ON THE CYCLOTOTOMIC TRACE MAP $K(\mathbb{Z}[C_p]) \to TC(\mathbb{Z}[C_p])$

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Abstract. Our main result gives an explicit calculation of the cyclotomic trace map $K(\mathbb{Z}[C_p]) \to TC(\mathbb{Z}[C_p])$ after two-adic completion under the assumption that $p$ is a two-regular prime. Let $X$ be a topological space whose fundamental group is of order $p$, where $p$ is as before. The main result, combined with a theorem of Dundas, gives an explicit calculation of the homotopy groups of the homotopy fibre of the trace map $A(X) \to TC(X)$ after two-adic completion.

1. Introduction and Summary of Results

Background and motivation. It is a nagging truth that algebraic $K$-theory, with a few delightful exceptions, has from its very beginning been quite inaccessible to explicit calculations. Any theory related to algebraic $K$-theory via a nontrivial map is therefore by birthright an interesting object to study. A first example is $HH$, Hochschild homology, connected to algebraic $K$-theory via the Dennis trace map. Sharper approximations are $THH$, topological Hochschild homology and its refinement $TC$, topological cyclic homology, obtained by exposing the cyclotomic structure on $THH$. Let $F$ be a functor with smash product, an FSP. The theories we have mentioned fit into a commutative diagram:

```
\begin{tikzcd}
TC(F) \\
THH(F) \\
K(F) & HH(F)
\end{tikzcd}
```

The map trc from algebraic $K$-theory to topological cyclic homology is called the cyclotomic trace map. Note that $TC$ is a better approximation to algebraic $K$-theory than $HH$ and $THH$. Topological cyclic homology was introduced by the authors Bökstedt et al. in [BHM]. Since then $TC$ has confirmed its relevance for calculating algebraic $K$-groups. Reference [Ma] gives an excellent overview of the results in this direction. The Eilenberg–Mac Lane spectrum of a unital ring $R$, and the suspension spectrum of the based loop group of a topological space $X$ has the structure of an FSP. There result spectra $TC(R)$ and $TC(X)$, and we may consider their two-adic completions in the sense of [BK]. In this paper we will, for some
primes $p$, calculate the cyclotomic trace map for the group ring $\mathbb{Z}[C_p]$ after passing to the two-adic completion. Here $C_p$ is a cyclic group of order $p$.

Our motivation for doing this calculation is to say something about symmetries of geometric objects. Let $X$ be a connected topological space, and write $X_+$ for $X$ together with a disjoint basepoint. Let $A(X)$ denote the Waldhausen’s algebraic K-theory space of $X$. Algebraic K-theory of rings is defined via group completion with respect to direct sum of matrices. Likewise, one of the many possible ways to define $A(X)$ is as follows. Let $Q(\Omega X_+)$ denote $\text{colim}_k \Omega^k \Sigma^k (\Omega X_+)$ where $\Omega$ denotes the loop group functor and $\Sigma$ the suspension functor. Define $GL_n(Q(\Omega X_+))$ as the colim over $k$ of the spaces $\text{Aut}_{\Omega X} (\vee^n S^k \wedge \Omega X_+)$ of $\Omega X$-equivariant self-maps of $\vee^n S^k \wedge \Omega X_+$ that are weak homotopy equivalences. By passing to the colim over $n$ one obtains the infinite general linear group $GL(Q(\Omega X_+))$, and one defines $A(X)$ to be the Quillen plus construction of $BGL(Q(\Omega X_+))$.

Let $M$ be a smooth manifold. The following theorem of Waldhausen shows that the algebraic K-theory of $M$ contains a copy of the smooth Whitehead space $Wh^\text{Diff}(M)$ of $M$. This object is of fundamental interest in differential topology.

**Theorem.** ([Waldhausen],[Wa]) There is a natural splitting of infinite loop spaces

$$A(M) \simeq Q(M_+) \times Wh^\text{Diff}(M).$$

The linearization map from $A$-theory to algebraic K-theory

$$L : A(X) \to K(\mathbb{Z}[\pi_1 X])$$

provides a map where the target is algebraic K-theory of group rings. This indicates that algebraic K-theory of group rings can be useful in the study of algebraic K-theory of spaces, in the same way as $HH, THH$ and $TC$ are for algebraic K-theory.

There is also a trace map $\text{trc}_X : A(X) \to TC(X)$ for any space $X$. The following special case of a theorem of Dundas [Du] shows the importance of the cyclotomic trace map in the study of algebraic K-theory of spaces.

**Theorem.** ([Dundas]) Let $X$ be a pointed connected space and let $R = \mathbb{Z}[\pi_1 X]$. Then

$$\begin{array}{ccl}
A(X) & \xrightarrow{L} & K(R) \\
\text{trc}_X \downarrow & & \downarrow \text{trc}_R \\
TC(X) & \xrightarrow[L]{L} & TC(R)
\end{array}$$

is a homotopy Cartesian square of spectra after $p$-adic completion for any rational prime $p$.

**Recent progress.** Based on the recent flourishing activity in motivic cohomology, by works of Bloch, Friedlander, Lichtenbaum, Morel, Suslin and Voevodsky, the authors Rognes and Weibel [RW] calculated the two-primary algebraic K-groups of number rings. Examples of explicit calculations of these groups are given in [RØ]. Combined with results of Hesselholt and Madsen in [HM], and calculations by the author in [Os] we are now in position to calculate the cyclotomic trace map at the prime two for the group rings $\mathbb{Z}[C_p]$ for some primes $p$. The cyclotomic trace map for the important initial example $\mathbb{Z}$ has been calculated by Rognes [R1] at the prime two.
Some notation. For a spectrum $X$ we let $X\widehat{\;}$ denote its two-adic completion. Let $\zeta_p$ be a primitive $p$th root of unity and let $\mathbb{F}_p$ be the field with $p$ elements. We say $p$ is a two-regular prime if the two-rank of the Picard group of the two-integers in $Q(\zeta_p)$ equals zero and $2$ is a primitive root modulo $p$. The last condition implies that $p \equiv 3,5 \mod 8$. As examples of such primes we have the Sophie Germain primes $p$ with $2$ a primitive root modulo $p$, cf. [Es]. The first examples are $p = 5, 11, 59, 83, 107$ and $179$. For an integer $i$ we let $v_2(i)$ be the greatest number $k$ such that $2^k$ divides $i$. For a natural number $n$ let $a = 1 + v_2(n + 1)$ and $b = 1 + v_2(n)$.

**Theorem 1.1.** Let $n > 1$ and let $p$ be a two-regular prime. If $p \equiv 5 \mod 8$, then we exclude the cases $n \equiv 1, 5 \mod 8$. We have the following data for the cyclotomic trace map $\text{trc}(n)\widehat{\;} : K_n(Z[C_p])\widehat{\;} \to TC_n(Z[C_p])\widehat{\;}$.

<table>
<thead>
<tr>
<th>$n \mod 8$</th>
<th>$\ker \text{trc}(n)\widehat{;}$</th>
<th>$\text{im } \text{trc}(n)\widehat{;}$</th>
<th>$\text{cok } \text{trc}(n)\widehat{;}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>$\mathbb{Z}^2_2 \oplus \mathbb{Z}/2$</td>
<td>$\mathbb{Z}^2_2 \oplus \mathbb{Z}/2$</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z}/2$</td>
</tr>
<tr>
<td>3</td>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z}^2_2 \oplus \mathbb{Z}/8$</td>
<td>$\mathbb{Z}^2_2 \oplus \mathbb{Z}/8$</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>$(\mathbb{Z}/8)^2$</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>$\mathbb{Z}^2_2$</td>
<td>$\mathbb{Z}^2_2 \oplus (\mathbb{Z}/2)^2$</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>$(\mathbb{Z}/2)^2$</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>$\mathbb{Z}^2_2 \oplus \mathbb{Z}/2^a$</td>
<td>$\mathbb{Z}^2_2 \oplus \mathbb{Z}/2^a$</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>0</td>
<td>$(\mathbb{Z}/2^b)^2$</td>
</tr>
</tbody>
</table>

The cited theorem of Dundas and Theorem 1.1 give the following result.

**Corollary 1.2.** Assume the hypothesis of Theorem 1.1, and let $X$ be a pointed connected space with fundamental group $C_p$. Then the homotopy groups of the homotopy fibre $\text{hofib}(\text{trc}_X)\widehat{\;}$ of the cyclotomic trace map $\text{trc}_X\widehat{\;} : A(X)\widehat{\;} \to TC(X)\widehat{\;}$ are given by the following table where we write $\pi_n$ for $\pi_n \text{hofib}(\text{trc}_X)\widehat{\;}$.

<table>
<thead>
<tr>
<th>$n \mod 8$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi_n$</td>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z}^2_2 \oplus \mathbb{Z}/8$</td>
<td>$\mathbb{Z}/16 \oplus \mathbb{Z}/8$</td>
<td>$\mathbb{Z}^2_2 \oplus (\mathbb{Z}/2)^2$</td>
</tr>
<tr>
<td>$n \mod 8$</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>$\pi_n$</td>
<td>$(\mathbb{Z}/2)^2$</td>
<td>$\mathbb{Z}^2_2 \oplus \mathbb{Z}/2^a$</td>
<td>$(\mathbb{Z}/2^b)^2$</td>
<td>$\mathbb{Z}^2_2 \oplus \mathbb{Z}/2$</td>
</tr>
</tbody>
</table>

**Idea of proof.** The proof of Theorem 1.1 is purely algebraic. Our basic idea is to translate questions about maps in topological cyclic homology and algebraic K-theory into questions about maps between étale cohomology groups. Indeed, naturality of the Bloch–Lichtenbaum spectral sequence tells us that we may use étale cohomology to determine maps in algebraic K-theory. See Section 6 for precise statements.
2. Review of input

In this section we will give an explicit description of the groups involved in the proof of Theorem 1.1. This section is not meant as an overview of known results, but rather as a vehicle for our later calculations. We fix an odd prime \( p \). The following result follows from Theorem D in [HM], whose proof uses the work of McCarthy in [Mc].

**Theorem 2.1.** (McCarthy, Hesselholt–Madsen) The cyclotomic trace map

\[
K(\mathbb{Z}_2[C_p]) \to TC(\mathbb{Z}_2[C_p])
\]

induces a homotopy equivalence on connective covers. Moreover, the map

\[
TC(\mathbb{Z}[C_p]) \to TC(\mathbb{Z}_2[C_p])
\]

induced by the inclusion is a homotopy equivalence.

The cyclotomic trace map is a natural map of spectra, so we have a commutative diagram

\[
\begin{array}{ccc}
K(\mathbb{Z}[C_p]) & \longrightarrow & K(\mathbb{Z}_2[C_p]) \\
\text{trc}_{\mathbb{Z}[C_p]} & & \text{trc}_{\mathbb{Z}_2[C_p]} \\
TC(\mathbb{Z}[C_p]) & \longrightarrow & TC(\mathbb{Z}_2[C_p])
\end{array}
\]

of spectra. We have documented two–adic equivalences on connective covers for the right vertical map and the lower horizontal map. This leaves us to consider the upper horizontal map.

**Theorem 2.2.** (Charney, [Ch] and Weibel, [We]) The Milnor square induces the exact sequence

\[
0 \to K_{2n-1}(\mathbb{Z}[C_p]) \to K_{2n-1}(\mathbb{Z}[\zeta_p]) \oplus K_{2n-1}(\mathbb{Z}) \to K_{2n-1}(\mathbb{F}_p) \to \\
\to K_{2n-2}(\mathbb{Z}[C_p]) \to K_{2n-2}(\mathbb{Z}[\zeta_p]) \oplus K_{2n-2}(\mathbb{Z}) \to 0
\]

for all \( n > 1 \).

Take now for granted that the map \( K_{2n-1}(\mathbb{Z}[\zeta_p]) \to K_{2n-1}(\mathbb{F}_p) \) is an isomorphism on the torsion part. This holds under the assumptions in Theorem 1.1 by Theorem 1.1 in [Os]. We find the exact sequence commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & K_n(\mathbb{Z}[C_p]) \\
& & \longrightarrow \ K_n(\mathbb{Z}[\zeta_p]) \oplus K_n(\mathbb{Z}) \longrightarrow \ K_n(\mathbb{F}_p) \longrightarrow 0 \\
0 & \longrightarrow & K_n(\mathbb{Z}_2[C_p]) \\
& & \longrightarrow \ K_n(\mathbb{Z}_2[\zeta_p]) \oplus K_n(\mathbb{Z}_2) \longrightarrow \ 0
\end{array}
\]

where the upper sequence is split. We adopt the notation \( \alpha_n : K_n(\mathbb{Z}[\zeta_p]) \to K_n(\mathbb{Z}_2[\zeta_p]) \), \( \beta_n : K_n(\mathbb{Z}) \to K_n(\mathbb{Z}_2) \) and \( \gamma_n : K_n(\mathbb{Z}[C_p]) \to K_n(\mathbb{Z}_2[C_p]) \). The snake lemma delivers an exact sequence

\[
0 \to \ker(\gamma_n) \to \ker(\alpha_n) \oplus \ker(\beta_n) \to \\
(2.3) \ K_n(\mathbb{F}_p) \to \cok(\gamma_n) \to \cok(\alpha_n) \oplus \cok(\beta_n) \to 0.
\]

Last in this section we present the two–primary calculations of the algebraic K–groups involved in the proof of Theorem 1.1.
Theorem 2.4. Let \( n > 1 \) and assume 2 is a primitive root modulo \( p \). Then:

<table>
<thead>
<tr>
<th>( n \mod 8 )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K_n(\mathbb{Z}[[C_p]]) )</td>
<td>( \mathbb{Z}_2^p \oplus (\mathbb{Z}/2)^2 )</td>
<td>( (\mathbb{Z}/2)^2 )</td>
<td>( \mathbb{Z}_2^p \oplus (\mathbb{Z}/8)^2 )</td>
<td>( (\mathbb{Z}/8)^2 )</td>
</tr>
<tr>
<td>( n \mod 8 )</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>( K_n(\mathbb{Z}[[C_p]]) )</td>
<td>( \mathbb{Z}_2^p \oplus (\mathbb{Z}/2)^2 )</td>
<td>( (\mathbb{Z}/2)^2 )</td>
<td>( \mathbb{Z}_2^p \oplus (\mathbb{Z}/2^a)^2 )</td>
<td>( (\mathbb{Z}/2^b)^2 )</td>
</tr>
</tbody>
</table>

Proof. The rational prime 2 is inert in \( \mathbb{Q}(\zeta_p) \) from the cyclotomic reciprocity law, so \( [\mathbb{Q}_2(\zeta_p) : \mathbb{Q}_2] = p - 1 \). Theorem 3.7 in [RW], the ring isomorphism \( \mathbb{Z}_2[C_p] \cong \mathbb{Z}_2[\zeta_p] \oplus \mathbb{Z}_2 \) and the fact that

\[
\omega_i^{(2)}(\mathbb{Q}_2) = \frac{\omega_i^{(2)}(\mathbb{Q}_2(\zeta_p))}{2^{i + v_2(i)}}
\]

for \( i \) odd,

\[
\omega_i^{(2)}(\mathbb{Q}_2(\zeta_p)) = \begin{cases} 
2^{i + v_2(i)} & \text{for } i \text{ even},
\end{cases}
\]

give our claim. See Section 6 for the definition of these numbers. \( \square \)

Theorem 2.5. (Rognes–Ostvær, [RÖ]) Let \( n > 1 \) and let \( p \) be a two–regular prime. Then:

<table>
<thead>
<tr>
<th>( n \mod 8 )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K_n(\mathbb{Z}[[\zeta_p]]) )</td>
<td>( \mathbb{Z}_2^{p-1} \oplus \mathbb{Z}/2 )</td>
<td>( 0 )</td>
<td>( \mathbb{Z}_2^{p-1} \oplus \mathbb{Z}/8 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( n \mod 8 )</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>( K_n(\mathbb{Z}[[\zeta_p]]) )</td>
<td>( \mathbb{Z}_2^{p-1} \oplus \mathbb{Z}/2 )</td>
<td>( 0 )</td>
<td>( \mathbb{Z}_2^{p-1} \oplus \mathbb{Z}/2^a )</td>
<td>( 0 )</td>
</tr>
</tbody>
</table>

Theorem 2.6. ([Ös]) Let \( n > 1 \) and let \( p \) be a two–regular prime. Then:

<table>
<thead>
<tr>
<th>( n \mod 8 )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K_n(\mathbb{Z}[[C_p]]) )</td>
<td>( \mathbb{Z}_2^{p-1} \oplus \mathbb{Z}/2 )</td>
<td>( \mathbb{Z}/2 )</td>
<td>( \mathbb{Z}_2^{p-1} \oplus \mathbb{Z}/16 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( n \mod 8 )</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>( K_n(\mathbb{Z}[[C_p]]) )</td>
<td>( \mathbb{Z}_2^{p-1} \oplus \mathbb{Z}/2 )</td>
<td>( 0 )</td>
<td>( \mathbb{Z}_2^{p-1} \oplus \mathbb{Z}/2^a )</td>
<td>( 0 )</td>
</tr>
</tbody>
</table>

The map from \( K_n(\mathbb{Z}[[C_p]]) \) to \( K_n(\mathbb{Z}) \) in the Milnor square induces an isomorphism on the torsion part after two–adic completion.

3. On the map \( K_n(\mathbb{Z}) \to K_n(\mathbb{Z}_2) \)

The map \( K_n(\mathbb{Z}) \to K_n(\mathbb{Z}_2) \) has been studied by topological methods in [R1]. The following is Theorem 7.7 of loc. sit.

Theorem 3.1. (Rognes) The natural map \( K_n(\mathbb{Z}[[\zeta_p]]) \to K_n(\mathbb{Z}_2[[\zeta_p]]) \) is an isomorphism modulo torsion if \( n > 1 \) and \( n \equiv 1 \mod 4 \).

The torsion part is discussed in Lemma 9.1 of loc. sit. Here is an alternative argument. Theorems 3.7 and 6.13 in [RW] give us the commutative diagram

\[
\begin{array}{ccc}
K_{8n+1}(\mathbb{Z}) \{2\} & \longrightarrow & K_{8n+1}(\mathbb{Z}_2) \{2\} \\
\cong \downarrow & & \cong \downarrow \\
H^0_{\text{ét}}(\mathbb{Q}; \mathbb{Q}_2/\mathbb{Z}_2(4n+1)) & \longrightarrow & H^0_{\text{ét}}(\mathbb{Q}_2; \mathbb{Q}_2/\mathbb{Z}_2(4n+1))
\end{array}
\]
The lower map is an isomorphism due to the compatibility with the inclusion into $H^0_{\text{et}}(\mathbb{C}; \mathbb{Q}_2/\mathbb{Z}_2(4n + 1))$. The case $8n + 7$ is identical. For $n \equiv 3 \mod 8$ we use, from loc. sit., the diagram

$$
\begin{array}{c}
0 \longrightarrow \mathbb{Z}/2 \longrightarrow K_{8n+3}(\mathbb{Z})\{2\} \longrightarrow H^0_{\text{et}}(\mathbb{Q}_2; \mathbb{Q}_2/\mathbb{Z}_2(4n + 2)) \longrightarrow 0 \\
\downarrow \quad \alpha \quad \downarrow \quad \alpha
K_{8n+3}(\mathbb{Z}_2)\{2\} \quad \longrightarrow \quad H^0_{\text{et}}(\mathbb{Q}_2; \mathbb{Q}_2/\mathbb{Z}_2(4n + 2))
\end{array}
$$

to conclude that $K_{8n+3}(\mathbb{Z})\{2\} \longrightarrow K_{8n+3}(\mathbb{Z}_2)\{2\}$ is surjective. Theorem 6.8 shows that $K_{8n+2}(\mathbb{Z})\{2\}$ injects into $K_{8n+2}(\mathbb{Z}_2)\{2\}$, i.e., $\beta_{8n+2}$ is an isomorphism.

**Theorem 3.2.** (Rognes) Let $n > 1$. The map $\beta_n : K_n(\mathbb{Z})\{2\} \longrightarrow K_n(\mathbb{Z}_2)\{2\}$ is an isomorphism on the torsion part for $n \equiv 1, 2, 7 \mod 8$, surjective with kernel $\mathbb{Z}/2$ for $n \equiv 3 \mod 8$ and trivial otherwise. The following table summarizes these observations, and also Theorem 3.1.

<table>
<thead>
<tr>
<th>$n \mod 8$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ker \beta_n$</td>
<td>0</td>
<td>0</td>
<td>$\mathbb{Z}/2$</td>
<td>0</td>
</tr>
<tr>
<td>$\text{im } \beta_n$</td>
<td>$\mathbb{Z}_2 \oplus \mathbb{Z}/2$</td>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z}/8$</td>
<td>0</td>
</tr>
<tr>
<td>$\text{cok } \beta_n$</td>
<td>0</td>
<td>0</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}/8$</td>
</tr>
<tr>
<td>$n \mod 8$</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>$\ker \beta_n$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\text{im } \beta_n$</td>
<td>$\mathbb{Z}_2$</td>
<td>0</td>
<td>$\mathbb{Z}/2^a$</td>
<td>0</td>
</tr>
<tr>
<td>$\text{cok } \beta_n$</td>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}/2^b$</td>
</tr>
</tbody>
</table>

4. On the map $K_n(\mathbb{Z}[\zeta_p]) \longrightarrow K_n(\mathbb{Z}_2[\zeta_p])$

**Proposition 4.1.** Let $n > 1$. The map $\alpha_n : K_n(\mathbb{Z}[\zeta_p])\{2\} \longrightarrow K_n(\mathbb{Z}_2[\zeta_p])\{2\}$ is split injective for $p$ a two–regular prime. With a table:

<table>
<thead>
<tr>
<th>$n \mod 8$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ker \alpha_n$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\text{im } \alpha_n$</td>
<td>$\mathbb{Z}_2^{p-1} \oplus \mathbb{Z}/2$</td>
<td>0</td>
<td>$\mathbb{Z}_2^{p-1} \oplus \mathbb{Z}/8$</td>
<td>0</td>
</tr>
<tr>
<td>$\text{cok } \alpha_n$</td>
<td>$\mathbb{Z}_2^{p-1}$</td>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z}_2^{p-1}$</td>
<td>$\mathbb{Z}/8$</td>
</tr>
<tr>
<td>$n \mod 8$</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>$\ker \alpha_n$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\text{im } \alpha_n$</td>
<td>$\mathbb{Z}_2^{p-1} \oplus \mathbb{Z}/2$</td>
<td>0</td>
<td>$\mathbb{Z}_2^{p-1} \oplus \mathbb{Z}/2^a$</td>
<td>0</td>
</tr>
<tr>
<td>$\text{cok } \alpha_n$</td>
<td>$\mathbb{Z}_2^{p-1}$</td>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z}_2^{p-1}$</td>
<td>$\mathbb{Z}/2^b$</td>
</tr>
</tbody>
</table>

**Proof.** Theorem 6.8 tells us that $K_n(\mathbb{Z}[\zeta_p])\{2\} \longrightarrow K_n(\mathbb{Z}_2[\zeta_p])\{2\}$ is split injective for $n$ odd since $p$ is a two–regular prime. Theorems 2.4 and 2.5 conclude the proof.
5. Proofs of Theorem 1.1 and Corollary 1.2

To prove Theorem 1.1 we combine (2.3), Theorems 2.4–2.6, Theorem 3.2 and Proposition 4.1. If \( n > 1 \) is even, then \( K_n(F_p) \) is the trivial group and our assertion is clear. Now let \( n > 1 \) be odd. For \( n \equiv 3 \mod 8 \) we find the exact sequence

\[
0 \to \ker(\gamma_n) \to \mathbb{Z}/2 \to \mathbb{Z}/8 \to \cok(\gamma_n) \to \mathbb{Z}_2^{\mathbb{F}_p} \to 0
\]

where Theorem 2.6 explains why \( \ker(\beta_n) \) maps trivially to \( K_n(F_p) \). For \( n \equiv 5 \mod 8 \) we have an exact sequence

\[
0 \to \mathbb{Z}/2 \to \cok(\gamma_n) \to \mathbb{Z}_2^{\mathbb{F}_p} \oplus \mathbb{Z}/2 \to 0.
\]

This extension is split since \( K_n(\mathbb{Z}[C_p])_2 \) is torsion free and \( \gamma_n \) is injective. The cases \( n \equiv 1, 7 \mod 8 \) are similar.

Corollary 1.2 follows from Theorem 1.1 and the short exact sequence

\[
0 \to \cok(\trc(n+1)_2^\ast) \to \pi_n \hofib(\trc_X)_2^\ast \to \ker(\trc(n)_2^\ast) \to 0
\]

from Dundas’ theorem. There is an extension issue for \( n \equiv 3 \mod 8 \). However, we know from Corollary 3.6 in [R2] that \( \pi_n \hofib(\trc_x)_2^\ast \) is cyclic of order sixteen in this case.

6. Input from étale cohomology

In this appendix we will justify the remaining claims of this paper, in particular the important Theorem 6.8. Let \( \ell \) be a rational prime and \( F \) a number field with ring of \( \ell \)-integers \( R_F \). Write \( S \) for the union of the \( \ell \)-adic primes \( S_\ell \) and the infinite primes \( S_\infty \) of \( F \). Denote the completion of \( F \) at a prime \( \mathfrak{p} \) by \( F_{\mathfrak{p}} \), and its underlying valuation ring by \( \mathcal{O}_{F_{\mathfrak{p}}} \). Let \( r_1 \) be the number of real embeddings of \( F \), and \( r_2 \) the number of pairs of complex embeddings of \( F \).

Let \( H^a_{\text{ét}}(R_F; M) \) denote the \( n \)th étale cohomology group of \( R_F \) with coefficients in \( M \). For us \( M \) equals \( \mathbb{Z}_l(i) \) or \( \mathbb{Q}_l/\mathbb{Z}_l(i) \) where the integer \( i \) denotes the \( i \)th Tate twist. Define \( \mathbb{Z}_l(i)' = \mathbb{Q}_l/\mathbb{Z}_l(1-i) \) and \( \mathbb{Q}_l/\mathbb{Z}_l(i)' = \mathbb{Z}_l(1-i) \). Let \( A^\# \) denote the Pontrjagin dual \( \text{Hom}(A, \mathbb{Q}/\mathbb{Z}) \) of an Abelian group \( A \). Let \( k \) be a field of characteristic zero and \( \zeta_n \) a primitive \( n \)th root of unity. For each integer \( i \), \( w^{(\ell)}_i(k) \) is defined as the maximal power \( \ell^n \) of \( \ell \) such that \( \text{Gal}(k(\zeta_{\ell^n})/k) \) has exponent dividing \( i \). Then \( w^{(\ell)}_i(k) \) equals the number of elements in \( H^0_{\text{ét}}(k; \mathbb{Q}_l/\mathbb{Z}_l(i)) \). We denote Tate cohomology with \( \hat{\mathcal{H}} \). The main tool for us will be Tate–Poitou duality. References are [Mi] and [Ta].

**Theorem 6.1.** (Tate, Poitou) There exists a natural 9-term exact sequence:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & H^0_{\text{ét}}(R_F; M) & \beta^0(M) & H^0_{\text{ét}}(F_p; M) & \gamma^0(M) & H^2_{\text{ét}}(R_F; M')^# \\
& & \oplus_{p \in S} \Pi^0_{\text{ét}}(F_p; M) & \longrightarrow & H^2_{\text{ét}}(R_F; M')^# & & \\
& & \downarrow & & \downarrow & & \\
& & H^1_{\text{ét}}(R_F; M')^# & \gamma^1(M) & H^1_{\text{ét}}(F_p; M) & \beta^1(M) & H^1_{\text{ét}}(R_F; M) \\
& & \downarrow & & \downarrow & & \\
& & H^2_{\text{ét}}(R_F; M) & \beta^2(M) & H^2_{\text{ét}}(F_p; M) & \gamma^2(M) & H^0_{\text{ét}}(R_F; M')^# & \longrightarrow & 0
\end{array}
\]
Definition 6.2. Let \( \alpha^n(R_F; M) \) denote the natural map from \( H^0_{et}(R_F; M) \) to the direct sum \( \bigoplus_{n=1}^{\infty} H^n_{et}(R; M) \) induced by the real embeddings of \( F \), and let \( \lambda^n_{S}(R_F; M) \) denote the kernel of the localization map with source \( H^n_{et}(R_F; M) \) and target the direct sum \( \bigoplus_{\nu \in S} H^n_{et}(F_{\nu}; M) \).

The following result is Theorem 4.10 in [RW].

Theorem 6.3. (Rognes–Weibel) Let \( i \geq 1 \). Then \( \lambda^n_{S}(R_F; Q_{\ell}/\mathbb{Z}_{\ell}(i)) = 0 \).

Next we consider negative twists. A number field \( F \) is called two-regular if the ideal \( (2) \) does not split in \( F \) and the narrow Picard group of \( R_F \) has odd order. Proposition 2.2(d) in [RÖ] tells us that \( F \) is two-regular if and only if the map \( \alpha^2(R_F; \mathbb{Z}_2(i)) \) is an isomorphism for \( i \neq 0, 1 \). Recall that \( H^2_{et}(R; \mathbb{Z}_2(i)) \cong \mathbb{Z}/2 \) for \( i \) even, and trivial for \( i \) odd. Clearly, also \( H^2_{et}(R_F; \mathbb{Z}_2(1)) = 0 \) if \( F \) is two-regular.

Corollary 6.4. Let \( F \) be a two-regular number field, and let \( i \neq 0 \). Then the group \( \lambda^n_{S}(R_F; Q_{\ell}/\mathbb{Z}_{\ell}(i)) \) is trivial. Let \( r = r_1 + r_2 \) for \( i \) odd, and let \( r = r_2 \) for \( i \) even. Then the \( \mathbb{Z}_2 \)-rank of \( H^1_{et}(R_F; \mathbb{Z}_2(i)) \) equals \( r \).

Proof. The argument for Theorem 4.10 in [RW] shows that \( \lambda^n_{S}(R_F; Q_{\ell}/\mathbb{Z}_{\ell}(i)) = 0 \). For the \( \mathbb{Z}_2 \)-rank calculation we just note that the two-rank of the torsion part of \( H^1_{et}(R_F; \mathbb{Z}_2(i)) \) equals one, and employ the short exact sequence

\[
0 \to H^1_{et}(R_F; \mathbb{Z}_2(i))/2 \to H^1_{et}(R_F; \mathbb{Z}/2(i)) \to 2 H^2_{et}(R_F; \mathbb{Z}_2(i)) \to 0.
\]

Our knowledge of the group \( H^1_{et}(R_F; \mathbb{Z}_2(i)) \) for a general \( F \) and negative \( i \) is quite poor. We refer to [KNF] for an updated discussion containing interesting conjectures.

Corollary 6.5. Let \( F \) be a two-regular number field, and let \( i \neq 0 \). Then the 9-term exact sequence of Tate–Poitou for \( M = Q_{\ell}/\mathbb{Z}_{\ell}(i) \) breaks up into three short exact sequences. In particular, if \( F \) is totally imaginary or \( i \) is odd, then the short exact sequence

\[
0 \to H^1_{et}(R_F; Q_{\ell}/\mathbb{Z}_{\ell}(i)) \to \bigoplus_{\nu \in S} H^1_{et}(F_{\nu}; Q_{\ell}/\mathbb{Z}_{\ell}(i)) \to H^1_{et}(R_F; \mathbb{Z}_2(1-i))^{(2)} \to 0
\]

is split.

Proof. In the following we use that \( \tilde{H}^0_{et}(R; Q_{\ell}/\mathbb{Z}_{\ell}(i)) \) is trivial for \( i \) even, and of order two for \( i \) odd. We must prove that \( \beta^1(Q_{\ell}/\mathbb{Z}_{\ell}(i)) \) is injective. The assertion is trivial for all \( i \) if \( F \) is totally imaginary, and also trivial if \( F \) is a real number field and \( i \) is even. Note that \( w_i^{(2)}(F) = w_i^{(2)}(F_{\nu}) \). By inspection we find that \( \gamma^0(Q_{\ell}/\mathbb{Z}_{\ell}(i)) \) is surjective for \( i \) odd.

Now for the splitting. Proposition 6.12 in [RW] states that \( H^1_{et}(R_F; \mathbb{Z}_2(i)) \) is a finitely generated \( \mathbb{Z}_2 \)-module for all \( n \) and all \( i \). Hence Proposition 2.4(a) in loc. cit. gives that \( H^1_{et}(R_F; Q_{\ell}/\mathbb{Z}_{\ell}(i)) \) is a divisible group. □

Next we translate from divisible coefficients to two-adic coefficients. The result we need is Theorem 3.1(a) from [Ta].
Theorem 6.6. (Tate, Poitou) There exists a natural perfect pairing

$$\mathcal{H}^2_{\eta}(R_F; M) \times \mathcal{H}^{3-n}_{\eta}(R_F; M') \to \mathbb{Q}/\mathbb{Z}$$

for \( n = 1, 2 \).

Corollary 6.7. Let \( F \) be a two-regular number field, and let \( i \neq 1 \). Then the 9-term exact sequence of Tate–Poitou for \( M = \mathbb{Z}_2(i) \) breaks up into three short exact sequences. In particular, if \( F \) is totally imaginary or \( i \) is even, then the short exact sequence

$$0 \to H^1_{\text{ét}}(R_F; \mathbb{Z}_2(i)) \to \bigoplus_{\varphi \in S} H^1_{\text{ét}}(F_\varphi; \mathbb{Z}_2(i)) \to H^1_{\text{ét}}(R_F; \mathbb{Q}_2/\mathbb{Z}_2(1-i))^\# \to 0$$

is split.

Proof. From Theorem 6.6; the first assertion is just the dual statement of Corollary 6.5. If \( F \) is totally imaginary or \( i \) is even, then \( H^2_{\text{ét}}(R_F; \mathbb{Z}_2(1-i)) = 0 \). Hence Proposition 2.4 in [RW] shows that \( H^1_{\text{ét}}(R_F; \mathbb{Q}_2/\mathbb{Z}_2(1-i))^\# \) is a free \( \mathbb{Z}_2 \)-module of finite rank, and the splitting is plain. \( \square \)

Next we translate these results into algebraic \( K \)-theory.

Theorem 6.8. Let \( F \) be a two-regular number field, and let \( n > 1 \). Then the naturally induced map

$$\varphi_n : K_n(R_F; \mathbb{Z}_2) \to K_n(O_{F_\varphi}; \mathbb{Z}_2)$$

is injective for all even \( n \) where \( \varphi \) is the dyadic prime in \( F \). Moreover, the following hold.

1) If \( F \) is totally imaginary, then \( \varphi_n \) is split injective for \( n \) odd.

2) If \( F \) is real, then \( \varphi_n \) is injective for \( n \equiv 1, 5 \mod 8 \) and split injective for \( n \equiv 7 \mod 8 \).

Proof. The slick way to prove this is to employ the extended integral Bloch–Lichtenbaum spectral sequence for the number ring and for the local number ring as constructed by Levine in [Le]. That spectral sequence is natural, and after tensoring with the two-adic integers we find commutative diagrams

$$\begin{array}{ccc}
K_{2n-1}(R_F; \mathbb{Z}_2) & \longrightarrow & K_{2n-1}(O_{F_\varphi}; \mathbb{Z}_2) \\
\downarrow & & \downarrow \\
H^1_{\text{ét}}(R_F; \mathbb{Z}_2(n)) & \longrightarrow & H^1_{\text{ét}}(O_{F_\varphi}; \mathbb{Z}_2(n))
\end{array}$$

and

$$\begin{array}{ccc}
K_{2n}(R_F; \mathbb{Z}_2) & \longrightarrow & K_{2n}(O_{F_\varphi}; \mathbb{Z}_2) \\
\downarrow & & \downarrow \\
H^2_{\text{ét}}(R_F; \mathbb{Z}_2(n+1)) & \longrightarrow & H^2_{\text{ét}}(O_{F_\varphi}; \mathbb{Z}_2(n+1))
\end{array}$$

where each vertical map is the resulting edge map. Next we apply Theorems 3.7 and 6.14 in [RW] and Corollary 6.7. \( \square \)
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