

The continuous functionals of finite types over the reals

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December 9, 1998

Abstract

We investigate a hierarchy of domains with totality where we close some selected base domains, including domains for the reals, the natural numbers and the boolean values, under cartesian products and restricted function spaces. We show that the total objects will be dense in the respective domains, and that our construction is equivalent to the analogue construction in the category of limit spaces. In order to obtain this we will consider a restricted function space construction.

1 Introduction

Algebraic domains are handy for interpreting types. Ershov [3] essentially used algebraic domains to give a characterisation of the Kleene-Kreisel continuous functionals [5, 6]. In this case one considers the finite types over the natural numbers.

One basic property of the continuous functionals is the density theorem, the total objects is a dense subset of the domain in question. The density theorem was essentially proved independently by Kleene and Kreisel. Ershov gave a proof in the setting of domains. Berger [1, 2] analysed the proof of the density theorem and isolated a dual property to density, called *totality*, and proved how density and totality are properties that are preserved in general through the function space construction. Totality has later been renamed *co-density*. There are two reasons for this. One reason is that there are

good examples of domains with a reasonable notion of totality, but where Berger's analysis does not apply. Thus we should not restrict the term 'total' to Berger's concept. The other reason is that Berger's 'totality' is a dual to 'density', so co-density is a natural term.

In this paper we will consider the possibility of using base types of various topological natures. For instance, we will include a domain with totality representing the set \mathbb{R} of reals as a base type. In a preliminary draft for this paper we showed that if we consider the typed hierarchy with \mathbb{R} as the only base type, we will also obtain the density theorem. However, if we consider e.g. $\mathbb{R} \rightarrow \mathbb{N}$ every total continuous function has to be constant while there will be compacts in the canonical domain interpretation that cannot be extended to any constant function.

There are of course partial, computable functionals of type $\mathbb{R} \rightarrow \mathbb{N}$ that are not constant. If we want to include any of those in our construction we cannot have a density theorem. In the main part of the paper, we will restrict our interest to the hereditarily total objects. With this aim in mind we will consider restrictions to the function spaces, restrictions that will lead to effective domains, but such that density is preserved.

The key observation is that the only obstacle to proving density is that we may try to map two path-connected objects to two objects separable by a closed-open set. In order to handle this observation technically we will consider domains with an extra relation \sim which we will call *connected*. In addition we will have to strengthen the density property and generalise Berger's co-density property in order to make the preservation theorem work.

In the final section we will consider the hereditarily total objects in the standard domain interpretations of the types, and compare this hierarchy with our main hierarchy.

We will assume familiarity with the theory of algebraic domains, e.g. as introduced in Stoltenberg-Hansen, Lindström and Griffor [14].

Remark 1 In this paper we will let a *domain* D be an algebraic domain, or a Scott-Ershov-domain in the sense of [14]. In addition all our domains will be coherence complete, i.e. a subset X of D is bounded in D if and only if any two-point subset of X is bounded in D . We will not always make a point of proving or stating this.

2 The types

Definition 1 We define the type terms inductively as follows:

1. The constants R , N , B and Ω are type terms.
2. If σ and τ are type terms, then $(\sigma \times \tau)$ and $(\sigma \rightarrow \tau)$ are type terms.

Remark 2 Without mentioning we will follow standard conventions for dropping parentheses, mainly the outermost.

R will essentially be interpreted as the reals, N as the natural numbers, B as the set \mathbb{B} of boolean values \mathbf{t} for *true* and \mathbf{f} for *false*. Ω will be interpreted as the generic convergent sequence with a limit point, the one point compactification of \mathbb{N} represented by the ordinal $\omega + 1$.

We of course have to use domain representations of these interpretations.

Definition 2 Let (D, \sqsubseteq) be a domain that is coherence complete.

1. We write $x \approx y$ if $\{x, y\}$ is bounded.
2. A *connection* on D will be a reflexive, symmetric relation \sim on D satisfying
 - i) If $x \sqsubseteq x_1$, $y \sqsubseteq y_1$ and $x_1 \sim y_1$, then $x \sim y$.
 - ii) If X and Y are two sets bounded in D , and $x \sim y$ for all $x \in X$ and $y \in Y$, then

$$\bigsqcup X \sim \bigsqcup Y.$$

If $x \sim y$ we say that x and y are *connected*.

If we operate with several domains, we will use appropriate indices to \approx and \sim to distinguish them.

Examples

1. \mathbb{N}_\perp and \mathbb{B}_\perp with $\sim = \approx$.

2. Let \mathcal{R}_0 be the set of closed intervals with endpoints in \mathbb{Q} , together with the full real (or equivalent, rational) line. Let these intervals be ordered by reversed inclusion, and let \mathcal{R} be the ideal completion.

We let $x \sim y$ for all x, y in \mathcal{R} .

We may consider \mathcal{R}_0 as the set of pairs (r, s) from \mathbb{Q} where $r \leq s$, together with \perp , when it is essential to have the finitary aspect available. This will be important in Definition 15.

3. Let Ω_0 consist of all natural numbers n and all "copies" n^* of natural numbers.

Let \sqsubseteq be the least partial ordering satisfying

- $n \sqsubseteq m$ if $n \leq m$.
- $n \sqsubseteq m^*$ if $n \leq m$.

Let Ω_c be the ideal completion of Ω_0 , and let $\sim = \approx$.

It is easy to prove that these relations satisfy the definition of a connection. We will of course use these domains as interpretations of N, B, R and Ω .

Definition 3 Let D and E be domains with connections \sim_D and \sim_E . We define \sim on $D \times E$ by

$$(x, y) \sim (u, v) \Leftrightarrow x \sim_D u \wedge y \sim_E v.$$

Lemma 1 *Let \sim be as in Definition 3. Then \sim is a connection.*

The proof is easy and is left for the reader.

Definition 4 Let D and E be two domains with connections \sim_D and \sim_E , and let $F, G : D \rightarrow E$ be continuous.

- a) $F \in D \xrightarrow{c} E$ if for all $x, y \in D$

$$x \sim_D y \rightarrow F(x) \sim_E F(y).$$

- b) If F and G are in $D \xrightarrow{c} E$ we let

$$F \sim G \Leftrightarrow \forall x \in D \forall y \in D (x \sim_D y \rightarrow F(x) \sim_E G(y)).$$

c) $D \xrightarrow{c} E$ is ordered by the pointwise ordering.

Lemma 2 Let $D \xrightarrow{c} E$ be as in Definition 4.

a) $D \xrightarrow{c} E$ is a domain that is coherence complete.

b) \sim is a connection on $D \xrightarrow{c} E$.

Proof

a): $D \xrightarrow{c} E$ will be a closed subset of $D \rightarrow E$ with

$$F \sqsubseteq G \in D \xrightarrow{c} E \Rightarrow F \in D \xrightarrow{c} E.$$

It follows that $D \xrightarrow{c} E$ is a domain. Moreover, if $F \in D \xrightarrow{c} E$, then F is compact in $D \xrightarrow{c} E$ if and only if F is compact in $D \rightarrow E$.

In order to prove a) we must also prove coherence completeness:

If $X \subseteq D \xrightarrow{c} E$ is pairwise bounded in $D \xrightarrow{c} E$, then X is pairwise bounded in $D \rightarrow E$. We will show that $\bigsqcup X \in D \xrightarrow{c} E$.

Let $x \sim_D y$. Since X is pairwise bounded in $D \xrightarrow{c} E$, we get that

$F_1(x) \sim_E F_2(y)$ for all F_1 and F_2 in X .

By property ii) of connections, it follows that

$$\bigsqcup \{F(x) \mid F \in X\} \sim_E \bigsqcup \{F(y) \mid F \in X\}.$$

b): Property i) follows from property i) for E and property ii) is proved as the last argument under a).

Remark 3 In general, $D \xrightarrow{c} E$ is not a subdomain of $D \rightarrow E$.

If p_1, \dots, p_n are compacts in D and q_1, \dots, q_n are compacts in E such that

- $p_i \approx_D p_j \Rightarrow q_i \approx_E q_j$
- $p_i \sim_D p_j \Rightarrow q_i \sim_E q_j$

then $\{(p_1, q_1), \dots, (p_n, q_n)\}$ will denote a compact

$$F(x) = \bigsqcup \{q_i \mid p_i \sqsubseteq x\}$$

in $D \xrightarrow{c} E$, and all compacts will be denoted in this way. For simplicity, we will identify a compact with this denotation.

Lemma 3 *Let D_1, D_2 and D_3 be domains with connections.*

- a) *If $f \in D_1 \xrightarrow{c} D_2$ and $g \in D_2 \xrightarrow{c} D_3$, then the composition $g \circ f \in D_1 \xrightarrow{c} D_3$.*
- b) *Let $f \in D_1 \xrightarrow{c} (D_2 \xrightarrow{c} D_3)$.
Then $g \in (D_1 \times D_2) \xrightarrow{c} D_3$ where $g(x, y) = f(x)(y)$*
- c) *Let $g \in (D_1 \times D_2) \xrightarrow{c} D_3$.
Then $f \in D_1 \xrightarrow{c} (D_2 \xrightarrow{c} D_3)$ where $f(x)(y) = g(x, y)$.*

The proofs are easy and are left for the reader.

Definition 5 To each type σ we interpret σ as a domain $D(\sigma)$ with a connection \sim_σ in the obvious way as follows:

1. N, B, R and Ω are interpreted according to the examples.
2. $D(\sigma \times \tau) = D(\sigma) \times D(\tau)$.
3. $D(\sigma \rightarrow \tau) = D(\sigma) \xrightarrow{c} D(\tau)$.

3 The hereditarily total objects

By recursion on the type σ we will now define the hereditarily total objects of type σ . Simultaneously we will define a binary relation \equiv that will turn out to be the consistency relation for total objects.

Definition 6 For each type σ we define the set $T(\sigma)$ of *total objects* together with the binary relation \equiv_σ on $T(\sigma)$ as follows:

1. $T(N) = \mathbb{N}$ and \equiv_N is the identity-relation.
2. $T(B) = \mathbb{B}$ and \equiv_B is the identity-relation.
3. Let x be an ideal of compacts in \mathcal{R}_0 . We let $x \in T(R)$ if $\cap x$ is a singleton. We let $x \equiv_R y$ if $\cap x = \cap y$.
4. $T(\Omega)$ will be the ideals generated from the compacts n^* together with the ideal $\omega = \mathbb{N}$. We let \equiv_Ω be the identity-relation.

5. We let $T(\sigma \times \tau) = T(\sigma) \times T(\tau)$ and $\equiv_{\sigma \times \tau} = \equiv_{\sigma} \times \equiv_{\tau}$.
6. We let $T(\sigma \rightarrow \tau) = \{F \in D(\sigma) \xrightarrow{\varepsilon} D(\tau) \mid \forall x \in T(\sigma)(F(x) \in T(\tau))\}$.
We let $F \equiv_{\sigma \rightarrow \tau} G$ if

$$\forall x \in T(\sigma) \forall y \in T(\sigma)(x \equiv_{\sigma} y \Rightarrow F(x) \equiv_{\tau} G(y)).$$

Lemma 4 (Essentially Longo and Moggi [8])

For each type σ , \equiv_{σ} is an equivalence relation and

$$x \equiv_{\sigma} y \Leftrightarrow x \sqcap y \in T(\sigma)$$

for x and y in $T(\sigma)$

This was proved for the pure types over \mathbb{N} in [8]. The property holds trivially for all our base types, it extends trivially to cartesian products and the argument of [8] is valid for our restricted function space construction.

Definition 7 By recursion on the type σ we define the set $Ct(\sigma)$ and the function $\rho_{\sigma} : T(\sigma) \rightarrow Ct(\sigma)$ as follows:

1. ρ_N and ρ_B are the identity-maps on \mathbb{N} and \mathbb{B} resp. with $Ct(N) = \mathbb{N}$ and $Ct(B) = \mathbb{B}$.
2. If $x \in T(R)$ we let $\rho_R(x)$ be the unique element in $\cap x$.
We let $Ct(R) = \mathbb{R}$.
3. We let $\rho_{\Omega}(x)$ be the ordinal number n if x is the ideal generated from n^* , while $\rho_{\Omega}(x)$ is the ordinal number ω if $x = \mathbb{N}$. We let $Ct(\Omega) = \omega + 1$.
4. For products we just let the definition of ρ commute with pairings, and $Ct(\sigma \times \tau) = Ct(\sigma) \times Ct(\tau)$.
5. We let $\rho_{\sigma \rightarrow \tau}(F)(\rho_{\sigma}(x)) = \rho_{\tau}(F(x))$ and we let $Ct(\sigma \rightarrow \tau)$ be the set of functions $\Psi : Ct(\sigma) \rightarrow Ct(\tau)$ that are of the form $\rho_{\sigma \rightarrow \tau}(F)$ for some $F \in T(\sigma \rightarrow \tau)$.

By Lemma 4 this definition is sound, ρ_{σ} is a map from $T(\sigma)$ onto $Ct(\sigma)$ that identifies exactly the \equiv_{σ} -equivalent total objects.

We will later show that $Ct(\sigma)$ has a natural definition within the category of limit spaces.

4 Density and co-density

Definition 8 Let D be a domain.

A *totality* on D will be a subset \bar{D} such that

- If $x \in \bar{D}$, $x \sqsubseteq y \in D$, then $y \in \bar{D}$.
- The relation

$$x \equiv y \Leftrightarrow x \sqcap y \in \bar{D}$$

is an equivalence relation on \bar{D} .

This concept corresponds to the category \mathcal{K}_2 in Normann [11]. All the domains with totality that we have encountered in this paper are of this sort.

Definition 9 Let D be a domain with a totality \bar{D} and a connection \sim . (D, \bar{D}, \sim) satisfies *strong density* if for all finite products R^n of the base type R , all compacts in $\mathcal{R}^n \hookrightarrow D$ can be extended to a total element in $\mathcal{R}^n \hookrightarrow D$.

Definition 10 Let D be a domain with a totality \bar{D} and a connection \sim . We say that (D, \bar{D}, \sim) satisfies *co-density* if

1. If p and q are inconsistent compacts in D there is a total, continuous map $t : D \hookrightarrow \mathcal{R}$ such that $t(p)$ and $t(q)$ are inconsistent.
2. If p and q are compacts in D with $p \not\sim q$ there is a total, continuous map $t : D \hookrightarrow \mathbb{B}_\perp$ such that $t(p) = \mathbf{t}$ and $t(q) = \mathbf{f}$.

Remark 4 When \bar{D} and \sim are determined by the context, we will just say that D satisfies strong density or co-density when (D, \bar{D}, \sim) does.

Lemma 5 \mathcal{R} will satisfy strong density.

Proof

Let $A = \{(R_1, I_1), \dots, (R_k, I_k)\}$ be a compact in $\mathcal{R}^n \rightarrow \mathcal{R}$.

We will show that there is a total, continuous function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $f[R_i] \subseteq I_i$ for all $i \leq k$.

Each R_i is a hyper-rectangle in \mathbb{R}^n where all corners have rational coordinates, possibly including ∞ and $-\infty$.

Let q be the least common denominator of all the rational coordinates of the

corners of the R_i 's. Let Δ_m be the set of m -dimensional cubes in \mathbb{R}^n with side lengths $\frac{1}{q}$ and corners with coordinates $(\frac{i_1}{q}, \dots, \frac{i_n}{q})$ for $i_1, \dots, i_n \in \mathbb{Z}$.

If $C \in \Delta_{m+1}$, then the boundary of C is a finite union of elements from Δ_m .

Let $G_m = \bigcup\{C \mid C \in \Delta_m\}$.

We may think of G_m as an m -dimensional grid in \mathbb{R}^n .

By recursion on $m \leq n$ we will define an increasing family of continuous functions $f_m : G_m \rightarrow \mathbb{R}$ such that if $C \in \Delta_m$ and $C \subseteq R_i$, then $f_m[C] \subseteq I_i$.

G_0 is a discrete set of points.

For $C \in G_0$, let $f_0(C) = 0$ if $\forall i \leq k(C \notin R_i)$ while $f_0(C) \in \bigcap\{I_i \mid C \in R_i\}$ otherwise.

The consistency requirement of compacts in $\mathcal{R}^n \rightarrow \mathcal{R}$ ensures that this is possible.

Now assume that $m < n$ and that $f_m : G_m \rightarrow \mathbb{R}$ is constructed.

If $C \in \Delta_{m+1}$ and C' is one of the boundary hyper-cubes of C , then

$$C \subseteq R_i \Rightarrow C' \subseteq R_i \Rightarrow f_m[C'] \subseteq I_i,$$

so the requirement is satisfied at the boundary of C .

If we extend f_m locally to a continuous function for each $C \in \Delta_m$, the global extension will also be continuous. It is easy to construct such an extension satisfying the requirement.

This ends the proof of the lemma.

Corollary 1 *All base types satisfy strong density and co-density.*

Proof

Strong density is established for \mathcal{R} . For the other base types σ , all compacts in $\mathcal{R}^n \rightarrow D(\sigma)$ will be sets of pairs with consistent right hand sides, so density will be trivial.

The co-density property is trivial in all cases.

Lemma 6 *Let D be a domain with a totality \bar{D} and a connection \sim .*

If D satisfies strong density and P is a finite product of \mathcal{R} and \mathbb{B}_\perp , then the total objects in $P \xrightarrow{e} D$ form a dense subset.

Proof

\bar{P} will be a discrete set of closed-open subsets isomorphic to $\bar{\mathcal{R}}^n$ for various n . The density problem for $P \xrightarrow{e} D$ then reduces to the density problems for

$\mathcal{R}^n \xrightarrow{c} D$, that by assumption have positive solutions.

Strong density of course implies ordinary density, considering $\mathcal{R}^0 \xrightarrow{c} D$. In the proofs we will not focus on this case, but all our arguments will also hold then.

Lemma 7 *Let (D, \bar{D}, \sim_D) and (E, \bar{E}, \sim_E) be two domains with totality and connection.*

- a) *If both D and E satisfy strong density, then $D \times E$ satisfies strong density.*
- b) *If both D and E satisfy co-density, then $D \times E$ satisfies co-density.*

The proof is trivial and is left for the reader.

Lemma 8 *Let (D, \bar{D}, \sim_D) satisfy co-density.*

Let (E, \bar{E}, \sim_E) satisfy strong density.

Then $D \xrightarrow{c} E$ with the induced totality and connection will satisfy strong density.

Proof

It is sufficient to prove that $D \xrightarrow{c} E$ satisfies density, because by Lemma 3,

$$\mathcal{R}^n \xrightarrow{c} (D \xrightarrow{c} E)$$

is isomorphic to

$$(\mathcal{R}^n \times D) \xrightarrow{c} E$$

and strong density will follow from the fact that the latter domain, by the general argument, will satisfy density.

Let $\{(p_1, q_1), \dots, (p_n, q_n)\}$ be a compact in $D \xrightarrow{c} E$. Let

$$K = \{(i, j) \mid 1 \leq i < j \leq n \wedge p_i \not\sim p_j\}.$$

To each $k = (i, j) \in K$ we let P_k be \mathcal{R} if $p_i \sim p_j$ and we let $P_k = \mathbb{B}_\perp$ otherwise.

Let $t_k \in D \xrightarrow{c} P_k$ be total such that $t_k(p_i)$ and $t_k(p_j)$ are inconsistent. For the sake of saving notation, and without loss of generality, we assume that $t_k(p_l)$ is a compact in P_k for all $l \leq n$ and $k \in K$.

Let

$$P = \prod_{k \in K} P_k.$$

For each $l \leq n$, let

$$R_l = \prod_{k \in K} t_k(p_l).$$

Consider the set

$$C = \{(R_1, q_1), \dots, (R_n, q_n)\}.$$

Claim

C is a compact in $P \xrightarrow{c} E$.

Proof of claim

Let q_i and q_j be inconsistent with $i < j$. Let $k = (i, j) \in K$.

Then $t_k(p_i)$ and $t_k(p_j)$ are inconsistent, so R_i and R_j are inconsistent in coordinate k . By the same argument, if q_i and q_j are disconnected, then R_i and R_j will be disconnected in the argument k . This ends the proof of the claim.

Now, by Lemma 6 let $G \in P \xrightarrow{c} E$ be a total extension of C . Let $F \in D \xrightarrow{c} E$ be defined by

$$F(x) = G(\lambda k \in K. t_k(x)).$$

(See Lemma 3.)

By construction, F will be a total extension of $\{(p_1, q_1), \dots, (p_n, q_n)\}$.

Lemma 9 *let (D, \bar{D}, \sim_D) and (E, \bar{E}, \sim_E) be domains with totalities and connections.*

If D satisfies strong density and E satisfies co-density, then $D \xrightarrow{c} E$ will satisfy co-density.

Proof

Let C_1 and C_2 be inconsistent compacts in $D \xrightarrow{c} E$. Then there are $(p, q) \in C_1$ and $(p', q') \in C_2$ where p and p' are consistent while q and q' are inconsistent.

Let $s \in E \xrightarrow{c} \mathcal{R}$ be total with $s(p)$ and $s(p')$ inconsistent.

Let $x \in \bar{D}$ extend both p and p' .

Let $t(f) = s(f(x))$ for $f \in D \xrightarrow{c} E$. Then $t \in (D \xrightarrow{c} E) \xrightarrow{c} \mathcal{R}$ is total and separates C_1 and C_2 .

Now assume that C_1 and C_2 are disconnected. Then there are $(p, q) \in C_1$ and $(p', q') \in C_2$ such that $p \sim_D p'$ but $q \not\sim_E q'$.

Let $s \in E \xrightarrow{c} \mathbb{B}_\perp$ be total such that $s(q) = \mathbf{t}$ and $s(q') = \mathbf{f}$.

Since D is strongly dense there is a total $\phi \in \mathcal{R} \xrightarrow{c} D$ such that $p \sqsubseteq \phi(0)$ and $p' \sqsubseteq \phi(1)$.

For $f \in D \xrightarrow{c} E$, let

$$t(f) = \sqcap \{s(f(\phi(x))) \mid 0 \leq x \leq 1\}.$$

We use the topological compactness of $[0, 1]$ to show that t is continuous and total. It is also easy to see that $t \in (D \xrightarrow{c} E) \xrightarrow{c} \mathbb{B}_\perp$.

We end the proof of the lemma by showing

Claim

$$t \sqcap \{(C_1, \mathbf{t}), (C_2, \mathbf{f})\} \in (D \xrightarrow{c} E) \xrightarrow{c} \mathbb{B}_\perp.$$

Proof of claim

Since C_1 and C_2 are inconsistent and disconnected,

$\{(C_1, \mathbf{t}), (C_2, \mathbf{f})\} \in (D \xrightarrow{c} E) \xrightarrow{c} \mathbb{B}_\perp$. We will show that (C_1, \mathbf{t}) agrees with t .

A similar argument will work for (C_2, \mathbf{f}) .

If f is consistent with C_1 we have that $p \sqsubseteq \phi(0)$, so $f(\phi(0))$ will be consistent with q .

Then $s(f(\phi(0)))$ is consistent with $s(q) = \mathbf{t}$.

If f is just connected with C_1 the same argument gives that

$t(f) \sqsubseteq s(f(\phi(0)))$ is connected with $s(q) = \mathbf{t}$.

This ends the proof of the claim and of the lemma.

These lemmas will give us

Theorem 1 *Let σ be a type. Then the domain $D(\sigma)$ with the induced totality $T(\sigma)$ and connection \sim_σ will satisfy both strong density and co-density.*

Corollary 2 *Let σ be a type, $x \in T(\sigma)$ and $y \in T(\sigma)$. Then*

$$x \equiv_\sigma y \Leftrightarrow x \approx_\sigma y.$$

This is proved by a trivial induction on the types, using density.

For each type σ , $Ct(\sigma)$ will have a canonical topology, the quotient topology of $(T(\sigma), \equiv_\sigma)$.

Corollary 3 *Let σ be a type and let $x, y \in Ct(\sigma)$. Then either x and y are path connected or x and y can be separated by a closed-open set.*

Proof

If x and y are disconnected there are disconnected compacts $x_0 \sqsubseteq x$ and $y_0 \sqsubseteq y$. There is a total map t from $D(\sigma)$ to \mathbb{B}_\perp separating x_0 and y_0 . Let t_ρ be the map from $Ct(\sigma)$ to \mathbb{B} obtained by factoring t through $Ct(\sigma)$ via ρ_σ . The inverses t_ρ^- of \mathbf{t} and \mathbf{f} will be closed-open sets separating x and y .

If x and y are connected we prove the result by induction on σ .

Continuously in $x \sim_\sigma y$ we construct a path $P_\sigma(x, y)$ from x to y , i.e. a continuous function $P = P_\sigma(x, y) : \mathbb{R} \rightarrow Ct(\sigma)$ such that $P(0) = x$ and $P(1) = y$. At base types we are either forced to use the constant, or, in case of $\sigma = R$, we just let $P(r) = ry + (1 - r)x$.

For product spaces we take the coordinatewise paths and for function spaces $\sigma \rightarrow \tau$ we let

$$P_{\sigma \rightarrow \tau}(x, y)(r) = \lambda z \in Ct(\sigma). P_\tau(x(z), y(z))(r).$$

Corollary 4 *Let $F \in D(\sigma) \rightarrow D(\tau)$ be total.*

Then $F \in D(\sigma \rightarrow \tau)$.

Proof

We have to show that $p_1 \sim_\sigma p_2 \Rightarrow F(p_1) \sim_\tau F(p_2)$. If p_1 and p_2 are consistent, this is trivial. If they are inconsistent, we use co-density to find path-connected total extensions of p_1 and p_2 which will be mapped to path-connected total extensions of $F(p_1)$ and $F(p_2)$. Then $F(p_1)$ and $F(p_2)$ must be connected.

The uniform proof of Corollary 3 also shows

Corollary 5 *Let $x \in Ct(\sigma)$.*

Then the path-connected component containing x is effectively contractible to x .

Using standard arguments for the Kleene-Kreisel continuous functionals combined with Corollary 4 gives us

Corollary 6 *If $F : Ct(\sigma) \rightarrow Ct(\tau)$ is continuous, there is a $\hat{F} \in T(\sigma \rightarrow \tau)$ with $\rho_{\sigma \rightarrow \tau}(\hat{F}) = F$.*

5 Limit spaces

In this section we will give a characterisation of the topological spaces $Ct(\sigma)$ in terms of limit spaces, introduced by Kuratowski [7]:

Definition 11 Let X be a set. X is a *limit space* if it is equipped with a relation

$$x = \lim_{n \rightarrow \infty} x_n$$

between elements x in X and sequences $\{x_n\}_{n \in \mathbb{N}}$ from X satisfying:

1. If $x_n = x$ for almost all n then $x = \lim_{n \rightarrow \infty} x_n$.
2. If $x = \lim_{n \rightarrow \infty} x_n$ and $f : \mathbb{N} \rightarrow \mathbb{N}$ is strictly increasing, then $x = \lim_{n \rightarrow \infty} x_{f(n)}$.
3. If $\neg(x = \lim_{n \rightarrow \infty} x_n)$, there is a strictly increasing $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for no strictly increasing $g : \mathbb{N} \rightarrow \mathbb{N}$ we have $x = \lim_{n \rightarrow \infty} x_{g(f(n))}$.

Remark 5 In plain words we will say that any almost constant sequence has the almost constant value as a limit, if a sequence has x as a limit, then any subsequence will also have x as a limit and finally, if a sequence does not have x as a limit, then there is a subsequence such that no further subsequence has x as a limit.

The convergent sequences with limit points of any topological space will satisfy these axioms.

A convergent sequence with limit may be viewed as a map from Ω_c into X . We included Ω as a base type in order to simplify the discussion of the limit space characterisation.

Definition 12 Let X and Y be two limit spaces.

- a) $X \times Y$ is organised to a limit space by coordinatewise limits.
- b) The *limit space* $X \rightarrow Y$ will consist of all functions $f : X \rightarrow Y$ such that

$$x = \lim_{n \rightarrow \infty} x_n \Rightarrow f(x) = \lim_{n \rightarrow \infty} f(x_n).$$

The limit structure is defined by

$$f = \lim_{n \rightarrow \infty} f_n \text{ if } f(x) = \lim_{n \rightarrow \infty} f_n(x_n) \text{ whenever } x = \lim_{n \rightarrow \infty} x_n.$$

Remark 6 Function convergency will imply pointwise convergency and as close to uniform convergency we can get with the vocabulary available.

It is well known that these constructions define new limit-spaces.

Scarpellini [12] characterised the Kleene-Kreisel continuous functionals of a given type as the canonical interpretation in the category of limit spaces. This is also proved in Normann [10]. Using Corollary 6 and the original proofs, we get

Theorem 2 *For each type σ , $Ct(\sigma)$ will be the interpretation of the type σ in the category of limit spaces.*

Moreover, the topology on $Ct(\sigma)$ inherited from the domain $D(\sigma)$ will be the same as the topology generated from the convergent sequences when σ is interpreted as a limit space.

Remark 7 Recently Menni and Simpson [9] established a connection between limit spaces and the equilogical spaces introduced by Bauer, Birkedal and Scott [13]. Their argument can also be used to prove Theorem 2

6 The hereditarily partial functionals

In the previous sections we have given a domain interpretation of the types σ in such a way that the total objects are dense. An equally natural approach would be to interpret the types in the standard way in the category of domains and then consider the hereditarily total objects. Using the method of Longi and Moggi [8] we still will have an equivalence relation on the total objects corresponding to extensional equality. We will show that this alternative hierarchy will be characterised in the category of limit spaces exactly as the previous hierarchy was. Thus we do essentially get the same total objects. In particular, all our results about the topology of the total objects will hold.

Definition 13 Let σ be a type

- a) By recursion on σ we let $E(\sigma)$ be the interpretation of σ in the category of algebraic domains with standard exponents and with the same interpretation of the base types as before.
- b) Let $S(\sigma)$ be the hereditarily total objects in $E(\sigma)$.

Following Longo and Moggi [8] we still have

Lemma 10 *The relation*

$$x \equiv^0 y \Leftrightarrow x \sqcap y \in S(\sigma)$$

is an equivalence relation on $S(\sigma)$ corresponding to hereditarily extensional equality, and in particular each total function will respect these relations.

Definition 14 We define $Pt(\sigma)$ and ρ_σ^0 from $S(\sigma)$ in complete analogy to the definition of $Ct(\sigma)$ and ρ_σ from $T(\sigma)$.

In order to establish the characterisation via limit spaces we need to improve the argument leading to Corollary 6.

Lemma 11 *Let X be a separable domain, let $\bar{X} \subseteq X$ be closed upwards and let $F : \bar{X} \rightarrow Pt(\sigma)$ be continuous.*

Then there is a continuous total function

$$\hat{F} : X \rightarrow E(\sigma)$$

such that $\hat{F} : \bar{X} \rightarrow S(\sigma)$ and

$$\forall x \in \bar{X} (\rho_\sigma^0(x) = F(x)).$$

Proof

Notice that \hat{F} combines a factorisation of F through ρ_σ^0 and an extension of this factor to all of X . The technical obstacle is that \bar{X} need not be dense in X .

We proof this lemma for $\sigma = R$ only. For the other base types, we use a similar, or even easier, argument. The extension to higher types is easy, and is left for the reader. See also Remark 7.

So, let $F : \bar{X} \rightarrow \mathbb{R}$ be continuous. Let p, q range over the compacts in X , and I, J range over the rational intervals, i.e. the compacts in \mathcal{R} .

Let $\{(p_n, I_n)\}_{n \in \mathbb{N}}$ be an enumeration of all pairs (p, I) where

1. $p \sqsubseteq x$ for some $x \in \bar{X}$.
2. $F(x) \in I$ for all $x \in \bar{X}$ with $p \sqsubseteq x$.

Let $(p, I) \in \Gamma$ if for some n

1. $p_n \sqsubseteq p \wedge I_n \subseteq I$ ($I \sqsubseteq I_n$).
2. If $i < n$ and $I_i \cap I_n = \emptyset$ then p and p_i are inconsistent.

We then obtain

Claim 1

If $(p, I) \in \Gamma$, $(q, J) \in \Gamma$ and p and q are consistent, then I and J are consistent.

Now let \hat{F} be the ideal generated from Γ , seen as a function.

Claim 2

\hat{F} is total.

Proof

Let $x \in \bar{X}$. Let $\epsilon > 0$ be given.

Then there is an n such that $p_n \sqsubseteq x$ and I_n has length $< \epsilon$.

Now assume that for some $i < n$, $I_i \cap I_n = \emptyset$.

If p_i is consistent with x , then $x \sqcup p_i$ is total and extends p_i , so $F(x \sqcup p_i) \in I_i$ contradicting that $F(x) \in I_n$.

Thus there is a compact $q \sqsubseteq x$ that is inconsistent with p_i .

As a consequence, there is a $q \sqsubseteq x$ such that $(q, I_n) \in \Gamma$. Then $I_n \sqsubseteq \hat{F}(x)$. Since $\epsilon > 0$ was arbitrary, this proves the claim.

It is easy to see that \hat{F} restricted to \bar{X} factorises F through ρ_R^0 . This ends the proof of the lemma.

Remark 8 This lemma was first proved for flat domains like \mathbb{N}_\perp . It was used by Waagbø [15, 16] proving a corresponding result for the transfinite hierarchy of domains with totality based on \mathbb{N}_\perp and $\{\perp\}$ with no total elements, and closed under dependent sums and products. The argument is also used in Normann [11] proving a corresponding lemma for domains with totality inherited from evaluation structures (see [11] for a definition).

It may be a challenge to extend the constructions in this paper to a transfinite hierarchy including the reals.

We may now argue as for Corollary 6 and for Theorem 2 and obtain

Corollary 7 *Let σ and τ be types.*

- a) $Pt(\sigma \rightarrow \tau)$ consists of exactly all continuous maps from $Pt(\sigma)$ to $Pt(\tau)$.

- b) $Pt(\sigma)$ with its topological limit structure is the interpretation of σ in the category of limit spaces.
- c) $Pt(\sigma) = Ct(\sigma)$.

A consequence of this corollary is of course that $Pt(\sigma)$ has the topological properties of $Ct(\sigma)$. In all fairness, we do not need the full density/co-density analysis to obtain this. If we consider the full evaluation trees of two total objects of the same type, there will either be a branch that ends with two different total objects in \mathbb{N} , \mathbb{B} or Ω_c , or, whenever a branch ends up in two different end nodes, these end nodes are real numbers. In the first case, we can separate the given objects by a closed-open set, and in the second case we can construct a path from one object to the other. In both cases, we seem to need Lemma 11 to extend from the total objects to all objects.

The fact that $Pt(\sigma) = Ct(\sigma)$ of course means that there must be some connection between $D(\sigma)$ and $E(\sigma)$. Lemma 11 will provide us with one connection. This is based on an underlying enumeration of all compacts involved, and may not be considered to be natural.

Corollary 8 *Let σ be a type. There are total, continuous maps*

$$\phi_\sigma : D(\sigma) \rightarrow E(\sigma)$$

and

$$\psi_\sigma : E(\sigma) \rightarrow D(\sigma)$$

such that

1. $\rho_\sigma = \rho_\sigma^0 \circ \phi_\sigma$.
2. $\rho_\sigma^0 = \rho_\sigma \circ \psi_\sigma$.

Proof

The existence of ϕ_σ is a direct consequence of Lemma 11 and the existence of ψ_σ is proved from an analogue result for the other hierarchy.

We will end this paper by indicating that there is a hierarchy of *kernel*-domains of the two hierarchies, a kernel in which both concepts of totality actually live.

In order to see this, we will view the compact elements of each hierarchy

as elements of HF , the hereditarily finite sets, using e.g. the empty set as representing \perp . We will let $D_0(\sigma)$ and $E_0(\sigma)$ be the ordered subsets of HF generating the domains $D(\sigma)$ and $E(\sigma)$ in the standard way.

Definition 15 For each type σ , let $K_0(\sigma) = D_0(\sigma) \cap E_0(\sigma)$. Let $K_0(\sigma)$ be ordered by the common subordering induced from $D_0(\sigma)$ and $E_0(\sigma)$. Let $K(\sigma)$ be the domain of ideals over $K_0(\sigma)$.

The following is proved by a tedious, but simple, induction over the types.

Theorem 3 *Let σ be a type. Let $\alpha_1 \in T(\sigma)$ and $\alpha_2 \in S(\sigma)$ such that*

$$\rho_\sigma(\alpha_1) = \rho_\sigma^0(\alpha_2).$$

Let $\alpha = \alpha_1 \cap \alpha_2$, let α_D be α extended to an ideal in $D(\sigma)$ and let α_E be α extended to an ideal in $E(\sigma)$. Then $\alpha_D \in T(\sigma)$ and $\alpha_E \in S(\sigma)$.

Corollary 9 *Let σ be a type and let α be an ideal in $K(\sigma)$. Then the following are equivalent*

1. *There is an $\alpha_1 \in T(\sigma)$ such that $\alpha = \alpha_1 \cap K_0(\sigma)$.*
2. *$\alpha_D \in T(\sigma)$.*
3. *$\alpha_E \in S(\sigma)$.*
4. *There is an $\alpha_2 \in S(\sigma)$ such that $\alpha = \alpha_2 \cap K_0(\sigma)$.*

Remark 9 In general it is not true that if α_1 is in $T(\sigma)$ (or $\alpha_2 \in S(\sigma)$), then $\alpha_1 \cap K_0(\sigma) \in K(\sigma)$ ($\alpha_2 \cap K_0(\sigma) \in K(\sigma)$), simply because two compacts in $K_0(\sigma)$ may be bounded in $D_0(\sigma)$ or in $E_0(\sigma)$ without being bounded in $K_0(\sigma)$. When, however, two elements of $K_0(\sigma)$ are bounded both in $D_0(\sigma)$ and in $E_0(\sigma)$, the join will be the same element of HF in the two cases. Thus the intersection of an element of $D(\sigma)$ and an element of $E(\sigma)$ will be in $K(\sigma)$.

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