpha^{n_j}(N)) = H(D \cap \bigvee_0^{n_j} \alpha^i(N)) = H(\bigvee_0^{n_j} \alpha^i(N)) ,$$

see [CS] or [CNT]. Thus

$$\begin{aligned} H(N, \alpha) &= \lim_{j\to\infty} \frac{1}{n_j+1} H(N, \ldots, \alpha^{n_j}(N)) \\ &= \lim \frac{1}{n_j+1} H(\bigvee_0^{n_j} \alpha^i(N)) \\ &\geq \liminf_n \frac{1}{n} (\bigvee_0^{n-1} \alpha^i(N)) \\ &\geq H(N, \alpha) , \end{aligned}$$

proving the lemma. □

As an immediate consequence of the above proof we have,

Corollary 3.7 If we in addition to the assumptions of Lemma 3.6 assume $\lim_n \frac{1}{n} H(\bigvee_0^{n-1} \alpha^i(N))$ exists, then N is a generator.

Example 3.8 Temperley-Lieb algebras
Let $(e_i)_{i\in\mathbb{Z}}$ be a sequence of projections with the properties

(a) $e_i e_{i\pm1} e_i = \lambda e_i$ for some $\lambda \in (0, \frac{1}{4}] \cup \{\frac{1}{4}\sec^2\frac{\pi}{m} : m \geq 3\}$

(b) $e_i e_j = e_j e_i$ for $|i-j| \geq 2$.

(c) $\tau(\omega e_i) = \lambda\tau(\omega)$, if ω is a word in 1 and e_j. $j < i$.

As is well-known [J] the von Neumann algebra R generated by the e_i is the hyperfinite II_1-factor. The shift automorphism θ_λ of R determined by $\theta_\lambda(e_i) = e_{i+1}$ has been studied by several authors [Pi, Po, Y, Ch, N]. Let $A[m,n] = C^*(e_i: m \le i \le n)$. Then $A[m,n]$ is finite dimensional. and the entropy of θ_λ is given by the formula

$$\begin{aligned} H(\theta_\lambda) &= \lim_n \frac{1}{2n+1} H(A[-n,n]) \\ &= \begin{cases} \eta(t) + \eta(1-t) & \text{for } \lambda \le \frac14,\ \lambda = t(1-t) \\ -\frac12 \log \lambda & \text{for } \frac14 \le \lambda < 1, \end{cases} \end{aligned}$$

where $\eta(t) = -t \log t$. In particular

$$\begin{aligned} H(\theta_\lambda) &= \lim_n \frac{1}{2n+1} H(\bigvee_{i=0}^{2n} \theta_\lambda^i(A_0)) \\ &= \lim_m \frac{1}{m} H(\bigvee_0^{m-1} \theta_\lambda^i(A_0)) , \end{aligned}$$

where $A_0 = C^*(e_0)$, because $H(\bigvee_0^k \theta_\lambda^i(A_0)) \le H(\bigvee_0^{k+1} \theta_\lambda^i(A_0))$, $k \in \mathbb{N}$ and $\lim \frac{1}{2n+1} H(\bigvee_0^{2n} \theta_\lambda^i(A_0))$ exists. It follows that A_0 is a mean generator.

Example 3.9 Non commutative Bernoulli shifts

Following [CS] let $\lambda_j > 0$, $j = 1, \ldots, d$, satisfy $\sum_1^d \lambda_j = 1$, where $d \ge 2$. Let $M_0 = M_d(\mathbb{C})$, and let ϕ_0 be the state on M_0 defined by $\phi_0(x) = \mathrm{Tr}(h_0 x)$, where h_0 is the diagonal operator

$$h_0 = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_d \end{pmatrix} .$$

Let $M_i = M_0$, $\phi_i = \phi_0$, and M be the factor obtained in the GNS-representation of $\bigotimes_{-\infty}^{\infty} M_i$ with respect to the product state $\phi = \bigotimes_{-\infty}^{\infty} \phi_i$. The shift σ on M is ϕ-invariant, hence so is its restriction α to the centralizer $R = M_\phi$. Let $\tau = \phi|R$. Then τ is a trace, and R is the hyperfinite II_1-factor [CS]. Let D_0 denote the diagonal in M_0, so $D_0 = \{h_0\}''$ in M_0. Let $D_i = D_0$, $i \in \mathbb{Z}$, $D = \bigotimes_{-\infty}^{\infty} D_i$, $M(m,n) = \bigotimes_{i=m}^{n} M_i$, $D(m,n) = \bigotimes_{i=m}^{n} D_i$, all considered as subalgebras of M. If $h_i = h_0$ is considered on an operator in M_i then the centralizer in $M_\phi(m,n)$ of $\phi|M(m,n)$ satisfies

$$M_\phi(m,n) = M_\phi \cap M(m,n) = M(m,n) \cap \{h_m \otimes \cdots \otimes h_n\}'.$$

and $D(m,n) = D \cap M(m,n)$ is maximal abelian in $M(m,n)$ for all $m < n$. In particular, it follows as in the proof of Lemma 3.6 that the sequence $(M_\phi(-n,n))$ is a family of generators for α in the sense of Remark 3.5.

Suppose we have found $n_0 \in \mathbb{N}$ such that

$$(*) \qquad \bigvee_{i=0}^{n} \alpha^i(M_\phi(0,n_0)) = M_\phi(0,n+n_0), \qquad n \in \mathbb{N}\,.$$

Then by translation of the indices, (i) and (ii) in Definition 3.1 hold. Since then furthermore

$$D \cap M_\phi(0,n+n_0) = \bigvee_0^n D \cap \alpha^i(M_\phi(0,n_0))$$

is maximal abelian in $M_\phi(0,n+n_0)$ it follows from Corollary 3.7 that $M_\phi(0,n_0)$ is a generator. We shall show $(*)$ for the case $d = 2$, $n_0 = 1$, hence that $M_\phi(0,1)$ is a generator. Note that since the shift θ_λ of the Temperley Lieb algebra is a Bernoulli shift for $\lambda \le \frac{1}{4}$ [PiPo] this shows the stronger result than 3.8 that θ_λ has a generator for $\lambda \le \frac{1}{4}$.

From now on $d = 2$. Denote by e_{ij}^0, $i,j = 1,2$, the matrix units in M_0, so that D_0 is the algebra generated by e_{ii}^0. Put $e_{ij}^k = \alpha^k(e_{ij}^0)$, $k \in \mathbb{Z}$, and let

$$N = C^*(e_{ii}^0, e_{jj}^1, e_{ij}^0 e_{ji}^1 \colon i,j = 1,2)$$

We shall show that $N = M_\phi(0,1)$, and that N is a generator for α. From the above remarks it remains to show axiom (i) in Definition 3.1. For this see also [PiPo, 5.5].

A straightforward computation shows that $e_{ij}^0 e_{j1}^1$ commutes with $h_0 \otimes h_1$, hence it belongs to $M_\phi(0,1)$. Thus $N \subset M_\phi(0,1)$. Since $\dim N = 6$, and a trivial computation shows $M_\phi(0,1) = 6$, $N = M_\phi(0,1)$. A similar computation shows $e_{ij}^p e_{ji}^q \in M_\phi$ for all $p \ne q$, hence products of such operators belong to M_ϕ.

We claim that $e_{ij}^q e_{ji}^q$ also belongs to $\bigvee_{i\in\mathbb{Z}} \alpha^i(N)$. Use induction, and assume it holds for $|p-q| \le N$. Then

$$e_{12}^0 e_{21}^{N+1} e_{22}^N = e_{12}^0 e_{21}^N e_{12}^N e_{21}^{N+1} \in \bigvee_i \alpha^i(N)$$

$$e_{12}^0 E_{21}^{N+1} e_{11}^N = e_{12}^N e_{21}^{N+1} e_{12}^0 e_{21}^N \in \bigvee_i \alpha^i(N),$$

hence

$$e_{12}^0 E_{21}^{N+1} = e_{12}^0 e_{21}^{N+1}(e_{11}^N + e_{22}^N) \in \bigvee_i \alpha^i(N),$$

completing the induction.

Thus in order to show $M_\phi = \bigvee_i \alpha^i(N)$ it suffices to show that M_ϕ is generated by the operators e^p_{ii} and $e^p_{ij}e^q_{ji}$, $i \neq j$, $i, j = 1, 2$, $p, q \in \mathbb{Z}$.

Let $f^0_1 = 1$, $f^0_2 = \lambda_2 e^0_{11} - \lambda_1 e^0_{22}$. Then f^0_i and e^0_{ij}, $i \neq j$, $i, j = 1, 2$ form an orthogonal basis for M_0 with respect to the inner product corresponding to $\phi|M_0 = \phi_0$. Similarly $f^p_i = \alpha^p(f^0_i)$ and $e^p_{ij} = \alpha^p(e^0_{ij})$ form an orthogonal basis for M_p. These operators are all eigenoperators for the modular automorphism, hence the operators of the form

$$f^{p_1}_{\alpha_1} \dots f^{p_k}_{\alpha_k} e^{q_1}_{i_1 j_1} \dots e^{q_k}_{i_k j_k} ,$$

where the p_i's are all distinct, the q_j's are all distinct, and $p_i \neq q_j$ for all i, j, form an orthogonal basis B for $L^2(M, \phi)$ consisting of eigenoperators for the modular automorphism. Furthermore, since $e^p_{ij}e^q_{ij} \in M_\phi$ for all $p \neq q$, the operators of the form

$$f^{p_1}_{\alpha_1} \dots f^{p_k}_{\alpha_k} (e^{q_1}_{i_1 j_1} e^{q_2}_{j_1 i_1}) \dots (e^{q_{n-1}}_{i_{\frac{n}{2}} j_{\frac{n}{2}}} e^{q_n}_{j_{\frac{n}{2}} i_{\frac{n}{2}}})$$

for n even form an orthogonal set C contained in M_ϕ.

We assert that C is an orthogonal basis for $L^2(M_\phi, \phi)$ as a subspace of $L^2(M, \phi)$. Indeed, let $x \in M_\phi$ and let $y \in B$, $y \notin M_\phi$. Since y is an eigenoperator for the modular automorphism an easy computation shows that the Fourier coefficient for x corresponding to y is zero. Thus the orthogonal series for x with respect to B contains only members with Fourier coefficients corresponding to elements in $B \cap M_\phi$. Thus $B \cap M_\phi$ is an orthogonal basis for $L^2(M_\phi, \phi)$. But the only elements in B which are invariant under the modular automorphism are those in C, thus $C = B \cap M_\phi$ is a basis for $L^2(M_\phi, \phi)$. Since $C \subset \bigvee_i \alpha^i(N)$, $M_\phi = \bigvee_i \alpha^i(N)$, and the proof is complete.

4 Binary shifts

If $X \subset \mathbb{N}$ we denote by $A(X)$ the C^*-algebra generated by a sequence $(s_n)_{n \in \mathbb{Z}}$ of symmetries satisfying the commutation relations

$$s_i s_j = (-1)^{g(|i-j|)} s_j s_i , \qquad i, j \in \mathbb{Z} ,$$

where g is the characteristic function of X considered as a subset of $\mathbb{Z}$. The canonical trace on $A(X)$ is the one which takes the value zero on all products $s_{i_1} \dots s_{i_n}$, where $i_1 < i_2 < \dots < i_n$, and $\tau(1) = 1$. Let π be the GNS-representation of τ, and put $R = \pi(A(X))''$. Then R is hyperfinite, and if $-X \cup \{0\} \cup X$ is a nonperiodic subset of $\mathbb{Z}$ then R is the hyperfinite II_1-factor

[PP]. In this case we say for simplicity that X is nonperiodic. We denote by α the automorphism determined by $\alpha(s_i) = s_{i+1}$. Let $A_n = C^*(s_0, \ldots, s_{n-1})$ for $n \in \mathbb{N}$, so that

$$A_n = \bigvee_{0}^{n-1} \alpha^i(C^*(s_0)).$$

We list some properties of A_n and $A(X)$ which will be used in the sequel, see [E, PP]. Denote by Z_n the center of A_n

4.1) There are $c_n, d_n \in \mathbb{N} \cup \{0\}$ such that $n = 2d_n + c_n$,
$A_n \cong M_{2^{d_n}}(\mathbb{C}) \otimes Z_n$,
and if $\mathbb{Z}_2^i = \mathbb{Z}_2$ then $Z_n \cong C^*(\prod_{i=1}^{c_n} \mathbb{Z}_2^i)$.

4.2) If e is a minimal projection in Z_n then $\tau(e) = 2^{-c_n}$.

4.3) If X is nonperiodic there is a sequence (m_i) in $\mathbb{N}$ such that (c_n) consists of the concatenation of the strings $(1, 2, \ldots, m_i - 1, m_i, m_i - 1, \ldots, 1, 0)$. In particular by 4.1 it follows that if A_n is a factor then n is even.

Note that by 4.1 and 4.2 all minimal projections in A_n have the same trace $2^{-d_n - c_n}$. Hence

4.4) $H(A_n) = \log \operatorname{rank} A = (c_n + d_n) \log 2$.

If X is nonperiodic it follows from 4.3 that $c_n = 0$ for an infinite number of n's. Hence by 4.1

$$\liminf_n \frac{1}{n} H(A_n) = \liminf \frac{1}{n}(c_n + d_n) \log 2 \leq \tfrac{1}{2} \log 2 .$$

However $d_n \leq \frac{1}{2}n$, so $\frac{1}{n}(d_n + c_n) \geq \frac{1}{2}$. Thus we have

4.5) $\liminf_n \frac{1}{n} H(A_n) = \frac{1}{2} \log 2$.

Lemma 4.6 With α as above and X nonperiodic

$$lHa(\alpha) = \tfrac{1}{2} \log 2 .$$

Proof. Let ω_{A_j} denote the set of matrix units in A_j, and let $\delta > 0$. By 4.5 we have

$$\begin{aligned} lHa(\alpha.\omega_{A_j}, \delta) &= \liminf_n \frac{1}{n} \inf\{H(A) \colon A \in F(R), \bigcup_{k=0}^{n-1} \alpha^k(\omega_{A_j}) \subset^\delta A\} \\ &\leq \liminf_n \frac{1}{n} H(A_{j+n-1}) \\ &= \liminf_n \frac{j+n-1}{n} \frac{1}{j+n-1} H(A_{j+n-1}) \\ &= \tfrac{1}{2} \log 2 . \end{aligned}$$

It follows that $lHa(\alpha) \leq \frac{1}{2}\log 2$. However, it is well-known that $H(\alpha \otimes \alpha) = \log 2$, hence by 2.3 and 2.5

$$\log 2 \leq lHa(\alpha \otimes \alpha) \leq 2lHa(\alpha) ,$$

proving equality. □

Lemma 4.7 $c_n = 0(n)$ if and only if $\lim_n \frac{1}{n} H(A_n) = \frac{1}{2}\log 2$.

Proof. If $\frac{c_n}{n} \to 0$ then $\frac{d_n}{n} \to \frac{1}{2}$, hence by 4.4 $\frac{1}{n}H(A_n) \to \frac{1}{2}\log 2$. Conversely, if $\lim \frac{1}{n}H(A_n) = \frac{1}{2}\log 2$ then by 4.4 $\frac{1}{n}(c_n + d_n) \to \frac{1}{2}$, hence by 4.1, $\frac{c_n}{n} \to 0$. □

Proposition 4.8 a) If $H(\alpha) = \frac{1}{2}\log 2$ then A_1 is a lower mean generator.

b) If moreover $c_n = 0(n)$ then A_1 is a mean generator.

Proof. Clearly axioms (i) and (ii) of Definition 3.1 hold for A_1. Axiom (iv) holds by 4.5). Thus a) holds.

Part b) follows from Lemma 4.7. □

If as before $ha(\alpha)$ denotes Voiculescu's approximation entropy then by 4.4 $Ha(\alpha) \leq ha) \leq \limsup \frac{1}{n}H(A_n)$, as is easily shown by the methods of the proof of Lemma 4.6. Thus it follows from Lemma 4.7 that if $c_n = 0(n)$ then $ha(\alpha) = Ha(\alpha) = \frac{1}{2}\log 2$. In the special case when α is a rational shift then (c_n) is bounded, so we recover the result in [N] that $ha(\alpha) = \frac{1}{2}\log 2$. Furthermore, for rational shifts, $H(\alpha) = \frac{1}{2}\log 2$ [Pr], so that by Proposition 4.8 b), if α is a rational shift, then $H(\alpha) = \lim_n \frac{1}{n}H(A_n)$. This result was shown in [GS] when X or $\mathbb{N} \setminus X$ is finite.

It was shown in [GS] that if either X is contained in the even or odd numbers then $H(\alpha) = \frac{1}{2}\log 2$. We next improve this result. We still assume X is nonperiodic.

Proposition 4.9 a) Suppose X is contained in the even numbers. Then A_2 is a lower generator.

b) Suppose X is contained in the odd numbers. Then A_1 is a lower generator.

Proof. a) It is clear that $A_2 = C^*(s_0, s_1)$ satisfies i) and ii) of Definition 3.1. It remains to show (iii).

Let $D = C^*(s_{2i}s_{2i+1}: i \in \mathbb{Z})$. Then D is abelian, as is easily computed using that $X \subset 2\mathbb{N}$. Let $D_n = C^*(s_{2i}s_{2i+1}: i = 0, \ldots, n-1)$. Then D_n is an abelian subalgebra of $D \cap A_{2n}$. Furthermore, $\dim D_n = 2^n$, and $\dim A_{2n} = 2^{2n}$. If A_{2n} is a factor then A is of type I_{2^n}, hence D_n is maximal abelian in

A_{2n}. By 4.3 there exists a sequence $(n_j)_{j\in\mathbb{N}}$ in $\mathbb{N}$ such that A_{2n_j} is a factor for each j. Note that we have

$$D \cap \bigvee_0^{n_j} \alpha^{2i}(A_2) = \bigvee_0^{n_j} D \cap \alpha^{2i}(A_2) = D_{n_j} \ .$$

Since (i) and (ii) of Definition 3.1 hold for A_2 with respect to the automorphism α^2, it follows from Lemma 3.6 that A_2 is a lower generator for α^2. We therefore have

$$\begin{aligned}
&\frac{1}{2k+1} H(A_2, \alpha(A_2), \ldots, \alpha^{2k}(A_2)) \\
&\geq \frac{1}{2k+1} H(A_2, \alpha^2(A_2), \ldots, \alpha^{2k}(A_2)) \\
&= \frac{k+1}{2k+1} \frac{1}{k+1} H(A_2, \ldots, (\alpha^2)^k(A_2)) \\
&\xrightarrow[k\to\infty]{} \tfrac{1}{2} H(A_2, \alpha^2) \\
&= \tfrac{1}{2} \liminf_n \frac{1}{n} H(\bigvee_{j=0}^{n-1} \alpha^{2j}(A_2)) \\
&= \liminf \frac{1}{2n} H(A_{2n}) \\
&= \tfrac{1}{2} \log 2 \ ,
\end{aligned}$$

using 4.5 and the fact that A_k is a factor only for even k, see 4.3. By 2.3 and Lemma 4.6 we have

$$H(A_2, \alpha) \leq H(\alpha) \leq lHa(\alpha) = \tfrac{1}{2} \log 2 \ ,$$

hence

$$H(A_2, \alpha) = \tfrac{1}{2} \log 2 = \liminf_n H(\bigvee_0^{n-1} \alpha^i(A_2)) \ ,$$

proving that A_2 is a lower generator.

b) Axioms i) and ii) of Definition 3.1 clearly hold for A_1. As above it suffices to show $H(A_1, \alpha) \geq \frac{1}{2} \log 2$.

Since X is contained in the odd numbers, $s_{2n} s_{2m} = s_{2m} s_{2n}$ for all $m, n \in \mathbb{Z}$. Thus the restriction

$$\alpha^2 \mid C^*(s_{2n} : n \in \mathbb{Z})$$

is the 2-shift, hence has entropy $\log 2$. In particular $H(\alpha) = \frac{1}{2}H(\alpha^2) = \frac{1}{2}\log 2$. Furthermore, we have for $n \in \mathbb{N}$

$$\begin{aligned}
&\frac{1}{2n+1}H(A_1, \alpha(A_1), \ldots, \alpha^{2n}(A_1)) \\
&\geq \frac{1}{2n+1}H(A_1, \alpha^2(A_1), \ldots, \alpha^{2n}(A_1)) \\
&= \frac{1}{2n+1}\log 2^n \\
&\xrightarrow[n\to\infty]{} \tfrac{1}{2}\log 2 ,
\end{aligned}$$

proving that $H(A_1, \alpha) \geq \frac{1}{2}\log 2$. □

Corollary 4.10 Suppose $c_n = 0(n)$.
a) If X is contained in the even numbers then A_2 is a generator.
b) If X is contained in the odd numbers then A_1 is a generator.

Proof. By Lemma 4.7 $\lim \frac{1}{n}H(\bigvee_{i=0}^{n-1} \alpha^i(A_1)) = \frac{1}{2}\log 2$, hence $\lim_n \frac{1}{n}H(\bigvee_{i=0}^{n-1} \alpha^i(A_2)) = \frac{1}{2}\log 2$, hence the conclusion follows from the proof of Proposition 4.9. □

Remark 4.11 By a proof analogous to that of a) in Proposition 4.9 one can show that $A_1 \otimes A_1$ is a lower generator for $\alpha \otimes \alpha$ for every binary shift α for which X is nonperiodic.

Remark 4.12 If X is nonperiodic and contains the odd numbers then $N = C^*(s_1, s_2)$ is a lower generator. Indeed, let $t_j = s_{2j-1}s_{2j}$, $j \in \mathbb{Z}$. Then the t_j's all commute, and α^2 acts as the 2-shift on $C^*(t_j : j \in \mathbb{Z})$. Since $t_j \in \alpha^{2(j-1)}(N)$ it follows as in the proof of b) in Proposition 4.9 that N is a lower generator.

Remark 4.13 If (c_n) is a sequence in $\mathbb{N} \cup \{0\}$ satisfying the conditions of 4.3, then (c_n) is the center sequence for a binary shift [PP, Thm. 6.6]. One can therefore find $X \subset 2\mathbb{N}$ such that the center sequence (c_n) satisfies $\limsup_n \frac{c_n}{n} > 0$, hence by Lemma 4.7 $\limsup_n \frac{1}{n}H(A_n) > \frac{1}{2}\log 2$. Since we can choose (c_n) such that $\limsup_n \frac{1}{n}H(A_n)$ can take any value $t \in (\frac{1}{2}\log 2. \log 2]$, we can therefore by Proposition 4.9 find a binary shift α with a lower generator such that $\limsup_n \frac{1}{n}H(A_n) = t$.

5 Automorphisms without generators

In [NST] there was exhibited an example of a binary shift α for which $H(\alpha) = 0$, hence α has no generator, and in [GS] we pointed out how to find an uncountable number of nonconjugate examples. We shall in the present section obtain larger classes of dynamical systems with no generators, and in particular find automorphisms α for which $lHa(\alpha) > H(\alpha) > 0$.

We shall follow the approach to entropy of Sauvageot and Thouvenot [ST], which is done for C^*-algebras. However, by [CNT], since our invariant state is a trace and the C^*-algebra A is nuclear, the entropy will by the same as $H(\alpha)$ when we represent A in its GNS-representation and take its weak closure R. We shall therefore move freely back and forth between A and R and α and its extension to R.

Let the notation be as in section 4. $A = A(X)$ is the C^*-algebra generated by symmetries (s_n), α the corresponding binary shift, and τ the canonical trace. By [NST] we can choose X such that $H(\alpha) = 0$. Let $C = C^*(\mathbb{Z}_2^{\mathbb{Z}})$. Then C is an abelian C^*-algebra, and $D = A \otimes C$ is an AF-algebra. We denote by θ the right shift on C and μ the product measure which is the product of the same measure on $\mathbb{Z}_2$, and which takes the value p at $\{0\}$ and $q = 1-p$ at $\{1\}$. Then $\mu \circ \theta = \mu$. We identify μ and θ with the corresponding state and automorphism on C. From the theory of Bernoulli shifts $H(\theta) = \eta(p) + \eta(q)$, where $\eta(t) = -t \log t$, $t > 0$, $\eta(0) = 0$. We give D the trace $\nu = \tau \otimes \mu$ and the automorphism $\gamma = \alpha \otimes \theta$. Then $\nu \circ \gamma = \nu$.

Proposition 5.1 With the above notation $H(\gamma) = H(\theta) = \eta(p) + \eta(q)$.

Proof. Let B be an abelian C^*-algebra, ρ a state on B, and β a ρ-invariant automorphism of B, Let λ be a state on $D \otimes B$ such that $\lambda(d \otimes 1) = \nu(d)$, $d \in D$, $\lambda(1 \otimes b) = \rho(b)$, $b \in B$. If P is a finite partition of B and p_i is an atom in P consider the state on D

$$\nu_i(d) = \rho(p_i)^{-1}\omega_i(d), \qquad d \in D\,,$$

where $\omega_i(d) = \lambda(d \otimes p_i)$. We then have the decomposition

$$\nu = \sum \rho(p_i)\nu_i$$

The "mutual information" $\varepsilon_\lambda(D, P)$ is defined by

$$(1) \qquad \varepsilon_\lambda(D, P) = \sum \rho(p_i) S(\nu, \nu_i),$$

where $S(\nu, \nu_i)$ is the quantum relative entropy for the states ν and ν_i, see [CNT] or [OP, §5]. Following the notation of [ST] let

$$(2) \qquad h'_\gamma(P, \lambda) = H_\rho(P \mid P^-) - H_\lambda(P \mid D),$$

where $P^- = \bigvee_{i=1}^{\infty} \beta^{-i}(P)$, $H_\lambda(P \mid D) = H_\rho(P) - \varepsilon_\lambda(D, P)$. By [ST, Lemma 3.2 and Prop. 4.1]

(3) $$H(\gamma) = H_\nu(\gamma) = \sup h'_\gamma(P, \lambda),$$

where the sup is taken over all finite partitions P of B and over all commutative dynamical systems $(B. \rho, \beta)$. Similarly

(4) $$h'_\theta(P \mid \lambda_1) = H_\rho(P \mid P^-) - H_{\lambda_1}(P \mid C),$$

where $H_{\lambda_1}(P \mid C) = H_\rho(P) - \varepsilon_{\lambda_1}(C, P)$ and λ_1 is a state on $C \otimes B$ such that $\lambda_1(c \otimes 1) = \mu(c)$, $c \in C$, $\lambda_1(1 \otimes b) = \rho(b)$, $b \in B$. Again we have

$$H(\theta) = H_\mu(\theta) = \sup h'_\theta(P, \lambda_1),$$

where the sup is taken as above.

If λ is as above then by [NST, Lemma 2.2] λ has the form $\lambda = \tau \otimes \lambda_2$, where λ_2 is a state on $C \otimes B$ such that $\lambda_2(1 \otimes b) = \rho(b)$, $b \in B$, $\lambda_2(c \otimes 1) = \mu(c)$, $c \in C$. It follows that

$$\nu_i(d) = \rho(p_i)^{-1} \tau \otimes \lambda_2(d \otimes p_i), \qquad d \in D\,.$$

If ϕ_1, ϕ_2, ϕ_3 are states on finite dimensional C^*-algebras, then it is a consequence of the definition of relative entropy that

$$S(\phi_1 \otimes \phi_2, \phi_1 \otimes \phi_3) = S(\phi_2, \phi_3).$$

By approximation this formula continues to hold for AF-algebras by [OP, Thm. 5.29]. Thus we have

$$S(\nu, \nu_i) = S(\mu, \rho_i),$$

where $\rho_i(c) = \rho(p_i)^{-1} \lambda_2(c \otimes \rho_i)$, $c \in C$. Hence, by (1)

$$\varepsilon_\lambda(D, P) = \varepsilon_{\lambda_2}(C, P).$$

Thus by (2) and (4)

$$h'_\gamma(P, \lambda) = h'_\theta(P, \lambda_2)$$

for all λ as above. Hence by (3) $H_\nu(\gamma) \leq H_\mu(\theta)$. But (C, μ, θ) is a subsystem of (D, ν, γ), so that $H_\mu(\theta) \leq H_\nu(\gamma)$, and we have equality. □

Theorem 5.2 With the above notation the automorphism γ of D'' satisfies

$$lHa(\gamma) \geq \tfrac{1}{2} \log 2 + H(\theta) > H(\gamma).$$

Proof. By Proposition 5.1 $H_\nu(\gamma) = H_\mu(\theta)$. By [V, Prop. 1.7] $lHa_\mu(\theta) = H_\mu(\theta)$, and $lHa(\alpha) = \frac{1}{2}\log 2$ by Lemma 4.6. Thus by 2.3 and 2.5

$$H_{\nu\otimes\nu}(\gamma\otimes\gamma) \le lHa_{\nu\otimes\nu}(\gamma\otimes\gamma) \le 2lHa_\nu(\gamma).$$

However, by [SV, Lemma 3.4] and [GS, Corollary 2.2]

$$\begin{aligned} H_{\nu\otimes\nu}(\gamma\otimes\gamma) &= H_{\tau\otimes\mu\otimes\tau\otimes\mu}(\alpha\otimes\theta\otimes\alpha\otimes\theta) \\ &\ge H_{\tau\otimes\tau}(\alpha\otimes\alpha) + H_{\mu\otimes\mu}(\theta\otimes\theta) \\ &= \log 2 + 2H_\mu(\theta). \end{aligned}$$

Thus

$$2lHa_\nu(\gamma) \ge \log 2 + 2H_\mu(\theta) > 2H_\mu(\theta) = 2H_\nu(\gamma).$$

□

It follows from Theorem 5.2 that γ cannot have generators in any of the senses described in section 3, i.e. generator, lower generator, mean generator, lower mean generator nor a family of generators.

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