K3-surfaces of genus 8 and varieties of sums of powers of cubic fourfolds

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Abstract. A general $K3$-surface S of genus 8 determines uniquely a pair of cubic 4-folds: The dual Pfaffian cubic $F' = F'(S)$ and the apolar cubic $F'' = F''(S)$. As Beauville and Donagi have shown, the Fano variety $\mathcal{F}_{F'(S)}$ of lines on the cubic $F'(S)$ is isomorphic to the Hilbert scheme $\mathrm{Hilb}_2 S$ of length two subschemes of S. The main result of this paper is that $\mathrm{Hilb}_2 S$ parameterizes the variety $VSP_G(F''(S), 10)$ of presentations of the cubic form $F''(S)$ as a sum of 10 cubes, which yields an isomorphism between $\mathcal{F}_{F'(S)}$ and $VSP_G(F''(S), 10)$. As another corollary of our and Beauville and Donagi's result we show that $VSP_G(F'', 10)$ sets up a $(6, 10)$ correspondence between $F'(S)$ and $\mathcal{F}_{F'(S)}$.

1. Pfaffian and apolar cubic 4-folds associated to $K3$-surfaces of genus 8

1.1. Let V be a 6-dimensional vector space over $\mathbf{C}$. Fix a basis $e_0, \ldots, e_5$ for V, then $e_i \wedge e_j$ for $0 \leq i < j \leq 5$ form a basis for the Plücker space of 2-spaces in V or lines in $\mathbf{P}^5 = \mathbf{P}(V)$. With Plücker coordinates x_{ij}, the embedding of the Grassmannian $G = \mathbf{G}(2, V)$ in $\mathbf{P}^{14} = \mathbf{P}(\Lambda^2 V)$ is precisely the locus of rank 2 skew symmetric 6×6 matrices

$$M = \begin{pmatrix} 0 & x_{01} & x_{02} & x_{03} & x_{04} & x_{05} \\ -x_{01} & 0 & x_{12} & x_{13} & x_{14} & x_{15} \\ -x_{02} & -x_{12} & 0 & x_{23} & x_{24} & x_{25} \\ -x_{03} & -x_{13} & -x_{23} & 0 & x_{34} & x_{35} \\ -x_{04} & -x_{14} & -x_{24} & -x_{34} & 0 & x_{45} \\ -x_{05} & -x_{15} & -x_{25} & -x_{35} & -x_{45} & 0 \end{pmatrix},$$

The secant variety K of G in $\mathbf{P}^{14}$ is the locus where M has rank 4, so it is a cubic hypersurface defined by the 6×6 Pfaffian of M. The dual variety of G is a cubic hypersurface $K^* \cong K$ in the dual space $\check{\mathbf{P}}^{14}$ [cf. Zak]. K^* is the secant variety of $G^* = \mathbf{G}(V, 2)$ the Grassmannian of rank 2 quotient spaces of V, and of course $G^* \cong G$.

1.2. A $K3$-surface is called general if its Picard group is isomorphic to $\mathbf{Z}$. A general $K3$-surface S with Picard group generated by a linebundle H of degree $H^2 = 14$ is embedded via $|H|$ into a Grassmannian $\mathbf{G}(6, 2)$, for convenience we choose $G^* = \mathbf{G}(V, 2)$ in $\check{\mathbf{P}}^{14}$. In fact S is the intersection of G^* with a linear space L_S of dimension 8 [Muk]. The dual space $P_S = L_S^\perp \subset \mathbf{P}^{14}$ is 5-dimensional, so $P_S \cap K$ is a Pfaffian cubic 4-fold which we denote by $F'(S)$.

1.3. Via duality $L_S \subset \check{\mathbf{P}}^{14}$ corresponds to the space of nine linear forms $h_0, \ldots, h_8$ on $\mathbf{P}^{14}$. The Plücker embedding of the Grassmannian $G = \mathbf{G}(2, V)$ in $\mathbf{P}^{14}$ is arithmetically Gorenstein. The homogeneous coordinate ring R_G has syzygies, easily computed with [MAC],

1	–	–	–	–	–	–
–	15	35	21	–	–	–
–	–	–	21	35	15	–
–	–	–	–	–	–	1 .

The Grassmannian variety has dimension 8, so $P_S = L^\perp$, defined by the linear forms h_i, does not intersect G, and the quotient $A = R_G/(h_0, \ldots, h_8)$ is an Artinian Gorenstein ring. Its Hilbert function is $(1, 6, 6, 1)$ and it has socledegree 3, so A is the apolar Artinian Gorenstein ring $A^{F''}$ for some cubic hypersurface $F'' \subset \check{P}_S$. We denote by $F''(S) = F''$ this apolar cubic 4-fold.

1.4 Lemma. *There is a 19-dimensional family of cubic 4-folds F'' whose apolar Artinian Gorenstein ring is a quotient of R_G.*

Proof. Macaulay showed that there is a $1:1$ correspondence between hypersurfaces of degree d and graded Artinian Gorenstein rings generated in degree 1 with socledegree d [Mac] (cf. also [E p. 527]). Now, an isomorphism between such rings is of course induced by a linear transformation on the generators. In our setting any such linear transformation is again induced by an automorphism of G^* and correspondingly of G. The isomorphism classes of general $K3$-surfaces of genus 8 correspond precisely to orbits of 8-dimensional subspaces L [cf. Muk]. There is a 19-dimensional family of $K3$-surfaces of genus 8, so the lemma follows. □

Remark 1.5. F'' is not a Pfaffian cubic. In fact the Pfaffian cubics also form a 19-dimensional family of cubic 4-folds, and the above correspondence determines at least a birationality between the family of Pfaffian cubics and the family of apolar cubics F''. On the other hand, computing the apolar quadrics to a Pfaffian cubic with [MAC], it can readily be checked that there are in general no quadratic relations between these apolar quadrics, while the apolar quadrics to a cubic F'' have nine quadratic relations: As we shall see in the next section, the apolar quadrics define the restriction to the 5-space $P = P_S = L^\perp$ of the Cremona transformation defined by all quadrics through G. The inverse Cremona transformation is defined by quadrics again, and since P has codimension 9 in $\mathbf{P}^{14}$, there are at least nine quadrics containing the image of P, i.e. at least nine quadratic relations between the apolar quadrics.

1.6 Problem. *Find an alternative description of the apolar cubic 4-folds F''.*

2. Geometry of $\mathbf{G}(2, V)$ and its associated Cremona transformation

2.1. In the Plücker coordinates x_{ij}, the equations of $G = \mathbf{G}(2, V)$ are the 4×4 Pfaffians of the matrix M. Denote the Pfaffians by $q_{ij} \quad 0 \le i < j \le 5$ and the 6×6 Pfaffian by m. While

$$\begin{aligned} m = {} & x_{05}x_{14}x_{23} - x_{04}x_{15}x_{23} - x_{05}x_{13}x_{24} \\ & + x_{03}x_{15}x_{24} + x_{04}x_{13}x_{25} - x_{03}x_{14}x_{25} \\ & + x_{05}x_{12}x_{34} - x_{02}x_{15}x_{34} + x_{01}x_{25}x_{34} \\ & - x_{04}x_{12}x_{35} + x_{02}x_{14}x_{35} - x_{01}x_{24}x_{35} \\ & + x_{03}x_{12}x_{45} - x_{02}x_{13}x_{45} + x_{01}x_{23}x_{45}, \end{aligned}$$

the quadrics q_{ij} are:

$$\begin{aligned} q_{45} &= x_{03}x_{12} - x_{02}x_{13} + x_{01}x_{23} \\ q_{35} &= x_{04}x_{12} - x_{02}x_{14} + x_{01}x_{24} \\ q_{34} &= x_{05}x_{12} - x_{02}x_{15} + x_{01}x_{25} \\ q_{25} &= x_{04}x_{13} - x_{03}x_{14} + x_{01}x_{34} \\ q_{24} &= x_{05}x_{13} - x_{03}x_{15} + x_{01}x_{35} \\ q_{23} &= x_{05}x_{14} - x_{04}x_{15} + x_{01}x_{45} \\ q_{15} &= x_{04}x_{23} - x_{03}x_{24} + x_{02}x_{34} \\ q_{14} &= x_{05}x_{23} - x_{03}x_{25} + x_{02}x_{35} \\ q_{13} &= x_{05}x_{24} - x_{04}x_{25} + x_{02}x_{45} \\ q_{12} &= x_{05}x_{34} - x_{04}x_{35} + x_{03}x_{45} \\ q_{05} &= x_{14}x_{23} - x_{13}x_{24} + x_{12}x_{34} \\ q_{04} &= x_{15}x_{23} - x_{13}x_{25} + x_{12}x_{35} \\ q_{03} &= x_{15}x_{24} - x_{14}x_{25} + x_{12}x_{45} \\ q_{02} &= x_{15}x_{34} - x_{14}x_{35} + x_{13}x_{45} \\ q_{01} &= x_{25}x_{34} - x_{24}x_{35} + x_{23}x_{45}. \end{aligned}$$

Notice that $(-1)^{i+j-1}q_{ij}$ is precisely the partial of m with respect to x_{ij} i.e.

$$3m = \sum_{0 \le i < j \le 5} (-1)^{i+j-1} x_{ij} q_{ij}.$$

The Pfaffians q_{ij}, define a Cremona transformation [cf. ES]

$$\varphi : \mathbf{P}^{14} - -- > \mathbf{P}^{14}.$$

In fact

$$q_{ij}(q_{st}) = m x_{ij},$$

so the Cremona transformation is its own inverse. Since the sum of two rank 2 matrices has rank at most 4, and any rank 4 skew symmetric matrix is the sum of two rank 2 skew symmetric matrices, the secant variety of G is the cubic hypersurface K defined by the 6×6 Pfaffian m of the matrix M. The Cremona transformation contracts precisely all secants to G. The exceptional divisor lying over G in the Cremona transformation is mapped to a cubic hypersurface K', the secant variety of a variety G' which in turn is isomorphic to G.

The Cremona transformation φ is a morphism on the complement of $G = \mathbf{G}(2,V)$, it is birational on the complement of K, while $K \setminus G$ is mapped to G'.

2.2. Under the Cremona transformation φ it is natural to set $G' = \mathbf{G}(V^*,2) \cong \mathbf{G}(4,V)$. In fact, the preimage under φ of a point $[U] \in G'$ is a 5-dimensional space P_U which intersects G in a quadric hypersurface of rank 6. Geometrically we may interpret this quadric as the Grassmannian $\mathbf{G}(2,U)$. This is readily checked with the equations above.

2.3. The preimage under φ of a line in G is a rational scroll ruled in 5-dimensional spaces. For this first note that the points where only the quadrics q_{01} and q_{02} are nonzero, are mapped to a line on G'. On the other hand, by inspection, all the quadrics q_{ij} except q_{01} and q_{02} vanish on the union of G and a cubic scroll defined by the 2×2 minors of

$$\begin{pmatrix} x_{13} & x_{14} & x_{15} \\ x_{23} & x_{24} & x_{25} \end{pmatrix}$$

inside the 8-dimensional space $Z(x_{01}, x_{02}, x_{03}, x_{04}, x_{05}, x_{12})$. By homogeneity on G' the preimage of any line is a 6-fold cubic scroll.

2.4. Next we consider a tangent space to G. Without loss of generality we may consider the line spanned by $L_{01} =< e_0, e_1 > \subset \mathbf{P}(V)$ corresponding to the point $p_{01} = (1, 0, \ldots, 0)$ on G. Let

$$N_{01} = \begin{pmatrix} x_{02} & x_{03} & x_{04} & x_{05} \\ x_{12} & x_{13} & x_{14} & x_{15} \end{pmatrix}$$

Lemma 2.5. *N_{01} has rank 1 on G precisely at the points which correspond to lines which meet L_{01}. In fact the tangent space to G at p_{01} is defined by $x_{ij} = 0 \quad 2 \leq i < j \leq 5$, and the 2×2 minors of N_{01} define the contact cone inside this tangent space.*

Proof. When $x_{ij} = 0$ for $2 \leq i < j \leq 5$, then the Plücker quadrics reduce to the minors of N_{01}. On the other hand when this matrix has rank 1, i.e.

$$\alpha(x_{02}, x_{03}, x_{04}, x_{05}) + \beta(x_{12}, x_{13}, x_{14}, x_{15}) = (0,0,0,0)$$

then it is the Grassmannian point of the line

$$(\beta e_0 + \alpha e_1) \wedge (x_{02}e_2 + x_{03}e_3 + x_{04}e_4 + x_{05}e_5)$$

which is a general line which meet L_{01}. $\square$

2.6. For a special tangent hyperplane section, i.e. a Schubert cycle corresponding to a point on G^*, we may consider the hyperplane $Z(x_{01})$. Notice that inside this hyperplane the 5-space $Z(x_{ij} \mid i \in \{0,1\})$ intersect G along the quadric hypersurface defined by q_{01}, which is naturally identified with $\mathbf{G}(2,U)$, where $U =< e_2, e_3, e_4, e_5 >$. Furthermore the points in the Schubert cycle $Z(x_{01}) \cap G$ correspond precisely to lines which meet $\mathbf{P}(U)$. Altogether the Schubert cycle forms the union of subvarieties $\mathbf{G}(2, U')$ where $\mathbf{P}(U')$ is a 3-space which intersect $\mathbf{P}(U)$ along a plane. Each of these subvarieties is a rank six 4-fold quadric hypersurface which span a fibre of the Cremona transformation. It follows from 2.5 that the matrix N_{01} has rank one inside the special hyperplane $Z(x_{01})$ along all these fibres.

2.7. Finally we investigate certain subvarieties associated to secant lines to G^*. For this, fix two disjoint sets of indicies ij and kl.
Let

$$V(ij, kl) =< q_{st} | (st) \notin \{ik, il, jk, jl\} >,$$

and

$$Z(ij, kl) = Z(V(ij, kl)) \cap Z(x_{ij}, x_{kl}).$$

Then $\dim V(ij, kl) = 11$ and $Z(ij, kl)$ is the locus inside the hyperplanes $Z(x_{ij})$ and $Z(x_{kl})$ where the matrices N_{ij} and N_{kl} defined as above, both drop rank.

2.8 Lemma. *$Z(ij, kl)$ is a subvariety of degree 10 and codimension 5 inside $Z(x_{ij}, x_{kl})$.*

Proof. This is a generic variety, with invariants easily computed in [MAC]. It is the intersection of two rational normal quartic scrolls inside a rank 4 quadric. For the degree we note that this subvariety is a degeneration of the intersection of two codimension 2 cycles of bidegree $(1, 3)$ on a rank 6 quadric. Thus the degree is 10. □

2.9. Clearly the two hyperplanes $Z(x_{ij})$ and $Z(x_{kl})$ correspond to points on G^*. Denote by $L(ij, kl)$ the line spanned by them. It is of course a secant line to G^*.

2.10. Comparing the two matrices N_{ij} and N_{kl} we see that inside the intersection of the corresponding tangent 8-spaces, i.e. in the 3-space $Z(x_{st} \mid st \neq ik, jl, il, jk)$, the two sets of minors reduce to the quadric

$$x_{ik}x_{jl} - x_{il}x_{jk}$$

which represents on G the intersection of the corresponding Schubert cycles, i.e. the lines which meet both L_{ij} and L_{kl}. On G this is of course a quadric surface in a 3-space. On the other hand, on G' the corresponding quadric surface has a preimage under the Cremona transformation which is precisely the variety $Z(ij, kl)$.

2.11 Corollary. *The preimage of a quadric surface under the Cremona transformation is projectively equivalent to $Z(ij, kl)$ of degree 10 and dimension 7 inside the intersection of two special tangent hyperplanes $Z(x_{ij})$ and $Z(x_{kl})$.*

3. The variety of sums of powers $\mathbf{VSP_G(F, 10)}$

3.1. For a homogeneous polynomial f in $n+1$ variables, which define the hypersurface $F = Z(f) \subset \mathbf{P}^n$, we define the variety of sums of powers as the closure of the set

$$VSP(F,s) = \overline{\{\{< l_1 >, \ldots, < l_s >\} \in Hilb_s(\check{\mathbf{P}}^n) \mid \exists \lambda_i \in \mathbf{C} : f = \lambda_1 l_1^d + \ldots + \lambda_s l_s^d\}}$$

of powersums presenting f in the Hilbert scheme [cf. RS]. It is often natural to identify a powersum of length s presenting f with some s-secant linear space to a projection of some e-uple embedding of $\check{\mathbf{P}}^n$. In our case $s = 10$, $e = 2$ and $n = 5$. When we take the closure in the corresponding Grassmannian, we get what we denote by $VSP_G(F, 10)$ and call a Grassmannian compactification of the set of powersums of f.

3.2. Given a general 5-dimensional space $P \subset \mathbf{P}^{14}$ defining the apolar Artinian Gorenstein ring $A^{F''}$ of some cubic $F'' \subset \check{P}$. The Cremona transformation above restricts to the projection from partials of P, and an element

$$\Gamma \in VSP_G(F'', 10)$$

is a subscheme of P of length 10 which spans a 3-dimensional space P_Γ in the image. The preimage $V_\Gamma \subset \mathbf{P}^{14}$ of P_Γ is defined by 11 quadrics.

3.3 Lemma. *With suitable choice of coordinates* $V_\Gamma = Z(ij, kl)$ *for some* (ij, kl).

Proof. This proof depends on the following lemma which is interesting on its own:

3.4 Lemma. *If* $\Gamma \in VSP_G(F'', 10)$, *then* $\Gamma \subset K \cap P$.

Proof. Let $\Gamma \in VSP_G(F'', 10)$. We may assume that Γ is smooth and spans P. Then $\Gamma \cap G \subset P \cap G = \emptyset$, so by the Cremona transformation any point of Γ is mapped to G' or to the complement of K'. The image $\varphi(\Gamma)$ spans a 3-space P_Γ. The inverse Cremona restricted to P_Γ is defined by the quadrics through $G' \cap P_\Gamma$. Assume that $K' \cap P_\Gamma \neq P_\Gamma$. Then the restriction of the inverse Cremona transformation to P_Γ is birational onto its image. Since the image intersects P in Γ which in turn span P, the image must span at least an 8-space. But in that case the degree of the image is 7 or 8 so it cannot contain Γ. Therefore $P_\Gamma \subset K'$, and $\Gamma \subset K$. $\square$

For Lemma 3.3 we note that Γ is mapped to G' by the Cremona transformation. Now, first note by 2.2 that a 3-space meets each fiber of the Cremona transformation in a linear space. Furthermore the restriction of the inverse Cremona transformation to the 3-space P_Γ is not birational. Therefore the fibers meet P_Γ in lines or planes unless it is all contained in a fiber. When the fibers meet P_Γ in planes, each plane must intersect G' in at least a conic section, so the intersection $G' \cap P_\Gamma$ must be the union of a plane and a line. On the other hand it is not hard to check that no 3-space intersect G' this way, so this is impossible. When the fibers meet P_Γ in lines, then $G' \cap P_\Gamma$ must be a curve with one secant line through each general point, i.e. a twisted cubic curve or the union of two lines. Both of these cases occur as intersections with the Grassmannian. Note also that there could be no extra points of intersection in addition to these curves, since these would lie

on proper trisecants to the Grassmannian, which is absurd since the Grassmannian is cut out by quadrics.
Now, the preimage under φ of a line is a 6-fold cubic scroll by 2.3, so the preimage of a conic section must be a 6-fold scroll of degree 6 and the preimage of a twisted cubic curve must be a 6-fold scroll of degree 9. But Γ is the intersection of the 5-space P with the preimage under the Cremona transformation of $G' \cap P_\Gamma$. Since Γ is 0-dimensional, the length of this intersection cannot exceed the degree of the corresponding preimages. Since these all have degree less than 10, these cases are excluded and we conclude that $G' \cap P_\Gamma$ is a quadric surface.
The intersection of G' with the 3-space $Z(q_{st}|st \notin \{ik, il, jk, jl\})$, is the quadric surface $Z(q_{ik}q_{jl} - q_{il}q_{jk})$. Corollary 2.11 implies that the preimage of this surface under the Cremona transformation is $Z(ij, kl)$. By homogeneity Lemma 3.3 follows.

3.5 Theorem. *$VSP_G(F'', 10)$ is isomorphic to the family of secant lines to $G^* \cap L$, i.e. to $\mathrm{Hilb}_2(S)$ where S is the K3 surface $S = G^* \cap L$.*

Proof. Clearly $P \cap Z(ij, kl)$ is a subscheme of degree 10 if and only if P is contained in the span $Z(x_{ij}, x_{kl})$ of $Z(ij, kl)$. Furthermore, by 2.9, $P \subset Z(x_{ij}, x_{kl})$ if and only if the secant line $L(ij, kl) \subset L = P^\perp$. Thus Lemma 3.3 gives an inclusion $VSP_G(F'', 10) \subset \mathrm{Hilb}_2(S)$. On the other hand consider a secant line to S, which we may choose to be $L(ij, kl)$ after a suitable change of coordinates. Assume that $\Gamma = Z(ij, kl) \cap P$ is a finite smooth scheme of length 10.

3.6 Lemma. *If Γ is smooth and the ideal of Γ is contained in the ideal of the 15 quadrics apolar to F'', then $\Gamma \in VSP_G(F'', 10)$.*

Proof. [RS, 1.3]. □

Smoothness and finiteness of $\Gamma = Z(ij, kl) \cap P$ are open conditions, so the theorem follows. □

3.7 Question. Does $VSP_G(F'', 10)$ and $VSP(F'', 10)$ coincide for general F in the family? For this it remains only to check whether every point in the closure of the family of secant lines actually correspond to finite subschemes of P, i.e. that the closure correspond to the closure in the Hilbert scheme of length 10 subschemes of P.

3.8. Now recall from Beauville and Donagi, that the variety of secant lines to $S = L \cap G^*$ is isomorphic to the Fano variety of lines $\mathcal{F}(F')$ of the Pfaffian cubic fourfold $F' = F'(S) = P \cap K$, where $P = L^\perp$ [cf. BD]. Thus we have an incidence correspondence

$$I = \{(p, \Gamma) | p \in \Gamma\} \subset F'(S) \times VSP_G(F''(S), 10) \cong F' \times \mathcal{F}(F').$$

The second projection is clearly $10 : 1$.

3.9 Proposition. *The projection of the incidence correspondence*

$$I \subset F'(S) \times VSP_G(F''(S), 10)$$

onto the first factor is generically $6 : 1$.

Proof. Consider a general point $p \in P \cap K$. Since $p \in K \setminus G$, the Cremona transformation φ is defined in p and the fiber containing p is a 5-space P_p. The space P_p intersects G in a quadric Q_p corresponding to all lines in some 3-space U_p, and the parameter point of U_p is the image point $\varphi(p)$ of the Cremona transformation. There is a unique point $p^* \in G^* = \mathbf{G}(V, 2)$ whose corresponding nullspace is U_p. In the dual space $P_p^\perp \subset \check{\mathbf{P}}^{14}$ is the tangent 8-space T_{p^*} to G^* at p^*. Now, the point p is in the image of the projection from the incidence correspondence, i.e. is contained in some $\Gamma \in VSP_G(F, 10)$, if P_p is contained in two special tangent hyperplanes, corresponding to two points on G^*. These special tangent hyperplanes which contain P_p are parametrized by all 3-spaces that intersect U_p in a plane. On G^* this is, by 2.5, precisely what is defined by the minors of a matrix N_p equivalent to N_{01} inside the tangent space $T_{p^*} = P_p^\perp \subset \check{\mathbf{P}}^{14}$. The hyperplanes which contain both P and P_p form $P^\perp \cap P_p^\perp$ in $\check{\mathbf{P}}^{14}$. This is a 3-space, since the two 5-spaces P and P_p intersect. Inside T_{p^*} the minors of N_p define a rational quartic 5-fold scroll, so the intersection with $P^\perp$ is 4 points. The secants between the 4 points are secant lines to $L \cap G^*$, corresponding to sets Γ that contain p. □

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