Proceedings of the Symposium on Mathematical Finance (ed. Edward Lungu), held at the University of Botswana in December 1997.

Stochastic differential games with controls – discussion of a specific example

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Abstract

We give a brief introduction to stochastic differential games with controls and Nash equilibria for such. To illustrate the concepts and methods involved in this theory we study in detail the Nash equilibrium in a specific example.

1 Introduction

Game theory was first formulated in 1944, in the seminal book by the mathematician John von Neumann and the economist Oskar Morgenstern [vNM]. Since then the theory has been extended and refined to cover a number of important applications, particularly in biology and economics. The theory is now widely recognized as a substantial part of both economics and mathematics. In 1994 John Nash, John Harsanyi and Reinhard Selten were awarded the Nobel Prize in Economics for their fundamental contributions to game theory. For more information on game theory we refer to [FT] and the references therein.

In this paper we will concentrate on a special branch of game theory which involves stochastic differential equations: *Stochastic differential games with controls.* This is a part of game theory which is relatively unknown, in spite of its great potential with respect to applications. We first describe the subject in general and then in Section 2 we illustrate the concepts and methods in a specific example. Stochastic differential games where the decision variables are *stopping times* are studied in [BF].

Suppose we are given a pair $X(t) = X(t,\omega) = (X_1(t,\omega), X_2(t,\omega)) \in \mathbf{R}^n$ of stochastic processes $X_1(t,\omega) \in \mathbf{R}^{n_1}$, $X_2(t,\omega) \in \mathbf{R}^{n_2}$ of the form

$$(1.1)\qquad \begin{cases} dX_i(t) = b_i(t, X(t), u(t))dt + \sigma_i(t, X(t), u(t))dB(t)\ ; & i = 1,2 \\ X_i(0) = x_i\ ; \quad i = 1,2 \end{cases}$$

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where $b_i : \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}^{n_i}$ and $\sigma_i : \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}^{n_i \times m}$ are given functions, $n = n_1 + n_2$, $B(t) = B(t, \omega)$ is m-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, $P(B(0) = 0) = 1$. We may regard $X_i(t)$ as the *state* at time t of person number i; $i = 1, 2$. Person number i is at any time t free to choose the value $u_i(t) = u_i(t, \omega)$ from a given set $Z_i \subset \mathbf{R}^{k_i}$, $k = k_1 + k_2$, in order to influence the development of the states. Thus $u_i(t)$ is the *control* process of person number i. We assume that $u_i(t, \cdot)$ is measurable with respect to the σ-algebra $\mathcal{F}_t$ generated by $\{B(s, \cdot); s \leq t\}$, i.e. that the processes $u_i(t)$ are $\mathcal{F}_t$-adapted. We have put $u(t) = (u_1(t), u_2(t))$.

Associated to each choice u_1, u_2 of control processes and initial point $y = (s, x_1, x_2)$ we are given two *payoff functionals* of the form

$$(1.2) \qquad J_i^u(y) = E^y \Big[\int_0^\tau f_i(t, X(t), u(t)) dt + K_i(\tau, X(\tau)) \Big]; \qquad i = 1, 2$$

where E^y denotes the expectation when the process $Y(t) = (s + t, X_1(t), X_2(t))$ starts at $y = (s, x_1, x_2)$ and $\tau = \inf\{t > 0; Y(t) \notin \Gamma\}$ is the first exit time of $Y(t)$ from a given set $\Gamma \subset \mathbf{R}^{1+n_1+n_2}$. Here $f_i : \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}$ and $K_i : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ are given functions called the *payoff rate* and the *bequest function* of person number i. Thus $J_i^u(y)$ represents the *total expected payoff* to person number i if the process starts at y and the two persons apply the controls $u_1(t, \omega), u_2(t, \omega)$, respectively.

Definition 1.1 The system (1.1), together with the payoff functionals (1.2), is called *a stochastic differential game with controls.*

Depending on the situation the family of controls u_1, u_2 considered are usually subject to additional restrictions to the ones mentioned above. In particular, we will here assume that the controls $u_i \in \mathcal{A}_i$ are *Markovian*, in the sense that u_i has the form

$$u_i(t, \omega) = U_i(Y(t, \omega))$$

for some function $U_i : \mathbf{R}^{1+n_1+n_2} \to Z_i$.

This is not a serious restriction. See e.g. [Ø, Chapter 11]. We will not specify the families of controls further in this general introduction, but simply assume that two such sets $\mathcal{A}_1, \mathcal{A}_2$ of *admissible* controls for person number 1 and 2, respectively, are given. For simplicity we will not distinguish between u_i and U_i in the following (although this is a slight abuse of notation).

It is natural to ask what will be the most likely outcome of a stochastic differential game (1.1)–(1.2). One candidate for such an outcome is a *Nash equilibrium*, first formulated (in a different setting) by G. Nash [N].

Definition 1.2 A pair $u^* = (u_1^*(t, \omega), u_2^*(t, \omega))$ is called a *Nash equilibrium* of the stochastic differential game (1.1)–(1.2) if

$$(1.3) \qquad J_1^{u^*}(x) \geq J_1^{u_1, u_2^*}(x) \qquad \text{for all } x,$$

for all admissible stochastic controls u_1 for person number 1 and

$$J_2^{u^*}(x) \geq J_2^{u_1^*,u_2}(x) \qquad \text{for all } x, \tag{1.4}$$

for all admissible stochastic controls u_2 for person number 2.

In other words, $u^* = (u_1^*, u_2^*)$ is a Nash equilibrium if, given that the second person applies her component u_2^* of this control u^*, it is optimal for the first person to apply her component u_1^* of u^*, and vice versa.

In order to find a Nash equilibrium we argue as follows: Suppose that for any choice of admissible control u_2 of person 2 it is possible for person 1 to find a corresponding optimal control $\widehat{u}_1 \in \mathcal{A}_1$ solving the stochastic control problem

$$J_1^{\widehat{u}_1,u_2}(x) = \sup_{u_1} J_1^{u_1,u_2}(x) \tag{1.5}$$

This optimal control u_1 may not depend uniquely on u_2, but let us assume that there exists a measurable selection

$$\widehat{u}_1 = F(u_2)\,. \tag{1.6}$$

Thus F is a measurable function from the space $\mathcal{A}_2$ of admissible controls for person 2 into the space $\mathcal{A}_1$ of admissible controls for person 1. $F(u_2)$ is the optimal response to u_2 for person number 1. Similarly, suppose that for any $u_1 \in \mathcal{A}_1$ there exists an optimal control $\widehat{u}_2$ for the corresponding stochastic control problem for person 2:

$$J_2^{u_1,\widehat{u}_2}(x) = \sup_{u_2} J_2^{u_1,u_2}(x) \tag{1.7}$$

Again we assume that the control $\widehat{u}_2$ is a (measurable) function of u_1:

$$\widehat{u}_2 = G(u_1) \tag{1.8}$$

where $G : \mathcal{A}_1 \to \mathcal{A}_2$.

We now make the following useful observation:

Lemma 1.3 Suppose there is a fixed point $u_1^* \in \mathcal{A}_1$ of the composed map $F \circ G : \mathcal{A}_1 \to \mathcal{A}_1$, i.e. there exists $u_1^* \in \mathcal{A}$, such that

$$F(G(u_1^*)) = u_1^* \tag{1.9}$$

Then

$$u^* := (u_1^*, G(u_1^*))$$

is a Nash equilibrium of the game.

Proof. The proof is immediate: Suppose person number 1 uses the control u_1^*. Then $G(u_1^*)$ is the optimal control for person number 2, by definition of G. Hence (1.4) holds. Similarly, if person number 2 uses the control $G(u_1^*)$ then $F(G(u_1^*)) = u_1^*$ is the optimal control for person number 1. Hence (1.3) holds. □

Remarks. **1)** Note that if (1.9) holds, then the composed map $G \circ F : \mathcal{A}_2 \to \mathcal{A}_2$ also has a fixed point, namely

$$u_2^* = G(u_1^*) \ .$$

This follows from (1.9), which gives

$$G(F(u_2^*)) = G(F(G(u_1^*))) = G(u_1^*) = u_2^* \ .$$

2) Lemma 1.3 confirms – on a heuristic level – that a Nash equilibrium is a likely outcome of a game. This is because fixed points can often be achieved as limits of repeated applications of the given function. If this holds for the function $F \circ G$ we can write

$$u_1^* = \lim_{n\to\infty} (F \circ G)^{(n)}(u_1)$$

where $(F \circ G)^{(n)}$ denotes n times composition of the function $F \circ G$ by itself and u_1 is an arbitrary first choice for player 1. Thus the Nash equilibrium appears as the limiting strategy if each player in turn reacts optimally to the other player's choice, for any initial strategy chosen.

2 An example

In general it is very difficult to find explicitly the Nash equilibrium for stochastic differential games with control. Indeed, from Section 1 we see that such a task involves solving two stochastic control problems and then finding the fixed points of a usually complicated function.

However, there are nontrivial cases which are – more or less – explicitly solvable and to illustrate the concepts and methods involved we present one such example here.

Suppose the processes $X_1(t), X_2(t)$ are 1-dimensional and given by

$$\begin{cases} dX_1(t) = u_1(t)dt + \sigma_{11}dB_1(t) + \sigma_{12}dB_2(t) \\ X_1(0) = x_1 \in \mathbf{R} \end{cases} \tag{2.1}$$

and

$$\begin{cases} dX_2(t) = u_2(t)dt + \sigma_{21}dB_1(t) + \sigma_{22}dB_2(t) \\ X_2(0) = x_2 \in \mathbf{R} \end{cases} \tag{2.2}$$

where $\sigma_{ij} \in \mathbf{R}$ are constants. Here $u_1(t) \in \mathbf{R}$, $u_2(t) \in \mathbf{R}$ are the controls of person number 1, 2, respectively.

Suppose the payoff to person number 1 is given by

$$J_1^{u_1,u_2}(x_1,x_2) = E^{x_1,x_2}\Big[\int_0^T -\alpha_1 u_1^2(t)X_2^2(t)dt + \gamma_1 X_1^2(T)X_2^2(T)\Big] \tag{2.3}$$

and that the payoff to person number 2 is given by

$$J_2^{u_1,u_2}(x_1,x_2) = E^{x_1,x_2}\Big[\int_0^T -\alpha_2 u_2^2(t)X_1^2(t)dt + \gamma_2 X_1^2(T)X_2^2(T)\Big] \tag{2.4}$$

where $\alpha_1, \alpha_2, \gamma_1, \gamma_2$ and T are positive constants.

An interpretation of this stochastic differential game could be the following: $X_i(t)$ represents the state of company i at time t. By investing at the rate $u_i(t)$ the value of $X_i(t)$ is influenced according to (2.1), (2.2). On the other hand, the presence of the other company j in the system "heats up" the economy in such a way that both the *terminal payment* $X_i^2(T)$ to company i and the *energy cost rate* $-u_i^2(t)$ for company i is proportional to the square of the state of company j. This gives the expressions (2.3), (2.4).

To find a Nash equilibrium for this game, let us try controls of the form

$$\widehat{u}_1(t) = \widehat{h}_1(t)X_1(t) \qquad \text{and} \qquad \widehat{u}_2(t) = \widehat{h}_2(t)X_2(t) \tag{2.5}$$

where $\widehat{h}_1(t), \widehat{h}_2(t)$ are deterministic functions of t, to be determined.

Suppose company 2 adopts such a control $\widehat{u}_2(t)$, say $\widehat{u}_2(t) = h_2(t)X_2(t)$. What is the optimal respons for company 1? To answer this, company 1 must solve the stochastic control problem

$$\Phi(s,x_1,x_2) = \sup_{u_1}\Big\{E^{s,x_1,x_2}\Big[\int_s^T -\alpha_1^2 u_1^2(t)X_2^2(t)dt + \gamma_1 X_1^2(T)X_2^2(T)\Big]\Big\}; \qquad s \le T \tag{2.6}$$

when the system is given by

$$dY(t) = (dt, dX_1(t), dX_2(t))$$

with

$$dX_1(t) = u_1(t)dt + \sigma_{11}dB_1(t) + \sigma_{12}dB_2(t)\,; \qquad X_1(0) = x_1 \tag{2.7}$$

$$dX_2(t) = h_2(t)X_2(t)dt + \sigma_{21}dB_1(t) + \sigma_{22}dB_2(t)\,; \qquad X_2(0) = x_2 \tag{2.8}$$

The corresponding Hamilton-Jacobi-Bellman equation for this problem is (see e.g. [Ø, Th. 11.2.2])

$$\sup_{v\in\mathbf{R}}\Big\{-\alpha_1 v^2 x_2^2 + \frac{\partial\phi}{\partial t} + v\frac{\partial\phi}{\partial x_1} + h_2(t)x_2\frac{\partial\phi}{\partial x_2} + \tfrac{1}{2}\sigma_1^2 x_1^2\frac{\partial^2\phi}{\partial x_1^2} + \sigma_1\cdot\sigma_2 x_1 x_2\frac{\partial^2\phi}{\partial x_1\partial x_2} + \tfrac{1}{2}\sigma_2^2 x_2^2\frac{\partial^2\phi}{\partial x_2^2}\Big\} = 0 \qquad \text{for } t < T\,, \tag{2.9}$$

with boundary condition

$$\phi(T,x_1,x_2)=\gamma_1 x_1^2 x_2^2 \tag{2.10}$$

Here $\phi(t,x_1,x_2)$ is a candidate for the value function $\Phi(t,x_1,x_2)$ and σ_1,σ_2 denotes row number 1 and 2, respectively, of the matrix σ, while $\sigma_1\cdot\sigma_2=\sigma_{11}\sigma_{12}+\sigma_{21}\sigma_{22}$ denotes their inner product.

The value of v which maximizes the expression in (2.9) is clearly given by

$$-2\alpha_1 v x_2^2+\frac{\partial\phi}{\partial x_1}=0$$

or

$$v=v^*=\frac{1}{2\alpha_1 x_2^2}\frac{\partial\phi}{\partial x_1} \qquad \text{if } x_2\neq 0\,. \tag{2.11}$$

Substituted into (2.9) this gives

$$\begin{aligned}&\frac{1}{4\alpha_1 x_2^2}\left(\frac{\partial\phi}{\partial x_1}\right)^2+\frac{\partial\phi}{\partial t}+h_2(t)x_2\frac{\partial\phi}{\partial x_2}\\ &+\tfrac{1}{2}\sigma_1^2x_1^2\frac{\partial^2\phi}{\partial x_1^2}+\sigma_1\cdot\sigma_2 x_1x_2\frac{\partial^2\phi}{\partial x_1\partial x_2}+\tfrac{1}{2}\sigma_2^2x_2^2\frac{\partial^2\phi}{\partial x_2^2}=0 \qquad \text{for } t<T\,.\end{aligned} \tag{2.12}$$

Let us try a solution of the form

$$\phi(t,x_1,x_2)=g_1(t)x_1^2x_2^2\,.$$

Then (2.12) and (2.10) give the following equation for $g_1(t)$:

$$g_1^2(t)+g_1'(t)+g_1(t)[2h_2(t)+\sigma_1^2+4\sigma_1\cdot\sigma_2+\sigma_2^2] = 0\,; \qquad t<T \tag{2.13}$$

$$g_1(T) = \gamma_1\,. \tag{2.14}$$

Now suppose this equation has a unique solution $g_1(t)=:(F[h_2])(t)$. Then the function

$$\phi(t,x_1,x_2)=g_1(t)x_1^2x_2^2$$

satisfies all the requirements of the verification theorem for stochastic control [Ø, Th. 11.2.2]. Hence the value function Φ is given by

$$\Phi(s,x_1,x_2)=\phi(s,x_1,x_2)=g_1(s)x_1^2x_2^2 \tag{2.15}$$

and the corresponding optimal control u_1^* of the stochastic control problem (2.6)–(2.8) is by (2.11) given by

$$u_1^*(t)=\frac{1}{\alpha_1}g_1(t)X_1(t) \tag{2.16}$$

Note that this control u_1^* is again of the form (2.5) with $\widehat{h}_1(t)=\frac{1}{\alpha_1}g_1(t)$.

We can now make a similar computation for company 2, assuming that company 1 adopts a control $\widehat{u}_1(t) = h_1(t)X_1(t)$ of the form (2.5). The result is that it is optimal for company 2 to use the control

$$u_2(t) = u_2^*(t) = \frac{1}{\alpha_2} g_2(t) X_2(t)\,, \tag{2.17}$$

where $g_2(t) =: (G[h_1])(t)$ solves the differential equation

$$g_2^2(t) + g_2'(t) + g_2(t)[2h_1(t) + \sigma_1^2 + 4\sigma_1 \cdot \sigma_2 + \sigma_2^2] = 0\,; \qquad t < T \tag{2.18}$$

$$g_2(T) = \gamma_2\,. \tag{2.19}$$

assuming that such a solution exists.

Let $C^1[0,T]$ denote the family of (real) continuously differentiable functions on $[0,T]$.

Combining Corollary 1.3 with the above calculation we now get the following conclusion:

Theorem 2.1 Suppose that for all $h_2 \in C^1[0,T]$ there exists a unique solution $g_1 = F[h_2] \in C^1[0,T]$ of (2.13)–(2.14) and that for all $h_1 \in C^1[0,T]$ there exists a unique solution $g_2 = G[h_1] \in C^1[0,T]$ of (2.18)–(2.19). Moreover, suppose there exists a function $h_1^* \in C^1[0,T]$ such that

$$\frac{1}{\alpha_1} F\Big[\frac{1}{\alpha_2} G[h_1^*]\Big] = h_1^*\,. \tag{2.20}$$

(i.e. h_1^* is a fixed point of $\frac{1}{\alpha_1}F \circ \frac{1}{\alpha_2}G$). Then the pair (u_1^*, u_2^*) of stochastic controls given by

$$u_1^*(t) = h_1^*(t)X_1(t), \qquad u_2^*(t) = \frac{1}{\alpha_2}(G[h_1^*])(t)X_2(t)$$

is a Nash equilibrium of the stochastic differential game (2.1)–(2.4).

Finally we turn to the question of the existence of a fixed point h_1^*. First, suppose we try to find $\widehat{h}_1$ such that

$$\frac{1}{\alpha_2} G[\widehat{h}_1] = k\,\widehat{h}_1 \qquad \text{for some } k \in \mathbf{R} \setminus \{0\}. \tag{2.21}$$

Then, with $y_1 = \frac{1}{\theta}\widehat{h}_1$, $\theta = \frac{1}{k\alpha_2}$ this gives

$$G[\theta\, y_1] = y_1$$

To find such a function y_1 we must solve the differential equation (2.18)–(2.19) with $h_1 = \theta\, y_1$ and $g_2 = y_1$. The following useful result is easy to verify:

Lemma 2.2 Suppose θ, λ, γ and T are constants, $\lambda \neq 0$, $\theta > 0$. The differential equation

$$\begin{cases} y^2(t) + y'(t) + y(t)[2\theta y(t) + \lambda] = 0 \quad ; \quad t < T \\ y(T) = \gamma \end{cases} \tag{2.22}$$

has the unique solution

$$y(t) = \frac{\lambda\gamma}{[(1+2\theta)\gamma + \lambda]e^{\lambda(t-T)} - (1+2\theta)\gamma} \ ; \qquad t \leq T \tag{2.23}$$

Applying this to equation (2.21) with $\lambda = \sigma_1^2 + 4\sigma_1 \cdot \sigma_2 + \sigma_2^2$ we get the solution

$$\widehat{h}_1 = \theta\, y_1 \ , \qquad \theta = \frac{1}{k\alpha_2} \tag{2.24}$$

where

$$y_1(t) = \frac{\lambda\gamma_2}{[(1+2\theta)\gamma_2 + \lambda]e^{\lambda(t-T)} - (1+2\theta)\gamma_2} \tag{2.25}$$

Suppose we search for a solution $h_1^* \in C^1[0,T]$ of the equation (2.20) with the property that

$$\frac{1}{\alpha_2} G[h_1^*] = k h_1^* \qquad \text{for some } k\ . \tag{2.26}$$

Then (2.20) gives

$$\frac{1}{\alpha_1} F\Big[\frac{1}{\alpha_2} G[h_1^*]\Big] = \frac{1}{\alpha_1} F[k h_1^*] = h_1^*\ , \tag{2.27}$$

or, with $y_2 = \alpha_1 h_1^*$,

$$F\Big[\frac{k}{\alpha_1} y_2\Big] = y_2\ . \tag{2.28}$$

By Lemma 2.2 the equation (2.28) has the solution

$$y_2(t) = \frac{\lambda\gamma_1}{[(1+\frac{2k}{\alpha_1})\gamma_1 + \lambda]e^{\lambda(t-T)} - (1+\frac{2k}{\alpha_1})\gamma_1} \ ; \qquad t \leq T \tag{2.29}$$

Hence

$$h_1^* = \frac{1}{\alpha_1} y_2 \tag{2.30}$$

is the solution of (2.26).

Combining (2.24) and (2.30) we conclude that a fixed point h_1^* with the additional property (2.26) exists if and only if $\widehat{h}_1 = h_1^*$, i.e.

$$\frac{1}{k\alpha_2} y_1 = \frac{1}{\alpha_1} y_2$$

i.e.

$$\alpha_1 y_1 = k\alpha_2 y_2 \,. \tag{2.31}$$

Substituting (2.25) and (2.29) in (2.31) this gives

$$\frac{\alpha_1\lambda\gamma_2}{[(1+\frac{2}{k\alpha_2})\gamma_2+\lambda]e^{\lambda(t-T)}-(1+\frac{2}{k\alpha_2})\gamma_2} = \frac{k\alpha_2\lambda\gamma_1}{[(1+\frac{2k}{\alpha_1})\gamma_1+\lambda]e^{\lambda(t-T)}-(1+\frac{2k}{\alpha_1})\gamma_1}\,, \tag{2.32}$$

which holds iff

$$k = \frac{\alpha_1\gamma_2}{\alpha_2\gamma_1} \quad \text{and} \quad \gamma_1(2-\alpha_1)\alpha_2 = \gamma_2(2-\alpha_2)\alpha_1 \tag{2.33}$$

We summarize this as follows:

Theorem 2.3 Suppose $\lambda = \sigma_1^2 + 4\sigma_1 \cdot \sigma_2 + \sigma_2^2 \neq 0$ and that

$$\gamma_1(2-\alpha_1)\alpha_2 = \gamma_2(2-\alpha_2)\alpha_1 \tag{2.34}$$

Then the pair (u_1^*, u_2^*) of stochastic controls given by

$$u_1^*(t) = h_1^*(t)X_1(t)\,, \qquad u_2^*(t) = h_2^*(t)X_2(t)$$

is a Nash equilibrium of (2.1)–(2.4), where

$$h_1^*(t) = \frac{\lambda\gamma_1\alpha_2}{\alpha_1([\gamma_1\alpha_2 + 2\gamma_2 + \alpha_2\lambda]e^{\lambda(t-T)} - \gamma_1\alpha_2 - 2\gamma_2)}\,, \tag{2.35}$$

$$h_2^*(t) = \frac{\alpha_1\gamma_2}{\alpha_2\gamma_1}h_1^*(t)\,. \tag{2.36}$$

It is natural to ask if this is the *only* Nash equlibrium of (2.1)–(2.4).

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