Solution to a class of monotone stochastic control problems by variational inequality methods

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Abstract

A simple model of a firm with irreversible investment opportunities acting in a market with uncertainty leads to a singular stochastic control problem with monotone controls. The value function is characterized by a set of differential inequalities and the optimal strategy is obtained as the minimal nondecreasing process keeping the state process outside a "forbidden region". The derivative of the value function w.r.t. the control variable is shown to satisfy a family of variational inequalities. General results from the theory of variational inequalities are applied in the analysis of the problem.

Key words. Singular stochastic control, monotone stochastic control, variational inequalities.

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1 Introduction

In this article we study a class of stochastic control problems in connection with a simple model of a firm acting in a "stochastic" market and with only irreversible investment opportunities. The model is described as follows. The financial state of the firm is represented by a pair (x, u) where $x \in \mathbb{R}^d$ describes the market situation, and where $u \in [0, \infty)$ represents the firms production capacity. The net profit rate in the state (x, u) is given by $f(x, u)$, where $f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is a given function, possibly taking negative values. Clearly the firm wants to adapt u to x in order to maximize profit. The value of u can be increased by any amount at any time. If u is increased by an amount Δu when the market state is x, a cost of $c(x)\Delta u$ is incurred. Here $c : \mathbb{R}^d \to [0, \infty)$ is a given function. However, u can never be reduced, which means that investments have irreversible effects on the firm. We model the market by a d-dimensional diffusion process X given as the solution of a SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t,$$

where W is a Brownian motion. Hence, at time t the market state is X_t. The firms production capacity is modeled by a nondecreasing stochastic process U, i.e. at time t the production capacity is U_t. The process U is the firms control process, and we may associate to it the expected profit $J(U)$ earned by using U to match the market process X. (This is all made precise in section 2.) The control problem we study is then the problem of choosing, in an appropriate class, a process U that maximizes $J(U)$ in that class. We shall focus on the situation where business is stopped when X exits from a bounded domain $\mathcal{O} \subset \mathbb{R}^d$, hence control is exercised up to a stopping time, and after that time nothing happens. The state space for the firm is then $\mathcal{O} \times [0, \infty)$.

A bit of thought leads to the idea that under fairly general conditions optimal strategies U for capacity control should have the following simple structure. There should be a function $\varphi : \mathcal{O} \to [0, \infty)$ dividing the state space $\mathcal{O} \times [0, \infty)$ into two disjoint regions

$$\mathcal{A} = \left\{ (x, u) \;\middle|\; u < \varphi(x) \right\}, \quad \mathcal{B} = \left\{ (x, u) \;\middle|\; u \geq \varphi(x) \right\},$$

below and above the graph of φ, in such a way that it is always optimal to increase u when $(x, u) \in \mathcal{A}$ and never optimal to increase u when $(x, u) \in \mathcal{B}$. Then an optimal strategy $U_t^\star$ can be described as the minimal nondecreasing process satisfying for all t

$$(X_t, U_t^\star) \in \mathcal{B}, \quad \text{or equivalently,} \quad U_t^\star \geq \varphi(X_t).$$

We will call this $U^\star$ process the φ-push-up of X, and we study such processes in section 3.1. We show in this paper that the problem we study has this type of solution under quite general conditions. We also give a good characterization of φ in (38), a form which is useful for numerical calculations.

The paper is organized as follows. In section 2 we give a precise formulation of the problem. In section 3 we give some preliminary results and then prove the verification theorem 3.4, which gives sufficient conditions for the existence of a solution of the type described above. Then in section 4 we show how the value function and the function φ can be constructed. The major tool used here is the theory of Variational Inequalities. The main result of the paper is theorem 4.13. Section 4 terminates with a discussion of the results. In section 5, the appendix, we have placed two rather long proofs which would probably distract most readers if placed in their logical place.

Several related problems has been studied over the years. The closest relative to this paper is [Kob93], where a very similar problem is studied, assuming the market process X is a one-dimensional geometric Brownian Motion. The fact that the state space is then two-dimensional allows special Green's function methods to be used, and a quite explicit description of the optimal strategies can be given with this method. In the present paper a different method is employed in that we rely heavily on the general theory of Variational Inequalities. This seems more useful since the state space can have any dimension in our problem. The proof of theorem 3.4 is largely based on the proof of the verification theorem in [Kob93], and several ideas presented in that paper has eased the work with the present paper. In [Arn96] a variant of the problem where the capacity can also be reduced was studied by use of the methods developed in [Kob93].

A problem quite similar to the present one is studied in [SS90]. There the state process is a d-dimensional Brownian Motion which is controlled by adding a monotone process to the last component. The presence of diffusion also in the control component of the state process gives a different analytical characterization of the value function. The value function is shown to be in $C^2(\mathbb{R}^d)$ and they prove considerable smoothness of the boundary of the "region of action". Results of the same type is obtained for a nonstationary version of this problem in [SS91].

In general the structure of the solution with a "forbidden region" for the pair (X_t, U_t) is very common in singular stochastic control problems, which has been studied by many authors. In [FS93] a general introduction to singular control problems, with a good set of references can be found.

2 Precise formulation of the problem

We let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ be a filtered probability space, with an m-dimensional Brownian Motion W. We assume $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual hypotheses.[1] For $x \in \mathbb{R}^d$, let X^x denote the strong solution to the SDE

$$(1) \qquad dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t, \quad X_0 = x,$$

where

$$b : \mathbb{R}^d \to \mathbb{R}^d, \quad \sigma : \mathbb{R}^d \to M_{d\times m}(\mathbb{R})$$

satisfy the Lipshitz condition

$$(2) \qquad |b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq K|x - y|,$$

which in this case implies that b, σ satisfies also a linear growth condition.

Additional assumptions on b, σ will have to be given later on.

We shall often need to take expected values of expressions involving the process X^x. As is customary we then indicate the dependence on the initial condition x by writing for instance $E^x[g(X_t)]$ instead of $E[g(X_t^x)]$.

We take as given a bounded, connected domain $\mathcal{O} \in \mathbb{R}^d$, with a boundary of class C^2 if $d \geq 2$. We write (x, u) for points in $\mathcal{O} \times [0, \infty)$, and if $f : \mathcal{O} \times [0, \infty) \to \mathbb{R}$ we write $\partial_{x_1} f, \ldots, \partial_{x_d} f, \partial_u f$ for the first order partial derivatives of f.

We assume $f \in C^2(\mathbb{R}^d \times \mathbb{R})$ and $c \in C^2(\mathbb{R}^d)$ are functions satisfying

(3) $\qquad c \geq 0$

(4) $\qquad$ for some $a > 0$, $\partial_{uu} f < -a$.

Note that (4) implies

(5) $\qquad$ there is an $M > 0$ such that $f(x, u) < 0$ for all $x \in \mathcal{O}, u \geq M$

We let $r > 0$ be a constant representing the discount rate.

${\tau_{\mathcal{O}}}^x : \Omega \to [0, \infty]$ denotes the first exit time of X^x from $\mathcal{O}$. Again we will write $E^x[\tau_{\mathcal{O}}]$ for $E[{\tau_{\mathcal{O}}}^x]$. We shall need to have

$$(6) \qquad E^x[\tau_{\mathcal{O}}] < \infty.$$

[1] i.e. that $\{\mathcal{F}_t\}_{t\geq 0}$ is right continuous and $\mathcal{F}_0$ contains all P-null sets in $\mathcal{F}$.

This is satisfied if the matrix $\sigma\sigma^T$ is uniformly positive definite, an assumption we make in (28).

If $u \in [0, \infty)$, let $\mathcal{A}_u$ denote the set of processes U on Ω that are RCLL, nondecreasing, bounded, adapted to $\{\mathcal{F}_t\}$ and such that $U_0 \geq u$. Note that if $u \leq v$, then $\mathcal{A}_u \supset \mathcal{A}_v$. $\mathcal{A}_u$ represents the set of admissible controls when the firm starts out with an initial production capacity u.

If $x \in \mathcal{O}$, and $U \in \mathcal{A}_u$ we consider the performance of U,

$$J(x, u; U) = E^x\Big[\int_0^{\tau_\mathcal{O}} e^{-rs} f(X_s, U_s)\, ds - \int_0^{\tau_\mathcal{O}} e^{-rs} c(X_s)\, dU_s - c(x)(U_0 - u)\Big] \tag{7}$$

and the value function

$$V(x, u) = \sup_{U \in \mathcal{A}_u} J(x, u; U); \tag{8}$$

The two first terms in (7) correspond respectively to the expected profit and the expected cost associated with increases in U. The last term represents the cost of choosing $U_0 > u$.

Using (5) and (6), we find

$$V(x, u) < \infty, \quad \text{for all } (x, u) \in \mathcal{O} \times [0, \infty).$$

A process $U \in \mathcal{A}_u$ will be called a *u-strategy*, and if $U^\star \in \mathcal{A}_u$ satisfies

$$V(x, u) = J(x, u; U^\star),$$

we call $U^\star$ an *optimal u-strategy* for X^x.

Now we may formulate the questions that will be considered in this paper.

Q1: Given (x, u), does an optimal u-strategy for X^x exist?

Q2: How can the optimal strategies be characterized?

Q3: What is the analytical characterization of the value function?

We discuss how these questions are answered in section 4.5.

Regarding the conditions, (3) and (4) are crucial for the analysis carried out in this paper. Indeed, if for some $x \in \mathcal{O}$ we had $c(x) < 0$, the firm could profit beyond any limits by carrying out sufficiently large initial capacity expansions. Clearly this situation, where no optimal strategy exists, must be ruled out. However, we shall allow for the case that $c = 0$. The condition (4) is essential in proving that the function φ mentioned in the introduction is continuous. One should also note that by (5), it will never be optimal to expand capacity u beyond the level M. Thus it suffices to consider bounded strategies.

We conclude this section with some of the notation we shall use in addition to the notation introduced above.

- $C^m(\overline{\mathcal{O}})$ denotes the space of real functions on $\mathcal{O}$ with uniformly continuous partial derivatives of all orders $\leq m$.
- $C^{m,\alpha}(\overline{\mathcal{O}})$ denotes the space of functions in $C^m(\overline{\mathcal{O}})$ with the m'th order partial derivatives being Hölder continuous of order α.
- $W^{m,p}(\mathcal{O})$ are the Sobolev spaces defined in section 4.1.
- $a(u, v)$ is a bilinear form, defined in section 4.1.
- A denotes the differential operator in formula (14), section 3.3.
- $\mathcal{I}(\mathcal{O} \times [0, \infty))$ is the space defined in section 3.3.

3 A verification theorem

The main result in this section is theorem 3.4, a result giving the analytical characterization of the value function and also giving a good description of the optimal strategies. First we need a few preliminary constructions.

3.1 The φ-push-up of X.

Let X^x be the solution to (1), and assume $\varphi : \mathcal{O} \to [0, \infty)$ is a bounded, continuous function. If $u \geq 0$, we define the *φ-push-up of X^x starting at u* by

$$U_t^{\varphi,x,u} = u \vee \sup_{s \leq t} \varphi(X_s^x). \tag{9}$$

Clearly $U^{\varphi,x,u} \in \mathcal{A}_u$ and as we shall see, the optimal strategies will be of this type. We shall need a certain property of such processes, given in lemma 3.1 below.

If we now fix an ω such that $s \mapsto X_s^x(\omega)$ is continuous and define

$$\begin{aligned} y(s) &\triangleq \varphi(X_s^x(\omega)) \\ v(s) &\triangleq U_t^{\varphi,x,u}(\omega) = u \vee \sup_{s \leq t} y(s) \\ T &\triangleq \tau_{\mathcal{O}}{}^x(\omega) \end{aligned}$$

Then y is continuous and v is the unique function satisfying

$$\begin{aligned} &\text{(i)} \quad v \text{ is continuous and nondecreasing, } v(0) \geq u \\ &\text{(ii)} \quad v(t) \geq y(t) \quad \text{for all } t \in [0, T] \\ &\text{(iii)} \quad \int_0^T 1_{\{v(s) > y(s)\}} dv(s) = 0. \end{aligned}$$

This result is usually known under the name *Skorohod equation* see [KS91] p. 210. Since $U^{\varphi,x,u}$ is a nondecreasing process, integration can be taken ω-wise and for any x we get a.s.

$$\begin{aligned} &\text{(i)} \quad U^{\varphi,x,u} \text{ is nondecreasing and continuous, } U_0^{\varphi,x,u} \geq u \\ &\text{(ii)} \quad U_t^{\varphi,x,u} \geq \varphi(X_t^x) \quad \text{for } t < \tau_{\mathcal{O}}{}^x \\ &\text{(iii)} \quad \int_0^{\tau_{\mathcal{O}}{}^x} 1_{\{U_t^{\varphi,x,u} > \varphi(X_t^x)\}} dU_t^{\varphi,x,u} = 0. \end{aligned}$$

3.1. Lemma. *If $g \in C([0,\infty) \times \mathbb{R}^d \times [0,\infty))$ satisfies $g(s, x, \varphi(x)) = 0$ whenever $\varphi(x) > 0$, we have a.s.*

$$\int_0^{\tau_{\mathcal{O}}{}^x} g(s, X_s^x, U_s^{\varphi,x,u}) dU_s^{\varphi,x,u} = 0.$$

Proof. By property (iii) of $U^{\varphi,x,u}$ and the condition on g we get if $u > 0$,

$$\int_0^{\tau_O{}^x} g(s, X_s^x, U_s^{\varphi,x,u})dU_s^{\varphi,x,u} = \int_0^{\tau_O{}^x} g(s, X_s^x, U_s^{\varphi,x,u})1_{\{U_s^{\varphi,x,u} > \varphi(X_s^x)\}}dU_s^{\varphi,x,u}$$
$$+ \int_0^{\tau_O{}^x} g(s, X_s^x, \varphi(X_s^x))1_{\{U_s^{\varphi,x,u} = \varphi(X_s^x)\}}dU_s^{\varphi,x,u} = 0 + 0 = 0.$$

If $u = 0$, let $\varepsilon > 0$. Note then that if $U_s^{\varphi,x,0} \geq \varepsilon$, then $U_s^{\varphi,x,0} = U_s^{\varphi,x,\varepsilon}$. Then let

$$\sigma = \inf\left\{ s \geq 0 \,\middle|\, U_s^{\varphi,,x,\varepsilon} = \varepsilon \right\} \wedge \tau_O{}^x.$$

now we get for some $M > 0$, independent of ε,

$$\left|\int_0^{\tau_O{}^x} g(s, X_s^x, U_s^{\varphi,x,0})dU_s^{\varphi,x,0}\right| \leq M\varepsilon + \left|\int_\sigma^{\tau_O{}^x} g(s, X_s^x, U_s^{\varphi,x,\varepsilon})dU_s^{\varphi,x,\varepsilon}\right| = M\varepsilon.$$

Since this holds for arbitrary $\varepsilon > 0$, we get the result also for $u = 0$. □

3.2 Reduction to continuous strategies

We show that for any u-strategy U, there are *continuous* strategies with performance arbitrarily close to the performance of U. Recall that any u-strategy is a bounded process.

3.2. Lemma. *Let $u \geq 0$, $x \in \mathcal{O}$ and $U \in \mathcal{A}_u$ be given. Then there is a sequence $\{U^m\}$ of continuous $U^m \in \mathcal{A}_u$ such that*

$$\lim_{m\to\infty} |J(x, u; U) - J(x, u; U^m)| = 0. \tag{10}$$

Proof. Fix an M such that $U_s \leq M$ for all s. We define U^m by requiring that $U_0^m = U_0$, and that U^m is the maximal process satisfying

$$U^m \leq U \text{ and the growth rate of } U^m \text{ is everywhere} \leq m.$$

It follows that U^m increases with rate m when $U_t^m < U_t$ and otherwise $U_t^m = U_t$. Clearly U^m is continuous. We recall the definition of $J(x, u; U)$ from (7) and treat the terms involving f first. Define

$$A_m(\omega) \triangleq \left\{ t \geq 0 \,\middle|\, U_t(\omega) \neq U_t^m(\omega) \right\}.$$

Then, since f is C^2, it is Lipshitz on $\overline{\mathcal{O}} \times [0, M]$, and we obtain

$$\text{(11)}\quad E^x\left[\int_0^{\tau_O} |f(X_s, U^s) - f(X_s, U_s^m)|\, ds\right]$$
$$\leq E^x\left[\int_0^{\tau_O} C1_{A_m}|U_s - U_s^m|\, ds \leq CME^x[\text{meas}([0, \tau_O] \cap A_m)]\right.$$

where meas denotes Lebesgue measure. Now let us write

$$\psi_m(\omega) \triangleq \text{meas}([0, \tau_{\mathcal{O}}(\omega)] \cap A_m(\omega)).$$

Note that for any m and almost any ω, $\text{meas}(A_m(\omega)) \leq \frac{M}{m}$. Since we have

$$0 \leq E^x[\psi_m] \leq E^x[\tau_{\mathcal{O}}] < \infty \quad \text{and} \quad \lim_{m\to\infty} \psi_m = 0 \quad \text{pointwise,}$$

we get by bounded convergence that

$$\lim_{m\to\infty} E^x[\text{meas}([0, \tau_{\mathcal{O}}] \cap A_m)] = \lim_{m\to\infty} E^x[\psi_m] = 0. \tag{12}$$

Now we treat the terms involving integrals w.r.t. U and U^m. For a fixed ω we write

$$\sigma(s) = U_s(\omega), \quad \sigma_m(s) = U_s^m(\omega), \quad f(s) = e^{-rs}c(X_s(\omega)), \quad T = \tau_{\mathcal{O}}(\omega).$$

It is clear that $f \in C_b([0, T])$ and that $\sigma_m(s) \to \sigma(s)$ whenever σ is continuous at s. Then the measures $d\sigma_m$ converges weakly to $d\sigma$. It follows that

$$\begin{aligned} Z_m &\triangleq \int_0^{\tau_{\mathcal{O}}} e^{-rs}c(X_s)\, dU_s^m \\ &= \int_0^T f(s)\, d\sigma_m(s) \to \int_0^T f(s)\, d\sigma(s) = \int_0^{\tau_{\mathcal{O}}} e^{-rs}c(X_s)\, dU_s \triangleq Z. \end{aligned}$$

i.e. $Z_m \to Z$ a.s. P^x for any x. Then note that

$$E^x[|Z_m - Z|] \leq 2M\|c\|_{\infty,\mathcal{O}} E^x[\tau_{\mathcal{O}}] < \infty,$$

hence by bounded convergence,

$$E^x[|\int_0^{\tau_{\mathcal{O}}} e^{-rs}c(X_s)\, dU_s^m - \int_0^{\tau_{\mathcal{O}}} e^{-rs}c(X_s)\, dU_s|] = E^x[|Z_m - Z|] \to 0. \tag{13}$$

Combining (11),(12) and (13) we arrive at (10). □

3.3 The space $\mathcal{I}(\mathcal{O} \times [0, \infty))$

Given b, σ as in (1), let

$$a(x) = \frac{1}{2}\sigma(x)\sigma(x)^T.$$

Also let A denote the differential operator given by

$$Aw(x) = -\sum_{i,j=1}^{d} a_{ij}(x)\partial_{x_i x_j} w\,(x) - \sum_{j=1}^{d} b_j(x)\partial_{x_j} w\,(x) + rw(x). \tag{14}$$

For a function $v : \mathcal{O} \times [0, \infty) \to \mathbb{R}$, when we write $Av(x, u)$ we shall mean A applied to the map $x \mapsto v(x, u)$ with u fixed.

3.3. Definition. *We denote by* $\mathcal{I}(\mathcal{O} \times [0,\infty))$ *the set of functions* $v \in C(\overline{\mathcal{O}} \times [0,\infty))$ *such that*

1. $\partial_u v$ *is continuous and for each* $u \geq 0$, $Av(\cdot, u)$ *is a measurable function on* $\mathcal{O}$.

2. *For any stopping time* $\tau \leq \tau_{\mathcal{O}}$, $x \in \mathcal{O}$ *and any continuous* $U \in \mathcal{A}_u$, *we have*

(15)

$$E^x[e^{-r\tau} v(X_\tau, U_\tau)] = v(x, U_0) - E^x[\int_0^\tau e^{-rs} Av(X_s, U_s)\, ds - \int_0^\tau e^{-rs} \partial_u v\,(X_s, U_s)\, dU_s]$$

Thus the space $\mathcal{I}(\mathcal{O} \times [0,\infty))$ consist of those functions for which the weak form (15) of Itô's formula is valid.

3.4 The verification theorem

We may now state and prove the verification theorem.

3.4. Theorem. *Assume there is a* $v \in \mathcal{I}(\mathcal{O} \times [0,\infty))$ *and a* $\varphi \in C(\mathcal{O}) \cap L^\infty(\mathcal{O})$ *such that* $\varphi \geq 0$ *and*

(i) $\partial_u v(x,u) \leq c(x)$ *with equality if* $u \leq \varphi(x)$ *and* $\varphi(x) > 0$.
(ii) $Av(x,u) \geq f(x,u)$ *with equality if* $u \geq \varphi(x)$.
(iii) $v(x,u) = 0$ *if* $x \in \partial\mathcal{O}$.

Then for any (x,u) *and any* $U \in \mathcal{A}_u$ *we have*

(16) $$J(x,u;U) \leq v(x,u).$$

If $U^{\varphi,x,u}$ *is the* φ*-push-up of* X^x, *i.e.* $U_t^{\varphi,x,u} = u \vee \sup_{s\leq t} \varphi(X_s^x)$ *then*

(17) $$J(x,u;U^{\varphi,x,u}) = v(x,u),$$

hence $U^{\varphi,x,u}$ *is an optimal u-strategy for* X^x *and* v *is the value function* V.

3.5. Remark. Since $Av(\cdot,u)$ is only assumed to be measurable, we take (ii) in the sense that for all u, the statement holds for a.a. x.

Proof. Throughout the proof, let (x,u) be fixed. By lemma 3.2 it suffices to consider a continuous $U \in \mathcal{A}_u$. Using that $v \in \mathcal{I}(\mathcal{O} \times [0,\infty))$ and *(iii)*, we find

$$\begin{aligned} 0 &= E^x[e^{-r\tau_{\mathcal{O}}} v(X_{\tau_{\mathcal{O}}}, U_{\tau_{\mathcal{O}}})] \\ &= v(x,U_0) - E^x[\int_0^{\tau_{\mathcal{O}}} e^{-rs} Av(X_s,U_s)\, ds - \int_0^{\tau_{\mathcal{O}}} e^{-rs} \partial_u v\,(X_s,U_s)\, dU_s]. \end{aligned}$$

Hence

$$(18) \qquad v(x,U_0) = E^x[\int_0^{\tau_O} e^{-rs} Av(X_s,U_s)\,ds - \int_0^{\tau_O} e^{-rs}\partial_u v\,(X_s,U_s)\,dU_s]$$

Now using that $U_0 \geq u$, and (i), (ii), we get

$$\begin{aligned} v(x,u) &\geq v(x,U_0) - c(x)(U_0-u) \\ &\geq E^x[\int_0^{\tau_O} e^{-rs} f(X_s,U_s)\,ds - \int_0^{\tau_O} e^{-rs} c(X_s)\,dU_s] - c(x)(U_0-u) \\ &= J(x,u;U), \end{aligned}$$

and so we get (16).

Now let us consider $U^{\varphi,x,u}$. For ease of notation let us write $U^\varphi = U^{\varphi,x,u}$. Since $U_s^\varphi \geq \varphi(X_s^x)$ for all s, we get from (ii)

$$(19) \qquad E^x[\int_0^{\tau_O} e^{-rs} Av(X_s,U_s^\varphi)\,ds] = E^x[\int_0^{\tau_O} e^{-rs} f(X_s,U_s^\varphi)\,ds].$$

Now using lemma 3.1 with

$$g(s,x,u) = e^{-rs}(\partial_u v\,(x,u) - c(x)),$$

together with condition (i), we get

$$E^x[\int_0^{\tau_O} e^{-rs}(\partial_u v\,(X_s,U_s^\varphi) - c(X_s))\,dU_s^\varphi] = 0,$$

hence

$$(20) \qquad E^x[\int_0^{\tau_O} e^{-rs}\partial_u v\,(X_s,U_s^\varphi)\,dU_s^\varphi] = E^x[\int_0^{\tau_O} e^{-rs} c(X_s)\,dU_s^\varphi].$$

Together, (18), (19) and (20) gives

$$(21) \qquad v(x,U_0^\varphi) = E^x[\int_0^{\tau_O} e^{-rs} f(X_s,U_s^\varphi)\,ds - \int_0^{\tau_O} e^{-rs} c(X_s)\,dU_s^\varphi.]$$

If $u \geq \varphi(x)$, then $U_0^\varphi = u$, and (21) reads

$$v(x,u) = J(x,u;U^\varphi).$$

If $u < \varphi(x)$, then $U_0^\varphi = \varphi(x)$, and by (i) and (21) we get once again

$$v(x,u) = v(x,U_0^\varphi) - c(x)(U_0^\varphi - u) = J(x,u:U^\varphi).$$

We have verified (17). □

3.6. Remark. The conditions (i) and (ii) tells us that v should solve the PDE

$$\max\{f(x,u) - Av(x,u), \partial_u v\,(x,u) - c(x)\} = 0$$

in $\mathcal{O} \times [0,\infty)$. This is similar to the PDE studied in [SS90] which is

$$\max\{u(x) - \Delta u(x) - h(x), \partial_{x_d} u\,(x) - 1\} = 0$$

for $x \in \mathbb{R}^d$. The major difference is that the operator A here does not operate at all in the u variable. This fact affects the regularity results for the value function and for the boundary function φ in theorem 3.4.

4 Construction of the value function and the optimal strategies.

To solve the optimization problem posed in section 2 , we need to find a pair (v, φ) satisfying the conditions of theorem 3.4. In this section we use the theory of Variational Inequalities to show how such a pair can be constructed.

4.1 Variational inequalities

The analytical problem posed by the conditions (i) - (iii) is a bit awkward as it stands, because it involves both derivatives with respect to the space variables $x_1, \dots, x_d$ and with respect to the control variable u. For this reason, we shall try to obtain v on the form

$$v(x,u) = \int_0^u w(x,r)\, dr + h(x).$$

Then we have

$$w(x,u) = \partial_u v\,(x,u),$$

and we shall consider this w first. Heuristic considerations leads one to expect that w satisfies the following, with $g(x,u) = \partial_u f\,(x,u)$:

(22) $\qquad w(x,u) \le c(x)$ with equality if $u \le \varphi(x)$ and $\varphi(x) > 0$,

(23) $\qquad Aw(x,u) \le g(x,u)$ with equality if $u \ge \varphi(x)$,

(24) $\qquad w(x,u) = 0$ if $x \in \partial\mathcal{O}$.

We consider the problem of obtaining a pair w, φ satisfying (22)-(24) with φ being continuous and bounded. Now note that this problem does not involve any derivatives of w w.r.t. the u variable, hence it is tempting to study the problem for fixed u. The role of φ is then merely to describe subsets of $\mathcal{O}$ where at least one of the inequalities is satisfied with equality. We shall see that we can obtain a pair w, φ solving (22)-(24) by first finding $w(\cdot, u)$ for each fixed u, and then defining φ so as to satisfy (22)-(23).

For a fixed u let us write $w(x) = w(x,u)$, $g(x) = g(x,u)$. The problem we consider is then to find w such that

$$w \le c, \quad Aw \le g, \quad (w-c)(Aw-g) = 0 \quad \text{a.e. in } \mathcal{O} \text{ and } w\big|_{\partial\mathcal{O}} = 0.$$

or more compactly

(25) $$\max(Aw - g, w - c) = 0 \quad \text{a.e. and } w\big|_{\partial\mathcal{O}} = 0.$$

This kind of problem is usually called an *obstacle problem* for w, with obstacle c. The standard framework for the study of problems of this type is the theory of variational inequalities. We will now introduce the necessary terminology from this theory.

If $1 \leq p \leq \infty$ and m is a positive integer we denote by $W^{m,p}(\mathcal{O})$ the Sobolev space of L^p-functions with all derivatives up to order m in L^p :

$$W^{m,p}(\mathcal{O}) = \left\{ v \in L^p(\mathcal{O}) \;\middle|\; \partial^\alpha v \in L^p(\mathcal{O}) \quad \text{for all } \alpha \text{ with } |\alpha| \leq m \right\},$$

where multi-index notation is used. This space is a Banach space equipped with the norm (for $p < \infty$)

$$\|v\|_{m,p} = \|v\|_{m,p,\mathcal{O}} = \Big(\sum_{|\alpha| \leq m} \|\partial^\alpha v\|^p_{L^p(\mathcal{O})} \Big)^{\frac{1}{p}}.$$

$W^{m,2}(\mathcal{O})$ is a Hilbert space, usually denoted by $H^m(\mathcal{O})$. If $\mathcal{O}$ is bounded there are natural inclusions

$$W^{m,p}(\mathcal{O}) \subset W^{m,q}(\mathcal{O}),$$

for $p > q$. In particular, if $p > 2$ then

$$W^{m,p}(\mathcal{O}) \subset H^m(\mathcal{O}).$$

We denote by $W_0^{m,p}(\mathcal{O})$ the completion of $C_0^\infty(\mathcal{O})$ in $W^{m,p}(\mathcal{O})$. If $p = 2$ the notation $H_0^m(\mathcal{O}) = W_0^{m,2}(\mathcal{O})$ is used. If $v \in H^1(\mathcal{O})$, it is not meaningful to say that $v(x) = 0$ for $x \in \partial\mathcal{O}$. The statement $v \in H_0^1(\mathcal{O})$ is the natural weak formulation of boundary conditions on functions in $H^1(\mathcal{O})$. More generally, if ψ is given, $v - \psi \in H_0^1(\mathcal{O})$ replaces the boundary condition $v = \psi$ on $\partial\mathcal{O}$. In $H_0^1(\mathcal{O})$ one often uses the norm $\|v\|_{H_0^1(\mathcal{O})} = \|\nabla v\|_{L^2(\mathcal{O})}$.

The operator A of (14) may be written in divergence form as

$$Aw = -\sum_{i,j} \partial_{x_i} (a_{ij} \partial_{x_j} w) + \sum_j f_j(x) \partial_{x_j} w + rw,$$

where now

$$f_j = \sum_i \partial_{x_i} a_{ij} - b_j.$$

The bilinear form on $H^1(\mathcal{O})$ associated with A is then

$$a(u,v) = \sum_{i,j} \int_{\mathcal{O}} a_{ij}(x) \partial_{x_j} u(x)\, \partial_{x_i} v(x)\, dx + \sum_j \int_{\mathcal{O}} f_j(x) \partial_{x_j} u(x)\, v(x)\, dx + \int_{\mathcal{O}} ru(x)v(x)\, dx.$$

For $u \in H^2(\mathcal{O})$, $v \in H_0^1(\mathcal{O})$ we have

$$a(u,v) = (Au \,|\, v),$$

where

$$(f \,|\, g) = \int_{\mathcal{O}} fg\,dx$$

is the inner product in $L^2(\mathcal{O})$. For $\xi, \psi \in H^1(\mathcal{O})$, let $K_{\xi,\psi}$ be the convex set

$$K_{\xi,\psi} = \left\{ \; v \in H^1(\mathcal{O}) \;\middle|\; v \leq \xi \text{ a.e. and } v - \psi \in H_0^1(\mathcal{O}) \; \right\}.$$

Now we consider the problem

(26) Find $w \in K_{c,0}$ such that $a(w, v - w) \geq (g \,|\, v - w)$ for all $v \in K_{c,0}$.

A problem of this type is called a *Variational Inequality (V.I.)* and it is a reformulation of (25) corresponding to the usual variational formulation of boundary value problems. If $w \in H^2(O)$ satisfies (26) then it will also satisfy (25). There are several texts on variational inequalities and related problems available. We used [BL82], [Rod87].

The existence and uniqueness of a solution to (26) can be established under quite general conditions, (see any of the above references.) We will denote the solution to (26) by w^g to indicate the dependence on the data g. We shall make the following assumptions for the problem.

(27) $$a_{ij} \in C^2(\overline{\mathcal{O}}), \;\; b_j \in C^1(\overline{\mathcal{O}}).$$

(28) $$\text{for some } \alpha > 0, \quad \sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j \geq \alpha|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^d.$$

(29) $$\text{for some } \nu > 0 \text{ and all } v \in H_0^1(\mathcal{O}), \quad a(v,v) \geq \nu \|v\|_{H_0^1(\mathcal{O})}.$$

4.1. Remark. The condition (29) is called a *coercivity* condition. If (27) and (28) holds and

$$r + \lambda \geq \frac{1}{2} \sum_{j=1}^{d} \partial_{x_j} f_j$$

in $\mathcal{O}$, then (29) follows. Here $\lambda > 0$ is the principal eigenvalue of the second order part of A on $\mathcal{O}$ with Dirichlet boundary condition. See [BL82] p. 190. In particular, (29) holds if σ, b are constant, or more generally if σ is constant and $\partial_{x_j} b_j \geq 0$ for $j = 1, \ldots, d$.

Assuming (27)-(29), we have the following regularity result which is direct from theorem 5.2.5 and theorem 5.6.1 in [Rod87].

4.2. Theorem. *Assume (27), (28) and (29), and let w^g be the solution to (26). Then,*

(30) $$\text{if } g \in L^p(\mathcal{O}), \text{ we have } w^g \in W^{2,p}(\mathcal{O}).$$

(31) $$\text{if } g \in C^{0,\eta}(\mathcal{O}), \text{ for some } \eta > 0, \text{ we have } w^g \in W^{2,\infty}_{loc}(\mathcal{O}).$$

4.3. Remark. One can also obtain $w^g \in W^{2,\infty}(\mathcal{O})$ under more restrictive conditions, but w^g is usually not in $C^2(\mathcal{O})$ regardless of the smoothness of the data.

4.4. Corollary. *The assumptions are those of theorem 4.2. Then if $g \in L^p(\mathcal{O})$ and $p > d$, we have*

$$w^g \in C^{1,\alpha}(\overline{\mathcal{O}}),$$

with $\alpha = 1 - \frac{d}{p}$.

Proof. Sobolev imbedding theorem. □

4.2 Further properties of w^g.

We shall first consider a probabilistic representation of w^g, the solution to (26).

4.5. Lemma. *Assume $g \in L^p(\mathcal{O})$ for some $p > \frac{d}{2}$. Then*

$$w^g(x) = \inf_\sigma E^x\left[\int_0^{\sigma\wedge\tau_{\mathcal{O}}} e^{-rs} g(X_s)\, ds + e^{-r\sigma} c(X_\sigma) 1_{[\sigma<\tau_{\mathcal{O}}]}\right],$$

the infimum being taken over all stopping times w.r.t. $\{\mathcal{F}_t\}$

Proof. This is theorem 3.3.1. in [BL82]. □

Then we consider monotonicity of the solution map $g \mapsto w^g$.

4.6. Lemma. *If $g_1, g_2 \in L^2(\mathcal{O})$ and $g_1 \leq g_2$ a.e. then*

$$w^{g_1} \leq w^{g_2}.$$

Proof. This is theorem 3.1.4 in [BL82]. □

We will also take interest in some continuity properties of this solution map.

4.7. Lemma. *If $g, g_j \in C(\overline{\mathcal{O}})$ for $j = 1, 2, \ldots$, and $\|g_j - g\|_{\infty,\mathcal{O}} \to 0$, then*

$$\|w^{g_j} - w^g\|_{\infty,\mathcal{O}} \to 0.$$

Proof. Clearly $g_j \to g$ in $L^p(\mathcal{O})$ for all $p \geq 1$. From Theorem 5.4.5 in [Rod87], there is a $p_0 \geq 1$, such that for any $p > p_0$,

$$\|w^{g_j} - w^g\|_{2,p} \to 0.$$

If p is large enough, $W^{2,p}(\mathcal{O})$ is continuously imbedded in $C(\overline{\mathcal{O}})$, hence

$$\|w^{g_j} - w^g\|_{\infty,\mathcal{O}} \to 0.$$

□

4.8. Remark. An estimate of the type

$$\|w^{g_j} - w^g\|_{\infty,\mathcal{O}} \leq C\|g_j - g\|_{\infty,\mathcal{O}}$$

is possible via lemma 4.5. Thus, as a map $C(\overline{\mathcal{O}}) \to C(\overline{\mathcal{O}})$, the solution map is Lipshitz continuous.

We also state an important bound on Aw^g.

4.9. Lemma. *Asssume (27), (28) and (29). If $g \in L^2(\mathcal{O})$, then a.e. in $\mathcal{O}$*

$$g \wedge Ac \leq Aw^g \leq g.$$

Proof. The right hand side inequality is trivial, for the inequality on the left, see [BL82] Thm. 3.1.21, or [Rod87] Thm. 5.3.3. □

4.10. Remark. One special feature of the problem (26), or (25), is the splitting of $\mathcal{O}$ into two regions

$$(32) \qquad C_g = \left\{ x \in \mathcal{O} \;\middle|\; w^g(x) < c(x) \right\} \quad \text{and} \quad I_g = \left\{ x \in \mathcal{O} \;\middle|\; w^g(x) = c(x) \right\}.$$

C_g is open, I_g is relatively closed, and $Aw^g = g$ a.e. in C_g. Moreover, in interior points in I_g, we have $Aw^g = Ac$, which shows that Aw^g is not likely to be continuous across the common boundary of C_g and I_g. Thus w^g is not in $C^2(\mathcal{O})$. We also note that C_g is the continuation region for the stopping problem in lemma 4.5.

Now let us return to the dependence on the u variable in the problem. For $u \geq 0$, we consider the problem of finding $w_u \in K_{c,0}$, such that

$$(33) \qquad a(w_u, v - w_u) \geq (g \,|\, v - w_u) \quad \text{for all } v \in K_{c,0}$$

now with $g(x) = g(x, u) = \partial_u f\ (x, u)$. We define $w : \mathcal{O} \times [0, \infty) \to \mathbb{R}$ by

$$w(x, u) = w_u(x). \tag{34}$$

In consistence with the notation w^g used before, we have

$$w(x, u) = w^{g(\cdot, u)}(x).$$

We will see that this function w is actually equal to the derivative $\partial_u V$ of the value function.

By our initial assumptions, $g(\cdot, u) \in C^1(\overline{\mathcal{O}})$ so by theorem 4.2 and corollary 4.4 we have for all $u \geq 0$, $2 \leq p < \infty$ and all $\alpha \in (0, 1)$,

$$w(\cdot, u) \in W^{2,p}(\mathcal{O}) \quad \text{and} \quad w(\cdot, u) \in C^{1,\alpha}(\overline{\mathcal{O}}) \cap W^{2,\infty}_{loc}(\mathcal{O}). \tag{35}$$

We wish to have some control on the joint dependence on (x, u). First note that since $\partial_u g\ (x, u) = \partial_{uu} f\ (x, u) < 0$, we have for any $\varepsilon > 0$

$$g(x, u + \varepsilon) < g(x, u).$$

Then from lemma 4.6 we get

$$w(x, u + \varepsilon) \leq w(x, u). \tag{36}$$

Note also that by our assumption (4), for some $u_0 > 0$ we have $g(x, u_0) < 0$ for all $x \in \mathcal{O}$. Using lemma 4.5 and the fact that $c \geq 0$, we get

$$\begin{aligned} w(x, u_0) &= \inf_\sigma E^x[\int_0^{\sigma \wedge \tau_\mathcal{O}} e^{-rs} g(X_s, u_0)\, ds + e^{-r\sigma} c(X_\sigma) 1_{[\sigma < \tau_\mathcal{O}]}] \\ &= E^x[\int_0^{\tau_\mathcal{O}} e^{-rs} g(X_s, u_0)\, ds] < 0, \end{aligned} \tag{37}$$

for all $x \in \mathcal{O}$.

We have thus proved half of the following lemma.

4.11. Lemma. *$w : \mathcal{O} \times [0, \infty) \to \mathbb{R}$ is jointly continuous, nonincreasing in the u variable, and there is a $u_0 > 0$ s.t. $w(x, u) < c(x)$ for all $u > u_0$.*

Proof. Taking into account (36) and (37), only the continuity remains to be demonstrated. Fix $(x, u) \in \mathcal{O} \times [0, \infty)$. By lemma 4.4,

$$w(\cdot, u) \in C^1(\overline{\mathcal{O}}).$$

In particular $w(\cdot, u)$ is Lipschitz. Let $(x_n, u_n) \in \mathcal{O} \times [0, \infty)$ s.t $x_n \to x$, $u_n \to u$. Then

$$\begin{aligned} |w(x_n, u_n) - w(x, u)| &\leq |w(x_n, u_n) - w(x_n, u)| + |w(x_n, u) - w(x, u)| \\ &\leq \|w(\cdot, u_n) - w(\cdot, u)\|_{\infty, \mathcal{O}} + C(u)|x_n - x|. \end{aligned}$$

We clearly have $\|g(\cdot, u_n) - g(\cdot, u)\|_{\infty, \mathcal{O}} \to 0$. Then by lemma 4.7 we get as $n \to \infty$,

$$\|w(\cdot, u_n) - w(\cdot, u)\|_{\infty, \mathcal{O}} \to 0,$$

and we also have $C(u)|x_n - x| \to 0$. Hence

$$|w(x_n, u_n) - w(x, u)| \to 0.$$

$\square$

In the problem (22) - (24) we refer to a function

$$\varphi : \mathcal{O} \to [0, \infty),$$

with a certain relation to w. Having defined w as in (34), we should define φ as follows in order to satisfy (22) - (24). We let

$$\varphi(x) = \sup\left\{ u \geq 0 \;\middle|\; w(x, u) = c(x) \right\}, \tag{38}$$

with the convention $\sup \emptyset = 0$.

To get a better picture of this φ, note that for each $u \geq 0$ we get a splitting of $\mathcal{O}$ corresponding to (32), which we may now write

$$C_u = \left\{ x \in \mathcal{O} \;\middle|\; w(x, u) < c(x) \right\} \quad \text{and} \quad I_u = \left\{ x \in \mathcal{O} \;\middle|\; w(x, u) = c(x) \right\}.$$

The monotonicity of $u \mapsto w(x, u)$ means that for $\varepsilon > 0$,

$$I_u \supset I_{u+\varepsilon},$$

and we have

$$\varphi(x) = \sup\left\{ u \geq 0 \;\middle|\; x \in I_u \right\}.$$

It also follows that

$$C_u = \left\{ x \in \mathcal{O} \;\middle|\; u > \varphi(x) \right\} \quad \text{and} \quad I_u = \left\{ x \in \mathcal{O} \;\middle|\; u \leq \varphi(x) \right\}$$

Using the continuity of w one then verifies (22) - (24). (In fact we have to take (23) in an a.e. sense, since $Aw(\cdot, u)$ is generally a discontinuous function in $L^p(\mathcal{O})$. This will however cause no trouble as we shall see.)

It is essential in theorem 3.4 that φ is continuous and bounded. The proof of this fact can be found in the appendix, we state the result here.

4.12. Lemma. *The function φ defined in (38) satisfies*

$$\varphi \in C(\mathcal{O}) \cap L^\infty(\mathcal{O}).$$

4.3 The value function and the main theorem

Let w be as defined in (34). We want a function v, satisfying the requirements in theorem 3.4 together with φ, defined in (38). We define for $(x,u) \in \overline{\mathcal{O}} \times [0,\infty)$

$$v(x,u) = \int_0^u w(x,r)\, dr + h(x), \tag{39}$$

where $h \in H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$ is the solution to

$$Ah(x) = f(x,\varphi(x)) - \varphi(x)Ac(x). \tag{40}$$

Since $Ah = \xi$ with ξ continuous and bounded, we have at least $h \in W^{2,p}(\mathcal{O})$ for any $p \in [2,\infty)$. We do not know whether φ is Hölder continuous and hence not whether h is in $C^2(\mathcal{O})$.

This v given in (39) is the value function for our problem, as we are about to see.

4.13. Theorem. *Let v be as in (39). Then*

a) $v \in \mathcal{I}(\mathcal{O} \times [0,\infty))$.
b) v satisfies (i) - (iii) in theorem 3.4.

Hence by theorem 3.4,

$$v(x,u) = \sup_{U \in \mathcal{A}_u} J(x,u;U),$$

and for any (x,u), $U^* = U^{\varphi,x,u}$ is an optimal u-strategy for X^x, where φ is given in (38)

Proof. The proof of a) is given in the appendix. We prove b) here.

(i): we clearly have $w(x,u) \le c(x)$ with equality if $\varphi(x) > 0$ and $u \le \varphi(x)$. Since $\partial_u v\,(x,u) = w(x,u)$, we have the result.

(ii): By lemma 5.16,

$$Av(x,u) = \int_0^u Aw(x,r)\, dr + Ah(x) = \int_0^u Aw(x,r)\, dr + f(x,\varphi(x)) - \varphi(x)Ac(x).$$

Also note that since φ is continuous, if $r < \varphi(x)$, we have $w(y,r) = c(y)$ for all y in a neighborhood of x and in this case we get

$$Aw(x,r) = Ac(x).$$

Now, if $u \geq \varphi(x)$, we write

$$\int_0^u Aw(x,r)\,dr = \int_0^{\varphi(x)} Ac(x)\,dr + \int_{\varphi(x)}^u g(x,r)\,dr$$
$$= \int_0^{\varphi(x)} Ac(x)\,dr + \int_{\varphi(x)}^u \partial_u f\,(x,r)\,dr = \varphi(x)Ac(x) + f(x,u) - f(x,\varphi(x)).$$

hence $Av(x,u) = f(x,u)$. If $u < \varphi(x)$, we get

$$(41)\quad Av(x,u) = \int_0^u Ac(x)\,dr + f(x,\varphi(x)) - \varphi(x)Ac(x)$$
$$= uAc(x) + f(x,\varphi(x)) - \varphi(x)Ac(x) = f(x,\varphi(x)) - (\varphi(x)-u)Ac(x).$$

For any $r < \varphi(x)$,

$$Ac(x) = Aw(x,r) \leq g(x,r) = \partial_u f\,(x,r),$$

from which we get

$$f(x,\varphi(x)) - f(x,u) = \int_0^{\varphi(x)} \partial_u f\,(x,r)\,dr \geq \int_0^{\varphi(x)} Ac(x)\,dr = (\varphi(x)-u)Ac(x).$$

Then by (41) we get

$$Av(x,u) \geq f(x,u).$$

(iii) If $x \in \partial\mathcal{O}$, we have $w(x,r) = 0$ for all $r \geq 0$, and $h(x) = 0$. (Because $w(\cdot,r)$ and h are continuous functions in $H_0^1(\mathcal{O})$.) Then (iii) follows directly. □

4.4 Regularity of the value function

Let us collect what we know about the regularity of the value function given by (39). First, since $\partial_u v = w$, we get the regularity of $\partial_u v$ from (35) and lemma 4.11. The function h from (40) is in $W^{2,p}(\mathcal{O})$ for any $p \in [2,\infty)$. From lemma 5.16 we get that

$$v_0(x,u) = \int_0^u w(x.r)\,dr$$

satisfies $v_0(\cdot,u) \in W^{2,p}(\mathcal{O})$ for $1 < p < \infty$. (v_0 is denoted by v in that lemma). Thus, $v(\cdot,u) \in W^{2,p}(\mathcal{O})$ for $2 \leq p < \infty$. In particular, v and $\partial_u v$ are jointly continuous in $\mathcal{O} \times [0,\infty)$. We do not know in general if $v(\cdot,u) \in C^2(\mathcal{O})$. However, we have the following result inspired by the regularity theorem in [SS90].

4.14. Theorem. *Assume for some $\alpha \in (0,1)$ that $\partial\mathcal{O}$ is of class $C^{2,\alpha}$, $c \in C^{2,\alpha}(\overline{\mathcal{O}})$ and that φ from equation (38) is in $C^{0,\alpha}(\overline{\mathcal{O}})$. Then for any $u \geq 0$, we have*

$$v(\cdot, u) \in C^{2,\alpha}(\overline{\mathcal{O}}).$$

Proof. Let u be fixed. As in the proof of theorem 4.13, property (ii), we have

$$Av(x,u) = \int_0^u Aw(x,r)\,dr + Ah(x).$$

From 40 we have

$$Ah(x) = f(x, \varphi(x)) - \varphi(x)Ac(x),$$

so $Ah \in C^{0,\alpha}(\overline{\mathcal{O}})$ under the above assumptions. Next let

$$F(x) = \int_0^u Aw(x,r)\,dr.$$

We show that F is also Hölder continuous. We consider only the case that $u > \varphi(x)$ for all $x \in \mathcal{O}$, the other cases are simpler. If $x, y \in \mathcal{O}$, we estimate as follows.

$$\begin{aligned} |F(x) - F(y)| &\leq \int_0^u |Aw(x,r) - Aw(y,r)|\,dr \\ &= \int_0^{\varphi(x)\wedge\varphi(y)} |Ac(x) - Ac(y)|\,dr + \int_{\varphi(x)\wedge\varphi(y)}^{\varphi(x)\vee\varphi(y)} |Aw(x,r) - Aw(y,r)|\,dr \\ &+ \int_{\varphi(x)\vee\varphi(y)}^{u} |g(x,r) - g(y,r)|\,dr \\ &\leq \|\varphi\|_{\infty,\mathcal{O}} M_1 |x-y|^\alpha + |\varphi(x) - \varphi(y)| M_2 + u M_3 |x-y|, \end{aligned}$$

for constants M_1, M_2, M_3. The estimate involving M_2 uses the inequality

$$g(x,r) \wedge Ac(x) \leq Aw(x,r) \leq g(x,r)$$

following from lemma 4.9. The two other estimates follow directly from the smoothness of c and g. It then easily follows that for some constant C independent of $x.y$,

$$|F(x) - F(y)| \leq C|x-y|^\alpha.$$

Hence we have

$$Av(\cdot, u) \in C^{0,\alpha}(\overline{\mathcal{O}}).$$

From general elliptic regularity theory we then have

$$v(\cdot, u) \in C^{2,\alpha}(\overline{\mathcal{O}}).$$

See e.g. [GT77], thm. 6.14. □

4.5 Discussion

Our treatment of the problem posed by Q1,Q2,Q3 in section 2 consists of two parts. The first part, in section 3, resulted in theorem 3.4. This result gives conditions under which the answer to Q1 is yes, and we also have an answer to Q2 and Q3 there. In section 3 we use only the conditions from section 2.

In the second part, developed in section 4, we impose the additional assumptions (27) - (29). Under these assumptions we show that there is a pair (v, φ) satisfying the conditions of theorem 3.4. This part concludes with theorem 4.13. This theorem shows that we can answer Q1 in the affirmative, under fairly simle conditions imposed directly on the problem data.

Section 4 also provides a way in which v and φ may be computed. First find w by solving the V.I. (33) for all u. Then compute φ by (38). Next, use φ to obtain the h in (40). Then v is given by (39). In practice one would do these computations for $u \in [0, M]$ where M could be as in (5), since nothing of interest happens for $u > M$.

A connection between singular stochastic control and optimal stopping was displayed by Karatzas and Shreve in [KS84] and [KS85]. A similar connection exist for our problem. Indeed, lemma 4.5 shows that the derivative $w = \partial_u v$ of the value function in (8) is the value function for the stopping problem of lemma 4.5 with $g(x) = g(x, u) = \partial_u f\ (x, u)$. If we consider the approach in section 4 in wiew of this connection, we have found a solution to our control problem by solving first a stack of stopping problems, and then piecing these solutions together to build a solution for the original problem.

Concerning the appendix, in section 5.1 the crucial result is lemma 5.3. For an obstacle problem of the type (25), this result says something about how the region of contact between the solution and the obstacle depends on the data of the problem. See remark 5.4. In section 5.2 we show that the value function is sufficiently regular to allow for the use of the Itô formula in (15). The proof here is based on approximation of the value function by smoother functions. The proof is complicated by the fact that the process (X, U) is not a diffusion in its last component.

5 Appendix

5.1 The continuity of φ.

We let w and φ be as defined in (34) and (38). We use the assumptions in section 2, and also (27) - (29). We prove lemma 4.12.

First, from lemma 4.11 we have for some $u_0 > 0$ that

$$0 \leq \varphi \leq u_0,$$

i.e. $\varphi \in L^\infty(\mathcal{O})$.

The continuity is harder to prove. We approach this problem now.

5.1. Lemma.

a) If $\varphi(x) > 0$, then $w(x, \varphi(x)) = c(x)$.
b) φ is upper semicontinuous in $\mathcal{O}$.

Proof. a) follows from the continuity of $u \mapsto w(x,u)$. For b), take $x \in \mathcal{O}$. By the definition of φ and the fact that $w(x,u) \leq c(x)$ for all (x,u), we get for any $\varepsilon > 0$,

$$w(x, \varphi(x) + \varepsilon) < c(x).$$

Since w and c are continuous, there is a $\delta > 0$ s.t.

$$w(y, \varphi(x) + \varepsilon) < c(y)$$

for all $y \in B(x, \delta)$. Then we have

$$\varphi(y) \leq \varphi(x) + \varepsilon$$

for all $y \in B(x, \delta)$. Thus φ is upper semicontinuous at x. ☐

5.2. Corollary. *φ is continuous at any x where $\varphi(x) = 0$.*

Proof. Combine the upper semicontinuity with the fact that $\varphi \geq 0$. ☐

It remains to show that φ is lower semicontinuous at any x where $\varphi(x) > 0$. For this purpose, note that by lemma 5.1 a), $w(x, \varphi(x)) = c(x)$. If $\alpha > 0$ is given, we need to show there is a $\delta > 0$ s.t.

$$\varphi(y) \geq \varphi(x) - \alpha$$

for all $y \in B(x,\delta)$. This is equivalent to showing

$$w(y, \varphi(x) - \alpha) = c(y)$$

for all $y \in B(x,\delta)$. To establish this we need a deeper investigation of the variational inequalities that determine w. A crucial property in this regard will be the strict monotonicity of $u \mapsto g(x,u) = \partial_u f\,(x,u)$ assumed in (4).

As in section 4.1 we let w^g denote the solution to (26). Recall that in this notation we have

$$w(x,u) = w^{g(\cdot,u)}(x).$$

For a given $g \in C^1(\overline{\mathcal{O}})$ and an $\varepsilon > 0$, we shall compare w^g to $w^{g+\varepsilon}$. Precisely we will show

5.3. Lemma. *Let $w^g, w^{g+\varepsilon}$ be as above. Assume $x_0 \in \mathcal{O}$ and*

$$w^g(x_0) = c(x_0).$$

Then there is a $\delta > 0$ such that

$$w^{g+\varepsilon}(x) = c(x)$$

for all $x \in B(x_0,\delta)$.

We shall prove this through some lemmas.

5.4. Remark. Lemma 5.3 shows an important property of the set I_g of (32). It shows that if $x_0 \in I_g$, then x_0 is an interior point in $I_{g+\varepsilon}$ for any $\varepsilon > 0$. Roughly speaking, this means that the set of contact, I_g is strictly increasing w. r. t. g. Our proof uses the assumption that the obstacle c is in C^2. The result does not hold if we only assume c to be continuous.

Considering the optimal stopping interpretation of the variational inequality given in lemma 4.5, we get the following interpretation of lemma 5.3: If it is optimal to stop at x_0 when the running cost is g, then it is optimal to stop at all y in a neighborhood of x_0 if the running cost is increased to $g + \varepsilon$.

Recall the set $K_{\xi,\psi}$ from section 4.

5.5. Lemma. *let $w^g, w^{g+\varepsilon}$ be as above, and let*

$$w_0 = w^g - c, \quad w_0^\varepsilon = w^{g+\varepsilon} - c.$$

Then $w_0 \in K_{0,-c}$ satisfies

$$a(w_0, v - w_0) \geq (h \mid v - w_0) \quad \textit{for all } v \in K_{0,-c} \tag{42}$$

where $h = g - Ac$. w_0^ε satisfies the corresponding V.I. with h replaced by $h + \varepsilon$.

Proof. Since $w^g \in K_{c,0}$ clearly $w_0 \in K_{0,-c}$. If $v \in K_{0,-c}$ then $u = v + c \in K_{c,0}$. Then

$$\begin{aligned} a(w_0, v - w_0) &= a(w^g - c, (u - c) - (w^g - c)) \\ &= a(w^g - c, u - w^g) \\ &= a(w^g, u - w^g) - a(c, u - w^g) \\ &\geq (g \mid u - w^g) - (Ac \mid u - w^g) \\ &= (g - Ac \mid u - w^g) \\ &= (g - Ac \mid v - w_0). \end{aligned}$$

□

It is clear that

$$w^g(x) = c(x) \iff w_0(x) = 0, \quad w^{g+\varepsilon}(x) = c(x) \iff w_0^\varepsilon(x) = 0.$$

We may therefore consider the problem for w_0, w^ε which is a bit simpler. We will use the concept of subsolutions to V.I.'s which we now define.

5.6. Definition. *Consider the following V.I: Find $w \in K_{0,\psi}$, s.t.*

(43) $$a(w, v - w) \geq (h \mid v - w) \quad \textit{for all } v \in K_{0,\psi}.$$

$v_s \in K_{0,\psi}$ is called a subsolution to this problem if for all $v \in H_0^1(\mathcal{O})$ with $v \geq 0$,

$$a(v_s, v) \leq (h \mid v).$$

5.7. Remark. If $v_s \in H^2(\mathcal{O}) \cap K_{0,\psi}$ satisfies

$$Av_s \leq h, \quad v_s \leq 0$$

a.e. then v_s is a subsolution. This follows from the relation

$$a(v_s, v) = (Av_s \mid v).$$

5.8. Lemma. *If v_s is a subsolution to (43) and w is the solution, then $v_s \leq w$ a.e.*

Proof. The proof follows the proof of thm. 4.5.7. in [Rod87]. We have

(44) $$a(w, v - w) \geq (h \mid v - w) \quad \text{for all } v \in K_{0,\psi}.$$

and

(45) $$a(v_s, v) \leq (h \mid v) \quad \text{for all } v \in H_0^1(\mathcal{O}) \text{ with } v \geq 0.$$

Now let $v = w \vee v_s$. Then $v \in K_{0,\psi}$, and

$$v = w + (v_s - w)^+.$$

Inserting v in (44) we get

$$a(w, (v_s - w)^+) \geq (h \,|\, (v_s - w)^+). \tag{46}$$

By (45) we get

$$a(v_s, (v_s - w)^+) \leq (h \,|\, (v_s - w)^+). \tag{47}$$

Subtracting (46) from (47) we get

$$a(v_s - w, (v_s - w)^+) \leq 0. \tag{48}$$

Now for $u \in H^1(\mathcal{O})$ one has

$$a(u^-, u^+) = 0,$$

Then, since a is coercive,

$$a(v, v^+) = a(v^+, v^+) \geq \nu \|v^+\|_{H^1_0(\mathcal{O})}.$$

Now (48) implies $(v_s - w)^+ = 0$, i.e $v_s \leq w$ □

We would like to produce subsolutions of a particular kind. Then it is useful to be able to construct perturbations of the identity map on $\mathcal{O}$ as in the following lemma.

5.9. Lemma. *Assume we are given $x_0 \in \mathcal{O}$. Then there are constants δ_0, C s.t. for any $\delta < \delta_0$ and any $y_0 \in B(x_0, \delta)$ there is a C^∞ map $p : \mathcal{O} \to \mathcal{O}$ such that*

$$\begin{aligned}
&p(y_0) = x_0, \quad \textit{and writing } p(x) = x + \sigma(x) \textit{ we have}\\
&|\sigma(x)| \leq \delta \quad \forall x \in \mathcal{O}\\
&\operatorname{supp} \sigma_j \subset B(x_0, 2\delta_0) \quad \textit{for } j = 1, \ldots, d\\
&|\partial_{x_i} \sigma_j| \leq C\delta \quad \textit{for } i, j = 1, \ldots, d\\
&|\partial_{x_i x_i} \sigma_j| \leq C\delta \quad \textit{for } i, j = 1, \ldots, d
\end{aligned}$$

Proof. Let $\delta_0 = \frac{1}{3} \operatorname{dist}(x_0, \partial\mathcal{O})$. Then let $\xi \in C_0^\infty(\mathcal{O})$ be s.t.

$$\begin{aligned}
&\xi = 1 \quad \text{on } B(x_0, \delta_0),\\
&0 \leq \xi \leq 1 \quad \text{and}\\
&\operatorname{supp} \xi \subset B(x_0, 2\delta_0).
\end{aligned}$$

Let

$$C = \max_{1 \le |\alpha| \le 2} \|\partial^\alpha \xi\|_\infty.$$

Now, if $\delta < \delta_0$ and $y_0 \in B(x_0, \delta)$, define

$$\sigma(x) = \xi(x)(x_0 - y_0).$$

Then let $p(x) = x + \sigma(x)$. The properties asserted in the lemma are then easily verified. □

We are now ready to prove the following central lemma.

5.10. Lemma. *Let $w_0, w_0^\xi \in K_{0,-c}$ be as in lemma 5.5. Then, if $x_0 \in \mathcal{O}$ and $w_0(x_0) = 0$ there is a $\delta > 0$ such that $w_0^\xi(x) = 0$ for all $x \in B(x_0, \delta)$.*

Proof. Let δ_0 be the constant from lemma 5.9. Let $\delta < \delta_0$, and let $y_0 \in B(x_0, \delta)$. Let $p : \mathcal{O} \to \mathcal{O}$ be a map with the properties given in lemma 5.9. We shall also write $p(x) = x + \sigma(x)$. Now we define $v \in H^2(\mathcal{O})$ by

$$v(x) = w_0(p(x)).$$

Then v is as regular as w_0, in particular v is continuous. The idea now is to show that v is a subsolution to the V.I. for which w_0^ξ is the solution, thus obtaining $v \le w_0^\xi$. Then we use that $v(y_0) = w_0(x_0) = 0$ to obtain

$$0 = v(y_0) \le w_0^\xi(y_0) \le 0.$$

We show that this can be done provided δ is sufficiently small.

To show that v is such a subsolution we compute Av and use remark 5.7. Direct computation yields

$$\partial_{x_j} v\,(x) = \sum_{k=1}^{d} \partial_{x_k} w_0\,(p(x)) \partial_{x_j} p_k\,(x)$$

and

$$\partial_{x_i x_j} v\,(x) = \sum_{k,l=1}^{d} \partial_{x_l x_k} w_0\,(p(x)) \partial_{x_i} p_l\,(x) \partial_{x_j} p_k\,(x) + \sum_{k=1}^{d} \partial_{x_k} w_0\,(p(x)) \partial_{x_i x_j} p_k\,(x).$$

We may now use that

$$\partial_{x_i} p_k\,(x) = \begin{cases} 1 + \partial_{x_i} \sigma_i\,(x) & \text{if } i = k, \\ \partial_{x_i} \sigma_k\,(x) & \text{if } i \ne k. \end{cases}$$

and that $w_0 \in W^{2,\infty}_{loc}(\mathcal{O})$ to write

$$\partial_{x_j} v\,(x) = \partial_{x_j} w_0\,(p(x)) + \sum_{k=1}^{d} \partial_{x_k} w_0\,(p(x))\partial_{x_j}\sigma_k\,(x) = \partial_{x_j} w_0\,(p(x)) + r_j(x)$$

and

$$\begin{aligned}\partial_{x_i x_j} v\,(x) &= \partial_{x_i x_j} w_0\,(p(x)) \\ &\quad + \sum_{(k,l)\neq(i,j)} \partial_{x_l x_k} w_0\,(p(x))\partial_{x_i} p_l\,(x)\partial_{x_j} p_k\,(x) + \sum_{k=1}^{d} \partial_{x_k} w_0\,(p(x))\partial_{x_i x_j}\sigma_k\,(x) \\ &= \partial_{x_i x_j} w_0\,(p(x)) + R_{ij}(x),\end{aligned}$$

where for some constant M, (depending on $\|w_0\|_{2,\infty,B(x_0,\delta_0)}$, not on δ, i, j), we have

$$|R_{ij}| < M\delta, \quad |r_j| < M\delta$$

for all i, j.

We have used that $\operatorname{supp}\sigma_k \subset B(x_0, \delta_0)$, for $k = 1, \ldots, d$, and that $\partial_{x_i} p_l\,(x)\partial_{x_j} p_k\,(x)$ involves at least one factor $\partial_{x_i}\sigma_l\,(x)$ or $\partial_{x_j}\sigma_k\,(x)$ if $(k, l) \neq (i, j)$. We also use the estimates from lemma 5.9.

Thus

$$\begin{aligned}Av(x) &= (Aw_0)(p(x)) + \sum_{i,j=1}^{d} a_{ij}(x)R_{ij}(x) + \sum_{j=1}^{d} b_j(x)r_j(x) \\ &= (Aw_0)(p(x)) + R(x).\end{aligned}$$

Since a_{ij}, b_j are bounded there is a $\hat{\delta} > 0$ such that

$$\|R\|_\infty \leq \frac{\varepsilon}{2}$$

when $\delta < \hat{\delta}$. Since $h = g - Ac$ is uniformly continuous, there is a $\tilde{\delta} > 0$ such that for any $x \in \mathcal{O}$,

$$h(p(x)) = h(x + \sigma(x)) \leq h(x) + \frac{\varepsilon}{2}$$

when $\delta < \hat{\delta}$. Then if $\delta < \hat{\delta} \wedge \tilde{\delta}$, we get

$$\begin{aligned}Av(x) &= Aw_0(p(x)) + R(x) \\ &\leq h(p(x)) + R(x) \\ &\leq h(x) + \frac{\varepsilon}{2} + R(x) \\ &\leq h(x) + \varepsilon.\end{aligned}$$

Thus we have

$$v \leq 0, \quad Av \leq h + \varepsilon.$$

This means v is a subsolution to (42) with h replaced by $h + \varepsilon$. By lemma 5.8 $v \leq w_0^\varepsilon$. Hence, if $\delta < \hat{\delta} \wedge \tilde{\delta}$ and $y_0 \in B(x_0, \delta)$, we have a continuous v satisfying

$$v(y_0) = w_0(p(y_0)) = w_0(x_0) = 0.$$

and thus

$$0 = v(y_0) \leq w_0^\varepsilon(y_0) \leq 0.$$

We have shown $w_0^\varepsilon(y_0) = 0$ for all $y_0 \in B(x_0, \delta)$. □

It is now easy to prove lemma 5.3.

Proof of lemma 5.3.

Proof. Recall that we have

$$w^g = w_0 + c, \quad w^{g+\varepsilon} = w_0^\varepsilon + c.$$

So, if $w^g(x_0) = c(x_0)$, then $w_0(x_0) = 0$. Then by lemma 5.10, $w_0^\varepsilon(x) = 0$ for all x in a ball around x_0, and then $w^{g+\varepsilon}(x) = c(x)$ on that ball. □

With the aid of lemma 5.3 we can complete the proof that φ is continuous.

5.11. Lemma. *φ is continuous.*

Proof. There remains to show that φ is lower semicontinuous at any x_0 where $\varphi(x_0) > 0$. Pick such x_0. Let $u_0 = \varphi(x_0)$. Then $w(x_0, u_0) = c(x_0)$. By our assumptions in section 2, if $\alpha > 0$, there is some $\varepsilon > 0$, such that

$$g(x, u_0 - \alpha) > g(x, u_0) + \varepsilon$$

for all $x \in \mathcal{O}$. By monotonicity, we have

$$c(x) \geq w(x, u_0 - \alpha) = w^{g(\cdot, u_0 - \alpha)}(x) \geq w^{g(\cdot, u_0) + \varepsilon}(x) \tag{49}$$

for all $x \in \mathcal{O}$. Since $w(x_0, u_0) = w^{g(\cdot, u_0)}(x_0) = c(x_0)$ we have by lemma 5.3 that for some $\delta > 0$,

$$w^{g(\cdot, u_0) + \varepsilon}(x) = c(x)$$

for all $x \in B(x_0, \delta)$. By (49) we get

$$w(x, u_0 - \alpha) = c(x)$$

for all $x \in B(x_0, \delta)$ and this gives

$$\varphi(x) \geq u_0 - \alpha = \varphi(x_0) - \alpha$$

for all $x \in B(x_0, \delta)$. So, φ is lower semicontinuous at x_0. □

5.2 An Itô formula for the value function.

The purpose of this section is to establish part a) of theorem 4.13. It is clear that v as defined in (39) satisfies part 1. of definition 3.3. We have to show (15) for our v.

The proof is split into several lemmas starting with a uniform boundedness result on w.

5.12. Lemma. *Let* $1 < p < \infty$, $M > 0$. *Then for all* $u \in [0, M]$,

$$w(\cdot, u) \in W^{2,p}(\mathcal{O}),$$

and there is a constant $C = C(p, M)$, *such that*

$$\|w(\cdot, u)\|_{2,p} \leq C. \tag{50}$$

Proof. In the notation of section 4.1 we have

$$w(x, u) = u^{g(\cdot,u)}(x).$$

The first statement is just (35) again. By lemma 4.9 we have for any u

$$g(\cdot, u) \wedge Ac \leq Aw(\cdot, u) \leq g(\cdot, u)$$

holding a.e. in $\mathcal{O}$. By our initial assumptions, $Ac \in C(\overline{\mathcal{O}})$ and $g \in C(\overline{\mathcal{O}} \times [0, M]))$. Then there is a constant C_0 independent of u s.t.

$$\|Aw(\cdot, u)\|_{L^p(\mathcal{O})} \leq C_0$$

for all $u \in [0, M]$. By a Sobolev inequality (see [Fri76] theorem 10.3.2.) there is a C satisfying (50). □

We shall also use the following general result.

5.13. Lemma. *Let* $M > 0$.

(a) *If* $\psi \in L^p(\mathcal{O})$, $p > \frac{d}{2} + 1$ *and* $k \in \mathbb{N}$ *there is a* $C > 0$ *s.t. for any stopping time* $\tau \leq \tau_{\mathcal{O}}$ *and any* $x \in \mathcal{O}$,

$$|E^x[\int_0^{\tau\wedge k} e^{-rs}\psi(X_s)\, ds]| \leq C\|\psi\|_{L^p(\mathcal{O})}$$

(b) *If* $v : \mathcal{O} \times [0, M] \to \mathbb{R}$ *is measurable,* $p > \frac{d}{2} + 1$ *and for some* $H \in L^p(\mathcal{O})$ *we have* $|v| \leq H$ *then there is a* $C > 0$ *s.t. for any stopping time* $\tau \leq \tau_{\mathcal{O}}$, *any* $x \in \mathcal{O}$ *and any measurable process* U *with values in* $[0, M]$,

$$|E^x[\int_0^{\tau\wedge k} e^{-rs}v(X_s, U_s)\, ds]| \leq C\|H\|_{L^p(\mathcal{O})}$$

(c) *If $\psi \in L^\infty(\mathcal{O} \times [0, M])$ then for any nondecreasing RCLL process with values in $[0, M]$ we have*

$$|\int_0^{\tau_\mathcal{O}} \psi(X_s, U_s)\, dU_s| \leq M\|\psi\|_{L^\infty(\mathcal{O}\times[0,M])}.$$

Proof. (a) is direct from [BL82], lemma 2.8.1. (b) follows from (a). (c) is trivial. □

We shall first assume that $h = 0$ in (39) i.e.

$$v(x, u) = \int_0^u w(x, r)\, dr. \tag{51}$$

In order to prove (15) for this v, we approximate v by smoother functions v_n for which (15) is well known. Some care has to be exercised when constructing v_n, our construction goes as follows.

If $\xi : \mathcal{O} \times [0, \infty) \to \mathbb{R}$, and $\psi : \mathbb{R}^d \to \mathbb{R}$ we write

$$\psi \star_d \xi(x, u) \triangleq \int_{\mathbb{R}^d} \psi(x - y)\tilde{\xi}(y, u)\, dy,$$

where $\tilde{\xi}$ is the zero extension of ξ to $\mathbb{R}^d \times [0, \infty)$. Hence $\star_d$ denotes the convolution product in the first d variables $x_1, \dots, x_d$.

Let $\rho \in C_0^\infty(\mathbb{R}^d)$ be nonnegative and s.t.

$$\int_{\mathbb{R}^d} \rho(x)\, dx = 1, \quad \operatorname{supp} \rho \subset B(0, 1).$$

For $n \in \mathbb{N}$, let $\rho_n(x) \triangleq n\rho(nx)$. Then we define

$$w_n(x, u) \triangleq \rho_n \star_d w(x, u) \quad \text{and } v_n(x, u) \triangleq \int_0^u w_n(x, r)\, dr. \tag{52}$$

5.14. Lemma.

(a) $v_n \in C^{2,1}(\mathbb{R}^d \times [0, \infty))$.

(b) *For any $M > 0$, $v_n \to v$ and $\partial_u v_n \to \partial_u v$ uniformly on $\overline{\mathcal{O}} \times [0, M]$ as $n \to \infty$.*

Proof. Recall that $w : \overline{\mathcal{O}} \times [0, \infty) \to \mathbb{R}$ is continuous and $w(x.u) = 0$ for $x \in \partial\mathcal{O}$. Let $\{(x_k, u_k)\}$ be a sequence in $\mathbb{R}^d \times [0, \infty)$ converging to (x, u) as $k \to \infty$. Then

$$\begin{aligned} w_n(x_k, u_k) &= \int_{\mathbb{R}^d} \rho_n(x_k - y)\tilde{w}(y, u_k)\, dy \\ &= \int_{\mathcal{O}} \rho_n(x_k - y) w(y, u_k)\, dy \\ &\to \int_{\mathcal{O}} \rho_n(x - y) w(y, u)\, dy = w_n(x, u) \end{aligned}$$

by bounded convergence. (Note that $\exists$ a compact K and $M > 0$ s.t $\forall k,\ (x_k, u_k) \in K \times [0, M]$. Then $\forall k$

$$|\rho_n(x_k - y) w(y, u_k)| \leq \|\rho_n\|_{\infty, K} \|w\|_{\infty, \overline{\mathcal{O}} \times [0, M]} \leq C.$$

Together with the continuity of $(x, u) \mapsto \rho_n(x - y) w(y, u)$, this justifies the above.)

Hence $w_n = \partial_u v_n$ is continuous on $\mathbb{R}^d \times [0, \infty)$. As is well known, for any $\alpha \in \mathbb{N}_0^d$,

$$\partial_x^\alpha w_n(x, u) = (\partial_x^\alpha \rho_n) \star_d w(x, u).$$

As above one sees that $\partial_x^\alpha w_n$ is jointly continuous in (x, u), and we get

$$\partial_x^\alpha v_n(x, u) = \partial_x^\alpha \int_0^u w_n(x, r)\, dr = \int_0^u \partial_x^\alpha w_n(x, r)\, dr,$$

showing that $\partial_x^\alpha v_n$ is continuous. This proves (a). For part (b), if $(x, u) \in \overline{\mathcal{O}} \times [0, M]$ we get

$$\begin{aligned} |w_n(x, u) - w(x, u)| &\leq \int_{\mathbb{R}^d} \rho_n(y) |\tilde{w}(x - y, u) - \tilde{w}(x, u)|\, dy \\ &\leq \sup_{\substack{y \in B(0, \frac{1}{n}) \\ u \in [0, M]}} |\tilde{w}(x - y, u) - \tilde{w}(x.u)| \leq \varepsilon(n), \end{aligned}$$

with $\varepsilon(n)$ independent of (x, u). The last inequality follows from the fact that $\tilde{w}$ is uniformly continuous on $\mathbb{R}^d \times [0, M]$. For the same reason we may take $\varepsilon(n)$ converging to 0 as $n \to \infty$. Thus $w_n \to w$, uniformly on $\overline{\mathcal{O}} \times [0, M]$. Now (b) follows easily. $\square$

We also want to have Av_n approximating Av in some suitable sense. It turns out to be necessary to have the convergence given in lemma 5.17 b) below. This is why we have chosen v_n the way we have. It is fairly easy to obtain the convergence

$$Av_n \to Av \quad \text{in } L^p(\mathcal{O} \times [0, M])$$

for any p, by a simpler approximation, but this is not suitable for the result we want. We need to go throuh some extra complications to get the stronger convergence as mentioned.

5.15. Lemma. *Assume $\xi : \mathcal{O} \times [0, M] \to \mathbb{R}$ is measurable and s.t.*

$$\|\xi(\cdot, u)\|_{L^p(\mathcal{O})} \le C$$

for all $u \in [0, M]$, with C independent of u. let

$$\xi_n(x, u) = \rho_n \star_d \xi(x, u).$$

If $\mathcal{O}_0 \subset \mathcal{O}$ is a domain, let

$$\Psi(x, u) = \int_0^u \xi(x, r)\, dr\ ,\ \ \Psi_n(x, u) = \int_0^u \xi_n(x, r)\, dr$$

for $x \in \mathcal{O}_0$. Then there are $H_n \in L^p(\mathcal{O}_0)$, s.t. $H_n \to 0$ in $L^p(\mathcal{O}_0)$ and s.t.

$$|\Psi(x, u) - \Psi_n(x, u)| \le H_n(x) \tag{53}$$

for a.a. $x \in \mathcal{O}_0$, all $u \in [0, M]$.

Proof. We prove this in two steps.

Step 1. For any $u \in [0, M]$ and any n we have

$$\begin{aligned} G_n(u) &\triangleq \|\xi_n(\cdot, u) - \xi(\cdot, u)\|^p_{L^p(\mathcal{O}_0)} \\ &\le (\|\xi_n(\cdot, u)\|_{L^p(\mathcal{O}_0)} + \|\xi(\cdot, u)\|_{L^p(\mathcal{O}_0)})^p \\ &\le (\|\xi(\cdot, u)\|_{L^p(\mathcal{O}_0)} \|\rho_n\|_{L^1(\mathbf{R}^d)} + \|\xi(\cdot, u)\|_{L^p(\mathcal{O}_0)})^p \le 2^p C^p, \end{aligned}$$

where we have used Young's inequality.

It is well known that for each u, $\xi_n(\cdot, u) \to \xi(\cdot, u)$ in $L^p(\mathcal{O})$ and so in $L^p(\mathcal{O}_0)$ as $n \to \infty$. Hence $G_n(u) \to 0$ boundedly on $[0, M]$.

Step 2. Let

$$H_n(x) = \int_0^M |\xi_n(x, r) - \xi(x, r)|\, dr \ge |\Psi_n(x, r) - \Psi(x, r)|,$$

so that (53) is satisfied. Using Hölders inequality in the form

$$\|f\|^p_{L^1([0,M])} \le \|f\|^p_{L^p([0,M])} \|1\|^p_{L^q([0,M])},$$

where $p^{-1} + q^{-1} = 1$, we find

$$\begin{aligned}
\|H_n\|^p_{L^p(\mathcal{O}_0)} &= \int_{\mathcal{O}_0} \Big| \int_0^M |\xi_n(x,r) - \xi(x,r)|\,dr\Big|^p\,dx \\
&\le \int_{\mathcal{O}_0} \Big(\int_0^M |\xi_n(x,r) - \xi(x,r)|^p\,dr\Big)\Big(\int_0^M 1^q\,dr\Big)^{\frac{p}{q}}\Big)\,dx \\
&= M^{\frac{p}{q}} \int_0^M \int_{\mathcal{O}_0} |\xi_n(x,r) - \xi(x,r)|^p\,dx\,dr \\
&= M^{\frac{p}{q}} \int_0^M \|\xi_n(\cdot,u) - \xi(\cdot,u)\|^p_{L^p(\mathcal{O}_0)}\,dr \\
&= M^{\frac{p}{q}} \int_0^M G_n(r)\,dr.
\end{aligned}$$

From step 1 and the dominated convergence theorem, it now follows that

$$\|H_n\|_{L^p(\mathcal{O}_0)} \to 0$$

as $n \to \infty$. □

5.16. Lemma. *For $1 < p < \infty$ we have $v(\cdot,u) \in W^{2,p}(\mathcal{O})$ and for any multi-index $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le 2$ we have*

$$\partial_x^\alpha v(x,u) = \int_0^u \partial_x^\alpha w(x,r)\,dr.$$

Proof. Pick a $\varphi \in C_0^\infty(\mathcal{O})$. Then if $|\alpha| \le 2$,

$$\begin{aligned}
(-1)^\alpha \int_{\mathcal{O}} v(x,u)\partial_x^\alpha \varphi(x)\,dx &= (-1)^\alpha \int_{\mathcal{O}} \int_0^u w(x,r)\partial_x^\alpha\varphi(x)\,dr\,dx \\
&= \int_0^u (-1)^\alpha \int_{\mathcal{O}} w(x,r)\partial_x^\alpha\varphi(x)\,dx\,dr \\
&= \int_0^u \int_{\mathcal{O}} \partial_x^\alpha w(x,r)\varphi(x)\,dx\,dr \\
&= \int_{\mathcal{O}} \varphi(x) \int_0^u \partial_x^\alpha w(x,r)\,dr\,dx
\end{aligned}$$

The order of integration may be changed as above, since by Hölders inequality

$$\int_0^u \int_{\mathcal{O}} |\partial_x^\alpha w(x,r)\varphi(x)|\,dx\,dr \le \int_0^u \|\partial_x^\alpha w(\cdot,r)\|_{L^p(\mathcal{O})} \|\varphi\|_{L^q(\mathcal{O})}\,dr \le MCD < \infty.$$

The above calculations show that if we let

$$\eta(x,u) = \int_0^u \partial_x^\alpha w(x,r)\,dr,$$

then $\eta(\cdot, u)$ is the α'th distributional derivative of $v(\cdot, u)$. To see that $\eta(\cdot, u) \in L^p(\mathcal{O})$ we once again use Hölders inequality as follows.

$$
\begin{aligned}
\|\eta(\cdot, u)\|^p_{L^p(\mathcal{O})} &= \int_{\mathcal{O}} |\eta(x,u)|^p \, dx \\
&= \int_{\mathcal{O}} |\int_0^u \partial_x^\alpha w(x,r)\, dr|^p \, dx \\
&\leq \int_{\mathcal{O}} (\int_0^u |\partial_x^\alpha w(x,r)|\, dr)^p \, dx \\
&\leq \int_{\mathcal{O}} (\int_0^u |\partial_x^\alpha w(x,r)|^p\, dr)(\int_0^u 1^q \, dr)^{\frac{p}{q}} \, dx \\
&\leq u^{\frac{p}{q}} \int_{\mathcal{O}} \int_0^u |\partial_x^\alpha w(x,r)|^p \, dr\, dx \\
&\leq M^{\frac{p}{q}} \int_0^u \|\partial_x^\alpha w(x,r)\|^p_{L^p(\mathcal{O})} \, dr \\
&\leq M^{\frac{p}{q}+1} C^p.
\end{aligned}
$$

This shows that $v(\cdot, u) \in W^{2,p}(\mathcal{O})$ for all $u \in [0, M]$.

□

5.17. Lemma. *Let v, v_n be as before. Let $M > 0$ and let $\mathcal{O}_0 \subset\subset \mathcal{O}$ be an open domain. Then if $1 \leq p < \infty$,*

(a) *For all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq 2$ there are $H_n^\alpha \in L^p(\mathcal{O}_0)$ s.t. $H_n^\alpha \to 0$ in $L^p(\mathcal{O}_0)$, and s.t*

$$|\partial_x^\alpha v(x,u) - \partial_x^\alpha v_n(x,u)| \leq H_n^\alpha(x)$$

for a.a. $x \in \mathcal{O}_0$ and all $u \in [0, M]$.

(b) *There are $H_n \in L^p(\mathcal{O}_0)$ s.t. $H_n \to 0$ in $L^p(\mathcal{O}_0)$, and s.t*

$$|Av(x,u) - Av_n(x,u)| \leq H_n(x)$$

for a.a. $x \in \mathcal{O}_0$ and all $u \in [0, M]$.

Proof. (a). Let $\xi(x,u) = \partial_x^\alpha w(x,u)$ for $(x,u) \in \mathcal{O} \times [0, M]$. Then let

$$\xi_n(x,u) = \rho_n \star_d \xi(x,u) = \rho_n \star_d \partial_x^\alpha w(x,u).$$

We have

$$w_n(x,u) = \rho_n \star_d w(x,u),$$

and if $\frac{1}{n} < \mathrm{dist}(x.\partial\mathcal{O})$ we get

$$\partial_x^\alpha w_n(x,u) = \rho_n \star_d \partial_x^\alpha w(x,u) = \rho_n \star_d \xi(x,u).$$

(The first of these equations does not hold on all of $\mathcal{O}$ in general!)[2] so for n large we have

$$\partial_x^\alpha w_n(x,u) = \rho_n \star_d \xi(x,u) = \xi_n(x,u) \tag{54}$$

for a.a. $x \in \mathcal{O}_0$. Now let

$$\Psi(x,u) = \partial_x^\alpha v(x,u), \quad \text{and } \Psi_n(x,u) = \partial_x^\alpha v_n(x,u).$$

By lemma 5.16,

$$\Psi(x,u) = \int_0^u \partial_x^\alpha w(x,r)\,dr, \quad \Psi_n(x,u) = \int_0^u \partial_x^\alpha w_n(x,r)\,dr,$$

then by the definition of ξ and by (54), we get for large n and $x \in \mathcal{O}_0$

$$\Psi(x,u) = \int_0^u \xi(x,r)\,dr, \quad \Psi_n(x,u) = \int_0^u \xi_n(x,r)\,dr.$$

Now by lemma 5.15 we know there is $H_n^\alpha \in L^p(\mathcal{O}_0)$, converging to 0 and s.t.

$$|\partial_x^\alpha v(x,u) - \partial_x^\alpha v_n(x,u)| = |\Psi(x,u) - \Psi_n(x,u)| \le H_n^\alpha(x).$$

(b) Follows from (a) since the coefficients in A are bounded. □

Now we may start to prove (15).

5.18. Lemma. *Assume $\mathcal{O}_0 \subset\subset \mathcal{O}$, and let $\tau \le \tau_{\mathcal{O}_0}$ be a stopping time. Then, for any $k \in \mathbb{N}$,*

$$\begin{aligned} &E^x[e^{-r(\tau\wedge k)} v(X_{\tau\wedge k}, U_{\tau\wedge k})] \\ &\quad = v(x,U_0) - E^x\Big[\int_0^{\tau\wedge k} e^{-rs} A v(X_s,U_s)\,ds - \int_0^{\tau\wedge k} e^{-rs}\partial_u v\,(X_s,U_s)\,dU_s\Big] \end{aligned} \tag{55}$$

for any continuous $U \in \mathcal{A}_u$.

Proof. Let $M > 0$ be s.t. $U_t \in [0,M]$ for all t a.s. w is jointly continuous on $\overline{\mathcal{O}} \times [0,\infty)$ and $w(x,u) = 0$ if $x \in \partial\mathcal{O}$, so by lemma 5.14 (a) and Itô's formula,

$$\begin{aligned} &E^x[e^{-r(\tau\wedge k)} v_n(X_{\tau\wedge k}, U_{\tau\wedge k})] \\ &\quad = v_n(x,u) - E^x\Big[\int_0^{\tau\wedge k} e^{-rs} A v_n(X_s,U_s)\,ds + \int_0^{\tau\wedge k} e^{-rs}\partial_u v_n\,(X_s,U_s)\,dU_s\Big] \end{aligned}$$

[2]example: $\mathcal{O} = (0,1)$, $w = 1$ on $\mathcal{O}$.

We then verify that each of the above terms converges to the corresponding term in (55) as $n \to \infty$.

By lemma 5.14 (b) and dominated convergence we get

$$E^x[e^{-r(\tau\wedge k)}v_n(X_{\tau\wedge k}, U_{\tau\wedge k})] \to E^x[e^{-r(\tau\wedge k)}v(X_{\tau\wedge k}, U_{\tau\wedge k})].$$

The same lemma also gives

$$v_n(x, u) \to v(x, u).$$

lemma 5.14 (b) and lemma 5.13 (c) give

$$E^x[\int_0^{\tau\wedge k} e^{-rs}|\partial_u v_n\,(X_s, U_s) - \partial_u v\,(X_s, U_s)|\, dU_s] \leq M\|\partial_u v_n\; - \partial_u v\;\|_{L^\infty(\mathcal{O}\times[0,M])} \to 0$$

and so

$$E^x[\int_0^{\tau\wedge k} e^{-rs}\partial_u v_n\,(X_s, U_s)\, dU_s] \to E^x[\int_0^{\tau\wedge k} e^{-rs}\partial_u v\,(X_s, U_s)\, dU_s].$$

Finally we have by lemma 5.17 b),

$$E^x[\int_0^{\tau\wedge k} e^{-rs}|Av_n(X_s, U_s) - Av(X_s, U_s)|\, ds] \leq E^x[\int_0^{\tau\wedge k} e^{-rs}H_n(X_s)\, ds]$$

for some $H_n \in L^p(\mathcal{O}_0)$ with $H_n \to 0$ as $n \to \infty$. By lemma 5.13 (a), the right hand side $\to 0$ and so

$$E^x[\int_0^{\tau\wedge k} e^{-rs}Av_n(X_s, U_s)\, ds] \to E^x[\int_0^{\tau\wedge k} e^{-rs}Av(X_s, U_s)\, ds]$$

as $n \to \infty$. □

Next we let $k \to \infty$.

5.19. Lemma. *With notation as in lemma 5.18 we have*

$$(56)\quad E^x[e^{-r\tau}v(X_\tau, U_\tau)] = v(x, U_0) - E^x[\int_0^\tau e^{-rs}Av(X_s, U_s)\, ds - \int_0^\tau e^{-rs}\partial_u v\,(X_s, U_s)\, dU_s]$$

for any continuous $U \in \mathcal{A}_u$.

Proof. Let $M > 0$ be s.t. $U_t \in [0, M]$ for all t a.s. Since $\tau < \infty$ a.s., we get

$$e^{-r(\tau\wedge k)}v(X_{\tau\wedge k}, U_{\tau\wedge k}) \to e^{-r\tau}v(X_\tau, U_\tau)$$

a.s. v is bounded on $\mathcal{O} \times [0, M]$ and by dominated convergence

$$E^x[e^{-r(\tau\wedge k)}v(X_{\tau\wedge k}, U_{\tau\wedge k})] \to E^x[e^{-r\tau}v(X_\tau, U_\tau)].$$

By lemma 5.16

$$Av(x, u) = \int_0^u Aw(x, r)\, dr$$

on $\mathcal{O} \times [0, M]$. Then by lemma 4.9 it follows that Av is bounded on $\mathcal{O} \times [0, M]$. Hence

$$E^x[\int_{\tau\wedge k}^{\tau} e^{-rs}|Av(X_s, U_s)|\, ds] \leq CE^x[\tau - (\tau \wedge k)] \to 0,$$

by dominated convergence, using $E^x[\tau] < \infty$.

For the last integral, we use that $\partial_u v = w$ is bounded on $\mathcal{O} \times [0, M]$ and that U is bounded and continuous, to obtain

$$E^x[\int_{\tau\wedge k}^{\tau} e^{-rs}|\partial_u v\,(X_s, U_s)|\, dU_s] \leq CE^x[U_\tau - U_{\tau\wedge k}] \to 0$$

where once again dominated convergence is applied.

It then follows that each term in (55) converges to the corresponding term in (56) as $k \to \infty$. hence (56) is valid. □

Finally, we restate and prove the major result here.

5.20. Theorem. *For any stopping time $\tau \leq \tau_{\mathcal{O}}$,*

(57) $$E^x[e^{-r\tau}v(X_\tau, U_\tau)] = v(x, U_0) - E^x[\int_0^\tau e^{-rs}Av(X_s, U_s)\, ds + \int_0^\tau e^{-rs}\partial_u v\,(X_s, U_s)\, dU_s]$$

for any continuous $U \in \mathcal{A}_u$.

Proof. Let $\tau \leq \tau_{\mathcal{O}}$ and $x \in \mathcal{O}$. For $k \in \mathbb{N}$, pick open sets $\mathcal{O}_k \subset\subset \mathcal{O}$ s.t for all $x \in \partial\mathcal{O}_k$, $\text{dist}(x, \partial\mathcal{O}) < \frac{1}{k}$. The formula in (57) is valid if τ is replaced with $\tau_k = \tau \wedge \tau_{\mathcal{O}_k}$, by lemma 5.19.

Assume we can show $\tau_k \to \tau$ a.s. Then by continuity

$$e^{-r\tau_k}v(X_{\tau_k}, U_{\tau_k}) \to e^{-r\tau}v(X_\tau, U_\tau)$$

a.s. By dominated convergence,

$$E^x[e^{-r\tau_k}v(X_{\tau_k}, U_{\tau_k})] \to E^x[e^{-r\tau}v(X_\tau, U_\tau)].$$

Moreover, since Av is bounded on $\mathcal{O} \times [0, M]$,

$$E^x[\int_{\tau_k}^{\tau} e^{-rs}|Av(X_s, U_s)|\, ds] \leq CE^x[\tau - \tau_k] \to 0.$$

Similarly we get by continuity and boundedness of U,

$$E^x[\int_{\tau_k}^{\tau} e^{-rs}|\partial_u v\,(X_s, U_s)|\, dU_s] \leq CE^x[U_\tau - U_{\tau_k}] \to 0.$$

Thus, passing to the limit in the formula with τ_k gives (57).

Now we show $\tau_k \to \tau$. Introducing

$$\rho(x) = E^x[\tau_{\mathcal{O}}], \quad \rho_k(x) = E^x[\tau_{\mathcal{O}_k}],$$

we have that

$$(i) \quad L\rho = 1 \quad \text{in } \mathcal{O} \text{ and } \rho(x) = 0 \text{ for } x \in \partial\mathcal{O},$$

and

$$(ii) \quad L\rho_k = 1 \quad \text{in } \mathcal{O}_k \text{ and } \rho_k(x) = 0 \text{ for } x \in \partial\mathcal{O}_k,$$

where $L = -A + r$ is the infinitesimal generator of X. If now $\psi_k = \rho - \rho_k$ in $\mathcal{O}_k$, then $\psi_k \geq 0$ satisfies

$$(iii) \quad L\psi_k = 0 \quad \text{in } \mathcal{O}_k \text{ and } \psi(x) = \rho(x) \text{ for } x \in \partial\mathcal{O}_k.$$

From (i) ρ is small on $\partial\mathcal{O}_k$ for large k. Then by (iii) and the maximum principle,[3]

$$\sup_{x\in\mathcal{O}_k} \psi_k(x) \leq \sup_{x\in\partial\mathcal{O}_k} \rho(x) \leq \varepsilon(k),$$

where $\varepsilon(k) \to 0$ as $k \to \infty$, because $\rho \in C(\overline{\mathcal{O}})$.

We can then conclude that for any $x \in \mathcal{O}$,

$$E^x[\tau_{\mathcal{O}} - \tau_{\mathcal{O}_k}] = \psi_k(x) \to 0$$

[3]See [GT77] theorem 8.1

as $k \to \infty$. Now if $\tau \leq \tau_{\mathcal{O}}$,

$$\tau - \tau_k = (\tau - \tau_{\mathcal{O}_k})1_{[\tau > \tau_{\mathcal{O}_k}]} \leq \tau_{\mathcal{O}} - \tau_{\mathcal{O}_k}.$$

Then

$$E^x[\tau - \tau_k] \to 0$$

And since $\tau_k \leq \tau_{k+1}$ (assuming, as we may, that $\mathcal{O}_k \subset \mathcal{O}_{k+1}$), we get

$$\tau_k \to \tau$$

a.s. □

We have shown (57) for $v(x,u)$ given by (39) when $h = 0$. In the general case h is given by (40), and to obtain (57) in the general case we need only show

$$E^x[e^{-r\tau}h(X_\tau)] = h(x) - E^x[\int_0^\tau e^{-rs}Ah(X_s)]\,ds, \tag{58}$$

for stopping times $\tau \leq \tau_{\mathcal{O}}$. This is a lot easier since we need not consider the U process now. We conclude with a proof of equation (58).

Proof. First of all, h solves a dirichlet problem $Ah = \xi$ where $\xi \in L^p(\mathcal{O})$ for any $p \in (1, \infty)$. Using theorem 2.5.6. [BL82], which is a general regularity result for elliptic equations, we get $h \in W^{2,p}(\mathcal{O})$ for $2 \leq p < \infty$. Then, if $k \in \mathbb{N}$, we get (58) for stopping times of the form $\tau \wedge k$, by the use of theorem 2.8.5 in [BL82]. Using that h and Ah are in $C(\overline{\mathcal{O}})$, we may let $k \to \infty$, and obtain (58) in the general case as in the proof of lemma 5.19. □

That completes the proof of theorem 4.13 a).

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