Abstract

We modify the Fock representation construction of the CAR-algebra to obtain a new proof for that the Binary Shift Algebra is isomorphic to the CAR-algebra.

Introduction

If $S$ is a bitstream, i.e. a sequence of 0's and 1's, we consider $S$ as the subset of $\mathbb{N}$ given by $i \in S$ if and only if the $i$'th term in the sequence is 1.

Let $S \subset \mathbb{N}$ be a given bitstream and $B$ the corresponding bitstream algebra, i.e. the C*-algebra generated by a sequence of self-adjoint unitaries that either commute or anticommute in a certain way depending on the bitstream. More specifically, if this sequence is denoted by $\{s_i\}_{i \in \mathbb{N}}$, then $s_i$ and $s_j$ anticommute if $|i - j| \in S$ and commute otherwise. The notation for the bitstream algebra used here is not common, and in e.g. [Po] the bitstream algebra is denoted by $\mathfrak{A}(S)$. Since the bitstream can be thought of as fixed throughout this work, we suppress the dependence and simply denote this algebra by $B$.

If $C(\mathcal{D})$ is the algebra of countinous functions on the Cantor set, it is known from a paper by Powers and Price [PP1] that $B$ will be of the form $M_n(C) \otimes C(\mathcal{D})$ if $S$ satisfies a certain periodicity condition, and the CAR-algebra otherwise. In this work we will see that by imitating the Fock representation construction of the CAR-algebra we can give an alternative proof for this result.

We will now sketch the approaches in [PP1] and this work to see their main differences. Whether the family of self-adjoint unitaries above is indexed over $\mathbb{N}$ or $\mathbb{Z}$ does not affect the results, and since many papers (e.g. those dealing with entropy) use the latter, we state all results with respect to this.

In both approaches the bitstream algebra $B$ is considered as an AF-algebra, i.e. $B = \bigcup_{n=1}^{\infty} B_n$, where $B_n$ is the finite-dimensional subalgebra generated by $\{s_i\}_{i=1}^n$. We denote the center of $B_n$ by $Z(B_n)$. To describe the embeddings $B_n \subset B_{n+1}$ (and thereby the Bratteli diagram corresponding to the AF-algebra $B$) it is essential to know the dimension of $Z(B_n)$ (for all $n \in \mathbb{N}$). In [PP] this is done by studying a sequence of matrices with entries in $\mathbb{F}_2$. More specifically, if $n \in \mathbb{N}$, the $n$'th Toeplitz matrix is given by

$$T_n = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_{n-2} & a_{n-1} \\ a_1 & a_0 & a_1 & a_2 & \cdots & a_{n-3} & a_{n-2} \\ a_2 & a_1 & a_0 & a_1 & \cdots & a_{n-4} & a_{n-3} \\ \vdots & & & & & & \\ a_{n-1} & a_{n-2} & a_{n-3} & a_{n-4} & \cdots & a_1 & a_0 \end{pmatrix},$$

where $a_0 = 0$ and $a_i$ is the $i$'th term in the bitstream. It is shown that $\dim Z(B_n) = 2^{\text{null}(T_n)}$, so the nullity of $T_n$ determines the dimension of
$Z(B_n)$. The sequence $\{\text{null}(T_n)\}_{n \in \mathbb{N}}$, called the center sequence, is then calculated from results on the ranks of the matrices above.

In this work we will avoid all discussions of Toeplitz matrices. The idea is to mimic the Fock representation construction of the CAR-algebra (see e.g. [HJ] and [BR]) to obtain an algebra $A$, with commutation-relations that depend on the bitstream in such a way that the bitstream algebra occurs as a subalgebra of $A$. If the bitstream consists only of 1's, the construction of $A$ should be identical to the Fock representation construction of the CAR-algebra. We find that $A$ is an AF-algebra with finite-dimensional subalgebras $A_n$ isomorphic to $M_{2^n}(\mathbb{C})$, i.e. $A$ is isomorphic to the CAR-algebra. If $B_n$ is defined as above, $B_n$ is a subalgebra of $A_n$. Under the isomorphism above we consider $B_n$ as a subalgebra of $M_{2^n}(\mathbb{C})$. We also obtain a nice description of the commutant of $B_n$ in $M_{2^n}(\mathbb{C})$. Next we find how $B_n$ is decomposed as a direct sum of matrix algebras, and we calculate the center sequence in a quite straightforward manner. With these two results the theorem follows quite easily.

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1 The Hilbert spaces $\mathcal{H}^\otimes n$ and $\bigwedge^n \mathcal{H}$

Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $(\xi_i)_{i \in \mathbb{Z}}$. For $n \in \mathbb{Z}^+$ define

$$\mathcal{H}^\otimes n = \begin{cases} \mathbb{C} & \text{if } n = 0, \\ \mathcal{H} \otimes \cdots \otimes \mathcal{H} \text{ (n copies)} & \text{if } n > 0. \end{cases}$$

Let $\mathcal{S} \subset \mathbb{N}$ be identified with the bitstream $(\chi_{\mathcal{S}}(n))_{n \in \mathbb{N}}$ and $\mathcal{S}' \subset \mathbb{Z}^+$ be given by $\chi_{\mathcal{S}'}(0) = 1$ and $\chi_{\mathcal{S}'}(n) = \chi_{\mathcal{S}}(n)$ $\forall n \in \mathbb{N}$. The reason for introducing $\mathcal{S}'$ is technical and will soon be clear. Note that, by convention, $\chi_{\mathcal{S}}(0) = 0$.

Let $\mathcal{S}_n$ denote the symmetric group on $n$ letters, and for $n \geq 2$ and $i = 1, \ldots, n - 1$ define a unitary operator on $\mathcal{H}^\otimes n$ by

$$u_i \xi_{k_1} \otimes \cdots \otimes \xi_{k_i} \otimes \xi_{k_{i+1}} \otimes \cdots \otimes \xi_{k_n} = (\xi_{k_{i}} \otimes \cdots \otimes \xi_{k_{i}} \otimes \xi_{k_{i+1}} \otimes \cdots \otimes \xi_{k_{n}} \cdot (\xi_{k_{i}} \otimes \cdots \otimes \xi_{k_{i}} \otimes \xi_{k_{i+1}} \otimes \cdots \otimes \xi_{k_{n}}.

If $n \in \mathbb{Z}^+$, the group generated by $\{\text{id}_{\mathcal{H}^\otimes n}, u_i\}_{i=1}^{n-1}$ contains $n!$ elements, each of which can be indexed by a corresponding permutation in $\mathcal{S}_n$ ($\mathcal{S}_0$ is understood to be the trivial group). If $u_\sigma$ denotes the element in this group corresponding to $\sigma \in \mathcal{S}_n$, then

$$u_\sigma \xi_{k_1} \otimes \cdots \otimes \xi_{k_n} = h_n(\sigma, k_1, \ldots, k_n) \xi_{k_{\sigma(1)}} \otimes \cdots \otimes \xi_{k_{\sigma(n)}}.
where \( h_n : \mathcal{S}_n \times \mathbb{Z}^n \to \{\pm 1\} \). \( h_n \) is uniquely determined by writing \( u_\sigma \) as a product of \( u_i \)'s, because if \( 1 \leq i < j \leq n \) and \( u_\sigma = \prod u_i \) is a factorization of \( u_\sigma \), the number of times \((-1)^{\chi_{S_n}(|k_i - k_j|)}\) contributes to the sign in

\[
\xi_{k_1} \otimes \cdots \otimes \xi_{k_n} = (\prod u_i) \xi_{k_1} \otimes \cdots \otimes \xi_{k_n} = \pm \xi_{k_{\sigma(1)}} \otimes \cdots \otimes \xi_{k_{\sigma(n)}}
\]

is either odd or even, independent of how \( u_\sigma \) is factorized. Since \( h_n \) is unique, the mapping \( \sigma \mapsto u_\sigma \) is indeed a group-isomorphism. We will now give a recursive expression for the sign function \( h_n \). It is clear that \( h_0 = 1 \) and \( h_1 \) are constant equal to 1, so let \( n \geq 2 \). If \( \sigma \in \mathcal{S}_n \), let \( j = \sigma(n) \) and \( \sigma' \in \mathcal{S}_{n-1} \) be given by \( \sigma'(i) = \sigma(i) \) if \( \sigma(i) < j \) and \( \sigma'(i) = \sigma(i) - 1 \) if \( \sigma(i) > j \). Then

\[
h_n(\sigma, k_1, \ldots, k_n) = -\prod_{i=j}^n (-1)^{\chi_{S_n}(|k_i - k_j|)} h_{n-1}(\sigma', k_1, \ldots, \check{k}_j, \ldots, k_n),
\]

where \( \check{k}_j \) means that \( k_j \) is removed. The reason for the minus sign in front of the product is to eliminate the extra minus caused by \((-1)^{\chi_{S_n}(|k_j - k_j|)}\).

For \( n \in \mathbb{Z}^+ \) define

\[
\bigwedge^n \mathcal{H} = \{ \xi \in \mathcal{H}^{\otimes n} : u_\sigma \xi = \xi \ \forall \sigma \in \mathcal{S}_n \},
\]

so \( \bigwedge^n \mathcal{H} \) is a closed subspace of \( \mathcal{H}^{\otimes n} \) (note that \( \bigwedge^0 \mathcal{H} = \mathbb{C} \) since \( \mathcal{S}_0 \) is trivial). If \( n \in \mathbb{Z}^+ \), define an operator \( P_n \) on \( \mathcal{H}^{\otimes n} \) by

\[
P_n = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} u_{\sigma}.
\]

If \( \xi \in \bigwedge^n \mathcal{H} \), then \( P_n \xi = \xi \), and since \( u_\sigma P_n = P_n \ \forall \sigma \in \mathcal{S}_n \), it follows that \( P_n \) maps \( \mathcal{H}^{\otimes n} \) onto \( \bigwedge^n \mathcal{H} \) and \( P_n^2 = P_n \). Calculation of the adjoint of \( P_n \) gives that \( P_n^* = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} u_{\sigma}^{-1} = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} u_{\sigma^{-1}} = P_n \), hence \( P_n \) is the orthogonal projection of \( \mathcal{H}^{\otimes n} \) onto \( \bigwedge^n \mathcal{H} \).

Define the outer product \( \wedge \) by

\[
\wedge : \mathcal{H} \times \cdots \times \mathcal{H} \to \bigwedge^n \mathcal{H},
\]

\[
(\eta_1, \ldots, \eta_n) \mapsto \eta_1 \wedge \cdots \wedge \eta_n = \sqrt{n!} P_n \eta_1 \otimes \cdots \otimes \eta_n.
\]

Observe that \( \wedge \) is linear and continuous in each variable, and if \( \sigma \in \mathcal{S}_n \), then

\[
\xi_{k_{\sigma(1)}} \wedge \cdots \wedge \xi_{k_{\sigma(n)}} = \sqrt{n!} P_n \xi_{k_{\sigma(1)}} \otimes \cdots \otimes \xi_{k_{\sigma(n)}}
\]

\[
= \sqrt{n!} P_n h_n(\sigma, k_1, \ldots, k_n) u_{\sigma} \xi_{k_1} \otimes \cdots \otimes \xi_{k_n}
\]

\[
= h_n(\sigma, k_1, \ldots, k_n) \xi_{k_1} \wedge \cdots \wedge \xi_{k_n}.
\]

Since \( \chi_{S_n}(0) = 1 \) this implies that if two \( k_i \)'s are equal, then the outer product is 0. Because this is a property of the outer product in the Fock-representation construction of the CAR-algebra, the outer product defined
Lemma 1.1. If $\eta \in \mathcal{H}$ and $\sigma \in S_n$, then

$$\eta \wedge \xi_{k_{\sigma(1)}} \wedge \cdots \wedge \xi_{k_{\sigma(n)}} = h_n(\sigma, k_1, \ldots, k_n) \eta \wedge \xi_{k_1} \wedge \cdots \wedge \xi_{k_n}.$$

Proof. Let $\eta = \sum_{i \in \mathbb{Z}} \lambda_i \xi_i \in \mathcal{H}$ and $\sigma \in S_n$. Let $\sigma' \in S_{n+1}$ be given by $\sigma'(1) = 1$ and $\sigma'(j + 1) = \sigma(j) + 1$ for $1 \leq j \leq n$. For $i \in \mathbb{Z}$ define $l_i = (l^{(i)}_1, l^{(i)}_2, \ldots, l^{(i)}_{n+1}) = (i, k_1, \ldots, k_n) \in \mathbb{Z}^{n+1}$. Then

$$\eta \wedge \xi_{k_{\sigma(1)}} \wedge \cdots \wedge \xi_{k_{\sigma(n)}} = \sum_{i \in \mathbb{Z}} \lambda_i \xi_i \wedge \xi_{k_{\sigma(1)}} \wedge \cdots \wedge \xi_{k_{\sigma(n)}}$$

$$= \sum_{i \in \mathbb{Z}} \lambda_i \xi_{l^{(i)}_1} \wedge \xi_{l^{(i)}_2} \wedge \cdots \wedge \xi_{l^{(i)}_{n+1}}$$

$$= \sum_{i \in \mathbb{Z}} \lambda_i h_{n+1}(\sigma', l^{(i)}_1, \ldots, l^{(i)}_{n+1}) \xi_{l^{(i)}_1} \wedge \xi_{l^{(i)}_2} \wedge \cdots \wedge \xi_{l^{(i)}_{n+1}}$$

$$= \sum_{i \in \mathbb{Z}} \lambda_i h_n(\sigma, k_1, \ldots, k_n) \xi_i \wedge \xi_{k_1} \wedge \cdots \wedge \xi_{k_n}$$

$$= h_n(\sigma, k_1, \ldots, k_n) \eta \wedge \xi_{k_1} \wedge \cdots \wedge \xi_{k_n}.$$

Lemma 1.2. Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $(\xi_i)_{i \in \mathbb{Z}}$. Then $(\xi_{k_1} \wedge \cdots \wedge \xi_{k_n})_{k \in \mathbb{Z}^n}$ where $k = (k_1, k_2, \ldots, k_n)$ with $k_1 < k_2 < \cdots < k_n$ is an orthonormal basis for $\wedge^n \mathcal{H}$.

Proof. Let $k, l \in \mathbb{Z}^n$ where $k = (k_1, k_2, \ldots, k_n)$ with $k_1 < k_2 < \cdots < k_n$ and $l = (l_1, l_2, \ldots, l_n)$ with $l_1 < l_2 < \cdots < l_n$. Then

$$(\xi_{k_1} \wedge \cdots \wedge \xi_{k_n}, \xi_{l_1} \wedge \cdots \wedge \xi_{l_n}) = n! (\xi_{k_1} \otimes \cdots \otimes \xi_{k_n}, P_n \xi_{l_1} \otimes \cdots \otimes \xi_{l_n})$$

$$= \sum_{\sigma \in S_n} (\xi_{k_1} \otimes \cdots \otimes \xi_{k_n}, u_\sigma \xi_{l_1} \otimes \cdots \otimes \xi_{l_n})$$

$$= \sum_{\sigma \in S_n} h_n(\sigma, l_1, \ldots, l_n) \prod_{i=1}^n (\xi_{k_i}, \xi_{l^{(i)}_1})$$

$$= \begin{cases} 1 & \text{if } k = l, \\ 0 & \text{if } k \neq l \end{cases}$$
and

\[
\text{lin}\{\xi_{k_1} \land \cdots \land \xi_{k_n} : k = (k_1, \ldots, k_n) \in \mathbb{Z}^n, k_1 < \cdots < k_n\} \\
= \text{lin}\{\xi_{k_1} \land \cdots \land \xi_{k_n} : k = (k_1, \ldots, k_n) \in \mathbb{Z}^n\} \\
= \text{lin}\{P_n \xi_{k_1} \otimes \cdots \otimes \xi_{k_n} : k = (k_1, \ldots, k_n) \in \mathbb{Z}^n\} \\
= P_n \text{lin}\{\xi_{k_1} \otimes \cdots \otimes \xi_{k_n} : k = (k_1, \ldots, k_n) \in \mathbb{Z}^n\} \\
= P_n \mathcal{H}^\otimes_n = \bigwedge^n \mathcal{H}.
\]

For the next result the classical theory applies verbatim (see [PJ]).

**Lemma 1.3.** Let \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) be two Hilbert spaces, \( n \in \mathbb{Z}^+ \), and \( L \in \mathcal{B}(\mathcal{H}_1^\otimes_n, \mathcal{H}_2) \). If \( L \) satisfies \( L u_\sigma = L \forall \sigma \in \mathcal{S}_n \), then \( L \eta_1 \land \cdots \land \eta_n = \sqrt{n!} L \eta_1 \otimes \cdots \otimes \eta_n \).

**Proof.**

\[
L \eta_1 \land \cdots \land \eta_n = \sqrt{n!} P_n \eta_1 \otimes \cdots \otimes \eta_n \\
= \frac{\sqrt{n!}}{n!} L(\sum_{\sigma \in \mathcal{S}_n} u_\sigma \eta_1 \otimes \cdots \otimes \eta_n) \\
= \frac{\sqrt{n!}}{n!} \sum_{\sigma \in \mathcal{S}_n} L \eta_1 \otimes \cdots \otimes \eta_n \\
= \sqrt{n!} L \eta_1 \otimes \cdots \otimes \eta_n.
\]

\[\square\]

2 The operators \( A_n(\xi) \) and \( a_n(\xi) \)

Let \( \mathcal{H} \) be a Hilbert space with orthonormal basis \( \{\xi_i\} \in \mathbb{Z} \). For \( \xi \in \mathcal{H} \) and \( n \in \mathbb{Z}^+ \) define an operator \( A_n(\xi) \) by

\[
A_n(\xi) : \mathcal{H}^\otimes_n \to \bigwedge^n \mathcal{H}, \eta_1 \otimes \cdots \otimes \eta_n \mapsto \frac{1}{\sqrt{n!}} \xi \land \eta_1 \land \cdots \land \eta_n.
\]
If \( m \in \mathbb{N} \) and \( \{\lambda_i\}_{i=1}^{m} \subset \mathbb{C} \), then

\[
\left\| A_n(\xi) \sum_{i=1}^{m} \lambda_i \eta_1^{(i)} \otimes \cdots \otimes \eta_n^{(i)} \right\| = \left\| \frac{\sqrt{(n+1)!}}{\sqrt{n!}} P_{n+1} \sum_{i=1}^{m} \lambda_i \xi \otimes \eta_1^{(i)} \otimes \cdots \otimes \eta_n^{(i)} \right\|
\leq \sqrt{n+1} \left\| \sum_{i=1}^{m} \lambda_i \xi \otimes \eta_1^{(i)} \otimes \cdots \otimes \eta_n^{(i)} \right\|
= \sqrt{n+1} \|\xi\| \left\| \sum_{i=1}^{m} \lambda_i \eta_1^{(i)} \otimes \cdots \otimes \eta_n^{(i)} \right\|,
\]

so \( A_n(\xi) \) extends to a bounded operator on \( \mathcal{H}^{\otimes n} \) with \( \|A_n(\xi)\| \leq \sqrt{n+1} \|\xi\| \).

If \( \sigma \in S_n \), then

\[
A_n(\xi) u_\sigma \xi_{k_1} \otimes \cdots \otimes \xi_{k_n} = \frac{1}{\sqrt{n!}} h_n(\sigma, k_1, \ldots, k_n) \xi \wedge \xi_{k_{\sigma(1)}} \wedge \cdots \wedge \xi_{k_{\sigma(n)}}
= \frac{1}{\sqrt{n!}} \xi \wedge \xi_{k_1} \wedge \cdots \wedge \xi_{k_n}
= A_n(\xi) \xi_{k_1} \wedge \cdots \wedge \xi_{k_n},
\]

where the second equality follows from Lemma 1.1. It now follows from Lemma 1.3 that

\[
A_n(\xi) \eta_1 \wedge \cdots \wedge \eta_n = \sqrt{n!} A_n(\xi) \eta_1 \otimes \cdots \otimes \eta_n = \xi \wedge \eta_1 \wedge \cdots \wedge \eta_n.
\]

For \( n \in \mathbb{Z}^+ \) let \( \alpha_n(\xi) = A_n(\xi) \mid_{\Lambda^n \mathcal{H}} \), so

\[
\alpha_n(\xi) : \bigwedge^n \mathcal{H} \rightarrow \bigwedge^{n+1} \mathcal{H},
\eta_1 \wedge \cdots \wedge \eta_n \mapsto \xi \wedge \eta_1 \wedge \cdots \wedge \eta_n
\]

and \( \|\alpha_n(\xi)\| \leq \sqrt{n+1} \|\xi\| \) (we will later see that this norm is independent of \( n \) when \( \xi = \xi_i \)). The adjoint of \( \alpha_n(\xi) \) will be denoted by \( \alpha_n^*(\xi) \).

**Proposition 2.1.** If \( i, j \in \mathbb{Z} \), then

\[
a_{n+1}(\xi_i) a_n(\xi_j) - (-1)^{\nu(i-j)} a_{n+1}(\xi_j) a_n(\xi_i) = 0
\]

and

\[
a_n^*(\xi_i) a_n(\xi_j) - (-1)^{\nu(i-j)} a_{n-1}(\xi_j) a_n^*(\xi_i) = (\xi_j, \xi_i) \text{id}_{\Lambda^n \mathcal{H}}.
\]

The first equation is valid for \( n \in \mathbb{Z}^+ \) and the second for \( n \in \mathbb{N} \).
Proof. It is sufficient to show that the equations are valid for an orthonormal basis for $\bigwedge^n \mathcal{H}$ (see Lemma 1.2), so let $n \in \mathbb{N}$ and $k = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$ with $k_1 < k_2 < \cdots < k_n$. Then
\[
a_{n+1}(\xi_i) a_n(\xi_j) \chi_{k_1} \wedge \cdots \wedge \chi_{k_n} = \xi_i \wedge \xi_j \wedge \chi_{k_1} \wedge \cdots \wedge \chi_{k_n}
\]
and
\[
a_{n+1}(\xi_j) a_n(\xi_i) \chi_{k_1} \wedge \cdots \wedge \chi_{k_n} = \xi_j \wedge \xi_i \wedge \chi_{k_1} \wedge \cdots \wedge \chi_{k_n}
= (-1)^{\chi_{\mathcal{E}}(i-j)} \xi_i \wedge \xi_j \wedge \chi_{k_1} \wedge \cdots \wedge \chi_{k_n},
\]
so $a_{n+1}(\xi_i) a_n(\xi_j) = (-1)^{\chi_{\mathcal{E}}(i-j)} a_{n+1}(\xi_j) a_n(\xi_i)$ (in the case $n = 0$ exchange $\chi_{k_1} \wedge \cdots \wedge \chi_{k_n}$ with $1 \in \bigwedge^0 \mathcal{H}$ in the computations above).

To show the second equation we need an expression for $a_n^*(\xi_i)$. We will use the following notation: If an element is marked with an $\tilde{}$, then this element is to be omitted, and if $i \in \mathbb{Z}$, $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$, then $\delta_{i,k_m}$ shall equal 1 if there exists an $m$ such that $i = k_m$ and 0 otherwise. If $\delta_{i,k_m} = 0$, then all expressions where $k_m$ is present are set to zero. Let $h'_n : \mathbb{Z}^n \to \{\pm 1, 0\}$ be given by
\[
h'_n(k_1, \ldots, k_n) = \begin{cases} h_n(\sigma, k_1, \ldots, k_n) & \text{if there exists } \sigma \in S_n \text{ such that } k_{\sigma(1)} < k_{\sigma(2)} < \cdots < k_{\sigma(n)}, \\ 0 & \text{otherwise.} \end{cases}
\]
Observe that $h'_n$ is well-defined since a permutation that strictly orders a given set of integers must be unique.

Let $k \in \mathbb{Z}^{n+1}$, $l \in \mathbb{Z}^n$ where $k = (k_1, k_2, \ldots, k_{n+1})$ with $k_1 < k_2 < \cdots < k_{n+1}$ and $l = (l_1, l_2, \ldots, l_n)$ with $l_1 < l_2 < \cdots < l_n$. Then
\[
(\chi_{k_1} \wedge \cdots \wedge \chi_{k_{n+1}}, a_n(\xi_i) \psi_{l_1} \wedge \cdots \wedge \psi_{l_n})
= (\chi_{k_1} \wedge \cdots \wedge \chi_{k_{n+1}}, \xi_i \wedge \psi_{l_1} \wedge \cdots \wedge \psi_{l_n})
= h'_{n+1}(i, k_1, \ldots, k_m, \ldots, k_{n+1})\begin{cases} 1 & \text{if } k_1 = l_1, k_2 = l_2, \ldots, k_m = i \\
0 & \text{if } k_{m+1} = l_m, \ldots, k_{n+1} = l_n, \text{otherwise} \end{cases}
= (\delta_{i,k_m} h'_{n+1}(i, k_1, \ldots, k_m, \ldots, k_{n+1}) \chi_{k_1} \wedge \cdots \wedge \chi_{k_m} \wedge \cdots \wedge \chi_{k_{n+1}}, \psi_{l_1} \wedge \cdots \wedge \psi_{l_n}),
\]
so if $n \in \mathbb{Z}^+$ (for $n = 0$ exchange $\psi_{l_1} \wedge \cdots \wedge \psi_{l_n}$ with 1 above), then
\[
a_n^*(\xi_i) \chi_{k_1} \wedge \cdots \wedge \chi_{k_{n+1}} = \delta_{i,k_m} h'_{n+1}(i, k_1, \ldots, k_m, \ldots, k_{n+1}) \chi_{k_1} \wedge \cdots \wedge \chi_{k_m} \wedge \cdots \wedge \chi_{k_{n+1}}.
\]
With this expression at hand, we are ready to verify the second equation in the proposition. Assume first that \( i = j \). If \( \delta_{i,k_m} = 1 \), then
\[
a_n^*(\xi_i) a_n(\xi_i) \chi_{k_1} \land \cdots \land \chi_{k_n} = 0
\]
and
\[
a_{n-1}(\xi_i) a_{n-1}^*(\xi_i) \chi_{k_1} \land \cdots \land \chi_{k_n}
\]
\[
= a_{n-1}(\xi_i) h'_n(i, k_1, \ldots, k_n) \chi_{k_1} \land \cdots \land \chi_{k_m} \land \cdots \land \chi_{k_n}
\]
\[
= h'_n(i, k_1, \ldots, k_n)^2 \chi_{k_1} \land \cdots \land \chi_{k_n}
\]
\[
= \chi_{k_1} \land \cdots \land \chi_{k_n},
\]
so \( a_n^*(\xi_i) a_n(\xi_i) + a_{n-1}(\xi_i) a_{n-1}^*(\xi_i) = \text{id}_{\bigwedge^n \mathcal{H}} \). The case \( \delta_{i,k_m} = 0 \) is similar, so assume that \( i < j \). Then
\[
a_n^*(\xi_i) a_n(\xi_i) \chi_{k_1} \land \cdots \land \chi_{k_n}
\]
\[
= a_n^*(\xi_i) h'_{n+1}(j, k_1, \ldots, k_n) \chi_{k_1} \land \cdots \land \chi_{k_m} \land \cdots \land \chi_{k_n}
\]
\[
= \delta_{i,k_m} h'_{n+1}(j, k_1, \ldots, k_n) h'_{n+1}(i, k_1, \ldots, k_m, \ldots, j, \ldots, k_n)
\]
\[
\chi_{k_1} \land \cdots \land \chi_{k_m} \land \cdots \land \chi_{k_n}
\]
and
\[
a_{n-1}(\xi_i) a_{n-1}^*(\xi_i) \chi_{k_1} \land \cdots \land \chi_{k_n}
\]
\[
= a_{n-1}(\xi_i) \delta_{i,k_m} h'_n(i, k_1, \ldots, k_m, \ldots, k_n) \chi_{k_1} \land \cdots \land \chi_{k_m} \land \cdots \land \chi_{k_n}
\]
\[
= \delta_{i,k_m} h'_n(i, k_1, \ldots, k_m, \ldots, k_n) h'_n(j, k_1, \ldots, k_m, \ldots, k_n)
\]
\[
\chi_{k_1} \land \cdots \land \chi_{k_m} \land \cdots \land \chi_{k_n}
\]
By observing that \( h'_{n+1}(j, k_1, \ldots, k_n) h'_{n+1}(i, k_1, \ldots, k_m, \ldots, j, \ldots, k_n) = (-1)^{\chi_g([i-j])} h'_n(i, k_1, \ldots, k_m, \ldots, j, \ldots, k_n) h'_n(j, k_1, \ldots, k_m, \ldots, k_n) \), we get the second equation. The case \( i > j \) is similar. \qed

3 The antisymmetric Fock space and the CAR-algebra

Let \( \mathcal{H} \) be an infinite-dimensional, separable Hilbert space. Define the Full Fock space of \( \mathcal{H} \) as
\[
EXP(\mathcal{H}) = \bigoplus_{n \geq 0} \mathcal{H}^\otimes n
\]
and the bitstream Fock space of \( \mathcal{H} \) as
\[
\mathcal{F}(\mathcal{H}) = \bigoplus_{n \geq 0} \bigwedge^n \mathcal{H},
\]
so \( \mathcal{F}(\mathcal{H}) \) can be identified with a closed subspace of \( EXP(\mathcal{H}) \). (If the bit-stream consists only of 1’s, then \( \mathcal{F}(\mathcal{H}) \) is the antisymmetric Fock space of \( \mathcal{H} \).) Proposition 2.1 gives that
\[
(a_n^*(\xi_i) a_n(\xi_i))^2 = a_n^*(\xi_i) (\id_{\bigwedge^{n+1} \mathcal{H}} - a_{n+1}^*(\xi_i) a_{n+1}(\xi_i)) a_n(\xi_i) = a_n^*(\xi_i) a_n(\xi_i),
\]
so \( a_n^*(\xi_i) a_n(\xi_i) \) is a projection (\( \neq 0 \)), hence \( ||a_n(\xi_i)|| = 1 \).

Let \( \eta \in \mathcal{F}(\mathcal{H}) \), so \( \eta = (\eta_n)_{n\in\mathbb{Z}^+} \), where \( \eta_n \in \bigwedge^n \mathcal{H} \), and for \( i \in \mathbb{Z} \), define the operators
\[
a(\xi_i), a^*(\xi_i) : \mathcal{F}(\mathcal{H}) \to \mathcal{F}(\mathcal{H}), \quad a(\xi_i) \eta = (a_n(\xi_i) \eta_n), \quad a^*(\xi_i) \eta = (a_n^*(\xi_i) \eta_{n+1}).
\]
Since \( ||a_n(\xi_i)|| \) is independent of \( n \), we get that
\[
||a(\xi_i) \eta||^2 = \sum_n ||a_n(\xi_i) \eta_n||^2 \\
\leq \sum_n ||a_n(\xi_i)||^2 ||\eta_n||^2 \\
\leq \sum_n ||\eta_n||^2 = ||\eta||^2,
\]
so \( a(\xi_i) \in \mathcal{B}(\mathcal{F}(\mathcal{H})) \). If \( \eta, \eta' \in \mathcal{F}(\mathcal{H}) \), then
\[
(a(\xi_i) \eta, \eta') = \sum_n (a_n(\xi_i) \eta_n, \eta'_{n+1}) \\
= \sum_n (\eta_n, a_n^*(\xi_i) \eta'_{n+1}) \\
= (\eta, a^*(\xi_i) \eta'),
\]
so \( a(\xi_i)^* = a^*(\xi_i) \). Furthermore, \( a(\xi_i) \) satisfies the same commutation-relations as \( a_n(\xi_i) \): Since \( a_0^*(\xi_i) a_0(\xi_i) = (\xi_i, \xi_i) \id_{\bigwedge^0 \mathcal{H}} \), it follows from Proposition 2.1 that
\[
a(\xi_i) a(\xi_i) - (-1)^{\chi_{w'((i-j))}} a(\xi_i) a(\xi_i) = 0
\]
and
\[
a^*(\xi_i) a(\xi_i) - (-1)^{\chi_{w'((i-j))}} a(\xi_i) a^*(\xi_i) = (\xi_i, \xi_i) \id_{\mathcal{F}(\mathcal{H})}.
\]
Define
\[
\text{CAR}_0(\mathcal{H}) = C^*(a(\xi_i) : i \in \mathbb{Z}),
\]
so \( \text{CAR}_0(\mathcal{H}) \) is a C*-subalgebra of \( \mathcal{B}(\mathcal{F}(\mathcal{H})) \).

Remark. If \( S = 111 \cdots \), \( \text{CAR}_0(\mathcal{H}) \) as defined here will equal the usual CAR-algebra, and in the next section we will see that \( \text{CAR}_0(\mathcal{H}) \cong \bigotimes_{i} M_2(\mathbb{C}) \) which justifies the notation. In the Fock representation construction of the CAR-algebra the canonical map \( a : \mathcal{H} \to \mathcal{B}(\mathcal{F}(\mathcal{H})) \), \( \xi \mapsto a(\xi) \) is an isometry, so \( \text{CAR}(\mathcal{H}) = C^*(a(\xi) : \xi \in \mathcal{H}) = C^*(a(\xi_i) : i \in \mathbb{Z}) \). Hence, if \( S = 111 \cdots \), \( ||a(\xi)|| = ||\xi|| \), but it is easy to see that this need not be the case when \( S \neq 111 \cdots \).
4 The AF-algebras $A$ and $B$

For $i \in \mathbb{Z}$ let $x_i = a(\xi_i)$ and $s_i = a(\xi_i) + a^*(\xi_i)$. If $i, j \in \mathbb{Z}$, then $s_i$ is a self-adjoint operator, and the commutation-relations give that

$$s_is_j = \begin{cases} (-1)^{xs(|i-j|)}s_js_i & \text{if } i \neq j, \\ \id_{\mathcal{H}} & \text{if } i = j, \end{cases}$$

so $\{s_i\}_{i \in \mathbb{Z}}$ is a family of self-adjoint, unitary operators satisfying $s_is_j = (-1)^{xs(|i-j|)}s_js_i$.

For notational reasons the Hilbert spaces will from now on have an orthonormal basis indexed over $\mathbb{N}$ instead of over $\mathbb{Z}$. The results below have corresponding proofs for $\mathbb{N}$ and $\mathbb{Z}$, and where there are differences they will be commented on. So for $i \in \mathbb{N}$, $x_i = a(\xi_i)$ and $s_i = x_i + x_i^*$.

For $n \in \mathbb{N}$, define

$$A_n = C^* (x_i : 1 \leq i \leq n), \quad A = C^* (x_i : i \in \mathbb{N})$$

and

$$B_n = C^* (s_i : 1 \leq i \leq n), \quad B = C^* (s_i : i \in \mathbb{N}),$$

so $A = CAR_0(\mathcal{H})$. Observe that $A$ and $B$ are AF-algebras with $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} B_n$. (If the index set is $\mathbb{Z}$, let $A_n = C^* (x_i : \frac{1-n}{2} \leq i \leq \frac{n}{2})$ and $A = C^* (x_i : i \in \mathbb{Z})$ etc.)

The commutation-relations will now read:

$$x_i^2 = 0, \quad x_ix_j = (-1)^{xs(|i-j|)}x_jx_i \quad (i \neq j)$$

and

$$x_i^*x_i + x_ix_i^* = 1, \quad x_i^*x_j = (-1)^{xs(|i-j|)}x_jx_i^* \quad (i \neq j).$$

**Proposition 4.1.** If $n \in \mathbb{N}$, then $A_n \cong M_{2^n}(\mathbb{C})$.

**Proof.** For $1 \leq i \leq n$ let

$$u_i = \prod_{j} (x_jx_j^* - x_j^*x_j),$$

where the product is taken over those $j < i$ which satisfy $x_ix_j = -x_jx_i$ (if no such $j$'s exist, set $u_i = 1$). Since

$$(x_jx_j^* - x_j^*x_j)^2 = x_jx_j^*x_jx_j^* + x_jx_j^*x_jx_j$$

$$= x_j(1 - x_jx_j^*)x_j^* + x_j^*(1 - x_j^*x_j)x_j = 1,$$
then
\[(x_j^* x_j - x_j^* x_j)(x_k^* x_k - x_k^* x_k) = \begin{cases} (x_k^* x_k - x_k^* x_k)(x_j^* x_j - x_j^* x_j) & \text{if } j \neq k, \\ 1 & \text{if } j = k. \end{cases}\]

From this it follows that each \(u_i\) is a self-adjoint unitary operator and that \(u_i^* u_j = u_j^* u_i\).

We will use the \(u_i\)'s to make \(n\) pairwise commuting systems of \(2 \times 2\) matrix units, so for \(i = 1, \ldots, n\) define
\[e_{11}^{(i)} = x_i^* x_i \quad e_{12}^{(i)} = u_i x_i^* \quad e_{21}^{(i)} = x_i u_i \quad e_{22}^{(i)} = x_i^* x_i.\]

It is easy to check that if \(i \in \{1, \ldots, n\}\), then \((e_{jk}^{(i)})_{j,k=1}^{2^i}\) is a system of \(2 \times 2\) matrix units. If \(j < i\), then
\[u_i x_j = (-1)^{x_{(i-j)}} x_j u_i\]
and
\[u_i x_j^* = (-1)^{x_{(i-j)}} x_j^* u_i\]
(since \((x_j x_j^* - x_j^* x_j)x_j = x_j x_j^* x_j = -x_j x_j^* x_j\) and \((x_j x_j^* - x_j^* x_j)x_j^* = -x_j^* x_j x_j^* = x_j^* (x_j x_j^* - x_j x_j^*)\), so the systems of matrix units commute pairwise. This allows us to, for \(i = 1, \ldots, n\), inductively make a system of \(2^i \times 2^i\) matrix units (by “taking the tensor product” of the matrix units).

Assume inductively that if \(i \in \{1, \ldots, n-1\}\), then \((e_{jk}^{(j)})_{j,k=1}^{2^j}\) is a system of \(2^j \times 2^j\) matrix units (which commutes with \((e_{jk}^{(j+1)})_{j,k=1}^{2^j}\)) and that \(x_1, \ldots, x_i, x_i^*, \ldots, x_i^* \in \text{lin}\{e_{jk}^{(j)}\}_{j,k=1}^{2^j}\). For \(j, k = 1, \ldots, 2^i\) define
\[f_{jk} = e_{jk}^{(i+1)} \quad f_{j(k+2^i)} = e_{jk}^{(i+1)}\]
\[f_{(j+2^i)k} = e_{jk}^{(i+1)} \quad f_{(j+2^i)(k+2^i)} = e_{jk}^{(i+1)}\].

It is easy to check that \((f_{jk})_{j,k=1}^{2^{i+1}}\) is a system of \(2^{i+1} \times 2^{i+1}\) matrix units (which commutes with \((e_{jk}^{(i+2)})_{j,k=1}^{2^{i+1}}\) if \(i < n-1\)). Since \(u_{i+1} \in \text{lin}\{f_{jk}\}_{j,k=1}^{2^{i+1}}\), it is also true that \(x_{i+1}, x_{i+1}^* \in \text{lin}\{f_{jk}\}_{j,k=1}^{2^{i+1}}\). By induction, there exists a complete system of \(2^n \times 2^n\) matrix units, hence \(A_n \cong M_{2^n}(\mathbb{C})\).

**Theorem 4.2.** If \(\mathcal{H}\) is an infinite-dimensional, separable Hilbert space, then
\[\text{CAR}_0(\mathcal{H}) \cong \bigotimes_1^{\infty} M_2(\mathbb{C}).\]
Proof. Let \((\xi_i)_{i\in\mathbb{N}}\) be an orthonormal basis for \(\mathcal{H}\). By Proposition 4.1
\[
CAR_0(\mathcal{H}) = C^*(a(\xi_i) : i \in \mathbb{N})
\]
\[
= \bigcup_{n=1}^{\infty} C^*(a(\xi_i) : 1 \leq i \leq n)
\]
\[
= \bigcup_{n=1}^{\infty} A_n \cong \bigotimes_1^{\infty} M_2(\mathbb{C}).
\]
\(\square\)

5 Description of \(B_n\) and \(B'_n\)

By the isomorphism in the proof of Proposition 4.1, \(B_n \subset A_n\) is identified with a subalgebra of \(M_{2^n}(\mathbb{C})\). By abuse of notation we will also denote this algebra by \(B_n\), and the elements in \(M_{2^n}(\mathbb{C})\) corresponding to \(s_i\) an \(u_i\) \((1 \leq i \leq n)\) by \(s_i\) and \(u_i\), respectively. Since \(B_n\) is generated by \(\{s_i\}_{i=1}^n\) and the \(s_i\)'s either commute or anticommute, it is clear that
\[
B_n = \operatorname{lin}\{s_{i_1} \cdots s_{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\},
\]
hence \(B_n\) is a \(2^n\)-dimensional subalgebra of \(M_{2^n}(\mathbb{C})\). In what follows we will give a closer description of this subalgebra.

Let \(a, b, c, d \in M_2(\mathbb{C})\) be given by
\[
a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]
and let \((a_i)_{i\in\mathbb{N}} \subset M_2(\mathbb{C})\) be defined by \(a_i = \begin{cases} a & \text{if } \chi_S(i) = 0, \\ c & \text{if } \chi_S(i) = 1. \end{cases}\)

For \(n \in \mathbb{N}\) the isomorphism in the proof of Proposition 4.1 gives that in \(M_{2^n}(\mathbb{C})\) we have that
\[
s_1 = b \otimes a \otimes \cdots \otimes a \otimes a \cdot u_1 \quad (n - 1 \text{ a's})
\]
\[
s_2 = a \otimes b \otimes \cdots \otimes a \otimes a \cdot u_2
\]
\[
\vdots
\]
\[
s_n = a \otimes a \otimes \cdots \otimes a \otimes b \cdot u_n,
\]
where
\[
u_1 = a \otimes a \otimes a \otimes \cdots \otimes a \otimes a
\]
\[
u_2 = a_1 \otimes a \otimes a \otimes \cdots \otimes a \otimes a
\]
\[
u_3 = a_2 \otimes a_1 \otimes a \otimes \cdots \otimes a \otimes a
\]
\[
\vdots
\]
\[
u_n = a_{n-1} \otimes a_{n-2} \otimes a_{n-3} \otimes \cdots \otimes a_1 \otimes a.
\]
Hence, if $1 \leq i \leq n$, then $s_i = a_{i-1} \otimes a_{i-2} \otimes \cdots \otimes a_1 \otimes b \otimes a \otimes \cdots \otimes a$.

**Remark.** If $1 \leq i < j \leq n$, then

$$s_is_j = a_{i-1} \otimes \cdots \otimes a_1 \otimes b \otimes a \otimes \cdots \otimes a \otimes a_{j-1} \otimes a_{j-i+1} \otimes a_{j-i} \otimes a_{j-i-1} \otimes \cdots \otimes b \otimes \cdots \otimes a$$

$$= (-1)^{(j-i)(j-i-1)} s_{j} s_{i},$$

because $b a_{j-i} = (-1)^{(j-i)(j-i-1)} a_{j-i} b$ and the rest of the terms commute.

If $B'_n$ denotes the commutant of $B_n$ in $M_{2^n}(\mathbb{C})$, define the map $\kappa$ to be linear and

$$\kappa : B_n \to B'_n, \quad s_i \mapsto \bar{s}_i,$$

where $\bar{s}_i$ means $s_i$ “read backwards”, i.e. $\bar{s}_i = a \otimes \cdots \otimes a \otimes b \otimes a_1 \otimes \cdots \otimes a_{i-2} \otimes a_{i-1}$.

Let $i, j$ be such that $i + j > n + 1$, and set $k = n - i$. Then

$$\bar{s}_i s_j = a \otimes \cdots \otimes a \otimes b \otimes a_1 \otimes \cdots \otimes a_{j-1-k} \otimes \cdots \otimes a_{i-1}$$

$$a_{j-1} \otimes \cdots \otimes a_j \otimes a_{j-2-k} \otimes \cdots \otimes b \otimes \cdots \otimes a$$

$$= s_j \bar{s}_i.$$

Since it is clear that $\bar{s}_i$ and $s_j$ commute when $i + j \leq n + 1$, $\kappa$ really maps $B_n$ into $B'_n$. It is clear that $\kappa$ is an injective $*$-homomorphism, so $\dim B'_n \geq \dim B_n$. We will soon see that $\dim B'_n = \dim B_n$, from which it follows that $\kappa$ is surjective.

### 6 The state $\omega_\Omega$

Let $\Omega$ denote the vector $1 \in \bigwedge^0 \mathcal{H} \subset \mathcal{F}(\mathcal{H})$, and for $n \in \mathbb{N}$ define

$$\mathcal{K}_n = B_n \Omega.$$

Since $\{\Omega, s_{i_1} \cdots s_{i_k} \Omega : 1 \leq i_1 < \cdots < i_k \leq n\}$ is an orthonormal basis for $\mathcal{K}_n$, it follows that $\mathcal{K}_n$ is a $2^n$-dimensional subspace of $\mathcal{F}(\mathcal{H})$.

**Remark.** $\mathcal{K}_n = A_n \Omega$, because if $1 \leq i_1 < \cdots < i_k \leq n$ and $1 \leq i \leq n$, then

$$x_{i_1} \cdots x_{i_k} \Omega = s_{i_1} \cdots s_{i_k} \Omega, \quad x_i^* \Omega = 0,$$

and $x_i^* (x_{i_1} \cdots x_{i_k} \Omega) = \pm x_{i_1} \cdots x_{i_m} \cdots x_{i_k} \Omega$ if there exists $m \in \{1, \ldots, k\}$ such that $i = i_m$, and 0 otherwise.

**Lemma 6.1.**

$$\bigcup_{n=1}^{\infty} \mathcal{K}_n = \mathcal{F}(\mathcal{H}).$$
Proof. Let \( \epsilon > 0 \) and \( \eta \in \mathcal{F}(\mathcal{H}) \), so \( \eta = (\eta_n)_{n \in \mathbb{Z}^+} \), where \( \eta_n \in \bigwedge^n \mathcal{H} \). Then there exists \( \eta' = (\eta'_n)_{n \in \mathbb{Z}^+} \in \mathcal{F}(\mathcal{H}) \) and \( N \in \mathbb{Z}^+ \) such that \( \| \eta - \eta' \| < \frac{\epsilon}{2} \) and \( \eta'_n = 0 \) for \( n > N \). Since \( s_{k_1} \cdots s_{k_n} \Omega = \xi_{k_1} \wedge \cdots \wedge \xi_{k_n} \) if \( 1 \leq k_1 < k_2 < \cdots < k_n \) (\( k_1 < k_2 < \cdots < k_n \) if the index set is \( \mathbb{Z} \)), and these vectors constitute a basis for \( \bigwedge^n \mathcal{H} \), then for each \( i \in \{0, \ldots, N\} \) there exists \( n_i \in \mathbb{N} \) and \( b^{(i)} \in \mathcal{B}_{n_i} \) with \( b^{(i)}_j = 0 \) for \( j \neq i \) such that \( \| \eta'_i - b^{(i)} \Omega \| < \frac{\epsilon}{2\sqrt{N+1}} \). Then \( \sum_{i=0}^N b^{(i)} \Omega \in \mathcal{K}_m \), where \( m = \max_i n_i \) and \( \| \eta' - \sum_{i=0}^N b^{(i)} \Omega \|^2 = \sum_{i=0}^N \| \eta'_i - b^{(i)} \Omega \|^2 < \sum_{i=0}^N \frac{\epsilon^2}{4} = \frac{\epsilon^2}{4} \). Hence \( \| \eta - \sum_{i=0}^N b^{(i)} \Omega \| \leq \| \eta - \eta' \| + \| \eta' - \sum_{i=0}^N b^{(i)} \Omega \| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \), so \( \eta \in \bigcup_{n=1}^\infty \mathcal{K}_n \). ■

Corollary 6.2. \( \Omega \) is cyclic for \( \mathcal{B} \) (and hence for \( \mathcal{A} \)).

Proof. \( \mathcal{B}\Omega = \bigcup_{n=1}^\infty \mathcal{B}_n \Omega = \mathcal{F}(\mathcal{H}) \). ■

Lemma 6.3. If \( n \in \mathbb{N} \), then \( \Omega \) is separating for \( \mathcal{B}_n \) on \( \mathcal{K}_n \).

Proof. Since \( \Omega \) is cyclic for \( \mathcal{B}_n \) on \( \mathcal{K}_n \), \( \Omega \) is separating for \( \mathcal{B}'_n \). This gives that the linear map \( \mathcal{B}'_n \to \mathcal{K}_n \), \( b \mapsto b \Omega \) is injective, so \( \dim \mathcal{B}'_n = \dim \mathcal{B}'_n \Omega \). Hence \( \dim \mathcal{K}_n = \dim \mathcal{A}_n \Omega = \dim \mathcal{B}'_n \Omega = \dim \mathcal{B}'_n \geq \dim \mathcal{B}_n = \dim \mathcal{K}_n \), where the last inequality follows from that \( \kappa \) is injective. Since \( \mathcal{K}_n \) is finite dimensional, this gives that \( \mathcal{B}'_n \Omega = \mathcal{K}_n \), so \( \Omega \) is cyclic for \( \mathcal{B}'_n \) on \( \mathcal{K}_n \), hence \( \Omega \) is separating for \( \mathcal{B}''_n = \mathcal{B}_n \). ■

Corollary 6.4. \( \kappa \) is a *-isomorphism.

Proof. The only thing left to prove is that \( \kappa \) is surjective, but that follows from the inequalities in the previous lemma. ■

Define the state \( \omega_{\Omega} \) on \( \mathcal{A} \) by \( \omega_{\Omega}(x) = (x \Omega, \Omega) \). Recall that by Proposition 4.1 \( \mathcal{A}_n \cong M_{2^n}(C) \), so \( \omega_{\Omega} \mid \mathcal{A}_n \) can be regarded as a state on the full matrix algebra \( M_{2^n}(C) \).

Proposition 6.5. \( \omega_{\Omega} \) is a pure state on \( \mathcal{A} \). Moreover, if \( n \in \mathbb{N} \) and \( \text{Tr} \) is the usual trace on \( M_{2^n}(C) \) with \( \text{Tr}(1) = 2^n \), then \( \omega_{\Omega} \mid \mathcal{A}_n = \phi_n \), where

\[
\phi_n : M_{2^n}(C) \to C, \ x \mapsto \text{Tr} \left( \bigotimes_{i=1}^n \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) \cdot x.
\]

Proof. It is enough to show that if \( n \in \mathbb{N} \), then \( \omega_{\Omega} \mid \mathcal{A}_n \) is a pure state on \( \mathcal{A}_n \), because \( \mathcal{A} = \bigcup_{n=1}^\infty \mathcal{A}_n \). By the commutation-relations we have

\[
A_n = \text{lin}\{1, x_{j_1}^* \cdots x_{j_s}^* x_{i_1} \cdots x_{i_r} : 1 \leq i_1 < i_2 < \cdots < i_r \leq n \text{ and } 1 \leq j_1 < j_2 < \cdots < j_s \leq n \text{ (either } r \text{ or } s \text{ can be } 0) \}.
\]
We evaluate $\omega_1$ on the basisvectors:

$$
\omega_1(x^*_j \cdots x^*_i x_i \cdots x_r) = (x_{i_1} \cdots x_{i_r}, \Omega, x_{j_1} \cdots x_{j_s}, \Omega)
= (\xi_{i_1} \wedge \cdots \wedge \xi_{i_r}, \xi_{j_1} \wedge \cdots \wedge \xi_{j_s})
= \begin{cases} 
1 & \text{if } i_1 = j_1, i_2 = j_2, \ldots, i_r = j_s, \\
0 & \text{otherwise.}
\end{cases}
$$

Let $\phi_n$ be defined as above. Since $\otimes^n_1 (1 \ 0)$ is a 1-dimensional projection, $\phi_n$ is a pure state on $M_{2^n}(\mathbb{C})$. We will show that if $n \in \mathbb{N}$, then $\omega_n|_{A_n} = \phi_n$, and to do so we inductively use the systems of matrix units from the proof of Proposition 4.1:

$$
\phi_1(x^*_1 x_1) = \text{Tr} \left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} e^{(1)}_{11} \right) = 1, \quad \phi_1(x^*_1) = \text{Tr} \left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} e^{(1)}_{12} \right) = 0
$$

and $\phi_1(x_1) = \text{Tr} \left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} e^{(1)}_{21} \right) = 0$,

so $\omega_n|_{A_1} = \phi_1$. Assume that $\omega_n|_{A_{n-1}} = \phi_{n-1}$ for an $n \geq 2$. We evaluate $\phi_n$ on the basisvector $x = x^*_j \cdots x^*_i x_i \cdots x_r$, where $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ and $1 \leq j_1 < j_2 < \cdots < j_s \leq n$ (either $r$ or $s$ can be 0). Assume first that $i_r, j_s \neq n$. Since $x \in A_{n-1} \cong M_{2^{n-1}}(\mathbb{C})$, then $x$ is identified with $x^*_j \cdots x^*_i x_i \cdots x_r \otimes (1 \ 0) \in M_{2^n}(\mathbb{C})$, so

$$
\phi_n(x) = \text{Tr} \left( \bigotimes^n_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x^*_j \cdots x^*_i x_i \cdots x_r \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)
= \text{Tr} \left( \bigotimes^n_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x^*_j \cdots x^*_i x_i \cdots x_r \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) \text{Tr} \left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) = \phi_{n-1}(x).
$$

If $i_r = j_s = n$, observe that $\otimes^n_1 (1 \ 0) = \prod^n_{i=1} x^*_i x_i$, so $x_n \otimes^n_1 (1 \ 0) x^*_n = (\prod^n_{i=1} x^*_i x_i) x_n x^*_n = (\prod^n_{i=1} x^*_i x_i) x_n x^*_n = \otimes^n_{i=1} (1 \ 0) \otimes (0 \ 0)$, so

$$
\phi_n(x) = \text{Tr} \left( \bigotimes^n_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x^*_n \cdots x^*_i x_i \cdots x_n \right)
= \text{Tr} \left( \bigotimes^n_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) x^*_n \cdots x^*_i x_i \cdots x_{i-1}
= \phi_{n-1}(x^*_j \cdots x^*_i x_i \cdots x_{i-1}).
$$
Assume at last that $j_s = n, i_r \neq n$ (the case $j_s \neq n, i_r = n$ is similar).
Observe that $x_n^* = e_\|_{12}^n u_n = \left( \bigotimes_{1}^{n-1} \frac{1}{0} \right) \otimes \left( \frac{1}{0} \frac{0}{0} \right) u_n$, so

$$\phi_n(x) = \text{Tr} \left( \bigotimes_{1}^{n} \frac{1}{0} \frac{0}{0} \right) x_n^* x_{j_s-1}^* \cdots x_{i_1}^* x_{i_r} \cdots x_{i_r}$$

$$= \text{Tr} \left( \bigotimes_{1}^{n} \frac{1}{0} \frac{0}{0} \right) \left( \bigotimes_{1}^{n-1} \frac{1}{0} \frac{0}{0} \right) \otimes \left( \frac{0}{1} \frac{0}{0} \right) u_n x_{j_s-1}^* x_{j_1}^* \cdots x_{i_1}^* \cdots x_{i_r}$$

$$= \text{Tr} \left( \bigotimes_{1}^{n-1} \frac{1}{0} \frac{0}{0} \right) u_n x_{j_s-1}^* x_{j_1}^* \cdots x_{i_1}^* \cdots x_{i_r} \text{Tr} \left( \frac{0}{1} \frac{0}{0} \right) = 0$$

From all this we get that $\phi_n(x_n^* x_{j_s-1}^* x_{j_1}^* \cdots x_{i_1}^* \cdots x_{i_r}) = \omega_{\Omega}(x_n^* x_{j_s-1}^* x_{j_1}^* \cdots x_{i_1}^* \cdots x_{i_r})$, so the result follows by induction. \[ \square \]

**Corollary 6.6.** $A$ acts irreducibly on $\mathcal{F}(\mathcal{H})$.

**Proof.** Since $\Omega \in \mathcal{F}(\mathcal{H})$ is a cyclic unit vector which is such that $\omega_{\Omega}(x) = (x \Omega, \Omega)$, the triple $(\text{id}_A, \mathcal{F}(\mathcal{H}), \Omega)$ satisfies the conditions in the GNS-construction. Since $\omega_{\Omega}$ is a pure state on $A$, $\text{id}_A$ is an irreducible representation, and the corollary follows. \[ \square \]

### 7 The center of $B_n$

We start this section with some definitions and explain the notation that will be used.

A word in $B_n$ is an element of the form $w = s_{1}^{k_1} s_2^{k_2} \cdots s_n^{k_n}$, where the vector $(k_1, k_2, \ldots, k_n) \in \mathbb{F}_2^n$. If at least one of the $k_i$'s is 1, the word $w$ is called non-trivial, and if all the $k_i$'s are 0, we define $w$ to be 1 (the trivial word). If $w$ occurs (e.g. in a computation) where the sign is not important, i.e. we can replace $w$ by $-w$ without affecting the result, we do so if it is convenient. If $C \subset B_n$ is a family of words, then $C$ is called an independent family (of words) if none of the words in $C$ can be written as a product (up to a sign) of the other words in $C$. Note that if $C$ is an independent family, then $1 \in C$ implies $C = \{1\}$. If $C = \{w_i\}_{i=1}^{m} \neq \{1\}$ and $w_i = s_{1}^{k_1} \cdots s_n^{k_n}$, then $C$ is an independent family if and only if $\{(k_1^{(i)}, \ldots, k_n^{(i)})\}_{i=1}^{m}$ is a set of linearly independent vectors in $\mathbb{F}_2^n$.

If $k = (k_1, \ldots, k_n)$ is a vector in $\mathbb{F}_2^n$, then let $\bar{k}$ denote the reversed vector $(k_n, k_{n-1}, \ldots, k_1) \in \mathbb{F}_2^n$.

A bitstream $S \subset \mathbb{N}$ is called mirror-periodic if the sequence $(\chi_S(|n|))_{n \in \mathbb{Z}}$ is periodic. Observe that $S$ is mirror-periodic if and only if there exists $m \in \mathbb{N}$ such that $\chi_S(j) = \chi_S(|m - j|) \, \forall j \in \mathbb{Z}^+$.

**Lemma 7.1.** If $n \in \mathbb{N}$, then $\omega_{\Omega}|_{B_n} = \text{tr}|_{B_n}$, where $\text{tr}$ is the normalized trace on $M_{2n}(\mathbb{C})$. Moreover, if $w$ is a non-trivial word in $B_n$, then $\omega_{\Omega}(w) = 0$.  

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Proof. Since $B_n$ is generated by the words it contains (see section 5), it is enough to check the equality for these elements. It is clear that $\omega(1) = \text{tr}(1)$, so let $w = s_1^{k_1} \cdots s_n^{k_n}$ be a non-trivial word in $B_n$. Let $j$ be such that $k_j = 1$ and $k_i = 0$ for $i > j$. Since $w = s_1^{k_1} \cdots s_{j-1}^{k_{j-1}} a_j s_{j-1}^{k_{j-1}} \cdots b_1 \otimes b_2 \otimes \cdots \otimes b_{j-1} \otimes b \otimes a \otimes \cdots \otimes a$, where $b_1, b_2, \ldots, b_{j-1} \in M_2(\mathbb{C})$, it follows that

$$\text{tr}(w) = \text{tr}(b_1) \text{tr}(b_2) \cdots \text{tr}(b_{j-1}) \text{tr}(b) \text{tr}(a) \cdots \text{tr}(a) = 0.$$ 

From Proposition 6.5 we get that

$$\omega(w) = \text{Tr}\left( \bigotimes_{i=1}^n \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) b_1 \otimes b_2 \otimes \cdots \otimes b_{j-1} \otimes b \otimes a \otimes \cdots \otimes a = 0.$$

The next lemma can be found in [PP]. We include the proof for completeness.

**Lemma 7.2.** Denote by $Z(B_n)$ the center of $B_n$. Every element in $Z(B_n)$ can be written as a linear combination of words in $Z(B_n)$.

**Proof.** Let $v \in Z(B_n)$. We may assume that $v \neq 0$, so there exists a family of words $\{w_i\}_{i=1}^m$ such that $v = \sum_i \lambda_i w_i$, where each $\lambda_i \neq 0$. Assume there exists a $j$ such that $w_j \notin Z(B_n)$, i.e. there exists a word $w \in B_n$ such that $ww_j = -w_j w$. Then $v = w^*vw = \sum_i \lambda_i w^* w_i w = \sum_i \pm \lambda_i w_i$ with minus for $i = j$, so $\lambda_j = -\lambda_j$, hence $\lambda_j = 0$. 

**Lemma 7.3.** Let $D_n(\mathbb{C})$ denote the subalgebra of $M_n(\mathbb{C})$ consisting of the diagonal matrices. If $n \in \mathbb{N}$, then there exist $c_n, d_n \in \mathbb{Z}^+$ such that

$$B_n \cong M_{2^n}(\mathbb{C}) \otimes D_{2^n}(\mathbb{C}).$$

**Proof.** If $Z(B_n) = C1$, then $B_n$ is isomorphic to a full matrix algebra, so in this case $n$ must be even since $\dim B_n = 2^n$. Hence $c_n = 0$ and $d_n = \frac{n}{2}$. If $Z(B_n) \neq C1$, then Lemma 7.2 gives that there exists an independent family of words, $C = \{w_j\}_{j=1}^m$, generating $Z(B_n)$. For $j = 1, \ldots, m$ define

$$q_j^+ = \frac{1}{2}(1 + \lambda_j w_j) \quad \text{and} \quad q_j^- = \frac{1}{2}(1 - \lambda_j w_j),$$

where $\lambda_j \in \mathbb{C}$ equals 1 if $w_j$ is self-adjoint and 0 otherwise (if $w_j$ is not self-adjoint, then $w_j^* = -w_j$). This gives rise to $2^m$ orthogonal central-projections of the form $q_1 q_2 \cdots q_m$, where each $q_i$ is either $q_j^+$ or $q_j^-$, so let $\{p_i\}_{i=1}^{2^m}$ be the family consisting of these projections. Since $Z(B_n)$ is a $2^n$-dimensional commutative algebra, it follows that the $p_i$'s are the minimal
projections in $Z(B_n)$. Hence, to each $p_i$ there exists a corresponding $n_i \in \mathbb{N}$ such that

$$B_n = \bigoplus_{i=1}^{2^m} M_{n_i}(C) \otimes C_{\mu_i} \subset M_{2^n}(C),$$

where $\mu_i \in \mathbb{N}$ is the multiplicity of the representation of $p_i B_n$ ($\cong M_{n_i}(C)$) in $B(p_i, C^{2^n})$ and $C_{\mu_i} = C1 \subset M_{\mu_i}(C)$. The commutant of $\bigoplus_{i=1}^{2^m} M_{n_i}(C) \otimes C_{\mu_i}$ is $\bigoplus_{i=1}^{2^m} C_{n_i} \otimes M_{\mu_i}(C)$, so the fact that $B_n$ is isomorphic to its commutant (Corollary 6.4) gives that $\bigotimes_{i=1}^{2^m} C_{n_i} \otimes M_{\mu_i}(C)$ has dimension $2^n$, hence

$$\sum_{i=1}^{2^m} \mu_i^2 = 2^n.$$

Since $B_n$ contains $1 \in M_{2^n}(C)$, we get that $\sum_{i=1}^{2^m} \mu_i n_i = 2^n$, and by calculating the dimension of $B_n$ we also get that $\sum_{i=1}^{2^m} n_i^2 = 2^n$. Moreover, if $p_i$ is a minimal projection in $Z(B_n)$, then $p_i = q_1 \ldots q_m = \frac{1}{2}(1 \pm \lambda_1 w_1) \ldots \frac{1}{2}(1 \pm \lambda_m w_m)$, which equals $\frac{1}{2^m}1$ plus a linear combination of products of the form $\prod_{i \in I} w_i$, where $I \subset \{1, \ldots, m\}$. C is an independent family, Lemma 7.1 implies that $\text{tr} \left( \prod_{i \in I} w_i \right) = 0$ for each $I \subset \{1, \ldots, m\}$, so $\text{tr} (p_i) = \frac{1}{2^m}$. This yields that the product $\mu_i n_i$ is independent of $i$, so $\sum_{i=1}^{2^m} \mu_i n_i = 2^n \mu_i n_i = 2^n$. Thus $2^n = \sum_{i=1}^{2^m} \frac{1}{n_i^2}$, so $\sum_{i=1}^{2^m} \frac{1}{n_i^2} = 2^{2m-n}$. Now the Cauchy-Schwarz inequality gives us that

$$2^{2m} = \left( \sum_{i=1}^{2^m} \frac{n_i - 1}{n_i} \right)^2 \leq \left( \sum_{i=1}^{2^m} \frac{1}{n_i} \right)^2 \left( \sum_{i=1}^{2^m} \frac{1}{n_i^2} \right) = 2^n \cdot 2^{2m-n} = 2^{2m}.$$

Equality in the Cauchy-Schwarz inequality implies that there exists a constant $c$ such that $(n_1, \ldots, n_{2^m}) = c(\frac{1}{n_1}, \ldots, \frac{1}{n_{2^m}})$, so all the $n_i$'s are equal. It follows that $c_n = m$ and $d_n = \frac{2^{2m} - m}{2^m}$, which concludes the proof.

\[\square\]

**Remark 1.** It follows from the proof of this lemma that if $n \in \mathbb{N}$, then there exists an independent family $C = \{w_i\}_{i=1}^{c_n}$ when $c_n \neq 0$ and $C = \{1\}$ when $c_n = 0$ which generates $Z(B_n)$, and it is clear that any independent family in $Z(B_n)$ consisting of $c_n$ words, generates $Z(B_n)$.

**Remark 2.** If $J : x \Omega \rightarrow x^* \Omega$, then $J p_i J = p_i$ and $J B_n J = B_n'$, see [Dix, Ch. I §5]. From this it follows that $\mu_i = n_i$ above.

**Remark 3.** For a shorter and quite different proof of this result (which also includes Lemma 7.5) see [EN].

**Corollary 7.4.** If $n \in \mathbb{N}$, then $n = 2d_n + c_n$.

**Proof.** $2^n = \dim B_n = \dim (M_{2^{d_n}}(C) \otimes D_{2^{c_n}}(C)) = (2^{d_n})^2 \cdot 2^{c_n}$.

\[\square\]

In what follows we will describe the behaviour of the sequence $(c_n)_{n=1}^{\infty}$ in order to understand the AF-algebra $B = \bigcup_{n=1}^{\infty} B_n$.

**Lemma 7.5.** If $n \in \mathbb{N}$, then $c_{n+1} = c_n \pm 1$. Moreover $c_{n+1} = c_n + 1 \iff d_{n+1} = d_n$. 

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Proof. Let \( n \in \mathbb{N} \). It follows from Lemma 7.3 that \( d_{n+1} \geq d_n \), so assume first that \( d_{n+1} = d_n \). Then Corollary 7.4 gives that \( 2d_{n+1} + c_{n+1} = 2d_n + c_n + 1 \), so \( c_{n+1} = c_n + 1 \) (this calculation also gives the reverse implication). If \( d_{n+1} > d_n \), then the same corollary gives that \( 2 \leq 2(d_{n+1} - d_n) = n + 1 - c_{n+1} - (n - c_n) \), so \( c_{n+1} \leq c_n - 1 \). From this it follows that \( c_n > 1 \), so there exists an independent family of words, \( C = \{ w_i \}_{i=1}^{c_n} \) that generates \( Z(B_n) \). Since \( c_{n+1} = c_n - 1 \), there exists \( j \in \{ 1, \ldots, c_n \} \) such that \( w_j \notin Z(B_{n+1}) \). If \( w_i \in C \setminus Z(B_{n+1}) \) (\( i \neq j \)), this means that \( w_i \) anticommutes with \( s_{n+1} \), and since the same is true for \( w_j \) it follows that \( w_j w_i \in Z(B_n) \). Hence, by replacing the words \( w_j \in C \setminus Z(B_{n+1}) \) (\( i \neq j \)) with \( w_j w_i \), we get that \( c_{n+1} \geq c_n - 1 \). (It is clear that the family obtained by this replacement is independent.)

Lemma 7.6. \( c_{n+1} = c_n + 1 \iff Z(B_n) \subset Z(B_{n+1}) \).

Proof. Assume that \( c_{n+1} = c_n + 1 \). If \( c_n = 0 \), the implication is trivial, so we may assume that \( c_n > 1 \). Let \( C = \{ w_i \}_{i=1}^{c_n} \) be an independent family of words generating \( Z(B_n) \), and assume that \( Z(B_n) \notin Z(B_{n+1}) \). The same argument as in the proof of Lemma 7.5 gives that \( w_i \notin Z(B_{n+1}) \) and \( \{ w_i \}_{i=1}^{c_n-1} \subset Z(B_{n+1}) \) (by modifying and rearranging some of the \( w_i \)’s if necessary). Now, since \( c_{n+1} = c_n + 1 \), there must exist two words \( w', w'' \in Z(B_{n+1}) \) such that \( \{ w', w'' \} \cup \{ w_i \}_{i=1}^{c_n-1} \) is an independent family of words generating \( Z(B_{n+1}) \). This, however, is impossible, since then at least one of the words \( w', w'' \) does not contain \( s_{n+1} \), i.e. is contained in \( Z(B_n) \). This violates the fact that \( C \) generates \( Z(B_n) \). The reverse implication is immediate from Lemma 7.5.

Lemma 7.7. If \( s_1^{k_1} s_2^{k_2} \cdots s_n^{k_n} \) is a word in \( B_n \), then \( s_1^{k_1} s_2^{k_2} \cdots s_n^{k_n} \in Z(B_n) \) if and only if \( s_1^{k_1} s_2^{k_2-1} \cdots s_n^{k_n} \in Z(B_n) \).

Proof. By symmetry it is enough to show one of the implications. Let \( w = s_1^{k_1} \cdots s_n^{k_n} \) be a word in \( Z(B_n) \) and \( 1 \leq i \leq n \). Observe that

\[
s_i s_1^{k_1} s_2^{k_2-1} \cdots s_n^{k_n-1} = \left( \prod_{j=1}^{n} (-1)^{x_{s_j}[i-j]} s_{n+1-j} \right) s_1^{k_1} s_2^{k_2-1} \cdots s_n^{k_n-1},
\]

so we must show that \( \prod_{j=1}^{n} (-1)^{x_{s_j}[i-j]} s_{n+1-j} = 1 \). Since \( w \in Z(B_n) \), then

\[
w s_{n+1-i} = s_{n+1-i} w = \left( \prod_{j=1}^{n} (-1)^{x_{s_j}[n+1-i-j]} k_j \right) w s_{n+1-i},
\]

hence \( \prod_{j=1}^{n} (-1)^{x_{s_j}[n+1-i-j]} k_j = 1 \). Now the lemma follows by the substitution \( k = n + 1 - j \).
The next proposition describes the sequence \((c_n)_{n=1}^\infty\) completely. To make the notation easier we define \(B_0 = C1\), so \(c_0 = 0\). Since \(c_1 = 1\), Lemma 7.5 is also valid for \(n = 0\).

**Proposition 7.8.** Let \(S \subseteq \mathbb{N}\) be a given bitstream. Then there exists a strictly increasing sequence \((n_r)_{r \in I}\) of even integers, where \(I = \{1, 2, \ldots, N\}\) \((N \in \mathbb{N}\) and set \(n_{N+1} = \infty\) or \(I = \mathbb{N}\), such that \(n_1 = 0\) and if \(m = \frac{n_{r+1} - n_r}{2}\), then

\[
    c_{n_r+j} = \begin{cases} 
        j & \text{if } 0 \leq j < m, \\
        2m - j & \text{if } m \leq j \leq 2m.
    \end{cases}
\]

Furthermore, \(S\) is mirror-periodic if and only if \(I\) is finite.

**Proof.** It is enough to show that if \(n \in \mathbb{Z}^+\) is such that \(c_n = 0\), then \(c_{n+j}\) will behave like in the proposition for all \(j \in \mathbb{N}\) until \(c_{n+j} = 0\). The result then follows by induction. Let \(n \in \mathbb{N}\) be such that \(c_n = 0\). By Lemma 7.5 there exists \(k = (k_1, \ldots, k_{n+1}) \in \mathbb{Z}_+^{n+1}\) such that the word \(w_1 = s_1^{k_1} \cdots s_{n+1}^{k_{n+1}}\) generates \(Z(B_{n+1})\). Lemma 7.7 implies that \(k = \{1, \ldots, \ell\}\), and if \(k_1 = k_{n+1} = 0\), then \(w_1 \in Z(B_{n+1})\), so \(k_1 = 1\). By Lemma 7.5 again, there exists \(m \in \mathbb{N} \cup \{\infty\}\) such that \(c_{n+m} = j\) for \(1 \leq j \leq m\) and \(c_{n+m+1} = m - 1\). For \(1 \leq i \leq m\) let

\[
    w_i = s_1^{k_1} \cdots s_{n+1}^{k_{n+1}}.
\]

We claim that \(\{w_i\}_{i=1}^m\) generates \(Z(B_{n+j})\) for \(1 \leq j \leq m\), so let \(1 \leq r < m\) be such that \(\{w_i\}_{i=1}^r\) generates \(Z(B_{n+r})\). Since \(c_{n+r+1} = r + 1\), Lemma 7.6 gives that \(Z(B_{n+r}) \subseteq Z(B_{n+r+1})\), so \(w_1 = s_1^{k_1} \cdots s_{n+1}^{k_{n+1}} s_{n+2}^{k_{n+2}} \cdots s_{n+r+1}^{k_{n+r+1}} \in Z(B_{n+r+1})\). Lemma 7.7 then imply that \(s_{r+1}^{k_{r+1}} \cdots s_{n+r+1}^{k_{n+r+1}} \in Z(B_{n+r+1})\), hence, since \(k = \{1, \ldots, \ell\}\), \(w_{r+1} \in Z(B_{n+r+1})\). Because \(\{w_i\}_{i=1}^m\) is an independent family of \(r + 1\) words in \(Z(B_{n+r+1})\), it generates \(Z(B_{n+r+1})\), and the claim follows by induction. If \(m = 1\) (which implies \(c_{n+m+1} = 0\) or \(m = \infty\), we are done, so assume \(1 < m < \infty\). Since \([w_i, s_{n+m+1}] = [w_{i-1}, s_{n+m}] = 0\) for \(2 \leq i \leq m\), it follows that \(\{w_i\}_{i=2}^m\) generates \(Z(B_{n+m+1})\). Hence \(w_1 \notin Z(B_{n+m+1})\), i.e., \(w_1\) anticommutes with \(s_{n+m+1}\), so if \(j \in \{1, \ldots, m\}\), then \(w_j\) anticommutes with \(s_{n+m+1}\). From this we get that \(w_1, \ldots, w_{j} \notin Z(B_{n+m+j})\). Because \([w_i, s_{n+m+1}] = [w_{i-j}, s_{n+m}] = 0\) for \(j < i \leq m\), it follows that \(\{w_i\}_{i=1}^m \subseteq Z(B_{n+m+j})\) (by induction, where \(\{w_i\}_{i=1}^m = C1\)). Lemma 7.6 and 7.5 now implies that \(\{w_i\}_{i=1}^m\) generates \(Z(B_{n+m+j})\).

Assume \(S\) is mirror-periodic with period \(p\). If \(j \in \mathbb{N}\), then

\[
    s_1 s_{p+1} s_j = (-1)^j s_{2p+1-j} (-1)^j s_{1-j} s_j s_1 s_{p+1} = (-1)^j s_{1-j} (-1)^j s_{1-j} s_j s_1 s_{p+1} = s_j s_1 s_{p+1},
\]

so \(s_1 s_{p+1} \in Z(B_n)\) for \(n \geq p + 1\), hence \(I\) is finite.
Assume \( I \) is finite. By the first part of the proposition there exists a word \( w_1 = s_1^{k_1} \cdots s_n^{k_n} \), where \( n \) is odd, \( k = (k_1, \ldots, k_n) \) satisfies \( k = \overline{k} \), and \( k_1 = 1 \), such that \( w_1 \in Z(B_{n+j}) \) for all \( j \in \mathbb{Z}^+ \). If \( n = 1 \), \( w_1 = s_1 \), so in this case \( S \) consists of only 0's. Since this \( S \) is mirror-periodic we may assume that \( n > 1 \). For \( j \in \mathbb{Z} \) define \( l_j = (\chi_1(|j|), \chi_1(|j+1|), \ldots, \chi_1(|j+n-1|)) \in \mathbb{F}_2^n \) and \( l'_j \in \mathbb{F}_2^{n-1} \) as the vector obtained from \( l_j \) by deleting its last entry. Let \( A \in M_{n-1}(\mathbb{F}_2) \) be given by

\[
A = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 1 & \ddots \\
k_1 & k_2 & k_3 & \cdots & k_{n-1}
\end{pmatrix},
\]

and let \( j \geq 1 - n \). By calculating the sign in \( s_{j+n}w_1 = \pm w_1s_{j+n} \) and using that \( w_1 \in Z(B_{n+j}) \) for all \( j \in \mathbb{Z}^+ \), we get that \( \sum_{i=1}^n \chi_1(|j+n-i|)k_i = 0 \).

Since \( k = \overline{k} \), this yields that \( 0 = \sum_{i=1}^n \chi_1(|j-1+n+1-i|)k_{n+1-i} = \sum_{i=1}^n \chi_1(|j-1+i|)k_i = l_j \cdot k \) for all \( j \geq 1 - n \) (where the middle equality follows by substituting \( n+1-i \)). If \( j < 1 - n \), then \( l_j \cdot k = \overline{l'_j} \cdot \overline{k} = l_{j-(n-1)} \cdot \overline{k} = l_{1-n-j} \cdot k = 0 \), because \( k = \overline{k} \), so \( l_j \cdot k = 0 \) for all \( j \in \mathbb{Z} \). Since \( k_n = k_1 = 1 \) this implies that \( l'_{j+1} = Al'_{j} \) for all \( j \in \mathbb{Z} \), and since \( A \) is invertible over \( \mathbb{F}_2 \), there exists \( m \in \mathbb{N} \) such that \( A^m = 1 \). From this it follows that \( l'_{j+m} = l'_j \) for all \( j \in \mathbb{Z} \), so \( S \) is mirror-periodic. \( \square \)

**Remark 1.** The calculation used to show that \( S \) is mirror-periodic implies that \( I \) is finite can also be found in [GS], and the proof of the reverse implication is due to [Pr1].

**Remark 2.** If \( B_n = C^*(s_i : \frac{1-n}{2} \leq i \leq \frac{n}{2}) \), the proof of Proposition 7.8 is the same, but we must exchange \( \{w_i\}_{i=1}^j \) with \( \{s_{i-rac{n-1}{2}}^{k_1} \cdots s_{i-rac{n+1}{2}}^{k_n} \}_{i-rac{n-1}{2} \leq i \leq \frac{n}{2}} \) and so on.

**Remark 3.** We see from the proof of Proposition 7.8 that for a given sequence \( (n_r)_{r \in I} \subset \mathbb{N} \), where \( I = \{1, 2, \ldots, N\} \) or \( I = \mathbb{N} \) which satisfies \( n_1 = 0 \), \( n_r \) even, and \( n_r < n_{r+1} \), there exists a bitstream \( S \subset \mathbb{N} \) giving rise to this sequence. This is Theorem 6.6. in [PP2].

**Theorem 7.9.** Let \( S \subset \mathbb{N} \) be a bitstream. Then there exists a family of self-adjoint, unitary operators, \( \{s_j\}_{j \in \mathbb{Z}} \), such that \( s_is_j = (-1)^{\chi_1(|j-i|)}s_j s_i \), and if \( B = C^*(s_i : i \in \mathbb{Z}) \), then

\[
B \cong \begin{cases}
M_{2^n}(\mathbb{C}) \otimes \bigotimes_1^\infty D_2(\mathbb{C}) & \text{if } S \text{ is mirror-periodic,} \\
\bigotimes_1^\infty M_2(\mathbb{C}) & \text{if } S \text{ is not mirror-periodic.}
\end{cases}
\]

**Proof.** Let \( I \subset \mathbb{N} \) be the index set given by Proposition 7.8. If \( I = \{1, \ldots, N\} \), there exists \( n \in \mathbb{N} \) such that \( c_{n+j} = j \) for all \( j \in \mathbb{Z}^+ \), so \( d_{n+j} = d_n \) for all \( j \in \mathbb{Z}^+ \).
$\mathbb{Z}^+$ by Lemma 7.5. Lemma 7.3 now implies that $B \cong M_{2^{2n}}(\mathbb{C}) \otimes \bigotimes_1^\infty D_2(\mathbb{C})$. If $I = \mathbb{N}$, Lemma 7.3 and Proposition 7.8 implies that $B_n \cong M_{2^{2n}}(\mathbb{C})$ for infinitely many $n$'s, so $B \cong \bigotimes_1^\infty M_2(\mathbb{C})$. □

References


