Optimal consumption and portfolio in a jump diffusion market with proportional transaction costs

by

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Abstract

We consider the problem of optimal consumption and portfolio in a jump diffusion market in the presence of proportional transaction costs, for an agent with constant relative risk aversion utility.

We show that the solution in the jump diffusion case has the same form as in the pure diffusion case first solved by Davis and Norman (1990). In particular, we show that (under some assumptions) there is a no transaction cone $D$ in the $xy$-plane such that it is optimal to make no transactions as long as the wealth position remains in $D$ and to sell/buy stocks according to local time on the boundary of $D$.

Keywords: Portfolio optimization, consumption optimization, transaction costs, viscosity solutions, free boundary problem.

JEL Classification: C44, C61, D81, D90, G11.


1 Introduction; the model

In this paper we study the problem of optimal consumption and investment policy in a jump diffusion market consisting of a bank account and a stock. A well established model for the stock price is the lognormal diffusion or geometric Brownian motion, which has several computational advantages; The rate of return is independent of the past, stationary (i.e., time-homogenous in law) and follows the normal distribution. In this paper, we will drop the latter assumption; returns will not be assumed to be Gaussian. A stochastic process with stationary increments independent of the past, and in addition satisfying the mild technical condition of continuity in probability (implying that it has no fixed jump times) is called a Lévy process, and is essentially a piecewise Brownian

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motion with both drift and Poisson jumps with uniform intensity. We briefly note that we pick the unique right continuous (with left limits) version of all processes, which is the natural choice from a stochastic integration point of view; this also applies to the control processes \((\mathcal{L}, \mathcal{M})\) below, even though that choice may seem less natural from an impulse control point of view.

In view of the above, we shall assume that the bank gives a fixed interest rate \(r\), and the bank deposit then follows the equation

\[
dP_1(t) = rP_1(t)dt \quad P_1(0) = p_1 \geq 0.
\]

The price \(P_2(t)\) at time \(t\) of the stock, is then assumed to be a geometric Lévy process, following the stochastic differential equation

\[
dP_2(t) = \alpha P_2(t)dt + \sigma P_2(t)dW(t) + P_2(t^-) \int_{-1}^\infty \eta \tilde{N}(dt, d\eta); \quad P_2(0) = p_2 \geq 0.
\]

Here \(\alpha \geq r\) and \(\sigma > 0\) are constants, \(W(t)\) is a Wiener process (Brownian motion) on a filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\). The \(\tilde{N}\) entity is \(\mathcal{F}_t\)-centered Poisson random measure,

\[
\tilde{N}(t, A) = N(t, A) - E[N(t, A)] = N(t, A) - tq(A)
\]

where \(N(t, A)\) is a Poisson random measure measuring the number of jumps with amplitude in \((a, b)\), \(A \subseteq (-1, \infty)\), up to and including time \(t\). \(\tilde{N}\) has a time-homogeneous intensity measure \(E[N(t, A)] = tq(A)\); \(q\) is then called the Lévy measure associated to \(\tilde{N}\). See e.g. Bensoussan and Lions (1984), Jacod and Shiryaev (1987) and Protter (1990) for more information about such stochastic differential equations.

Note that since we only allow jump sizes \(\eta\) which are bigger than \(-1\), the process \(P_2(t)\) will remain positive for all \(t \geq 0\), a.s., and will not violate limited liability. On the technical side, we shall assume that

\[
\int_{-1}^\infty 1 \vee \eta^2 dq(\eta) < \infty.
\]

We assume that at any time \(t\) the investor can choose a rate \(c(t) \geq 0\) of consumption taken from the bank account. We also assume that he can transfer money at any time from one asset to the other with a transaction cost which is proportional to the size of the transaction. Let \(X(t), Y(t)\) denote the amount of money invested in asset number 1, 2, respectively. Then the evolution equations for \(X(t), Y(t)\) are

\[
dx(t) = dX^{c_1, \mathcal{L}, \mathcal{M}}(t) \\
= (rX(t) - c(t))dt - (1 + \lambda)d\mathcal{L}(t) + (1 - \mu)d\mathcal{M}(t); \quad X(0^-) = x \in \mathbb{R} \\
dY(t) = dY^{\mathcal{L}, \mathcal{M}}(t) \\
= Y(t^-) (\alpha dt + \sigma dW(t) + \int_{-1}^\infty \eta \tilde{N}(dt, d\eta)) + d\mathcal{L}(t) - d\mathcal{M}(t); \quad Y(0^-) = y \in \mathbb{R}.
\]
Here $\mathcal{L}(t), \mathcal{M}(t)$ represent cumulative purchase and sale, respectively, of stocks up to time $t$. The coefficients $\lambda \geq 0, \mu \in [0, 1]$ represent the constants of proportionality of the transaction costs.

**Remark.** By multiplying all processes by $e^{-rt}$ and differentiating using the Itô formula, one will see that the problem only depends on $\alpha$ and $r$ through their difference, just like the Merton problem. It would in fact suffice to consider the case $r = 0$; In this case, $X(t)$ would be non-increasing except at the times we sell stocks. \(\triangle\)

Our controls will have to meet certain conditions. The *solvency region* $S$ is defined to be the set of states where the net wealth is nonnegative:

$$S = \{(x, y) \in \mathbb{R}^2; x + (1 + \lambda)y \geq 0 \text{ and } x + (1 - \mu)y \geq 0\}$$

(1.5)

with boundaries $\partial_1 S, \partial_2 S$ as in Figure 1:

![Diagram](image_url)

**Figure 1:** The solvency region.

It is natural to require that

If $(x, y) \in S$, then $(X(t), Y(t)) \in S$ for all $t$ (a.s.). \hspace{1cm} (1.6a)

Note that in the presence of a jump term, we need to make sure that we can cover any position we could happen to jump to. Hence, if we define $U' \subset S$ as

$$U' = \{(x, y) \in S; (x, y(1 + \eta)) \in S \text{ for all } \eta \in \text{supp } q\}$$

(1.6b)

then it is necessary and sufficient for (1.6a) to hold that

If $(x, y) \in S$, then $(X(t), Y(t)) \in U'$ for all $t$ (a.s.). \hspace{1cm} (1.6c)
Since we already have to deal with a cone contained in \( S \) (with equality iff \( q = 0 \)), we get the following generalization more or less for free: Let \( U \subseteq U' \) be a given open convex cone with vertex at the origin. It will later be convenient to characterize \( U \) in terms of polar coordinates; let \( \partial U \) be given by angles \( \theta_1 \in [\frac{-\pi}{4}, \frac{3\pi}{4}] \) and \( \theta_2 \in (\theta_1, \frac{3\pi}{4}) \) (and such that \( U \subseteq U' \)). Thus,

\[
U = \{(x, y) = Re^{i\theta}; \quad R \geq 0; \quad \theta_1 < \theta < \theta_2\}.
\]

So what we will require, is the following:

If \( (x, y) \in S \), then \( (X(t), Y(t)) \in \overline{U} \) for all \( t \) (a.s.). \hfill (1.7)

The restriction to a (possibly) smaller cone \( U \) may be given an economic interpretation as a (say, law enforced) limitations on short sale or leverage. Of particular interest is the case where \( U \) is the first quadrant. This serves as the authors' "moral justification" for the restrictive assumption of Theorem 4.2, that the no transaction region is contained in the first quadrant (eq. (4.5)) – an assumption we conjecture not to hold if the Merton line lies outside the first quadrant.

**DEFINITION 1.1.**

The set \( \mathcal{A} \) of admissible controls is the set of predictable consumption-investment policies \((c, \mathcal{L}, \mathcal{M})\) with \( c(t, \omega) \geq 0 \) (a.e. \((t, \omega)) \) and \( \mathcal{L}(t), \mathcal{M}(t) \) right-continuous, non-decreasing and \( \mathcal{L}(0^-) = \mathcal{M}(0^-) = 0 \), and such that (1.7) holds.

The intuition behind requiring (1.7), is that if a jump should bring us out of \( \overline{U} \), then an admissible control will bring us back into \( \overline{U} \) immediately. Now since we have chosen to work with the right-continuous version, then "out of \( \overline{U} \)" should be interpreted as

\[
(X(t^-), Y(t^-) + \Delta N Y(t)) \notin \overline{U},
\]

where

\[
\Delta N Y(t) := Y(t^-) \int_{-1}^{\infty} \eta N(\{\{\}, d\eta)
\]

and \( N(\{\{\}, \cdot) \) denotes the jump in the Poisson random measure occurring exactly at time \( t \).

Define the performance criterion by

\[
J_{c,\mathcal{L},\mathcal{M}}(x, y) = \mathbb{E}^x_y \left[ \int_0^{\infty} e^{-\delta t} c^r(t) dt \right] \hfill (1.9)
\]

where \( \delta > 0, \gamma \in (0, 1) \) are constants and \( \mathbb{E}^x_y \) is the expectation with respect to the probability law \( \mathbb{P}^x_y \) of \( (X(t), Y(t)) \) when \( (X(0^-), Y(0^-)) = (x, y) \in \mathbb{R}^2 \). The problem is to find \( V \) and (if exists) \((c^*, \mathcal{L}^*, \mathcal{M}^*) \in \mathcal{A} \) such that

\[
V(x, y) = \sup_{(c, \mathcal{L}, \mathcal{M}) \in \mathcal{A}} J_{c,\mathcal{L},\mathcal{M}}(x, y) = J_{c^*,\mathcal{L}^*,\mathcal{M}^*}(x, y). \hfill (1.10)
\]
Due to the choice of utility function, the solvency restriction is necessary for the problem to be well defined; obviously, the only concave extension of CRRA utility, is to put utility equal to minus infinity for negative consumption.

In the special case when the stock price is a geometric Brownian motion (i.e., \( q = 0 \)) and there are no transaction costs (i.e., \( \lambda = \mu = 0 \)) this problem was first studied by Merton (1971). He proved that if

\[
\delta > \gamma \left[ r + \frac{(\alpha - r)^2}{2\sigma^2(1 - \gamma)} \right] \quad (1.11)
\]

then the value function \( V_0(x, y) \) is given by

\[
V_0(x, y) = K_0(x + y)\gamma \quad (1.12)
\]

where

\[
K_0 = \frac{1}{\gamma} \left[ \frac{1}{1 - \gamma} \left( \frac{1}{\delta - \gamma r} - \frac{\gamma(\alpha - r)^2}{2\sigma^2(1 - \gamma)} \right) \right]^{\frac{1}{\gamma - 1}}. \quad (1.13)
\]

Moreover, the corresponding optimal consumption \( c^*_0 \) is given (in feedback form) by

\[
c^*_0(x, y) = (K_0\gamma)^{\frac{1}{\gamma - 1}}(x + y) \quad (1.14)
\]

and the corresponding optimal portfolio is to keep the fraction \( Y(t)/(X(t) + Y(t)) \) of wealth invested in the stocks constantly equal to the value

\[
u^*_0 = \frac{\alpha - r}{(1 - \gamma)\sigma^2} \quad (1.15)
\]
at all times. In other words, it is optimal to perform transactions in such a way that the state \((X(t), Y(t))\) is always situated on the line \( y = \frac{u^*_0}{1 - u^*_0}x \) in the \((x, y)\)-plane (the Merton line).

In Aase (1984) and later in Framstad et al. (1998) (see also Benth et al. (1999)) the results of Merton (1971) are extended to the case when the stock price is a geometric Lévy process, i.e. as (1.2), still assuming that there are no transaction costs, i.e., \( \lambda = \mu = 0 \). It is proved that the value function \( V(x, y) \) still has the same form, namely

\[
V(x, y) = K(x + y)^\gamma \quad (1.16)
\]

but with a different constant \( K \) (under an assumption similar to (1.11)). The corresponding optimal consumption \( c^* \) is given by

\[
c^*(x, y) = (K\gamma)^{\frac{1}{\gamma - 1}}(x + y) \quad (1.17)
\]
and it is still optimal to keep the fraction \( Y(t)/(X(t) + Y(t)) \) constantly equal to a value \( u^* \). See Framstad et al. (1998), Theorem 2.3.

In Framstad et al. (1998), Corollary 2.4 it is proved that if \( q \neq 0 \) then

\[
V(x, y) < V_0(x, y),
\]
\[
c^*(x, y) > c^*_0(x, y)
\]

and

\[
u^* < u^*_0,
\]

so the introduction of the jump term involving the integral with respect to \( \tilde{N} \) has the same effect on the solution as increasing the volatility \( \sigma \).

The purpose of this paper is to study the general case with the stock price given by a geometric Lévy process (1.2) and with proportional transaction costs. As the constant ratio portfolio in the no transaction cost case implies that one must rebalance continuously, it would lead to instant bankruptcy when transaction costs are nonzero, and so it cannot be optimal to keep the fraction of wealth invested in the risky asset constant. We shall however prove that it will be optimal to keep that fraction in a fixed interval, i.e. there exists a no transaction region \( D \) in the \((x, y)\)-plane with the shape of a cone with vertex at the origin, such that it is optimal to make no transactions as long as \((X(t), Y(t)) \in D\) and to sell stocks at the rate of local time (of the reflected process) at the upper/left boundary of \( D \) and purchase stocks at the rate of local time at the lower/right boundary. These results generalize the results of Davis and Norman (1990) who obtained similar results in the no jump case \((q = 0)\). Our paper is also inspired by the paper of Shreve and Soner (1994), who also considered the case \( q = 0 \). They used, as we do, a viscosity solution approach and were able to remove some of the assumptions in Davis and Norman (1990). Viscosity solutions of combined stochastic control and optimal stopping problems for jump diffusion processes are studied by Pham (1998). However, his conditions are not satisfied in the case we consider because our utility rate \( c^*_\gamma \) is not bounded as a function of \( c \geq 0 \).

## 2 Sufficient conditions

In this section we show how to associate an integro-variational inequality based on dynamic programming, to the value function \( V(x, y) \) defined by (1.10).

First note that is we apply a Markov control \( c(t) = c(X(t), Y(t)) \) and there are no transactions, then the generator \( \tilde{A} \) of the time-space process

\[
d\tilde{Z}(t) = (dt, dZ(t)) = (dt, dX(t), dY(t))
\]
is given by
\[
\bar{\mathcal{A}}\varphi(s, x, y) = \frac{\partial \varphi}{\partial s} + (r x - c) \frac{\partial \varphi}{\partial x} + \alpha y \frac{\partial \varphi}{\partial y} + \frac{1}{2} \sigma^2 y^2 \frac{\partial^2 \varphi}{\partial y^2} \\
+ \int_{-1}^{\infty} [\varphi(s, x, y + \eta) - \varphi(s, x, y) - \frac{\partial \varphi}{\partial y}(s, x, y) \cdot \eta \eta] d\eta. \tag{2.1}
\]

If \( \varphi \) has the form
\[
\varphi(s, x, y) = e^{-s\delta}\psi(x, y)
\]
then
\[
\bar{\mathcal{A}}\varphi(s, x, y) = e^{-s\delta} A^c \psi(x, y)
\]
where
\[
A^c \psi(x, y) = -\delta \psi(x, y) + (r x - c) \frac{\partial \psi}{\partial x} + \alpha y \frac{\partial \psi}{\partial y} + \frac{1}{2} \sigma^2 y^2 \frac{\partial^2 \psi}{\partial y^2} \\
+ \int_{-1}^{\infty} [\psi(x, y + \eta) - \psi(x, y) - \frac{\partial \psi}{\partial y}(x, y) \cdot \eta \eta] d\eta. \tag{2.2}
\]

If \((\mathcal{L}(t), \mathcal{M}(t))\) is an admissible control we will in the following let \( t_k \) denote the jumping times of \((\mathcal{L}(t), \mathcal{M}(t))\). The jumps of \( \mathcal{L}(t), \mathcal{M}(t) \) at \( t = t_k \) are
\[
\Delta \mathcal{L}(t_k) := \mathcal{L}(t_k) - \mathcal{L}(t_k^-), \quad \Delta \mathcal{M}(t_k) := \mathcal{M}(t_k) - \mathcal{M}(t_k^-), \tag{2.3a}
\]
respectively. And we let
\[
\mathcal{L}_c(t) := \mathcal{L}(t) - \sum_{0 \leq t_k \leq t} \Delta \mathcal{L}(t_k), \quad \mathcal{M}_c(t) := \mathcal{M}(t) - \sum_{0 \leq t_k \leq t} \Delta \mathcal{M}(t_k) \tag{2.3b}
\]
be the continuous part of \( \mathcal{L}(t), \mathcal{M}(t) \), respectively. If \( v \) is a continuous real function on \( S \) we let
\[
\Delta_{\mathcal{L}, \mathcal{M}}v(Z(t_k)) := v(Z(t_k)) - v(X(t_k^-), Y(t_k^-) + \Delta_N Y(t_k)) \tag{2.4}
\]
denote the jump in the value of \( v(Z(t)) \) caused by the jump of \( (\mathcal{L}(t), \mathcal{M}(t)) \) at \( t = t_k \). We emphasize that the possible jumps in \( Z(t) \) coming from \( \tilde{N} \) are not included in \( \Delta_{\mathcal{L}, \mathcal{M}}v(Z(t_k)) \).

First, we give a few properties of the value function:

**Lemma 2.1.**

a) \( V(x, y) \) is a non-decreasing function with respect to both \( x \) and \( y \).
b) \( V(x, y) \) is a concave function of \((x, y)\).

c) \( V(x, y) \) is homogeneous of degree \( \gamma \), i.e. \( V(\rho x, \rho y) = \rho^\gamma V(x, y) \) for all \( \rho > 0 \).

d) Outside \( U \), the value function has the form

\[
V(x, y) = K_1 \cdot (x + (1 + \lambda)y)^\gamma \quad \text{on } \theta < \theta_1, \text{ and}
\]
\[
V(x, y) = K_2 \cdot (x + (1 - \mu)y)^\gamma \quad \text{on } \theta > \theta_2.
\]

Proof.

a) If \( \bar{x} \geq x \) and \( \bar{y} \geq y \), then one can reach \((x, y)\) from \((\bar{x}, \bar{y})\) by an immediate transaction (possibly both buying and selling at the same time).

b) This follows from the concavity of the utility function \( c \mapsto \frac{1}{\gamma} c^\gamma \) and the linearity of (1.4), as in Akian et al. (1999), Prop. 3.1.

c) This follows as in Davis and Norman (1990), Th.3.1a.

d) Outside \( \bar{U} \), a transaction (at least) to \( \partial U \) is compulsory, and therefore the value of a state outside \( U \) is constant along the half-lines from \( U \) and parallel to the boundary of the solvency region. Then the claim follows from homogeneity.

Part d) above together with the continuity of the value function (Lemma 2.4 below), now characterize the value function outside \( U \).

THEOREM 2.2 (Integro-variational inequality verification theorem).

a) Suppose there exists a nonnegative function \( v(x, y) \in C^2(S) \) such that

\[
L_v := -(1 + \lambda) \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \leq 0 \tag{2.5a}
\]
\[
M_v := (1 - \mu) \frac{\partial v}{\partial x} - \frac{\partial v}{\partial y} \leq 0 \tag{2.5b}
\]
\[
A^c_v + \frac{1}{\gamma} c^\gamma \leq 0 \text{ on } U \text{ for all } c \geq 0. \tag{2.5c}
\]

Then

\[
v(x, y) \geq V(x, y). \tag{2.6}
\]

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b) Suppose, in addition to (2.5a) – (2.5c) that there exists \( \hat{c}(x, y) \geq 0 \) such that

\[
\max \{Lv(x, y), A^\hat{c}v(x, y) + \frac{1}{\gamma} \hat{c}^\gamma(x, y), \mathcal{M}v(x, y)\} = 0
\]

for all \((x, y) \in U\).

Define the no transaction region \( D \) by

\[
D = \{(x, y) \in U; \ Lv(x, y) < 0 \text{ and } \mathcal{M}v(x, y) < 0\}.
\]

Suppose there exist \( \hat{\mathcal{L}}(t), \hat{\mathcal{M}}(t) \) such that \( \hat{\omega} := (\hat{c}, \hat{\mathcal{L}}, \hat{\mathcal{M}}) \) is admissible and such that, if we put

\[
Z^{\hat{\omega}}(t) := (X^{\hat{\mathcal{L}}, \hat{\mathcal{M}}}(t), Y^{\hat{\mathcal{L}}, \hat{\mathcal{M}}}(t))
\]

then we have

\[
(-1 + \lambda) \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}(Z^{\hat{\omega}}(t))d\hat{\mathcal{L}}_c(t) = 0 \text{ for all } t,
\]

\[
((1 - \mu) \frac{\partial v}{\partial x} - \frac{\partial v}{\partial y})(Z^{\hat{\omega}}(t))d\hat{\mathcal{M}}_c(t) = 0 \text{ for all } t
\]

(with notation as in (2.3).)

Moreover, suppose that (see (2.4)) for all jumping times \( t_k \) of \((\hat{\mathcal{L}}(t), \hat{\mathcal{M}}(t))\) we have

\[
\Delta^{\hat{\mathcal{L}}, \hat{\mathcal{M}}}v(Z^{\hat{\omega}}(t_k)) = 0,
\]

\[
Z^{\hat{\omega}}(t) \in \overline{D} \text{ for almost all } t
\]

and

\[
\lim_{R \to \infty} \mathbb{E}^{v}[e^{-\delta T_R}v(Z^{\hat{\omega}}(T_R))] = 0
\]

where

\[
T_R = \min(R, \inf\{t > 0; |Z^{\hat{\omega}}(t)| \geq R\}).
\]

Then

\[
v(x, y) = V(x, y) \text{ for all } (x, y) \in U
\]

and the control \( \hat{\omega} = (\hat{c}, \hat{\mathcal{L}}, \hat{\mathcal{M}}) \) is optimal.
Proof. Several versions of this result are known, see e.g. Bensoussan and Lions (1984). For completeness we include a sketch of a proof in the Appendix.

\[ \square \]

Remark. Suppose the no transaction domain \( D \) defined by (2.8) is known and has a smooth boundary (e.g. Lipschitz). Then one can regard (2.7) as a Neumann boundary value problem in \( D \). In Section 4 we will discuss the relation between such problems and reflections/local time of diffusions in our jump diffusion case. Such a relation makes it possible to identify \( (\hat{\mathcal{L}}(t), \hat{\mathcal{M}}(t)) \) with the local time at \( \partial D \) of the process \((\hat{X}(t), \hat{Y}(t)) = (\bar{X}(t), \bar{Y}(t)) \) obtained by reflecting \((\bar{X}(t), \bar{Y}(t)) \) at \( \partial D \) in the directions indicated by (2.10) and (2.11).

\[ \triangle \]

Remark. Theorem 2.2 shows that it is natural to associate the integro-variational inequality

\[
\max \left\{ \mathcal{L}v(x, y), \sup_{c \geq 0} \left\{ A^c v(x, y) + \frac{1}{\gamma} c^\gamma \right\}, \mathcal{M}v(x, y) \right\} = 0
\]

(2.16) to the value function \( V \) defined in (1.10). However, a priori we do not know if \( V \) is smooth enough for (2.16) to make sense in the usual way. Nevertheless, in the next section we will prove that \( V \) solves an equation related to (2.16) in the weak sense of viscosity.

\[ \triangle \]

Before we do this we establish some other useful properties of \( V \):

**Lemma 2.3.**

a) Suppose (1.11) holds. Let \( K_0 \) be as in (1.13). Then

\[
V(x, y) \leq K_0(x + y)^\gamma \text{ for all } (x, y) \in U.
\]  

(2.17)

b) Let \( b \) be a constant such that

\[
1 - \mu \leq b \leq 1 + \lambda.
\]  

(2.18)

Suppose

\[
\delta > \gamma \alpha.
\]  

(2.19)

Then there exists \( K < \infty \) such that

\[
V(x, y) \leq K(x + by)^\gamma \text{ for all } (x, y) \in U.
\]  

(2.20)
**Proof.** See the Appendix.

**LEMMA 2.4.**

$V(x, y)$ is continuous on $\mathcal{S}$.

**Proof.** The continuity of $V$ on the interior of $\mathcal{S}$ follows from the fact that $V$ is concave (see e.g. Rockafeller (1970)). If $(\bar{x}, \bar{y}) \in \partial \mathcal{S}$ then the only admissible control is to take $(X_\bar{y}(t), Y_\bar{y}(t))$ to the origin immediately, because otherwise diffusion will bring the process out of $\mathcal{S}$, almost surely. Hence $V(\bar{x}, \bar{y}) = 0$ for $(\bar{x}, \bar{y}) \in \partial \mathcal{S}$. So it remains to prove that

$$\lim_{(x, y) \to (\bar{x}, \bar{y})} V(x, y) = 0 \text{ for all } (x, y) \in \partial \mathcal{S}.$$ 

This follows from Lemma 2.3b) by choosing $b = 1 + \lambda$ and $b = 1 - \mu$.

Suppose the system is in state $z = (x, y)$ and we decide to make an admissible transaction $\ell = \Delta \xi \geq 0$ and $m = \Delta \eta \geq 0$. Then the state jumps to $z' = (x', y')$, where (see Figure 2)

$$\begin{cases}
    x' &= x - (1 + \lambda)\ell + (1 - \mu)m \\
    y' &= y + \ell - m
\end{cases}$$

![Figure 2: Transactions from $(x, y)$ to $(x', y')$.](image)

Suppose we, when starting from $(x, y) \in \mathcal{S}$, make an immediate transaction which brings us to the state $(x', y') \in \mathcal{S}$ and from then on perform optimally from $(x', y')$. Then we get a performance which is at most as good as the best possible when starting from $(x, y)$. Therefore we have, with $(x', y'), l$ and $m$ as explained (see Figure 2):
**Lemma 2.5.**

\[ V(x, y) \geq V(x', y') \text{ for all } (x, y) \in \mathcal{S}, \quad \ell \geq 0, m \geq 0. \]

Now fix \((x, y) \in \mathcal{S}\) and consider the function

\[ g(\ell) = g_{x,y}(\ell) = V(x - (1 + \lambda)\ell, y + \ell); \quad 0 \leq \ell \leq \bar{\ell} \]

where

\[ \bar{\ell} = \sup \{\ell > 0; (x - (1 + \lambda)\ell, y + \ell) \in \mathcal{S}\}. \]

It follows from Lemma 2.5 that \(g\) is a decreasing function. Hence \(g'(\ell) \leq 0\) for a.a. \(\ell \in (0, \bar{\ell}).\) Now \(g'(\ell)\) is just the directional derivative of \(V\) in the direction \((- (1 + \lambda), 1)\) and since \(\frac{\partial V}{\partial x}\) exists for a.a. \(x\) for each \(y\) (by Lemma 2.1a)) we conclude that

\[ g'(\ell) = -(1 + \lambda) \frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} \text{ for a.a. } x, y, \ell. \]

This gives part a) of the following result:

**Lemma 2.6.**

a) \(-(1 + \lambda) \frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} \leq 0\) for a.a. \((x, y) \in \mathcal{S}\)

b) \((1 - \mu) \frac{\partial V}{\partial x} - \frac{\partial V}{\partial y} \leq 0\) for a.a. \((x, y) \in \mathcal{S}\).

Part b) of this Lemma is proved similarly to part a) by replacing \(g\) by the function

\[ h(m) := V(x + (1 - \mu)m, y - m); \quad 0 \leq m \leq \bar{m}, \]

where

\[ \bar{m} = \sup \{m > 0; (x + (1 - \mu)m, y - m) \in \mathcal{S}\}. \]

Note that \(g\) and \(h\) are concave functions, because they are just the restriction of the concave function \(V\) to straight lines.

For an arbitrary continuous function \(v : \mathcal{S} \mapsto \mathbb{R}\) we now define \(\hat{L}v\) and \(\hat{M}v\) by

\[
\hat{L}v(x, y) = \begin{cases} 
-1 & \text{if } v(x - (1 + \lambda)\ell, y + \ell) < v(x, y) \text{ for all } \ell \in (0, \bar{\ell}) \\
0 & \text{if } v(x - (1 + \lambda)\ell, y + \ell) = v(x, y) \text{ for some } \ell \in (0, \bar{\ell}) \\
+1 & \text{if } v(x - (1 + \lambda)\ell, y + \ell) > v(x, y) \text{ for all } \ell \in (0, \bar{\ell})
\end{cases}
\]

\[
\hat{M}v(x, y) = \begin{cases} 
-1 & \text{if } v(x + (1 - \mu)m, y - m) < v(x, y) \text{ for all } m \in (0, \bar{m}) \\
0 & \text{if } v(x + (1 - \mu)m, y - m) = v(x, y) \text{ for some } m \in (0, \bar{m}) \\
+1 & \text{if } v(x + (1 - \mu)m, y - m) > v(x, y) \text{ for all } m \in (0, \bar{m})
\end{cases}
\]
Then we have seen above that

\[ \hat{V}(x, y) \leq 0 \text{ and } \hat{M}(x, y) \leq 0 \text{ for all } (x, y) \in \hat{S}. \]

Moreover, if \( \hat{V}(x_0, y_0) < 0 \) then we must have \( g'(\ell) < 0 \) for arbitrary small \( \ell > 0 \) and hence for almost all \( \ell > 0 \) by concavity. A similar argument works for \( h(m) \). This proves:

**Lemma 2.7.**

a) If \( \hat{V}(x_0, y_0) < 0 \), then

\[-(1 + \lambda) \frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} < 0\]

at almost all points \((x, y) = (x_0 - (1 + \lambda)\ell, y_0 + \ell), \quad 0 < \ell < \ell_0.\]

b) If \( \hat{M}(x_0, y_0) < 0 \), then

\[(1 - \mu) \frac{\partial V}{\partial x} - \frac{\partial V}{\partial y} < 0\]

at almost all points \((x, y) = (x_0 + (1 - \mu)m, y_0 - m), \quad 0 < m < \bar{m}.\)

**Theorem 2.8.**

There exist real numbers \( \theta_1^*, \theta_2^* \) with \( \theta_1^* < \theta_1 < \theta_2^* < \theta_2 \) such that if we define

\[ B = \{(x, y) = R e^{i\theta} \in U; \quad R \geq 0; \quad \theta \leq \theta_1^* \} \quad \text{(the buy region)} \]

and

\[ S = \{(x, y) = R e^{i\theta} \in U; \quad R \geq 0; \quad \theta \geq \theta_2^* \} \quad \text{(the sell region)} \]

then

\[
\begin{align*}
\hat{V}(x, y) &:= -(1 + \lambda) \frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} \\
&\begin{cases}
= 0 & \text{a.e. on } B \\
< 0 & \text{a.e. on } U \setminus B
\end{cases}
\]

and

\[
\hat{M}(x, y) := (1 - \mu) \frac{\partial V}{\partial x} - \frac{\partial V}{\partial y} \\
\begin{cases}
= 0 & \text{a.e. on } S \\
< 0 & \text{a.e. on } U \setminus S.
\end{cases}
\]

**Proof.** See the Appendix.
3 Viscosity solutions

For the characterization of the value function in terms of the integro-differential equations (2.5), the appropriate smoothness condition of Theorem 2.2 is necessary. The concept of viscosity solutions serves the purpose of admitting a function only known to be continuous, as solution to (integro-) differential equations, usually admitting the removal of smoothness assumptions. See e.g. Fleming and Soner (1993). For existence and uniqueness of first order Hamilton-Jacobi integro-differential equations see Awatif (1991a), (1991b).

Let $L$, $M$ and $A^c$ be as in Theorem 2.2, $\hat{L}$, $\hat{M}$ as in (2.23), (2.24) and define

$$\hat{L}u(x, y) = \max (L(u(x, y), \hat{L}u(x, y))$$

$$\hat{M}u(x, y) = \max (M(u(x, y), \hat{M}u(x, y)).$$

We consider the following integro-variational inequality associated to (2.16):

$$\max \left\{ \hat{L}u(x, y), \sup_{c \geq 0} \left\{ A^c u(x, y) + \frac{1}{\gamma} \right\}, \hat{M}u(x, y) \right\} = 0 \text{ in } U. \quad (3.3)$$

We define viscosity solutions of (3.3) as follows:

**DEFINITION 3.1.**

a) A function $u \in C(U)$ is a viscosity subsolution in $U$ of (3.3) if, for all functions $\phi \in C^2(U)$ and all $z_0 = (x_0, y_0) \in U$ such that $\phi \geq u$ on $U$ and $\phi(z_0) = u(z_0)$ we have

$$\max \left\{ \hat{L}\phi(z_0), \sup_{c \geq 0} \left\{ A^c \phi(z_0) + \frac{1}{\gamma} \right\}, \hat{M}\phi(z_0) \right\} \geq 0. \quad (3.4)$$
b) A function $u \in C(U)$ is a viscosity supersolution of (3.3) if, for all functions $\phi \in C^2(U)$ and all $z_0 = (x_0, y_0) \in U$ such that $\phi \leq u$ on $U$ and $\phi(z_0) = u(z_0)$ we have

$$\max \left\{ \mathcal{L}\phi(z_0), \sup_{c \geq 0} \{ A^c\phi(z_0) + \frac{1}{\gamma} c^\gamma \}, \mathcal{M}\phi(z_0) \right\} \leq 0.$$

(3.5)

c) A continuous function $u$ on $U$ is called a viscosity solution of (3.3) if it is both a viscosity subsolution and a viscosity supersolution.

\[ \square \]

**THEOREM 3.2.**

Let $V$ be the value function for problem (1.10). Then $V$ is a viscosity solution of (3.3) on $U$.

**Proof.** The somewhat technical proof is left to the Appendix.

**COROLLARY 3.3.**

a) $V(x, y)$ is $C^1$ on $S \setminus \{(x, 0); x > 0\}$.

b) Let

$$B = \{(x, y) = Re^{i\theta} \in U; \theta \leq \theta_1^*, R \geq 0\}$$

(3.6)

and

$$S = \{(x, y) = Re^{i\theta} \in U; \theta \geq \theta_2^*, R \geq 0\}$$

(3.7)

be as in Theorem 2.8. Then, except possibly at $y = 0$, we have

$$LV(x, y) \leq 0 \text{ and } MV(x, y) \leq 0 \text{ everywhere in } S$$

(3.8)

and

$$LV(x, y) = 0 \iff (x, y) \in B$$

$$MV(x, y) = 0 \iff (x, y) \in S.$$ 

(3.9) (3.10)

c) Define

$$D := \{(x, y) = Re^{i\theta}; \theta^*_1 < \theta < \theta^*_2, R \geq 0\} = U \setminus (B \cup S)$$

(3.11)

Then $V \in C^2(D)$ and $V$ satisfies in the strong sense the equation

$$\sup_{c \geq 0} \{ A^cV(x, y) + \frac{1}{\gamma} c^\gamma \} = 0 \text{ for } (x, y) \in D.$$ 

(3.12)
Proof.

a) In the case without jumps \( q = 0 \) this is contained in Corollary 8.3 in Shreve and Soner (1994). See also Proposition VIII.7.1 in Fleming and Soner (1993); their proof works in our case with the extra integral term in the generator \( A^c \) (see (2.2)), because of assumption (1.3). We omit the details.

b) Since \( V(x, y) \) is \( C^1 \) on \( S \setminus \{(x, 0); x > 0\} \), this follows immediately from Theorem 2.8.

c) Since \( V(x, y) \) is \( C^1 \) on \( S \setminus \{(x, 0); x > 0\} \) we see that

\[
LV(x, y) < 0 \Rightarrow \tilde{L}V(x, y) < 0.
\]

Therefore

\[
LV(x, y) < 0 \iff \tilde{L}V(x, y) < 0
\]

and similarly with \( MV \) and \( \tilde{M}V \). Hence we may replace \( \tilde{L}, \tilde{M} \) by \( L, M \) in (3.3). Since \( V \) is a viscosity solution of (3.3) we deduce from b) that \( V \) is a viscosity solution of (3.12) in \( D \). Therefore, to complete the proof it suffices to prove that \( \frac{\partial^2}{\partial y^2} V \) exists and is continuous in \( D \setminus \{y = 0\} \) (if \( y = 0 \), the second derivative does not enter in \( L \)). This follows by the same argument as the proof of Proposition VIII.7.1 in Fleming and Soner (1993) (see also Corollary 8.6 in Shreve and Soner (1994)) because the hypoellipticity property is not destroyed by the extra integral term in our generator \( A^c \).

\( \square \)

4 Identification of the optimal portfolio as a reflected jump diffusion

In this section we identify the optimal portfolio with the local time of a reflected jump diffusion. We start with the following adaptation of Theorem 15 in Chaleyat-Maurel et al. (1980):

**THEOREM 4.1.**

Let \( c(t) \geq 0 \) be a given adapted process. Fix \( \hat{\theta}_1, \hat{\theta}_2 \) such that

\[
\theta_1 \leq \hat{\theta}_1 < \hat{\theta}_2 \leq \theta_2
\]

and define

\[
D = \{(x, y) = Re^{i\theta}; \quad R \geq 0; \quad \hat{\theta}_1 < \theta < \hat{\theta}_2 \}
\]

(4.1a)

\[
\partial_k D = \{(x, y) = Re^{i\theta}; \quad R \geq 0; \quad \theta = \hat{\theta}_k, \quad k = 1, 2
\]

(4.1b)

\[
B = \{(x, y) = Re^{i\theta} \in S; \quad R \geq 0; \quad \theta \leq \hat{\theta}_1 \}
\]

(4.1c)

\[
S = \{(x, y) = Re^{i\theta} \in S; \quad R \geq 0; \quad \hat{\theta}_2 \leq \theta \}
\]

(4.1d)
Then there exist unique càdlàg adapted processes $\tilde{X}(t), \tilde{Y}(t), \tilde{\mathcal{L}}(t), \tilde{\mathcal{M}}(t)$ satisfying the following Skorohod stochastic differential equation, given by the set of conditions (4.2a) – (4.2d):

\[
\begin{align*}
d\tilde{X}(t) &= (r\tilde{X}(t) - c(t))dt - (1 + \lambda)d\tilde{\mathcal{L}}(t) + (1 - \mu)d\tilde{\mathcal{M}}(t); \quad X(0^-) = x \in \mathbb{R} \\
d\tilde{Y}(t) &= \tilde{Y}(t^-)(\alpha dt + \sigma dW(t) + \int_{-1}^{\infty} \eta N(dt, d\eta)) + d\tilde{\mathcal{L}}(t) - d\tilde{\mathcal{M}}(t); \quad Y(0^-) = y \in \mathbb{R} \\
(\tilde{X}(t), \tilde{Y}(t)) &\in \tilde{D} \text{ for all } t > 0
\end{align*}
\]  

(4.2a)

(4.2b)

$\tilde{\mathcal{L}}(t), \tilde{\mathcal{M}}(t)$ are non-decreasing and their continuous parts, $\tilde{\mathcal{L}}_c(t), \tilde{\mathcal{M}}_c(t)$, increase only when $(\tilde{X}(t), \tilde{Y}(t)) \in \partial_1 D, \quad (\tilde{X}(t), \tilde{Y}(t)) \in \partial_2 D$, respectively

(4.2c)

\[
\begin{align*}
\Delta \tilde{\mathcal{L}}(t) > 0 & \text{ if and only if } (X(t^-), Y(t^-) + \Delta_N Y(t)) \in \overset{\circ}{B}, \\
\Delta \tilde{\mathcal{M}}(t) > 0 & \text{ if and only if } (X(t^-), Y(t^-) + \Delta_N Y(t)) \in \overset{\circ}{S}, \\
\end{align*}
\]

and if this is the case then

(4.2d)

\[
\begin{align*}
\Delta \tilde{\mathcal{L}}(t) &= \min\{\ell > 0; \ (X(t^-) - (1 + \lambda)\ell, Y(t^-) + \Delta_N Y(t) + \ell) \notin \overset{\circ}{B}\} \\
\Delta \tilde{\mathcal{M}}(t) &= \min\{m > 0; \ (X(t^-) + (1 - \mu)m, Y(t^-) + \Delta_N Y(t) - m) \notin \overset{\circ}{S}\}
\end{align*}
\]

with $\Delta_N Y(t)$ as in (1.8).

Remark. The process $(\tilde{X}(t), \tilde{Y}(t))$ is called the reflection of the process $(X^{(c,0,0)}(t), Y^{(c,0,0)}(t))$ in the directions $(-(1 + \lambda), 1)$ and $((1 - \mu), -1)$ at the two boundary curves $\partial_1 D$ and $\partial_2 D$ of $D$. Note that we only have $\Delta \tilde{\mathcal{L}}(t) > 0$ or $\Delta \tilde{\mathcal{M}}(t) > 0$ if either $t = 0$ and $Z(0^-) \notin \overset{\circ}{D}$ or if $Z(t)$ (by the jump in the random measure term) jumps out of $\overset{\circ}{D}$. In these cases we either buy ($\Delta \tilde{\mathcal{L}} > 0$) or sell ($\Delta \tilde{\mathcal{M}} > 0$) immediately to bring $Z(t)$ to $\partial D$. See Figure 4. We call the two processes $\tilde{\mathcal{L}}(t), \tilde{\mathcal{M}}(t)$ the Skorohod generalized local times of $(\tilde{X}(t), \tilde{Y}(t))$ at $\partial_1 D, \partial_2 D$, as they solve the Skorohod problem; The continuous parts $\tilde{\mathcal{L}}_c(t), \tilde{\mathcal{M}}_c(t)$ are local times at $\partial_1 D, \partial_2 D$ in the sense of e.g. Chaleyat-Maurel et al. (1980), paragraph 2.1. We see that $\tilde{\mathcal{L}}(t), \tilde{\mathcal{M}}(t)$ satisfy the equations (2.10) – (2.11).

Finally we use this to give an explicit description of the optimal control for problem (1.10), under some conditions:
Figure 4: A possible wealth sample path as a reflected jump diffusion. Notice the immediate transactions following the (Poisson) jumps.

**THEOREM 4.2.**

Let $\theta_1 \leq \theta_1^* < \theta_2^* \leq \theta_2$ be as in Theorem 2.8 and Corollary 3.3. Suppose that

\[
0 \leq \theta_1^* < \theta_2^* \leq \frac{\pi}{2}
\]  

(4.5)

and that

\[
\delta > \gamma \alpha - \frac{1}{2} \sigma^2 \gamma (1 - \gamma) - \gamma \|q\| + \int_{-1}^{\infty} ((1 + \eta)^\gamma - 1) \, dq(\eta)
\]  

(4.6)

with $\|q\| = q((-1, \infty))$ (finite by (1.3)). Define

\[
(X^*, Y^*, \mathcal{L}^*, \mathcal{M}^*) = (\hat{X}, \hat{Y}, \hat{L}, \hat{M})
\]  

(4.7)

where $(\hat{X}, \hat{Y}, \hat{L}, \hat{M})$ is the solution of the Skorohod equation in Theorem 4.1 with

\[
\hat{\theta}_1 = \theta_1^*, \quad \hat{\theta}_2 = \theta_2^*.
\]

Put

\[
c^*(x, y) = \left( \frac{\partial V}{\partial x} \right)^{\frac{1}{\gamma - 1}}
\]  

(4.8)

Then $w^* = (c^*, \mathcal{L}^*, \mathcal{M}^*)$ is an optimal control for problem (1.10) and $Z^*(t) = (X^*(t), Y^*(t))$ is the corresponding optimal state process.

**Proof.** See the Appendix.
Appendix

Proof of Theorem 2.2 (sketch): If \( v \) satisfies the conditions of a) then by the Itô formula for semimartingales (see e.g. Proter (1990), Th.II.7.33) we have, for any admissible \((c, \mathcal{L}, \mathcal{M})\) with corresponding state process \( Z(t) = (X(t), Y(t)) \),

\[
E^{x,y}[e^{-\delta T_R}v(Z(T_R))] = v(Z(0)) + E^{x,y} \left[ \int_0^{T_R} e^{-\delta t}\mathcal{A}v(Z(t))dt \right] + E^{x,y} \left[ \int_0^{T_R} e^{-\delta t} \left( \frac{\partial v}{\partial x}(Z(t^-)) \cdot (-1 + \lambda)d\mathcal{L}(t) + \frac{\partial v}{\partial y}(Z(t^-)) \cdot (d\mathcal{L}(t) - d\mathcal{M}(t)) \right) \right] + E^{x,y} \left[ \sum_{0 < t_k \leq T_R} e^{-\delta t_k} \{ \Delta_{\mathcal{L},\mathcal{M}}v(Z(t_k)) \} \right] - \frac{\partial v}{\partial x}(Z(t^-)) \cdot \Delta_{\mathcal{L},\mathcal{M}}X(t_k) - \frac{\partial v}{\partial y}(Z(t^-)) \cdot \Delta_{\mathcal{L},\mathcal{M}}Y(t_k) \right] \right],
\]

where \( t_k \) denotes the times of jumps for \((\mathcal{L}(t), \mathcal{M}(t))\); \( \Delta_{\mathcal{L},\mathcal{M}}Y(t_k) = Y(t_k) - (Y(t_k^-) + \Delta_N Y(t_k)) \) and similarly for \( \Delta_{\mathcal{L},\mathcal{M}}X(t_k), \Delta_{\mathcal{L}}(t_k), \) and \( \Delta_{\mathcal{M}}(t_k) \).

This can be written

\[
E^{x,y}[e^{-\delta T_R}v(Z(T_R))] = v(x, y) + E^{x,y} \left[ \int_0^{T_R} e^{-\delta t}\mathcal{A}v(Z(t))dt \right] + E^{x,y} \left[ \int_0^{T_R} e^{-\delta t} \cdot (-1 + \lambda)\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}(Z(t^-))d\mathcal{L}_c(t) \right] + E^{x,y} \left[ \int_0^{T_R} e^{-\delta t} \cdot ((1 - \mu)\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y})(Z(t^-))d\mathcal{M}_c(t) \right] + E^{x,y} \left[ \sum_{0 < t_k \leq T_R} e^{-\delta t_k} \Delta_{\mathcal{L},\mathcal{M}}v(Z(t_k)) \right].
\]

Note that by the mean value theorem we have

\[
\Delta_{\mathcal{L},\mathcal{M}}v(Z(t_k)) = \frac{\partial v}{\partial x}(\hat{Z}(t_k))\Delta_{\mathcal{L},\mathcal{M}}X(t_k) + \frac{\partial v}{\partial y}(\hat{Z}(t_k))\Delta_{\mathcal{L},\mathcal{M}}Y(t_k) = \left( - (1 + \lambda)\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \right)(\hat{Z}(t_k))\Delta_{\mathcal{L}}(t_k) + \left( (1 + \mu)\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \right)(\hat{Z}(t_k))\Delta_{\mathcal{M}}(t_k),
\]

where \( \hat{Z}(t_k) \) is some point on the line segment between \( Z(t^-_k) \) and \( Z(t_k) \). Hence if (2.5a), (2.5b) and (2.5c) hold then by (A.2) and (A.3) we get

\[
v(x, y) \geq \lim_{R \to \infty} E^{x,y}[\int_0^{T_R} e^{-\delta t} \frac{1}{\gamma}c^\gamma(t)dt + e^{-\delta T_R}v(Z(T_R))] \geq J_c^{\mathcal{L},\mathcal{M}}(x, y).
\]
Since this holds for all admissible \((c, \mathcal{L}, \mathcal{M})\) we conclude that (2.6) holds.

To prove b) we apply the above argument to the control \((\hat{c}, \hat{\mathcal{L}}, \hat{\mathcal{M}})\). Then by (2.10) – (2.13) we get equality in the first part of (A.3), so that

\[
v(x, y) = E^x, y [\int_0^{T_R} e^{-\delta t} \frac{1}{\gamma} \hat{c}^\gamma(t) dt + e^{-\delta T_R} v(\hat{Z}(T_R))] .
\]

By (2.14) this gives

\[
v(x, y) = E^x, y [\int_0^\infty e^{-\delta t} \frac{1}{\gamma} \hat{c}^\gamma(t) dt] \leq V(x, y).
\]

Combined with (2.6) this gives the result.

\[\square\]

Proof of Lemma 2.3. Let \(b\) be as in (2.18) and choose \(K > 0\). We will apply Theorem 2.2a) to the function

\[
v(x, y) := K(x + by)^\gamma . \quad (A.5)
\]

First note that by (2.18) we have

\[
\begin{align*}
Lv &= -(1 + \lambda) \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} = -(1 + \lambda) + b)K(x + by)^{\gamma - 1} \leq 0 \quad (A.6)
\end{align*}
\]

and

\[
\begin{align*}
Mv &= (1 - \mu) \frac{\partial v}{\partial x} - \frac{\partial v}{\partial y} = ((1 - \mu) - b)K(x + by)^{\gamma - 1} \leq 0, \quad (A.7)
\end{align*}
\]

so clearly (2.5a) and (2.5b) hold.

To verify (2.5c) we first note that

\[
\begin{align*}
\int_{-1}^\infty & [v(x, y + y\eta) - v(x, y) - \frac{\partial v}{\partial y}(x, \eta) y\eta] d\eta(x) \\
&= K \int_{-1}^\infty [(x + by + b\eta\gamma) - (x + by)^\gamma - \gamma(x + by)^{\gamma - 1} by\eta] d\eta(x) \\
&= K\gamma \int_{-1}^\infty [(x + by + \theta b y\eta)^{\gamma - 1} - (x + by)^{\gamma - 1}] by\eta d\eta(x)
\end{align*}
\]

for some \(\theta \in (0, 1)\) by the mean value theorem. This integrand is negative if \(\eta > 0\) (since \(\gamma < 1\)) and also if \(\eta < 0\). Thus this integral is always nonpositive. Hence, for all \(c \geq 0\) we have

\[
A^c v(x, y) + \frac{1}{\gamma} c^\gamma \leq -\delta v(x, y) + (rx - c) \frac{\partial v}{\partial x} + \alpha y \frac{\partial v}{\partial y} + \frac{1}{2} \sigma^2 y^2 \frac{\partial^2 v}{\partial y^2} + \frac{1}{\gamma} c^\gamma.
\]

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For $b = 1$ and $K = K_0$ the last expression is nonpositive, for all $c \geq 0$, because we know that $v_0(x, y) := K_0(x + y)^\gamma$ solves the Hamilton-Jacobi-Bellman equation for the Merton problem ($q = \lambda = \mu = 0$). We conclude that $v_0$ satisfies all the conditions of Theorem 2.2a) and a) follows.

To prove b) it suffices to verify that there exists $K < \infty$ such that with $v$ as in (A.5),

$$H(c, x, y) := -\delta v + (rx - c) \frac{\partial v}{\partial x} + \alpha y \frac{\partial u}{\partial y} + \frac{1}{2} \sigma^2 y^2 \frac{\partial^2 v}{\partial y^2} + \frac{1}{\gamma} c^\gamma \leq 0.$$ 

Since $c \mapsto H(c, x, y)$ is maximal when

$$c = c^* = \left( \frac{\partial v}{\partial x} \right)^{\frac{1}{1-\gamma}} > 0,$$

it suffices to obtain

$$H_0(x, y) := H(c^*, x, y) = -\delta v + x \frac{\partial v}{\partial x} + \alpha y \frac{\partial v}{\partial y} + \frac{1}{2} \sigma^2 y^2 \frac{\partial^2 v}{\partial y^2} + \frac{1}{\gamma} (K \gamma) \frac{1}{\gamma-1} (x + by)^\gamma \leq 0$$

which holds if

$$\left[ \frac{1}{\gamma} (K \gamma) \frac{1}{\gamma-1} - \delta K + K \gamma \alpha \right] (x + by)^2 \leq \frac{1}{2} \sigma^2 K \gamma (1 - \gamma) b^2 y^2$$

for all $x, y$. This holds if and only if the coefficient of $(x + by)^2$ is nonpositive, i.e., if and only if

$$\delta > \gamma \alpha + (K \gamma) \frac{1}{\gamma-1}.$$  \hspace{1cm} (A.8)

If (2.19) holds, then there exists $K$ such that (A.8) holds and therefore (2.5c) holds for $v$ given by (A.5). This completes the proof of b). \hfill \Box

**Proof of Theorem 2.8:** Following Akian et al. (1996) we introduce the new variables

$$\rho = x + (1 - \mu) y \quad \text{(net wealth)} \hspace{1cm} (A.9)$$

and

$$\beta = \frac{(1 - \mu) y}{\rho} \quad \text{(fraction of net wealth invested in stock).} \hspace{1cm} (A.10)$$

Then by homogeneity we can write

$$V(x, y) = V(\rho(1 - \beta), \rho \frac{\beta}{1 - \mu}) = \rho^\gamma V(1 - \beta, \frac{\beta}{1 - \mu}) = \rho^\gamma G(\beta),$$
where
\[ G(\beta) = V(1 - \beta, \frac{\beta}{1 - \mu}), \quad \beta > -\frac{1 - \mu}{\lambda + \mu}. \]

Note that \( G(\beta) \) is concave, because it is just the restriction of a concave function to a straight line. Moreover,
\[ G'(\beta) = (-1) \frac{\partial V}{\partial x} + \frac{1}{1 - \mu} \frac{\partial V}{\partial y} = -\frac{1}{1 - \mu} \left[ (1 - \mu) \frac{\partial V}{\partial x} - \frac{\partial V}{\partial y} \right]. \]

Suppose there exists \( \beta^* > -\frac{1 - \mu}{\lambda + \mu} \) such that \( G'(\beta^*) \leq 0 \). Then by concavity we must have \( G'(\beta) \leq 0 \) for a.a. \( \beta \geq \beta^* \). On the other hand, by Lemma 2.6b) we also have \( G'(\beta) \geq 0 \) for a.a. \( \beta \). We conclude that \( G'(\beta) = 0 \) for a.a. \( \beta \geq \beta^* \). Therefore, if we define \( \beta_2^* = \inf\{ \beta^*; G'(\beta^*) \leq 0 \} \) we have
\[ G'(\beta) > 0 \text{ for a.a. } \beta < \beta_2^* \text{ and } G'(\beta) = 0 \text{ for a.a. } \beta > \beta_2^*. \]

Since \( \beta > \beta_2^* \Leftrightarrow \theta > \theta_2^*, \text{ where } \tan \theta_2^* = \frac{\beta_2^*}{(1 - \beta_2^*)(1 - \mu)} \)
we conclude that (2.28) holds with \( \tan \theta_2^* = \frac{\beta_2^*}{(1 - \beta_2^*)(1 - \mu)} \).

Similarly, by using the coordinates \( \hat{\rho} = x + (1 + \lambda)y, \quad \hat{\beta} = \frac{(1 + \lambda)y}{\rho} \)
we deduce that (2.27) holds.

Proof of Theorem 3.2: We know from Lemma 2.4 that \( V \) is continuous.

(i): \( V \) is a subsolution.

Choose \( \phi \in C^2(U) \) and \( z_0 \in U \) such that \( \phi \geq V \) on \( U \) on \( \phi(z_0) = V(z_0) \). Let \( w = (c, \mathcal{L}, \mathcal{M}) \) be an admissible control. Define
\[ Z(t) := \hat{Z}(t) = (X(t^-), Y(t^-) + \Delta N Y(t^+)). \quad (A.11) \]
For a bounded domain $A \subseteq U$ and some constant $M \in (0, \infty)$, define
\begin{equation}
\bar{\tau}_A := M \land \inf \{ t > 0; \bar{Z}(t) \notin A \} \tag{A.12}
\end{equation}
(cf. (1.8)). We split the transactions as follows: Let $(d\mathcal{L}_0(t), d\mathcal{M}_0(t))$ be equal to $(d\mathcal{L}, d\mathcal{M})$ on $[0, \bar{\tau}_A)$, equal to zero for $t > \bar{\tau}_A$ and for $t = \bar{\tau}_A$, we let
\begin{equation}
(\Delta \mathcal{L}_0(\bar{\tau}_A), \Delta \mathcal{M}_0(\bar{\tau}_A)) = \begin{cases} 
\bar{b} \cdot (\Delta \mathcal{L}(\bar{\tau}_A), \Delta \mathcal{M}(\bar{\tau}_A)) & \text{if } \bar{Z}(\bar{\tau}_A) \in A \\
0 & \text{otherwise}
\end{cases} \tag{A.13}
\end{equation}
where $\bar{b}$ is defined as
\begin{equation}
\bar{b} = \inf \{ b \in [0, 1]; \bar{Z}(\bar{\tau}_A) + b \cdot (\Delta X(\bar{\tau}_A), \Delta Y(\bar{\tau}_A)) \notin A \}. \tag{A.14}
\end{equation}
In words, if we at time $\bar{\tau}_A$ perform a transaction taking the process from inside to outside the set $A$, then $(\mathcal{L}_0, \mathcal{M}_0)$ includes only the fraction $\bar{b}$ of $(\Delta \mathcal{L}(\bar{\tau}_A), \Delta \mathcal{M}(\bar{\tau}_A))$ taking the process to the boundary of $A$. In addition, $(\mathcal{L}_0, \mathcal{M}_0)$ include all transactions before time $\bar{\tau}_A$.

Similarly, put $c_0(t) := c(t)\chi_{\{t < \bar{\tau}_A\}}$ and define the controls $w_0$ and $w_1$ by
\begin{align*}
w_0(t) &= (c_0(t), \mathcal{L}_0(t), \mathcal{M}_0(t)) \\
w_1(t) &= w(t) - w_0(t).
\end{align*}
We remark that since transaction costs are linear, it makes no difference to split one transaction into two "simultaneous" smaller transactions. Therefore, by the (strong) dynamic programming principle (see e.g. Krylov (1980) p. 134, Theorem 9 and 11), we have
\begin{equation}
J^w(z_0) = E^{z_0} \left[ \int_0^\tau e^{-\delta t} \frac{S(t)}{\gamma} dt + e^{-\delta \tau} J^{w_1}(z_1(\omega)) \right] \tag{A.15}
\end{equation}
where $z_1(\omega) = Z^{w_0}(\tau)$ and $\tau = \bar{\tau}_A$. Choosing $w = w_0 + w_1$ as an $\epsilon$-optimal control and using Itô’s formula (see (A.2)) on the latter term, we get
\begin{align*}
0 &\leq \epsilon + E^{z_0} \left[ \int_0^\tau e^{-\delta t} \left( A^{w_0} \phi(Z^{w_0}(t)) + \frac{c_0(t)}{\gamma} \right) dt \right] \\
&+ E^{z_0} \left[ \int_0^\tau e^{-\delta t} \cdot (- (1 + \lambda) \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y})(Z^{w_0}(t))d\mathcal{L}_0(t) \right] \\
&+ E^{z_0} \left[ \int_0^\tau e^{-\delta t} \cdot ((1 - \mu) \frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial y})(Z^{w_0}(t))d\mathcal{M}_0(t) \right] \\
&+ E^{z_0} \left[ \sum_{0 \leq t_k \leq \tau} e^{-\delta t_k} \Delta \mathcal{L}_0, \mathcal{M}_0 \phi(Z^{w_0}(t_k)) \right] \tag{A.16}
\end{align*}
for any $\tau = \bar{\tau}_A$ as above.
To prove (3.4), suppose for contradiction that
\[ \hat{L}\phi(z_0) < 0, \quad \hat{M}\phi(z_0) < 0 \quad \text{and} \quad R\phi(z_0) := \sup_{c \geq 0} \{ A^c \phi(z_0) + \frac{1}{\gamma} c^\gamma \} < 0. \] (A.17)

Then necessarily \( \frac{\partial \phi}{\partial x}(z_0) > 0 \) and hence by continuity \( \frac{\partial \phi}{\partial x} > 0 \) on some neighborhood \( A' \) of \( z_0 \). But then
\[ R\phi(z_0) = \frac{\partial \phi}{\partial s} + (rx - \hat{c}) \frac{\partial \phi}{\partial x} + \alpha y \frac{\partial \phi}{\partial y} + \frac{1}{2} \sigma^2 \frac{\partial^2 \phi}{\partial y^2} + \frac{\hat{c}^\gamma}{\gamma} \] (A.18)
on \( A' \), with \( \hat{c} = \hat{c}(z) = \left( \frac{\partial \phi}{\partial x} \right)^{-\frac{1}{\gamma - 1}} \). It follows that \( R\phi \) is continuous on \( A' \), and so is \( \hat{L}\phi \) and \( \hat{M}\phi \). Therefore there exists a \( \beta > 0 \) and some \( \tau \)-neighborhood \( A = A_\tau \) of \( z_0 \) such that \( L\phi, M\phi \) and \( R\phi \) all are \( < -\beta e^{\delta M} \) on \( A \). Since we have defined \( w_0 \) such that all transactions take place on \( A \), then we get by (A.16),
\[ \epsilon \geq \beta E^{z_0}[\tau + \mathcal{L}_0(\tau) + \mathcal{M}_0(\tau)]. \] (A.19)
The stopping time \( \tau \) depends on \( A \) and on \( w_0 \), so define another stopping time \( \tau^{(0)} \) by choosing \( A_{\tau/2}, c = \hat{c} \) and \( \mathcal{L}_0 = \mathcal{M}_0 = 0 \). By (A.16) and (A.19), we have
\[ \epsilon \geq \beta E^{z_0}[\tau \chi_{\tau \geq \tau^{(0)}} + (\mathcal{L}_0(\tau) + \mathcal{M}_0(\tau))\chi_{\tau < \tau^{(0)}}] \geq \beta (E^{z_0}[\tau \chi_{\tau \geq \tau^{(0)}}] + a\tau P^{z_0}[\tau < \tau^{(0)}]) \] (A.20)
since if \( \tau(\omega) < \tau^{(0)} \), then \( \mathcal{L}_0(\tau) + \mathcal{M}_0(\tau) \geq a\tau \) for some constant \( a > 0 \), by the continuity of the coefficients of \( Z \). Now let \( \epsilon \to 0 \) and consider what happens. There are two possibilities:

Either there exists \( \rho > 0 \) such that \( \limsup_{\epsilon \to 0} P^{z_0}[\tau < \tau^{(0)}] \geq \rho \). Then \( a\rho \beta \leq \lim_{n} \epsilon_n = 0 \), which is impossible. In the other case, \( P^{z_0}[\tau < \tau^{(0)}] \to 0 \). Then \( \chi_{\tau \geq \tau^{(0)}} \to 1 \) a.s., and taking limits in (A.20), we have
\[ 0 \geq \beta E^{z_0}[\tau^{(0)}] > 0, \] (A.21)
a contradiction. We conclude that \( V \) is a viscosity subsolution.

(ii): \( V \) is a supersolution.

Choose \( \phi \in C^2(U) \) and \( z_0 \in U \) such that \( \phi \leq V \) on \( U \) and \( \phi(z_0) = V(z_0) \). Then as in (A.16) we get, for all admissible controls \( w = (c, \mathcal{L}, \mathcal{M}) \) and all \( \tau \leq \tau_M \),
\[ 0 \geq E^{z_0} \left[ \int_0^\tau e^{-\delta t} (A^c \phi(Z(t)) + \frac{c^\gamma(t)}{\gamma}) dt \right] + E^{z_0} \left[ \int_0^\tau e^{-\delta t} (- (1 + \lambda) \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y})(Z(t))d\mathcal{L}_c(t) \right] \]
\[ + E^{z_0} \left[ \int_0^\tau e^{-\delta t} (- (1 - \mu) \frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial y})(Z(t))d\mathcal{M}_c(t) \right] \]
\[ + E^{z_0} \left[ \sum_{0 \leq t_k \leq \tau} e^{-\delta t_k} \Delta \mathcal{L}_c(\Delta \phi(Z(t_k))) \right]. \] (A.22)
Now choose $M = 0$ and $L(t)$ to make an immediate jump to $\ell > 0$ at time $t = 0$. Then by (A.22) we get

$$E^\omega[\Delta_{L,M}\phi(Z(0))] \leq 0$$

i.e.,

$$\phi(x_0 - (1 + \lambda)\ell, y_0 + \ell) \leq \phi(x_0, y_0).$$

This implies that $L\phi(x_0, y_0) \leq 0$. Similarly we obtain $M\phi(x_0, y_0) \leq 0$.

Finally, by choosing $w = (c, 0, 0)$ for $t \leq \tau$ with $c \geq 0$ constant (and $\tau$ so small that $w$ is admissible) we get by (A.22) that

$$E^\omega \left[ \int_0^\tau e^{-\delta t} (A^c \phi(Z(t)) + \frac{c'}{\gamma}) dt \right] \leq 0.$$  

Dividing by $E^\omega[\tau]$ and letting $\tau \to 0$ we get

$$A^c \phi(z_0) + \frac{c'}{\gamma} \leq 0. \quad (A.23)$$

We conclude that (3.5) holds. So $V$ is a viscosity supersolution. \qed

**Proof of Theorem 4.2:** By Corollary 3.3 we know that

$$\sup_{c \geq 0} \left\{ A^c V(x, y) + \frac{1}{\gamma} c' \right\} = 0 \quad (A.24)$$

for all $(x, y) \in D = \{ Re^{i\theta}; \theta_1 < \theta < \theta_2 \}$. This implies that $\frac{\partial V}{\partial x}(x, y) > 0$ in $D$ and that the value of $c$ for which the supremum in (A.24) is attained is

$$c = c^*(x, y) = \left( \frac{\partial V}{\partial x} \right)^\frac{1}{\gamma}. $$

Hence

$$A^{c^*} V(x, y) + \frac{1}{\gamma} (c^*)' = 0 \quad \text{in } D. \quad (A.25)$$

If we apply the control $w^* = (c^*, L^*, M^*)$ the corresponding process $Z^*(t)$ will remain in $\overline{D}$ for almost all $t$. Therefore, we can apply the Itô formula to $V$ just as in (A.1) to conclude that (see (A.2))

$$V(x, y) = E^\omega[e^{-\delta T_R} V(Z_{T_R})] + E^\omega\left[ \int_0^{T_R} \frac{1}{\gamma} (c^*)'(t) dt \right]. \quad (A.26)$$

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So if (2.14) holds for \( \dot{w} = w^* \), we obtain by letting \( R \to \infty \) that

\[
V(x, y) = \mathbb{E}^x[y \int_0^\infty \frac{1}{\gamma^\gamma} e^{r(t)} dt]
\]  

(A.27)

and therefore \( w^* \) is optimal.

It remains to verify (2.14). Since \( V(x, y) \leq K_0(x + y)^\gamma \) (by Lemma 2.3a)) it suffices to prove that

\[
\lim_{R \to \infty} \mathbb{E}^x[y e^{-\delta T_R} (X^{w^*}(T_R) + Y^{w^*}(T_R))^\gamma] = 0.
\]  

(A.28)

To this end, note that from the dynamics of \( X \) and \( Y \), combined with (4.5) we have, with \( Z^* = Z^{w^*} \),

\[
dZ^*(t) \leq \alpha Z^*(t) dt + \sigma Z^*(t) dW(t) + Z^*(t^-) \int_{-1}^{\infty} \eta \tilde{N}(dt, d\eta).
\]  

(A.29)

Hence (see the proof of Theorem 2.3 in Framstad et al. (1998) for details)

\[
Z^*(t) \leq Z(0) \exp \left\{ (\alpha - \frac{1}{2} \sigma^2 - \|q\|) t + \sigma W(t) + \int_0^t \int_{-1}^{\infty} \ln(1 + \eta) N(ds, d\eta) \right\}
\]

and hence

\[
\mathbb{E} \left[ e^{-\delta T_R} (Z^*(T_R))^\gamma \right] \\
\leq Z^*(0) \mathbb{E} \left[ \exp \left\{ (\alpha + \gamma \alpha - \frac{1}{2} \sigma^2 \gamma (1 - \gamma) - \gamma \|q\| + \int_{-1}^{\infty} ((1 + \eta)^\gamma - 1) dq(\eta)) T_R \right\} \right]
\]

which goes to 0 as \( R \to \infty \) because by (4.6) the coefficient of \( T_R \) in the exponent is negative. □

**Conclusion**

We have obtained, in some sense, the simplest possible generalization to the Merton problem; As noted in the introduction, the optimal portfolio for the CRRA investor in the lognormal case with transaction costs, is obtained by keeping the fraction of wealth invested in the risky asset in a fixed interval, appearing as a “no transaction cone” in the \( xy \)-plane, and rebalancing according to the local time at its boundary. We have shown that this result does not depend on lognormality; since the process may jump out of the (closure of) the no transaction cone, we must allow for the generalization of the local time concept in order to have the connection to the Skorohod problem as in Theorem 4.1.

For future research, economic intuition suggests the following properties:
• The continuation region tends to the first quadrant as $\lambda \to \infty$ and $\mu \to 1$. It is tempting to
guess that the boundaries of the no transaction region tend monotonically to the axes. This
agrees with the conjecture in Shreve and Soner (1994), Remark 11.3 in that when leverage is
optimal in the Merton problem, then the presence of transaction cost will reduce the leverage.
We expect to see the similar for short-selling as well. Furthermore, if these properties hold,
then the $\theta_i$ boundary coincides with the $i$-th axis iff the Merton line does, again in accordance
with the remark in Shreve and Soner (1994).

• Let us note that if $\theta_2 = \pi/2$, then we face the following interesting situation: Once on the
$y$-axis, we have $dX = 0$ so that $d\mathcal{M} = c\,dt/(1 - \mu)$ is absolutely continuous for $t > 0$ and
we face a pure consumption optimization problem. (A similar thing happens on the $x$-axis
if $\theta_1 = 0$). It is fairly obvious that if the no transaction region has no boundaries coinciding
with axes, then $\mathcal{L}$ and $\mathcal{M}$ are $dt$-singular, while they are absolutely continuous (for $t > 0$) if
on the axes. This may be the explanation why it has turned out to be difficult to prove the
value function to be $C^2$ on the axes.

• In all cases, we conjecture that the assumptions made to ensure that the Merton line lies in
the first quadrant, are not needed.

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• Let us note that if \( \theta_2 = \pi/2 \), then we face the following interesting situation: Once on the \( y \)-axis, we have \( dX = 0 \) so that \( d\mathcal{M} = cdt/(1 - \mu) \) is absolutely continuous for \( t > 0 \) and we face a pure consumption optimization problem. (A similar thing happens on the \( x \)-axis if \( \theta_1 = 0 \). It is fairly obvious that if the no transaction region has no boundaries coinciding with axes, then \( \mathcal{L} \) and \( \mathcal{M} \) are \( dt \)-singular, while they are absolutely continuous (for \( t > 0 \)) if on the axes. This may be the explanation why it has turned out to be difficult to prove the value function to be \( C^2 \) on the axes.

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