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Co-Amenability of Compact Quantum Groups

E. Bédos* G.J. Murphy L. Tuset*

Abstract

We study the concept of co-amenability for a compact quantum group. Several conditions are derived that are shown to be equivalent to it. Some consequences of co-amenability that we obtain are faithfulness of the Haar integral and automatic norm-boundedness of positive linear functionals on the quantum group's Hopf $*$-algebra (neither of these properties necessarily holds without co-amenability).

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1 Introduction

In this paper we introduce and study a concept of co-amenability for compact quantum groups defined in the sense of S.L. Woronowicz [19, 20]—see also [14] for an exposition that provides much of the background for this paper. Co-amenability of so-called regular multiplicative unitaries has been introduced by S. Baaj and G. Skandalis [1, Appendix] [6]. One can then procede to define co-amenability of a compact quantum group by requiring that the regular multiplicative unitary associated to its reduced quantum group is co-amenable. However, the C*-algebra formulation of compact quantum groups is more accessible than the theory of multiplicative unitaries, which is technically quite involved. We therefore feel that it is worthwhile and appropriate to present a direct definition of co-amenability, which is perhaps more intrinsic to the C^*-algebra theory of compact quantum groups. The Baaj-Skandalis approach to co-amenability for compact quantum groups has been rephrased by T. Banica [2, 3] to accomodate this, but details are defered to Baaj-Skandalis' work. Our exposition starts from an elementary remark of Woronowicz [19, p. 623] and is aimed to be self-contained. To motivate our definition we briefly discuss here the concept of amenability for a discrete group and its equivalent formulations in terms of the group C*-algebras [16, 17].

If Γ is a discrete group, its reduced and full group C*-algebras $C^*_{\mathrm{r}}(\Gamma)$ and $C^*(\Gamma)$ can be endowed with co-multiplications Δ_{r} and Δ making them into compact quantum groups. Details are given in Section 2. We shall call these the *reduced* and *universal* compact quantum groups associated with Γ. The Haar integrals of $(C^*_{\mathrm{r}}(\Gamma), \Delta_{\mathrm{r}})$ and $(C^*(\Gamma), \Delta)$ are the canonical tracial states. Since the left kernel of the trace on $C^*(\Gamma)$ is the kernel of the canonical $*$-homomorphism θ from $C^*(\Gamma)$ onto $C^*_{\mathrm{r}}(\Gamma)$, faithfulness of the Haar integral of $(C^*(\Gamma), \Delta)$ is equivalent to amenability of Γ. Of course, we are using here the well known equivalence of amenability of Γ and injectivity of θ; this result is often called the Hulanicki-Reiter theorem in the literature. The co-unit of $(C^*(\Gamma), \Delta)$ is norm-bounded, but that of $(C^*_{\mathrm{r}}(\Gamma), \Delta_{\mathrm{r}})$ may not be. In fact, it is known that Γ is amenable if, and only if, the co-unit of the latter is norm-bounded. This is essentially a reformulation of the classical result that Γ is amenable if, and only if, the trivial 1-dimensional representation of Γ is weakly contained in the regular representation.

This discussion serves to motivate our introduction of the concept of co-amenability for a general compact quantum group and we shall frequently refer back to these examples for the purposes of illustration and motivation of the results we obtain in the sequel. We define a compact quantum group (A, Δ) to be *co-amenable* if the co-unit of its reduced compact quantum group $(A_{\mathrm{r}}, \Delta_{\mathrm{r}})$ is norm-bounded (see Section 2 for the definition of $(A_{\mathrm{r}}, \Delta_{\mathrm{r}})$). If a concept is to be a fruitful one in an abstract theory, it is desirable that it have a number of different formulations. Indeed we show that co-amenability is equivalent to several other conditions; one of these equivalences is an analog of the Hulanicki–Reiter theorem (see Theorem 3.6), which etablishes the link with Banica's definition. One particularly nice condition ensuring co-amenability of a compact quantum group is the existence of a non-zero multiplicative linear functional on its reduced quantum group (Corollary 2.9).

A co-amenable compact quantum group has a number of desirable properties not possessed by arbitrary compact quantum groups. We show, for example, that a co-amenable compact quantum group has a faithful Haar integral (it then follows that the Haar integral is a KMS state [11, 12]). If a compact quantum group is not co-amenable, then the co-unit on the Hopf $*$-algebra of its reduced compact quantum group provides an example of a positive linear functional that is *not* norm-bounded. However, we show that every positive linear functional on the Hopf $*$-algebras of a co-amenable compact quantum group is necessarily norm-bounded (Corollary 3.7).

The use of the word *co-amenability* deserves some explanation. First recall that amenability of Kac algebras [7] is defined in terms of the existence of an invariant state. If we define amenability of a compact quantum group in these terms, namely by requiring only the existence of an invariant state, then all compact quantum groups are trivially amenable, since the Haar integral is an invariant state. Thus, this is not a satisfactory definition. On the other hand, the natural concept of amenability for discrete quantum groups makes good sense—we study this notion in a forthcoming paper [5]. There is a relationship between co-amenability of a compact quantum group as defined in this paper and amenability of the associated dual discrete quantum group. The chosen terminology is aimed to reflect this dual relationship. It also fits with the one introduced by Baaj and Skandalis in [1] for regular multiplicative unitaries. Note however the slightly confusing fact that Banica [2, 3] uses most of the time the word amenability instead of co-amenability for compact quantum groups (which he calls "Woronowicz algebras").

The paper is organized as follows: In Section 2 we construct the reduced quantum group corresponding to a compact quantum group and use it to define co-amenability of the original compact quantum group. We then derive conditions equivalent to co-amenability and show it implies faithfulness of the Haar integral. As an application of the ideas in this section, we give a new proof of the theorem of G. Nagy on faithfulness of the Haar measure of quantum $SU(2)$. In Section 3 we consider the universal compact quantum group associated to a compact quantum group and obtain other conditions equivalent to co-amenability; in addition, we prove the norm-boundedness result for positive linear functionals alluded to above. Our final section, Section 4, is a short one in which we explore the idea of a bounded co-unit in the context of a compact quantum semigroup and show that if the latter admits a faithful Haar integral and a bounded co-unit, it is necessarily a co-amenable compact quantum group.

For the ease of the reader, our account is quite detailed and we provide proofs of several important results which are presented in a rather sketchy manner in the literature. Especially, we give in an appendix a proof of the uniqueness property of the associated dense Hopf $*$-algebra of a compact quantum group. This useful property is stated without proof in [11].

We shall use the convention that $X \otimes Y$ represents the algebraic tensor product when X and Y are simply linear spaces, or $*$-algebras that are not C*-algebras; if X and Y are Hilbert spaces, $X \otimes Y$ represents the Hilbert space tensor product and if X and Y are C*-algebras, $X \otimes Y$ represents the spatial C*-tensor product [13, Chapter 6].

2 The Reduced Quantum Group

Throughout this section (A, Δ) denotes a compact quantum group. Its Haar integral is denoted by h. The associated Hopf $*$-algebra is denoted by $\mathcal{A}$, the co-inverse by κ and the co-unit by ε. Recall that ε and κ are, in general, only defined on $\mathcal{A}$. One can describe $\mathcal{A}$ by saying it is the unique Hopf $*$-algebra for which $\mathcal{A}$ is a dense unital $*$-subalgebra of A and the co-multiplication of $\mathcal{A}$ is obtained by restriction of the co-multiplication of A. The reader may find some basic definitions and a proof of this uniqueness property in an appendix to this paper. We refer otherwise to [14] and [20] for the basic theory of compact quantum groups.

Let $(C(G), \Delta)$ be a commutative compact quantum group associated to a compact group G, the co-multiplication Δ being dual to the group multiplication operation $G \times G \to G$. In this case the Haar integral h is the integral with respect to the Haar measure on G. This has full support and therefore h is faithful. Faithfulness of the Haar integral no longer holds for an arbitrary compact quantum group. To illustrate this we return to the group C*-algebras of a discrete group and discuss them in a little more detail.

Let Γ be a discrete group and let $L\colon x \mapsto L_x$ be the left regular representation of Γ on $\ell^2(\Gamma)$. Thus, if $(\delta_x)_{x\in\Gamma}$ is the canonical orthonormal basis of $\ell^2(\Gamma)$, $L_x(\delta_y) = \delta_{xy}$. Let $A_{\mathrm{r}} = C^*_{\mathrm{r}}(\Gamma)$ be the reduced group C*-algebra of Γ; that is, A_{r} is the C*-subalgebra of $B(\ell^2(\Gamma))$ generated by the operators L_x $(x \in \Gamma)$. The linear map Δ_{r} defined on A_{r} by $\Delta_{\mathrm{r}}(L_x) = L_x \otimes L_x$, for all $x \in \Gamma$, is a co-multiplication of A_{r}. (To see that Δ_{r} is well defined, observe that there is a unitary operator W on $\ell^2(\Gamma) \otimes \ell^2(\Gamma)$ for which $L_x \otimes L_x = W^*(1 \otimes L_x)W$, for all $x \in \Gamma$; W is defined by setting $W(\delta_x \otimes \delta_y) = \delta_{x^{-1}y} \otimes \delta_y$, for all $x, y \in \Gamma$.) It is easy to see that $(A_{\mathrm{r}} \otimes 1)\Delta_{\mathrm{r}}A_{\mathrm{r}}$ and $(1 \otimes A_{\mathrm{r}})\Delta_{\mathrm{r}}A_{\mathrm{r}}$ each have closed linear span equal to $A_{\mathrm{r}} \otimes A_{\mathrm{r}}$. Hence, $(A_{\mathrm{r}}, \Delta_{\mathrm{r}})$ is a compact quantum group.

It is well known that $C^*_{\mathrm{r}}(\Gamma)$ admits a faithful tracial state tr given by $\mathrm{tr}(L_x) = 0$, if x is an element of Γ that is not equal to the unit of Γ. In fact, tr is the Haar integral of $(A_{\mathrm{r}}, \Delta_{\mathrm{r}})$ [14, Example 10.4]. The dense Hopf $*$-algebra $\mathcal{A}_{\mathrm{r}}$ of $(A_{\mathrm{r}}, \Delta_{\mathrm{r}})$ is the linear span of all the unitaries L_x $(x \in \Gamma)$. It may be identified with the group algebra $\mathbf{C}(\Gamma)$ of Γ equipped with its canonical Hopf $*$-algebra structure.

The full group C*-algebra $A_{\mathrm{u}} = C^*(\Gamma)$ is, by definition, the enveloping C*-algebra of the Banach $*$-algebra $\ell^1(\Gamma)$. By construction, $\mathbf{C}(\Gamma)$ is dense in A_{u}. Therefore, Γ admits a universal unitary representation, $V\colon \Gamma \to A_{\mathrm{u}}$, $x \mapsto V_x$ such that the linear span of the elements V_x is dense in A_{u}. A co-multiplication on A_{u} making it into a compact quantum group is determined by first setting $\Delta(V_x) = V_x \otimes V_x$, for all $x \in \Gamma$, and then extending Δ to A_{u} by its universal property. The Hopf $*$-algebra $\mathcal{A}_{\mathrm{u}}$ of (A_{u}, Δ) is the linear span of the elements V_x, and it too may be identified with $\mathbf{C}(\Gamma)$.

By the universal property of $C^*(\Gamma)$ there exists a canonical surjective $*$-homomorphism $\theta : A_{\mathrm{u}} \to A_{\mathrm{r}}$ mapping each V_x onto L_x, hence mapping $\mathcal{A}_{\mathrm{u}}$ onto $\mathcal{A}_{\mathrm{r}}$. The Haar integral on A_{u} is the canonical tracial state of A_{u} given by $h = \mathrm{tr} \circ \theta$. Its left kernel N_h is clearly the kernel of θ, so $A_{\mathrm{r}} = A_{\mathrm{u}}/N_h$. Again using the universal property of $C^*(\Gamma)$, we see there is a $*$-homomorphism ε from A_{u} to $\mathbf{C}$ such that $\varepsilon(V_x) = 1$, for all $x \in \Gamma$. A simple computation shows that ε is the co-unit for (A_{u}, Δ). (More precisely, the restriction of ε to the Hopf $*$-algebra of (A_{u}, Δ) is the co-unit.) The important point here is that ε is norm-bounded.

The group Γ is amenable if, and only if, θ is injective, and the co-unit of $C^*_{\mathrm{r}}(\Gamma)$ is therefore norm-bounded in this case. If Γ is not amenable, this co-unit is not norm-bounded, as pointed out in the Introduction. In the case that $\Gamma = \mathbf{F}_2$, the free group on two generators, one can see the co-unit of $C^*_{\mathrm{r}}(\Gamma)$ is not norm-bounded by means of the well known fact that $C^*_r(\Gamma)$ is simple (and not one-dimensional!) and therefore admits no $*$-homomorphism onto $\mathbf{C}$.

Suppose now that (A, Δ) is an arbitrary compact quantum group with associated Hopf $*$-algebra $\mathcal{A}$. It is known [20] that the Haar integral of (A, Δ) is faithful on $\mathcal{A}$, but as we have seen, in general, not on the C^*-algebra A. We will now furnish a C^*-algebra envelope of the Hopf $*$-algebra $\mathcal{A}$ for which the Haar integral is faithful. Recall that the left kernel N_h of h is a two-sided ideal of A [20]. Set $A_{\mathrm{r}} = A/N_h$ and let θ be the quotient map from A onto A_{r}. We shall make A_{r} into a compact quantum group. This reduction procedure is sketched in [19], but no details are given there, or anywhere else in the literature that we are aware of. Since this is an important construction for this paper we give the required details in the following result.

Theorem 2.1 *If (A, Δ) is a compact quantum group, then the C^*-algebra A_{r} can be made into a compact quantum group whose co-multiplication Δ_{r} is determined by $\Delta_{\mathrm{r}}(\theta(a)) = (\theta \otimes \theta)\Delta(a)$, for all $a \in A$. The Haar integral of $(A_{\mathrm{r}}, \Delta_{\mathrm{r}})$ is the unique state h_{r} of A_{r} such that $h = h_{\mathrm{r}} \circ \theta$. The state h_{r} is faithful. Also, the quotient map θ is faithful on $\mathcal{A}$ and the Hopf $*$-algebra of $(A_{\mathrm{r}}, \Delta_{\mathrm{r}})$ is $\theta(\mathcal{A})$, with co-unit ε_{r} and co-inverse κ_{r} determined by $\varepsilon = \varepsilon_{\mathrm{r}} \circ \theta$ and $\theta \circ \kappa = \kappa_{\mathrm{r}} \circ \theta$, respectively.*

Proof. To show that we can define a $*$-homomorphism $\Delta_{\mathrm{r}}: A_{\mathrm{r}} \to A_{\mathrm{r}} \otimes A_{\mathrm{r}}$ such that $\Delta_{\mathrm{r}}(\theta(a)) = (\theta \otimes \theta)\Delta(a)$, for all $a \in A$, we need only show that $\ker(\theta) \subseteq \ker(\theta \otimes \theta)\Delta)$. Clearly, it suffices to show that $\ker(\theta) \subseteq \ker((\mathrm{id} \otimes \theta)\Delta)$. To see this, we first observe that, by the Cauchy-Schwartz inequality, h vanishes on $\ker(\theta)$. Therefore it induces a unique state h_{r} on A_{r} such that $h = h_{\mathrm{r}} \circ \theta$. Since $\ker(\theta) = N_h$, it is clear that h_{r} is faithful. Using the fact that product states separate elements of $A_{\mathrm{r}} \otimes A_{\mathrm{r}}$, it easily follows that $\mathrm{id} \otimes h_{\mathrm{r}}: A_{\mathrm{r}} \otimes A_{\mathrm{r}} \to A_{\mathrm{r}}$ is faithful. Suppose now $\theta(a) = 0$. Then $h(a^*a) = 0$ and therefore, $(\mathrm{id} \otimes h_{\mathrm{r}})(\mathrm{id} \otimes \theta)\Delta(a^*a)$ $= (\mathrm{id} \otimes h)\Delta(a^*a) = h(a^*a)1 = 0$. Consequently, $(\mathrm{id} \otimes \theta)\Delta(a^*a) = 0$, and therefore $(\mathrm{id} \otimes \theta)\Delta(a) = 0$ as required. Thus, we can well define a $*$-homomorphism Δ_{r} as claimed above.

One can easily check now that Δ_{r} is a co-multiplication on A_{r}. Since the linear spans of $(1 \otimes A)\Delta(A)$ and $(A \otimes 1)\Delta(A)$ are dense in $A \otimes A$, it follows immediately that the linear spans of $(1 \otimes A_{\mathrm{r}})\Delta_{\mathrm{r}}(A_{\mathrm{r}})$ and $(A_{\mathrm{r}} \otimes 1)\Delta_{\mathrm{r}}(A_{\mathrm{r}})$ are dense in $A_{\mathrm{r}} \otimes A_{\mathrm{r}}$. Hence, $(A_{\mathrm{r}}, \Delta_{\mathrm{r}})$ is a compact quantum group.

If $a \in A$, then $(\mathrm{id} \otimes h_{\mathrm{r}})\Delta_{\mathrm{r}}(\theta(a)) = (\mathrm{id} \otimes h_{\mathrm{r}})(\theta \otimes \theta)\Delta(a) = \theta(\mathrm{id} \otimes h)\Delta(a) = \theta(h(a)1) = h_{\mathrm{r}}(\theta(a))\theta(1)$. Similarly, $(h_{\mathrm{r}} \otimes \mathrm{id})\Delta_{\mathrm{r}}(\theta(a)) = h_{\mathrm{r}}(\theta(a))\theta(1)$. Hence, h_{r} is the Haar integral of $(A_{\mathrm{r}}, \Delta_{\mathrm{r}})$.

The injectivity of θ on $\mathcal{A}$ follows readily: If $a \in \mathcal{A}$ and $\theta(a) = 0$, then $h(a^*a) = 0$. Since h is faithful on $\mathcal{A}$, we deduce that $a = 0$.

We can therefore define linear maps, $\varepsilon_{\mathrm{r}} : \theta(\mathcal{A}) \to \mathbf{C}$ and $\kappa_{\mathrm{r}} : \theta(\mathcal{A}) \to \theta(\mathcal{A})$, by setting $\varepsilon_{\mathrm{r}}(\theta(a)) = \varepsilon(a)$ and $\kappa_{\mathrm{r}}(\theta(a)) = \theta(\kappa(a))$, for all $a \in \mathcal{A}$. It is then clear that $\theta(\mathcal{A})$ is a dense Hopf $*$-subalgebra of $(A_{\mathrm{r}}, \Delta_{\mathrm{r}})$ with co-unit ε_{r} and co-inverse κ_{r}. Hence, by uniqueness, $\theta(\mathcal{A})$ is the Hopf $*$-algebra associated to $(A_{\mathrm{r}}, \Delta_{\mathrm{r}})$. $\square$

We call the compact quantum group (A_r, Δ_r) described in the theorem the *reduced quantum group* of (A, Δ) and we call θ the *canonical map* from A onto A_r. It is clear that θ is a $*$-isomorphism if, and only if, h is faithful.

If (A, Δ) is the universal compact quantum group associated to a discrete group Γ, then the reduced compact quantum group of (A, Δ) is equal to the reduced compact quantum group of Γ; that is, $(A_r, \Delta_r) = (C^*_r(\Gamma), \Delta_r)$. That $A_r = C^*_r(\Gamma)$ follows from the fact that the left kernel of the Haar integral of (A, Δ) is equal to the kernel of the canonical $*$-homomorphism θ from $C^*(\Gamma)$ onto $C^*_r(\Gamma)$, as we have observed before. The only other item that needs to be checked is that $\Delta_r\theta = (\theta \otimes \theta)\Delta$, and this easily follows from the definitions of the co-multiplications on $C^*(\Gamma)$ and $C^*_r(\Gamma)$.

If (A, Δ) is an arbitrary compact quantum group, we say it is *co-amenable* if the co-unit ε_r of $(\mathcal{A}_r, \Delta_r)$ is norm-bounded. We can then extend the co-unit to a $*$-homomorphism ε_r on A_r. A consequence is that A is never simple, if (A, Δ) is co-amenable, since the kernel of $\varepsilon_r\theta$ is a closed two-sided ideal of A of co-dimension one.

From our discussion above, it is evident that the reduced (resp. universal) compact quantum group associated to a discrete group Γ is co-amenable if, and only if, Γ is amenable. Note also that a finite quantum group—that is, a compact quantum group (A, Δ) for which A is finite dimensional—is necessarily co-amenable, since in this case $A = \mathcal{A}$.

It is perhaps of some interest to interpret the idea of co-amenability in the context of a commutative compact quantum group $(C(G), \Delta)$ associated to a classical compact group G. Since the Haar integral is faithful, as we observed before, $(C(G), \Delta)$ is co-amenable if its co-unit is norm-bounded. That this is the case is trivial, since the co-unit is given by the (restriction of) the evaluation map, $f \mapsto f(e)$, where e is the unit of G. Thus, a classical compact group is "co-amenable".

The following theorem allows us to verify co-amenability without reference to the reduced compact quantum group. However, its real importance is its assertion that faithfulness of the Haar integral is a consequence of co-amenability. In practice, it provides a useful method of showing such faithfulness (see Corollary 2.13 below).

The first paragraph of the proof of the theorem is taken from the proof of Theorem 8.1 of [14] (the exactness assumption on A used in [14] is not needed here).

Theorem 2.2 *A compact quantum group (A, Δ) is co-amenable if, and only if, its Haar integral is faithful and its co-unit is norm-bounded.*

Proof. Clearly, we need only show that if (A, Δ) is co-amenable, then h is faithful. Let $I = N_h$. If $a \in I$ and σ is a positive linear functional on A, then $(\sigma \otimes h)\Delta(a^*a) = \sigma(1)h(a^*a) = 0$, since $(\mathrm{id} \otimes h)\Delta(a^*a) = h(a^*a)1$. Hence, since $\sigma \otimes h$ is positive, $(\sigma \otimes h)(c\Delta(a)) = 0$, for all $c \in A \otimes A$. Because σ is an arbitrary positive linear functional on A, this implies $(\mathrm{id} \otimes h)(c\Delta(a)) = 0$. If $\tau \in A^*$ and $c = 1 \otimes b$, where $b \in A$, then we have $h(b(\tau \otimes \mathrm{id})(\Delta a)) = \tau((\mathrm{id} \otimes h)(c\Delta(a))) = 0$. Hence, $(\tau \otimes \mathrm{id})\Delta(a) \in I$.

The co-units ε_r and ε are norm-bounded, by co-amenability, so admit extensions ε_r and ε to A_r and A, respectively, which satisfy $\varepsilon = \varepsilon_r\theta$. It follows that $\tau(a) = \tau((\mathrm{id} \otimes \varepsilon)\Delta(a)) = \varepsilon_{\mathrm{r}}\theta((\tau \otimes \mathrm{id})\Delta(a)) = \varepsilon_{\mathrm{r}}(0) = 0$. Since τ was an arbitrary element of A^*, we must have $a = 0$. Hence, $N_h = I = 0$; that is, h is faithful. □

It follows from Theorem 2.2 that co-amenability is preserved under formation of the tensor product of two compact quantum groups. This is the quantum counterpart of the statement that a product of two discrete amenable groups is amenable. Recall that the *tensor product* of two compact quantum groups (A_i, Δ_i) is the compact quantum group $(A, \Delta) = (A_1 \otimes A_2, \Delta_1 \times \Delta_2)$ with co-multiplication defined by

$$\Delta_1 \times \Delta_2 = (\mathrm{id} \otimes F \otimes \mathrm{id})(\Delta_1 \otimes \Delta_2) : A \to A \otimes A,$$

where $F : A_1 \otimes A_2 \to A_2 \otimes A_1$ denotes the flip map given by $F(a_1 \otimes a_2) = a_2 \otimes a_1$, for $a_1 \in A_1$ and $a_2 \in A_2$. The Hopf $*$-algebra of (A, Δ) is $\mathcal{A}_1 \otimes \mathcal{A}_2$, where $\mathcal{A}_i$ is the Hopf $*$-algebra of (A_i, Δ); the Haar integral and the co-unit of (A, Δ) are $h_1 \otimes h_2$ and $\varepsilon_1 \otimes \varepsilon_2$, respectively, where h_i is the Haar integral and ε_i is the co-unit of (A_i, Δ_i).

If (A_i, Δ_i) are both co-amenable, then, by Theorem 2.2, their Haar integrals h_i are faithful, and therefore $h_1 \otimes h_2$ is also faithful. Hence (A, Δ) is equal to its reduced compact quantum group, so we only need to check that the co-unit $\varepsilon_1 \otimes \varepsilon_2$ is norm-bounded and this is obvious, since ε_i are both norm-bounded. Thus, (A, Δ) is co-amenable.

In the reverse direction, if (A, Δ) is co-amenable, then both (A_1, Δ_1) and (A_2, Δ_2) are co-amenable. For, faithfulness of $h_1 \otimes h_2$ trivially implies faithfulness of each of h_1 and h_2; equally easily, norm-boundedness of $\varepsilon_1 \otimes \varepsilon_2$ implies norm-boundedness of ε_1 and ε_2. Hence, co-amenability of (A_1, Δ_1) and (A_2, Δ_2) follows from Theorem 2.2.

This observation allows us to give an example of a compact quantum group (A, Δ) that is not co-amenable and that is neither co-commutative nor commutative: We set $A_1 = C^*(\mathbf{F}_2)$ and $A_2 = C(\mathbf{S}_3)$, where $\mathbf{F}_2$ is the free group on two generators and $\mathbf{S}_3$ is the finite (compact) group of permutations on three symbols. Then we let (A, Δ) be the tensor product of these two compact quantum groups.

We turn now to finding other conditions equivalent to co-amenability or, more generally, conditions equivalent to norm-boundedness of the co-unit ε.

Recall a finite-dimensional unitary co-representation $U \in M_N(\mathbf{C}) \otimes \mathcal{A}$ of (A, Δ) is said to be *fundamental* if its matrix elements U_{ij} (relative to some system of matrix units for $M_N(\mathbf{C})$) generate the Hopf $*$-algebra $\mathcal{A}$ associated to (A, Δ), as a $*$-algebra. The *compact matrix pseudogroups*, as defined by Woronowicz in [19], are precisely the compact quantum groups that admit a fundamental unitary co-representation.

The equivalence of Conditions (1) and (2) in the corollary of the following theorem can be regarded as a generalization of H. Kesten's classical characterization of the amenability of a finitely-generated discrete group in terms of the spectrum of the sum of the generators in the regular representation (see [9], and also [8]). This equivalence, which is due to G. Skandalis, is proved in [2]. Its connection to Kesten's result is explained in [3]. The proof of our more general result is somewhat different.

Theorem 2.3 *Suppose that (A, Δ) is a compact matrix pseudogroup and that $U \in M_N(\mathbf{C}) \otimes \mathcal{A}$ is a fundamental unitary co-representation of (A, Δ).*
We set $\chi_U = \sum_{i=1}^N U_{ii}$.
Of course, since $\|U_{ij}\| \leq 1$, for all indices i and j, $\|\mathrm{Re}\,\chi_U\| \leq N$.
The following are equivalent conditions:

(1) The co-unit ε of (A, Δ) is norm-bounded;
(2) N belongs to the spectrum of $\mathrm{Re}\,\chi_U$ in A;
(3) There exists a state τ on A such that $\tau(\mathrm{Re}\,\chi_U) = N$;
(4) There exists a state τ on A such that $\tau(U_{ii}) = 1$, for $i = 1, \ldots, N$.
(5) For all scalars $\lambda_0, \lambda_1, \ldots, \lambda_N$,

$$|\sum_{i=0}^N \lambda_i| \leq \|\lambda_0 1 + \sum_{i=1}^N \lambda_i U_{ii}\|. \tag{1}$$

Proof. Recall first from [19, Proposition 1.8] that ε is uniquely determined on $\mathcal{A}$ by $\varepsilon(U_{ij}) = \delta_{ij}$, for all indices i and j. Especially, $\varepsilon(U_{ii}) = 1$ for all i, so we have $\sum_{i=0}^N \lambda_i = \varepsilon(\lambda_0 1 + \sum_{i=1}^N \lambda_i U_{ii})$. The implication (1) $\Rightarrow$ (5) follows by noting that if ε is norm-bounded, its norm must be equal to one, and Inequality (1) is an immediate consequence. To see Condition (5) implies (4), we note that Inequality (1) implies that the linear functional τ_0, defined on the linear span of 1 and the elements U_{ii} by mapping all of these elements to 1 in $\mathbf{C}$, is well defined and has norm equal to 1. By the Hahn–Banach theorem, τ_0 extends to a norm-one linear functional τ on A. Since $\tau(1) = \|\tau\| = 1$, τ is a state of A.

Since a state is necessarily self-adjoint, the implication (4) $\Rightarrow$ (3) is clear.

Set $X_{ij} = U_{ij} - \delta_{ij}$ and $X = \sum_{i,j=1}^N X_{ij}^* X_{ij} + X_{ij} X_{ij}^*$. Using the fact that $\sum_{i=1}^N U_{ij}^* U_{ij} = \sum_{i=1}^N U_{ij} U_{ij}^* = 1$, we have $X = 4(N - \mathrm{Re}\,\chi_U)$. Hence, the element $N - \mathrm{Re}\,\chi_U$ is positive. Therefore, $N - \mathrm{Re}\,\chi_U$ is invertible if, and only if, there exists a positive number such that $N - \mathrm{Re}\,\chi_U \geq \delta$. Hence, N belongs to the spectrum of $\mathrm{Re}\,\chi_U$ if, and only if, $\tau(\mathrm{Re}\,\chi_U) = N$, for some state τ of A. That is, Conditions (2) and (3) are equivalent.

Thus, it remains only to show that (3) $\Rightarrow$ (1). Suppose Condition (3) holds, so that there exists a state τ on A such that $\tau(N - \mathrm{Re}\,\chi_U) = 0$ and therefore, $\tau(X) = 0$. Hence, $\tau(X_{ij}^* X_{ij}) = \tau(X_{ij} X_{ij}^*) = 0$. Let φ be the GNS representation associated to τ, acting on the Hilbert space H, and let x be the canonical cyclic vector associated to this representation, so that $\tau(a) = (\varphi(a)x \mid x)$ and $\varphi(A)x$ is dense in H. Clearly, $\varphi(X_{ij})x = \varphi(X_{ij}^*)x = 0$ and therefore $\varphi(U_{ij})x = \varphi(U_{ij}^*)x = \delta_{ij}x$. Hence, if a is product of matrix elements U_{ij} and U_{kl}^*, then $\varphi(a)x \in \mathbf{C}x$. Since U is a fundamental co-representation of (A, Δ), the closed linear span of such products is equal to A and therefore $\varphi(A)x \subseteq \mathbf{C}x$. Hence, $H = \mathbf{C}x$ and therefore $\dim(H) = 1$. It follows that φ is scalar-valued and therefore $\varphi(a) = \tau(a)1$, for all $a \in A$. Hence, τ is a norm-bounded $*$-homomorphism. Moreover, since $|\tau(X_{ij})|^2 \leq \tau(X_{ij}^* X_{ij}) = 0$, we have $\tau(U_{ij}) = \delta_{ij} = \varepsilon(U_{ij})$, for $i, j = 1, \ldots, N$. Hence, since the elements U_{ij} generate $\mathcal{A}$ as a $*$-algebra, $\tau = \varepsilon$ on $\mathcal{A}$ and therefore ε is norm-bounded. □

Corollary 2.4 *With the same assumptions as in the preceding theorem, the following are equivalent conditions:*

(1) (A, Δ) is co-amenable;

(2) *N belongs to the spectrum of* $\theta(\mathrm{Re}\,\chi_U)$ *in* A_{r}*;*
(3) *There exists a state* τ *on* A_{r} *such that* $\tau\theta(\mathrm{Re}\,\chi_U) = N$*;*
(4) *There exists a state* τ *on* A_{r} *such that* $\tau\theta(U_{ii}) = 1$*, for* $i = 1, \ldots, N$*.*
(5) *For all scalars* $\lambda_0, \lambda_1, \ldots, \lambda_N$*,*

$$|\sum_{i=0}^{N} \lambda_i| \leq \|\lambda_0 1 + \sum_{i=1}^{N} \lambda_i \theta(U_{ii})\|.$$

Proof. The result follows from the theorem by observing that $(\mathrm{id} \otimes \theta)(U)$ is a fundamental co-representation of $(A_{\mathrm{r}}, \Delta_{\mathrm{r}})$. □

If U is a a unitary co-representation of (A, Δ) on a Hilbert space H, so that $U \in M(K(H) \otimes A)$, the multiplier algebra of $K(H) \otimes A$, recall that its matrix elements are the elements of A of the form $(\omega \otimes \mathrm{id})(U)$, where ω is a strictly continuous linear map on $K(H)$. Not every compact quantum group admits a fundamental unitary co-representation but all admit a unitary co-representation for which the matrix elements generate its C*-algebra (for example, the matrix elements of the regular co-representation have dense linear span in the C*-algebra).

If U is any unitary co-representation of (A, Δ) on a Hilbert space H and the co-unit ε is norm-bounded, then $(\mathrm{id} \otimes \varepsilon)(U) = 1$ in $B(H)$. For, the equality $(\mathrm{id} \otimes \varepsilon)\Delta = \mathrm{id}$ implies $U = (\mathrm{id} \otimes (\mathrm{id} \otimes \varepsilon)\Delta)(U) = (\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon)(\mathrm{id} \otimes \Delta)(U) = (\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon)(U_{(12)}U_{(13)}) = U((\mathrm{id} \otimes \varepsilon)(U) \otimes 1)$. Since U is invertible, we deduce that $1 \otimes 1 = (\mathrm{id} \otimes \varepsilon)(U) \otimes 1$ and therefore $(\mathrm{id} \otimes \varepsilon)(U) = 1$, as required.

The following represents a partial generalization of Theorem 2.3.

Theorem 2.5 *Let* U *be a unitary co-representation of* (A, Δ) *whose matrix elements generate* A *as a C*-algebra. Then the following are equivalent conditions:*
(1) *The co-unit* ε *is norm-bounded;*
(2) *There exists a state* τ *of* A *for which* $(\mathrm{id} \otimes \tau)(U) = 1$*.*

Proof. Taking $\tau = \varepsilon$, the implication (1) $\Rightarrow$ (2) is immediate from the remarks preceding this theorem. To see the converse, suppose given a state τ of A for which $(\mathrm{id} \otimes \tau)(U) = 1$. Let φ be the GNS representation associated to τ. We suppose H is the Hilbert space on which φ acts and that x is the canonical cyclic vector for φ. As in the proof of Theorem 2.3, we shall show that $\varphi(a) = \tau(a)1$, for all $a \in A$. First, let $a = (\omega \otimes \mathrm{id})(U)$ be a matrix element of U, where ω is a strictly continuous linear map on $K(H)$. We shall show that $\varphi(a)x, \varphi(a)^*x \in \mathbf{C}x$. Since ω is linear combination of strictly continuous states on $K(H)$, to show this result we may suppose that ω is a state. Then $\|a\| \leq 1$ and $\tau(a) = \omega((\mathrm{id} \otimes \tau)(U)) = \omega(1) = 1$; hence, $0 \leq \tau((a-1)^*(a-1)) = \tau(a^*a) - \tau(a) - \tau(a)^- + \tau(1) \leq \tau(1) - 1 - 1 + \tau(1) = 0$. Consequently, $\tau((a-1)^*(a-1)) = 0$, from which it follows that $\varphi(a)x = x$. Similar reasoning shows that $\tau((a-1)(a-1)^*) = 0$ and therefore $\varphi(a)^*x = x$. Since the elements $a = (\omega \otimes \mathrm{id})(U)$ generate A, as a C*-algebra, we can now argue again as in the proof of Theorem 2.3 to deduce that $\varphi(A)x = \mathbf{C}x$. Hence, $\varphi(a) = \tau(a)1$, for all $a \in A$, as claimed. This implies that τ is a $*$-homomorphism on A.

Now we shall show that $(\mathrm{id} \otimes \tau)\Delta(a) = a$, for all $a \in A$. To see this, we may clearly suppose that a is a matrix element, $a = (\omega \otimes \mathrm{id})(U)$, say. Then

$$(\mathrm{id} \otimes \tau)\Delta(a) = (\mathrm{id} \otimes \tau)(\omega \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)(U)$$

$$
\begin{aligned}
&= (\mathrm{id} \otimes \tau)(\omega \otimes \mathrm{id} \otimes \mathrm{id})(U_{(12)}U_{(13)}) \\
&= (\omega \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \tau)(U_{(12)}U_{(13)}) \\
&= (\omega \otimes \mathrm{id})(U((\mathrm{id} \otimes \tau)(U) \otimes 1)) \\
&= (\omega \otimes \mathrm{id})(U(1 \otimes 1)) = a.
\end{aligned}
$$

We complete the proof now by showing that $\tau(a) = \varepsilon(a)$, for all $a \in \mathcal{A}$: We have $\tau(a) = \tau((\varepsilon \otimes \mathrm{id})(\Delta(a)) = \varepsilon((\mathrm{id} \otimes \tau)(\Delta(a)) = \varepsilon(a)$. Hence, τ is a norm-bounded linear map extending ε and therefore ε is norm-bounded. □

Let us note explicitly that our proof of the preceding theorem shows that if τ is as in Condition (2), then τ is the—necessarily unique—extension of ε to A.

Corollary 2.6 *Let U be a unitary co-representation of (A, Δ) whose matrix elements generate A as a C*-algebra. Then the following are equivalent conditions:*
(1) (A, Δ) is co-amenable;
(2) There exists a state τ of A_{r} for which $(\mathrm{id} \otimes \tau\theta)(U) = 1$.

Proof. The element $V = (\mathrm{id} \otimes \theta)(U)$ in the multiplier algebra $M(K(H) \otimes A_{\mathrm{r}})$ is a unitary co-representation of $(A_{\mathrm{r}}, \Delta_{\mathrm{r}})$ whose matrix elements $(\omega \otimes \mathrm{id})(V) = \theta((\omega \otimes \mathrm{id})(U))$ generate A_{r} as a C*-algebra. The result therefore follows directly from the theorem. □

We stated before the preceding theorem that it is a partial generalization of Theorem 2.3. To see why, let $U \in M_N(\mathbf{C}) \otimes \mathcal{A}$ be a co-representation a finite-dimensional unitary co-representation of (A, Δ) with matrix elements U_{ij} (relative to some system of matrix units for $M_N(\mathbf{C})$). The equation $(\mathrm{id} \otimes \tau)(U) = 1$ is clearly equivalent to the condition that $\tau(U_{ij}) = \delta_{ij}$, for all i and j. Reasoning as in the proof of Theorem 2.3, this is easily seen to be equivalent to the condition that $\tau(U_{ii}) = 1$, for all i. Hence, the preceding theorem implies the equivalence of Conditions (1) and (4) of Theorem 2.3.

We shall need the following result for the proof of Theorem 2.8.

Lemma 2.7 *Let (A, Δ) be a compact quantum group for which the Haar integral h is faithful. Let π be a non-zero $*$-homomorphism from A to a C*-algebra B. Then the $*$-homomorphism, $\hat{\pi}\colon A \to A \otimes B$, $a \mapsto (\mathrm{id} \otimes \pi)\Delta(a)$, is isometric.*

Proof. Let $a \in A$ and suppose that $\hat{\pi}(a) = 0$. Then $\hat{\pi}(a^*a) = 0$ and therefore $0 = (h \otimes \mathrm{id})\hat{\pi}(a^*a) = \pi((h \otimes \mathrm{id})\Delta(a^*a)) = \pi(h(a^*a)1) = h(a^*a)\pi(1)$. Consequently, since $\pi(1) \neq 0$, we have $h(a^*a) = 0$; faithfulness of h now gives $a = 0$. Hence, $\hat{\pi}$ is injective and therefore isometric. □

The corollary to the following theorem gives another characterization of co-amenability, this time in terms of a scalar-valued $*$-homomorphism on the C*-algebra of the reduced quantum group:

Theorem 2.8 *Let (A, Δ) be a compact quantum group for which the Haar integral h is faithful. Then the following are equivalent conditions:*
(1) The co-unit ε is norm-bounded;
(2) There exists a non-zero $*$-homomorphism $\tau\colon A \to \mathbf{C}$.

Proof. The implication (1) $\Rightarrow$ (2) is obvious. Suppose therefore that we have a non-zero $*$-homomorphism $\tau: A \to \mathbf{C}$. If U is an N-dimensional unitary co-representation of (A, Δ), then $(\mathrm{id} \otimes \tau)\Delta(U_{ij}) = (\mathrm{id} \otimes \tau)(\sum_{k=1}^{N} U_{ik} \otimes U_{kj}) = \sum_{k=1}^{N} U_{ik}\tau(U_{kj})$. Also, since the matrix $(\tau(U_{ij}))$ is a unitary, because τ is a $*$-homomorphism, $\sum_{j=1}^{N}(\mathrm{id} \otimes \tau)\Delta(U_{ij})\tau(U_{lj})^{-} = \sum_{k,j=1}^{N} U_{ik}\tau(U_{kj})\tau(U_{lj})^{-} = \sum_{k=1}^{N} U_{ik}\delta_{kl} = U_{il}$. Hence, recalling that $\mathcal{A}$ is the linear span of the matrix elements of finite-dimensional unitary co-representations of (A, Δ), it is clear that the $*$-homomorphism $\hat{\tau}: \mathcal{A} \to \mathcal{A}$, defined by setting $\hat{\tau}(a) = (\mathrm{id} \otimes \tau)\Delta(a)$, is surjective. Since h is assumed to be faithful, it follows from Lemma 2.7 that $\hat{\tau}$ is an isometry. Therefore, if $a \in \mathcal{A}$, $|\varepsilon(\hat{\tau}(a))| = |\tau((\varepsilon \otimes \mathrm{id})\Delta(a))| = |\tau(a)| \le \|a\| = \|\hat{\tau}(a)\|$. Therefore, ε is norm-bounded. Hence, (2) $\Rightarrow$ (1). □

Corollary 2.9 *If (A, Δ) is an arbitrary compact quantum group, the following are equivalent conditions:*

(1) (A, Δ) is co-amenable;

(2) There exists a non-zero $*$-homomorphism $\tau: A_{\mathrm{r}} \to \mathbf{C}$.

(3) The Haar integral on (A, Δ) is faithful and there exists a non-zero $*$-homomorphism $\tau: A \to \mathbf{C}$.

Proof. The equivalence between (1) and (2) follows immediately from the theorem, since the Haar integral of $(A_{\mathrm{r}}, \Delta_{\mathrm{r}})$ is faithful. The equivalence between (1) and (3) follows by combining the theorem and Theorem 2.2. □

As an immediate consequence of the equivalence between (1) and (2) above, we obtain the following corollary which is a special case of a result in [4].

Corollary 2.10 *Let Γ be a discrete group. The following are equivalent conditions:*

(1) Γ is amenable;

(2) There exists a non-zero $*$-homomorphism $\tau: C^*_{\mathrm{r}}(\Gamma) \to \mathbf{C}$.

If Γ is a discrete group, then its reduced group C*-algebra is given by a concrete faithful representation on the Hilbert space $\ell^2(\Gamma)$. Given a compact quantum group (A, Δ), there is a natural faithful representation of $(A_{\mathrm{r}}, \Delta_{\mathrm{r}})$ whose existence may be deduced from [1]. For completeness, we now present this representation in details. Let $\pi: A \to B(H)$ be the GNS representation of A associated to the Haar integral h of (A, Δ) and let z be its canonical cyclic vector, so that $\pi(A)z$ is dense in H and $h(a) = (\pi(a)z \mid z)$, for all $a \in A$. We denote by $\|\cdot\|_2$ the norm of H. We set $A_{\mathrm{rc}} = \pi(A)$ and $\mathcal{A}_{\mathrm{rc}} = \pi(\mathcal{A})$, so that A_{rc} is a unital C*-subalgebra of $B(H)$ and $\mathcal{A}_{\mathrm{rc}}$ is a dense unital $*$-subalgebra of A_{rc}. The map π is injective on $\mathcal{A}$. For, if $a \in \mathcal{A}$ and $\pi(a) = 0$, then $\|\pi(a)z\|_2^2 = h(a^*a) = 0$ and therefore, by faithfulness of h on $\mathcal{A}$, $a = 0$. Hence, we can define linear maps, $\Delta_{\mathrm{rc}} : \mathcal{A}_{\mathrm{rc}} \to \mathcal{A}_{\mathrm{rc}} \otimes \mathcal{A}_{\mathrm{rc}}$, $\varepsilon_{\mathrm{rc}} : \mathcal{A}_{\mathrm{rc}} \to \mathbf{C}$ and $\kappa_{\mathrm{rc}} : \mathcal{A}_{\mathrm{rc}} \to \mathcal{A}_{\mathrm{rc}}$ by setting $\Delta_{\mathrm{rc}}(\pi(a)) = (\pi \otimes \pi)\Delta(a)$, $\varepsilon_{\mathrm{rc}}(\pi(a)) = \varepsilon(a)$ and $\kappa_{\mathrm{rc}}(\pi(a)) = \pi(\kappa(a))$, for all $a \in \mathcal{A}$. Clearly, Δ_{rc} is a unital $*$-homomorphism.

Theorem 2.11 *Let (A, Δ) be a compact quantum group and retain the notation of the preceding paragraph. The map $\Delta_{\mathrm{rc}} : \mathcal{A}_{\mathrm{rc}} \to \mathcal{A}_{\mathrm{rc}} \otimes \mathcal{A}_{\mathrm{rc}}$ has a unique extension to a $*$-homomorphism $\Delta_{\mathrm{rc}} : A_{\mathrm{rc}} \to A_{\mathrm{rc}} \otimes A_{\mathrm{rc}}$. The pair $(A_{\mathrm{rc}}, \Delta_{\mathrm{rc}})$ is a compact quantum group with faithful Haar state h_{rc} given by $h_{\mathrm{rc}}(a) = (\, az \mid z\,)$,* *for all* $a \in A_{\mathrm{rc}}$. *The Hopf* $*$*-algebra associated to* $(A_{\mathrm{rc}}, \Delta_{\mathrm{rc}})$ *is* $\mathcal{A}_{\mathrm{rc}} = \pi(\mathcal{A})$, *with co-unit* $\varepsilon_{\mathrm{rc}}$ *and co-inverse* κ_{rc}. *The map* π *is a morphism of* (A, Δ) *onto* $(A_{\mathrm{rc}}, \Delta_{\mathrm{rc}})$ *and its kernel is equal to the left kernel of* h, *so that* π *induces a faithful representation of* A_{r} *on* H. *This representation is an isomorphism of the compact quantum groups* $(A_{\mathrm{r}}, \Delta_{\mathrm{r}})$ *and* $(A_{\mathrm{rc}}, \Delta_{\mathrm{rc}})$.

Proof. To prove that $\Delta_{\mathrm{rc}} : \mathcal{A}_{\mathrm{rc}} \to \pi(\mathcal{A}) \otimes \pi(\mathcal{A}) \subset B(H \otimes H)$ has an extension $\Delta_{\mathrm{rc}} : A_{\mathrm{rc}} \to B(H \otimes H)$, we construct a unitary W on $H \otimes H$. First, define the linear map $W : \mathcal{A} \otimes \mathcal{A} \subset H \otimes H \to \mathcal{A} \otimes \mathcal{A} \subset H \otimes H$ by setting $W(a \otimes b) = \Delta(b)(a \otimes 1)$, for all $a, b \in \mathcal{A}$. We claim that W is isometric. To see this, let $c = \sum_i a_i \otimes b_i \in \mathcal{A} \otimes \mathcal{A}$ and $\Delta(b_i) = \sum_k a_i^k \otimes b_i^k$ for finitely many elements $a_i^k, b_i^k \in \mathcal{A}$. Then

$$W(c)^* W(c) = \sum_{ij} (\Delta(b_i)(a_i \otimes 1))^* \Delta(b_j)(a_j \otimes 1)$$

$$= \sum_{ijkl} (a_i^* \otimes 1)((a_i^k)^* a_j^l \otimes (b_i^k)^* b_j^l)(a_j \otimes 1) = \sum_{ijkl} a_i^* (a_i^k)^* a_j^l a_j \otimes (b_i^k)^* b_j^l,$$

and therefore

$$\|W(c)\|_2^2 = (W(c)|W(c)) = (h \otimes h)(W(c)^* W(c))$$

$$= \sum_{ijkl} h(a_i^* (a_i^k)^* a_j^l a_j) h((b_i^k)^* b_j^l) = \sum_{ij} h(a_i^* [\sum_{kl} (a_i^k)^* a_j^l h((b_i^k)^* b_j^l)] a_j)$$

$$= \sum_{ij} h(a_i^* [(id \otimes h)\Delta(b_i^* b_j)] a_j) = \sum_{ij} h(a_i^* h(b_i^* b_j) 1 a_j) = \sum_{ij} h(a_i^* a_j) h(b_i^* b_j)$$

$$= (h \otimes h)(\sum_{ij} a_i^* a_j \otimes b_i^* b_j) = (h \otimes h)(c^* c) = (c|c) = \|c\|_2^2.$$

Hence W is isometric, as claimed. Since $\mathcal{A} \otimes \mathcal{A}$ is equal to the linear span of $\Delta\mathcal{A}(\mathcal{A} \otimes 1)$, we have $W(\mathcal{A} \otimes \mathcal{A}) = \mathcal{A} \otimes \mathcal{A}$. It follows that W extends from the dense subspace $\mathcal{A} \otimes \mathcal{A}$ to a unitary on $H \otimes H$. We shall denote this extension also by W.

We claim that, for all $a \in \mathcal{A}$,

$$\Delta_{\mathrm{rc}}(\pi(a)) = W(\pi(1) \otimes \pi(a))W^*; \tag{2}$$

equivalently, $\Delta_{\mathrm{rc}}(\pi(a))W = W(\pi(1) \otimes \pi(a))$. These operators are equal if they act identically on elementary tensors of the dense subspace $\mathcal{A} \otimes \mathcal{A}$ of $H \otimes H$. Thus, let $b, c \in \mathcal{A}$ and observe that

$$\Delta_{\mathrm{rc}}(\pi(a))W(b \otimes c) = \Delta_{\mathrm{rc}}(\pi(a))\Delta(c)(b \otimes 1) = ((\pi \otimes \pi)\Delta(a))\Delta(c)(b \otimes 1)$$

$$= \Delta(a)\Delta(c)(b \otimes 1) = \Delta(ac)(b \otimes 1) = W(b \otimes ac) = W(\pi(1) \otimes \pi(a))(b \otimes c).$$

Thus, Equation (2) holds and it follows that $\Delta_{\mathrm{rc}} : \mathcal{A}_{\mathrm{rc}} \to A_{\mathrm{rc}} \otimes A_{\mathrm{rc}} \subseteq B(H \otimes H)$ is norm decreasing. Consequently, it admits a $*$-homomorphism extension $\Delta_{\mathrm{rc}} : A_{\mathrm{rc}} \to A_{\mathrm{rc}} \otimes A_{\mathrm{rc}}$. That Δ_{rc} is a co-multiplication on A_{rc} is an obvious consequence of its restriction to $\mathcal{A}_{\mathrm{rc}}$ being one, and density of $\mathcal{A}_{\mathrm{rc}}$ in A_{rc}. It follows directly, from the fact that the linear spans of $(\mathcal{A} \otimes 1)\Delta\mathcal{A}$ and $(1 \otimes \mathcal{A})\Delta\mathcal{A}$

are each equal to $\mathcal{A} \otimes \mathcal{A}$, that $(A_{\mathrm{rc}}, \Delta_{\mathrm{rc}})$ is a compact quantum group. That π is a morphism of compact quantum groups is obvious.

Since $h = 0$ on $\ker(\pi)$, it induces a unique state h_{rc} on A_{rc} for which $h_{\mathrm{rc}} \circ \pi = h$. Therefore, $h_{\mathrm{rc}}(a) = (\, az \mid z\,)$, for all $a \in A_{\mathrm{rc}}$. It is easily verified that h_{rc} is the Haar state of $(A_{\mathrm{rc}}, \Delta_{\mathrm{rc}})$. Suppose now $a \in A_{\mathrm{rc}}$ and $h_{\mathrm{rc}}(a^*a) = 0$. Since $N_{h_{\mathrm{rc}}}$ is a two-sided ideal in A_{rc}, $ab \in N_{h_{\mathrm{rc}}}$, for all $b \in A_{\mathrm{rc}}$ and therefore $h_{\mathrm{rc}}(b^*a^*ab) = 0$. Hence, $(\, abz \mid abz\,) = 0$ for all $b \in A_{\mathrm{rc}}$, which shows that $a = 0$. Thus, h_{rc} is faithful. It clearly follows that the left kernel of h is equal to the kernel of π. Hence, the representation of A_{r} on H induced by π is faithful and then, by construction, an isomorphism of $(A_{\mathrm{r}}, \Delta_{\mathrm{r}})$ onto $(A_{\mathrm{rc}}, \Delta_{\mathrm{rc}})$.

Finally, it is clear that $\pi(\mathcal{A})$ is a dense Hopf $*$-subalgebra of $(A_{\mathrm{rc}}, \Delta_{\mathrm{rc}})$ with co-unit $\varepsilon_{\mathrm{rc}}$ and co-inverse κ_{rc}, and therefore it is the Hopf $*$-algebra associated to $(\mathcal{A}_{\mathrm{rc}}, \Delta_{\mathrm{rc}})$, by uniqueness . □

We turn now to an application of some of our results to the the prototypical example of a compact quantum group, the quantization of $SU(2)$ constructed by Woronowicz. We shall show that it is co-amenable, from which we shall obtain the known, and non-trivial, result that its Haar integral is faithful. It also follows from Banica's more general result [2, Corollary 6.2] which uses the theory of $\mathcal{R}^+$-deformations. Our quite elementary proof is totally different.

Let q be a real number for which $0 < |q| < 1$. Let $(A, \Delta) = SU_q(2)$, and let α and γ be the canonical generators of A, satisfying the conditions of Table 0 of [19]. Let $k \in \mathbf{Z}$ and $m, n \in \mathbf{N}$. Set $a_{kmn} = \alpha^{(k)}\gamma^m\gamma^{*n}$, where $\alpha^{(k)} = \alpha^k$, if $k \geq 0$ and $\alpha^{(k)} = (\alpha^{-k})^*$, if $k < 0$. Recall that these elements a_{kmn} form a linear basis for the Hopf *-algebra $\mathcal{A}$ associated to (A, Δ) and that $h(a_{kmn}) = 0$, if $k \neq 0$ or if $m \neq n$ [19, Equation A1.8].

Take U to be the fundamental irreducible co-representation of $SU_q(2)$ given by

$$U = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}.$$

Before stating the following theorem, we make an elementary observation: If V is the forward unilateral shift on a Hilbert space H with orthonormal basis $(e_n)_{n\in\mathbf{N}}$, so that $Ve_n = e_{n+1}$, then there exists a state τ on $B(H)$ such that $\tau(V) = 1$ and $\tau(K) = 0$, for all compact operators $K \in B(H)$. To see this, one observes that the image of V in the Calkin algebra C of H is a unitary containing 1 in its spectrum and therefore there exists a state on C whose value at this unitary is equal to 1. The required state on $B(H)$ is then the composition of the state on C and the quotient map from $B(H)$ to C.

Theorem 2.12 *The compact quantum group $SU_q(2)$ is co-amenable.*

Proof. As before, let $(A, \Delta) = SU_q(2)$ and let α and γ be the canonical generators of A. Set $c_n = (1 - q^{2n})^{1/2}$, for $n \in \mathbf{N}$. Recall from Appendix A.1 of [20] that A admits a representation φ on a Hilbert space H with an orthonormal basis $(e_{n,k})$, where $n \in \mathbf{N}$ and $k \in \mathbf{Z}$, such that

$$\varphi(\alpha)e_{nk} = c_n e_{n-1,k} \quad \text{and} \quad \varphi(\gamma)e_{nk} = q^n e_{n,k+1}$$

and that

$$h(a) = (1 - q^2)\sum_{n=0}^{\infty} (\, \varphi(a)e_{n0} \mid e_{n0}\,).$$

It follows immediately that $h(a^*a) = 0$ if, and only if, $\varphi(a)e_{n0} = 0$, for all $n \in \mathbf{N}$. Using the equations $\varphi(\gamma^m)e_{n0} = q^{nm}e_{nm}$ and $\varphi(\gamma^{*m})e_{n0} = q^{nm}e_{n,-m}$, for $m > 0$ and the fact that $a\gamma^m$ and $a\gamma^{*m}$ belong to N_h, if a does, we get that $h(a^*a) = 0$ if, and only if, $\varphi(a) = 0$. Hence, we get an induced faithful representation ψ of A_r on H given by $\psi\theta(a) = \varphi(a)$.

Now, for $k \in \mathbf{Z}$, let H_k be the Hilbert subspace of H with orthonormal basis $(e_{nk})_{n\in\mathbf{N}}$. Obviously, $H = \oplus_k H_k$, and $T = \varphi(\alpha)$ reduces each space H_k, so that $T = \oplus_k T_k$, where T_k is the restriction of T to H_k. We have $T_k e_{nk} = c_n e_{n-1,k}$, so that $T_k = U_k^* D_k$, where U_k is the forward unilateral shift on the basis $(e_{nk})_n$ of H_k and D_k is the diagonal norm-bounded linear operator on H_k defined by setting $D_k(e_{nk}) = c_n e_{nk}$. Since $\lim c_n = 1$, it is clear that $D_k = 1 + L_k$, where L_k is a compact operator on H_k. Hence, $T_k = U_k^* + U_k^* L_k$. By the remarks preceding this theorem, there exists a state $\tau_\kappa \in B(H_k)$ such that $\tau_k(T_k) = 1$. For $k \in \mathbf{Z}$, chose positive numbers t_k such that $\sum_{k\in\mathbf{Z}} t_k = 1$. Now define a state τ on the C*-algebra $\oplus_k B(H_k)$ containing T by setting $\tau(S) = \sum_{k\in\mathbf{Z}} t_k\tau_k(S_k)$, if $S = (S_k)_k \in \oplus_k B(H_k)$. Clearly, $\tau(T) = 1$. Now let τ' be the state $\tau\psi$ on A_r. Then $\tau'(\theta(\alpha)) = \tau(T) = 1$ and therefore $\tau'\theta(\mathrm{Re}\,\chi_U) = \tau'\theta(\alpha) + (\tau'\theta(\alpha))^- = 2$. Hence, (A, Δ) is co-amenable, by Condition (3) of Corollary 2.4. □

Corollary 2.13 (G. Nagy) *The Haar integral h of $SU_q(2)$ is faithful.*

Proof. This is a consequence of the preceding theorem and Theorem 2.2. □

There is an alternative way of proving $SU_q(2) = (A, \Delta)$ is co-amenable, using the fact that A is of Type I, as a C*-algebra [19, Theorem A2.3]. Since A_r is unital, it admits a maximal ideal I. Since A_r/I is a Type I simple unital C*-algebra, it is isomorphic to $M_N(\mathbf{C})$, for some positive integer N. Thus, we have a surjective $*$-homomorphism π from A_r onto $M_N(\mathbf{C})$. The existence of a faithful, tracial state on $M_N(\mathbf{C})$, together with the commutation relations of [19, Table 0] for the canonical generators α and γ, forces the image $\pi(\gamma)$ of γ in $M_N(\mathbf{C})$ to be equal to zero and $\pi(\alpha)$ to be a unitary. Since $\pi(\alpha)$ and $\pi(\gamma)$ generate $M_N(\mathbf{C})$, this implies that $M_N(\mathbf{C})$ is commutative. Hence, $N = 1$ and $M_N(\mathbf{C}) = \mathbf{C}$. Thus, A_r admits a $*$-homomorphism onto $\mathbf{C}$ and it now follows from Corollary 2.9 that $SU_q(2)$ is co-amenable.

3 The Universal Quantum Group

In this section we first give a detailed account on the construction of the universal compact quantum group associated to an arbitrary compact quantum group. One way to construct such an object relies on Baaj and Skandalis' theory of regular multiplicative unitaries [1]. A general construction for locally compact quantum groups has recently been given by J. Kustermans [10]. However, our approach, which is briefly sketched by Woronowicz in [21] for compact matrix pseudogroups, is much less technical and is therefore included. The reduced quantum group has the advantage that the Haar integral is always faithful, whereas its co-unit need not be norm-bounded. For the universal quantum group the situation is the opposite; its co-unit is always norm-bounded, whereas its Haar integral need not be faithful.

Let (A, Δ) be a compact quantum group. Define $\|\cdot\|_{\mathrm{u}}$ on $\mathcal{A}$ by

$$\|a\|_{\mathrm{u}} = \sup_{\pi} \|\pi(a)\|,$$

where the variable π runs over all unital $*$-homomorphisms π from $\mathcal{A}$ into $B(H_\pi)$, for a Hilbert space H_π (the *unital $*$-representations* of $\mathcal{A}$).

Lemma 3.1 *The function $\|\cdot\|_u : \mathcal{A} \to [0, \infty]$ is a C^*-norm on $\mathcal{A}$ which majorises any other C^*-norm on $\mathcal{A}$.*

Proof. We first need to show that $\|a\|_{\mathrm{u}}$ is finite, for all $a \in \mathcal{A}$. Let $(U^\alpha)_\alpha$ be a complete set of inequivalent, irreducible unitary co-representations of (A, Δ); then the matrix elements U^α_{ij} linearly span $\mathcal{A}$. Clearly, it suffices to show that $\|U^\alpha_{ij}\|_{\mathrm{u}} < \infty$, for all α and i, j. Suppose then $\pi : \mathcal{A} \to B(H)$ is a unital $*$-representation of $\mathcal{A}$ on some Hilbert space H. Since $\sum_k (U^\alpha_{ki})^* U^\alpha_{ki} = 1$, we have $(U^\alpha_{ji})^* U^\alpha_{ji} = 1 - \sum_{k \neq j} (U^\alpha_{ki})^* U^\alpha_{ki}$, and therefore

$$0 \leq (\pi(U^\alpha_{ji}))^* \pi(U^\alpha_{ji}) = \pi(1) - \sum_{k \neq j} (\pi(U^\alpha_{ki}))^* \pi(U^\alpha_{ki}) \leq \pi(1).$$

Hence, $\|\pi(U^\alpha_{ji})\|^2 = \|(\pi(U^\alpha_{ji}))^* \pi(U^\alpha_{ji})\| \leq \|\pi(1)\| = 1$. It follows that $\|U^\alpha_{ij}\|_{\mathrm{u}}$ is finite. (Note that although $0 \leq \|a^*a\| I - a^*a$, for any $a \in \mathcal{A}$, we cannot conclude $\|a^*a\| I - a^*a = b^*b$, for some element b belonging to $\mathcal{A}$. This is why the preceding argument had to be more careful than one might first expect and had to use the rather strong property that $\mathcal{A}$ is the linear span of the matrix elements U^α_{ij}.)

It is clear now that $\|\cdot\|_{\mathrm{u}}$ is a C*-seminorm on $\mathcal{A}$ and since A admits a faithful unital $*$-representation, $\|\cdot\|_{\mathrm{u}}$ is, in fact, a C*-norm. That $\|\cdot\|_{\mathrm{u}}$ majorises any other C*-norm on $\mathcal{A}$ is clear from its definition. □

We define A_{u} to be the C^*-algebra completion of $\mathcal{A}$ with respect to the C^*-norm $\|\cdot\|_{\mathrm{u}}$. As usual, we identify $\mathcal{A}$ with its canonical copy inside A_{u}. The C*-algebra A_{u} has the universal property that if $\pi \colon \mathcal{A} \to B$ is a unital $*$-homomorphism from $\mathcal{A}$ to a unital C*-algebra B, it extends uniquely to a $*$-homomorphism from A_{u} to B, since π is easily seen to be norm-decreasing on $\mathcal{A}$ equipped with its universal norm.

In particular, the $*$-homomorphism $\Delta \colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \subseteq A_{\mathrm{u}} \otimes A_{\mathrm{u}}$ extends to a $*$-homomorphism $\Delta \colon A_{\mathrm{u}} \to A_{\mathrm{u}} \otimes A_{\mathrm{u}}$. It is easily verified Δ is a co-multiplication on A_{u}. Since the linear spans of the sets $(\mathcal{A} \otimes 1)\Delta\mathcal{A}$ and $(1 \otimes \mathcal{A})\Delta\mathcal{A}$ are each equal to $\mathcal{A} \otimes \mathcal{A}$, it follows immediately that (A_{u}, Δ) is a compact quantum group. We call it the *universal* compact quantum group associated to (A, Δ).

Since $\mathcal{A}$ is, by construction, a dense Hopf $*$-subalgebra of (A_{u}, Δ), it is the Hopf $*$-algebra associated to (A_{u}, Δ), by uniqueness.

Note also that the co-unit ε of $\mathcal{A}$, being a $*$-homomorphism from $\mathcal{A}$ to $\mathbf{C}$, extends to a $*$-homomorphism ε_{u} from A_{u} to $\mathbf{C}$. By density of $\mathcal{A}$ in A_{u}, the equalities $(\varepsilon \otimes \mathrm{id})\Delta(a) = (\mathrm{id} \otimes \varepsilon)\Delta(a) = a$, which hold for all $a \in \mathcal{A}$, extend to $(\varepsilon_{\mathrm{u}} \otimes \mathrm{id})\Delta(a) = (\mathrm{id} \otimes \varepsilon_{\mathrm{u}})\Delta(a) = a$, for all $a \in A_{\mathrm{u}}$. Hence, ε_{u} must be the unique extension to A_{u} of the co-unit of (A_{u}, Δ). The important point we wish to emphasize here is that (A_{u}, Δ) has thus a norm-bounded co-unit.

Clearly, by the universal property of (A_{u}, Δ), there is a $*$-homomorphism ψ from A_{u} onto A extending the identity $*$-isomorphism from $\mathcal{A}$ to itself. Also,

$\Delta\psi = (\psi \otimes \psi)\Delta$. We call ψ the *canonical map* from $A_{\rm u}$ onto A. Likewise, if θ is the canonical map from A onto $A_{\rm r}$, we call the composition $\theta\psi$ the *canonical map* from $A_{\rm u}$ onto $A_{\rm r}$.

Clearly, $h\psi$ is the Haar integral $h_{\rm u}$ of $(A_{\rm u}, \Delta)$; hence, $h_{\rm u} = h_{\rm r}\theta\psi$. Since $h_{\rm r}$ is faithful, it follows that $N_{h_{\rm u}} = \ker(\theta\psi)$. From this it is immediate that the reduced compact quantum group of $(A_{\rm u}, \Delta)$ is (isomorphic to) $(A_{\rm r}, \Delta_{\rm r})$ and that $\psi\theta$ is the canonical map from $(A_{\rm u}, \Delta)$ onto $(A_{\rm r}, \Delta_{\rm r})$. Therefore, $(A_{\rm u}, \Delta)$ is co-amenable if, and only if, (A, Δ) is co-amenable.

We summarise the preceding discussion in the following theorem.

Theorem 3.2 *Let (A, Δ) be a compact quantum group. Then $\mathcal{A}$ is the Hopf $*$-algebra associated to the universal compact quantum group $(A_{\rm u}, \Delta)$. The co-unit of $(A_{\rm u}, \Delta)$ is norm-bounded. Finally, the reduced compact quantum group of $(A_{\rm u}, \Delta)$ is (isomorphic to) $(A_{\rm r}, \Delta_{\rm r})$, so that $(A_{\rm u}, \Delta)$ is co-amenable if, and only if, (A, Δ) is.*

It is quite obvious that the universal compact quantum group $(C^*(\Gamma), \Delta)$ associated to a discrete group Γ is its own universal compact quantum group; that is, if $(A, \Delta) = (C^*(\Gamma), \Delta)$, then $(A_{\rm u}, \Delta) = (A, \Delta)$. Moreover, if $(A, \Delta) = (C^*_{\rm r}(\Gamma), \Delta_{\rm r})$, then $(A_{\rm u}, \Delta) = (C^*(\Gamma), \Delta)$. This is the motivating example for the general definition of the universal compact quantum group.

Suppose now (A, Δ) is an arbitrary compact quantum group. It is easy to see that if (B, Φ) is a compact quantum group whose associated Hopf $*$ -algebra $(\mathcal{B}, \Phi)$ is isomorphic to $(\mathcal{A}, \Delta)$, then (B_u, Φ) is isomorphic to $(A_{\rm u}, \Delta)$. In particular, the universal compact quantum group associated to $(A_{\rm r}, \Delta_{\rm r})$, or to $(A_{\rm u}, \Delta)$, is isomorphic to $(A_{\rm u}, \Delta)$.

We call a compact quantum group (A, Δ) *universal* if $(A, \Delta) = (A_{\rm u}, \Delta)$, i. e. if the canonical map ψ from $A_{\rm u}$ onto A is injective. Equivalently, (A, Δ) is universal if, and only if, the given norm on $\mathcal{A}$ is its greatest C*-norm. We will show in Corollary 3.7 that any co-amenable compact quantum group is universal.

We prove now a striking automatic continuity result for positive linear functionals on the Hopf $*$-algebra of a universal compact quantum group. Recall that a linear functional τ on a $*$-algebra B is called *positive* if $\tau(b^*b) \geq 0$, for all $b \in B$.

Theorem 3.3 *Suppose that (A, Δ) is a universal compact quantum group. Then every positive linear functional τ on $\mathcal{A}$ is norm-bounded.*

Proof. We form the GNS representation of $\mathcal{A}$ with respect to τ: Since the map $(a, b) \mapsto \tau(b^*a)$ is sesquilinear, the inequality $|\tau(b^*a)|^2 \leq \tau(b^*b)\tau(a^*a)$ implies that the left kernel N_τ of τ is a left ideal of $\mathcal{A}$. Hence, the quotient space $\mathcal{A}/N_\tau$ is a inner product space with inner product given by $(a + N_\tau \mid b + N_\tau) = \tau(b^*a)$, where $a, b \in \mathcal{A}$. Denote the Hilbert space completion by H and its norm by $\|\cdot\|_2$. Define the operator $M_a : \mathcal{A}/N_\tau \to \mathcal{A}/N_\tau$ by setting $M_a(b + N_\tau) = ab + N_\tau$, for all $a, b \in \mathcal{A}$.

We shall show now that M_a is norm-bounded, for all $a \in \mathcal{A}$. Since the map, $a \mapsto M_a$, is linear, it suffices to show boundedness for $a = U^\alpha_{ij}$, where $(U^\alpha)_\alpha$ is a complete set of inequivalent, irreducible unitary representations of (A, Δ) and

U^α_{ij} are the matrix elements of U^α. We have, for all $b \in \mathcal{A}$,

$$b^*b - b^*(U^\alpha_{ji})^*U^\alpha_{ji}b = b^*(\sum_{k\neq j}(U^\alpha_{ki})^*U^\alpha_{ki})b = \sum_{k\neq j}(U^\alpha_{ki}b)^*(U^\alpha_{ki}b) \geq 0.$$

Hence, $\|U^\alpha_{ji}b+N_\tau\|_2^2 = \tau(b^*(U^\alpha_{ji})^*U^\alpha_{ji}b) \leq \tau(b^*b) = \|b+N_\tau\|_2^2$, so that $\|M_a\| \leq 1$. (This kind of argument was used tacitly in the proof of the generalized Tannaka-Krein theorem in [21].)

Hence, for all $a \in \mathcal{A}$, we may extend M_a to a norm-bounded operator $\pi(a)$ on H. The corresponding map, $\pi : \mathcal{A} \to B(H)$, $a \mapsto \pi(a)$, is obviously a unital $*$-representation of $\mathcal{A}$. By the universal property of A_{u}, this map extends to a $*$-homomorphism $\pi: A_{\mathrm{u}} \to B(H)$. Since, for all $a \in \mathcal{A}$, $\tau(a) = (\pi(a)x \mid x)$, where $x = 1 + N_\tau$, we have

$$|\tau(a)| = |(\pi(a)x \mid x)| \leq \|\pi(a)\|\|x\|_2^2 = \|\pi(a)\|\tau(1^*1) \leq \|a\|_u\tau(1).$$

Hence, τ is norm-bounded with respect to the universal C*-norm on $\mathcal{A}$. Since (A, Δ) is assumed to be universal, this norm is equal to the given norm on $\mathcal{A}$. □

When A is a unital C*-algebra, one may consider the C*-algebra invariant consisting of all non-zero $*$-homomorphisms from A to $\mathbf{C}$, i. e. of all unital multiplicative linear functionals on A. This (possibly empty) set is clearly compact in the relative weak* topology inherited from A^*. Of course, when A is commutative, it is precisely the Gelfand spectrum of A. For some other classes of (non-simple) C^*-algebras, this generally rather poor invariant is of some interest. For example, when A is the universal compact group associated to a discrete group Γ, it is easily identified with the dual group of the abelianized group of Γ (see [18]) and therefore it is computable in many cases. We will show below that this invariant is a compact group for any universal compact quantum group.

We need a lemma which may be known to specialists, but for which we could not find a suitable reference in the literature.

Lemma 3.4 *If (A, Δ) is a compact quantum group, the unital multiplicative linear functionals on $\mathcal{A}$ form a group under the multiplication, $(\tau, \sigma) \mapsto \tau * \sigma$, where $\tau * \sigma = (\tau \otimes \sigma)\Delta$. The unit is ε and the inverse of the element τ is $\tau\kappa$. Moreover, the $*$-homomorphisms from $\mathcal{A}$ onto $\mathbf{C}$ form a subgroup (which may be proper).*

Proof. That the operation is closed and associative and the co-unit is a unit for this operation is well known. We prove first that the inverse of the element τ is $\tau\kappa$. To see $\tau * (\tau\kappa) = \varepsilon$, let $a \in \mathcal{A}$, and observe that $(\tau * (\tau\kappa))(a) = (\tau \otimes \tau\kappa)\Delta(a) = \tau(m(\mathrm{id} \otimes \kappa)\Delta(a)) = \tau(\varepsilon(a)1) = \varepsilon(a)$. Here $m: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is the linearization of the multiplication $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$. We used the fact that $\tau \otimes \tau\kappa = \tau m(\mathrm{id} \otimes \kappa)$ which is a consequence of the multiplicative property enjoyed by τ. That $(\tau\kappa) * \tau = \varepsilon$ is similarly proved. Now if $\tau: \mathcal{A} \to \mathbf{C}$ is a $*$-homomorphism, then $\tau\kappa$ is also. We prove this indirectly. The map $\hat{\tau} = (\mathrm{id} \otimes \tau)\Delta: \mathcal{A} \to \mathcal{A}$ is a $*$-homomorphism, since τ is one. Moreover, $\hat{\tau}((\tau\kappa)\hat{}(a)) = (\tau * \tau\kappa)\hat{}(a) = \hat{\varepsilon}(a) = a$ and likewise $(\tau\kappa)\hat{}(\hat{\tau}(a)) = ((\tau\kappa) * \tau)\hat{}(a) = \hat{\varepsilon}(a) = a$. Hence, $(\tau\kappa)\hat{}$ is the inverse of $\hat{\tau}$ and it is therefore also a $*$-homomorphism.

Finally, since $\tau\kappa = \varepsilon \circ (\tau\kappa)\hat{}$ is a composition of $*$-homomorphisms, it is one also. Hence, the $*$-homomorphisms from $\mathcal{A}$ onto $\mathbf{C}$ form a subgroup, which may be proper since multiplicative linear functionals on a *-algebra do not necessarily preserves adjoints. □

Theorem 3.5 *If (A, Δ) is a universal compact quantum group, then the set G of unital multiplicative linear functionals on A forms a compact topological group under the relative weak* topology and the multiplication, $(\tau, \sigma) \mapsto \tau * \sigma$, where $\tau * \sigma = (\tau \otimes \sigma)\Delta$.*

Proof. As before, closure and associativity of the multiplication operation is well known and since the co-unit of $(\mathcal{A}, \Delta)$ is norm-bounded, its extension to A exists and provides a unit for G. If τ is a unital multiplicative linear functional on A, it is necessarily a $*$-homomorphism. Hence, if τ is its restriction to $\mathcal{A}$, the functional $\tau\kappa$ is also a $*$-homomorphism, by the preceding lemma. By universality of (A, Δ), $\tau\kappa$ admits an extension to a $*$-homomorphism, σ say, on A. Since $(\tau \otimes \sigma)\Delta(a) = (\sigma \otimes \tau)\Delta(a) = \varepsilon(a)$, for all $a \in \mathcal{A}$, the same equalities hold for all $a \in A$, by continuity. Thus, $\tau * \sigma = \sigma * \tau = \varepsilon$. It is straightforward to check that G is a weak* closed subset of the unit ball of A^* and therefore, by the Banach–Alaoglu theorem, G is weak* compact. It is also easily checked that the multiplication operation is weak* continuous, as is the inversion operation $\tau \mapsto \tau^{-1}$. This proves the theorem. □

As an example, let (A, Δ) be the compact quantum group $SU_q(2)$, where $q \in \mathbf{R}$ and $0 < |q| < 1$. Being co-amenable, (A, Δ) is universal. Let α and γ be the canonical generators of A. If τ belongs to the group G of multiplicative linear functional on A, then the equations $\alpha\alpha^* + \gamma\gamma^* = 1 = \alpha^*\alpha + q^2\gamma^*\gamma$ imply that $\tau(\gamma) = 0$ and $\tau(\alpha)$ belongs to the unit circle group $\mathbf{T}$. Conversely, given $\lambda \in \mathbf{T}$, the universal property enjoyed by A implies that there exists a—necessarily unique—element τ of G for which $\tau(\alpha) = \lambda$ (and $\tau(\gamma) = 0$). Since $\Delta\alpha = \alpha \otimes \alpha - q\gamma^* \otimes \gamma$, we have $(\tau * \sigma)(\alpha) = \tau(\alpha)\sigma(\alpha)$, for all $\tau, \sigma \in G$. Hence the map, $\tau \mapsto \tau(\alpha)$, is a group isomorphism from G onto $\mathbf{T}$. It is trivially continuous, so that it is also a homeomorphism (since the spaces are compact and Hausdorff). Thus, $G = \mathbf{T}$, as topological groups.

Lemma 3.4 can be used to give an alternative proof of Corollary 2.9: Let (A, Δ) be a compact quantum group and suppose given a $*$-homomorphism $\tau: A \to \mathbf{C}$. Of course, its restriction to $\mathcal{A}$ is therefore a $*$-homomorphism, from which it follows that $\tau\kappa$ is one also. Hence, by [14, Lemma 10.2], $(\tau\kappa)\hat{}$ is an isometry (we are retaining the notation used in the proof of Lemma 3.4). Since $\varepsilon = \tau \circ (\tau\kappa)\hat{}$ is the composition of two norm-bounded maps, it is norm-bounded and therefore (A, Δ) is co-amenable.

We now come to one of the main results of the theory. It especially confirms that the Haar integral of a co-amenable compact quantum group is faithful. The equivalence between (1) and (2) shows that our definition of co-amenability agrees with the one considered by Banica [2, 3].

Theorem 3.6 *The following are equivalent conditions for a compact quantum group (A, Δ):*

(1) (A, Δ) is co-amenable;

(2) The canonical map from A_{u} to A_{r} is a $*$-isomorphism;

(3) The canonical maps from A_{u} onto A and A onto A_{r} are $*$-isomorphisms;
(4) The Haar integral h_{u} of (A_{u}, Δ) is faithful.

Proof. If Condition (1) holds, then (A_{u}, Δ) is co-amenable, by Theorem 3.2 and therefore h_{u} is faithful, by Theorem 2.2. Thus, (1) $\Rightarrow$ (4). Since $h_{\mathrm{u}} = h\psi = h_{\mathrm{r}}\theta\psi$, it is clear that Condition (4) implies (2). The equivalence of Conditions (2) and (3) is trivial. Suppose now that (2) holds and let ε_{u} be the extension of the co-unit of (A_{u}, Δ) to A_{u} Then $\varepsilon_{\mathrm{u}}(\theta\psi)^{-1}$ is a non-zero $*$-homomorphism on A_{r} and therefore, by Corollary 2.9, (A, Δ) is co-amenable. Thus, (2) $\Rightarrow$ (1). This proves the theorem. □

The following is now immediate from the theorem, from Theorem 3.3 and from Theorem 3.5.

Corollary 3.7 *Let (A, Δ) be a co-amenable compact quantum group. Then (A, Δ) is universal. Especially, every unital $*$-homomorphism from $\mathcal{A}$ to a unital C^*-algebra is necessarily norm-decreasing. Further, every positive linear functional on $\mathcal{A}$ is norm-bounded. Finally, the unital multiplicative linear functionals on A form a compact group*

Note that co-amenability imposes a norm-boundedness condition on just a single postive linear functional (the co-unit of the reduced quantum group). However, the corollary shows it implies a much stronger norm-boundedness result.

The equivalence between (1) and (3) in Theorem 3.6 may be rephrased as saying that a compact quantum group (A, Δ) is co-amenable if, and only if, it is both universal and reduced. Note in this connection that $C^*(\mathbf{F}_2) \otimes C^*_r(\mathbf{F}_2)$ is an example of a compact quantum group which is neither universal nor reduced, since, obviously, its Haar integral is not faithful and its co-unit is not norm bounded.

If (A, Δ) is an arbitrary compact quantum group, we know that $\|\cdot\|_{\mathrm{u}}$ is the greatest C*-norm on the associated Hopf $*$-algebra $\mathcal{A}$. We define a C*-seminorm on $\mathcal{A}$ by setting $\|a\|_{\mathrm{r}} = \|\theta(a)\|$, for all $a \in \mathcal{A}$. This is, in fact, a C*-norm, since θ is injective on $\mathcal{A}$. Therefore we can regard not only (A_{u}, Δ) and (A, Δ) as compact quantum group completions of $\mathcal{A}$, but $(A_{\mathrm{r}}, \Delta_{\mathrm{r}})$ also. When we say that a compact quantum group $(A_{\mathrm{c}}, \Delta_{\mathrm{c}})$ is a *compact quantum group completion* of $\mathcal{A}$, we mean not only that $\mathcal{A}$ is a dense unital $*$-subalgebra of the C*-algebra, but also that the co-multiplication Δ_{c} extends the co-multiplication Δ of $\mathcal{A}$. We shall call a C*-norm $\|\cdot\|_{\mathrm{c}}$ on $\mathcal{A}$ *regular* if it is the restriction to $\mathcal{A}$ of the norm of a compact quantum group completion $(A_{\mathrm{c}}, \Delta_{\mathrm{c}})$ of $\mathcal{A}$. Thus, the given C*-norm on $\mathcal{A}$ and the norms $\|\cdot\|_{\mathrm{u}}$ and $\|\cdot\|_{\mathrm{r}}$ are regular.

We show now that $\|\cdot\|_{\mathrm{r}}$ is the least regular C*-norm on $\mathcal{A}$.

Theorem 3.8 *Let (A, Δ) be a compact quantum group and $\|\cdot\|_{\mathrm{c}}$ be a regular C^*-norm on $\mathcal{A}$. Then $\|a\|_{\mathrm{r}} \leq \|a\|_{\mathrm{c}} \leq \|a\|_{\mathrm{u}}$, for all $a \in \mathcal{A}$. If $(A_{\mathrm{c}}, \Delta_{\mathrm{c}})$ is the compact quantum group completion of $\mathcal{A}$ with respect to $\|\cdot\|_{\mathrm{c}}$, then there exist unique $*$-homomorphisms $\psi_{\mathrm{c}}\colon A_{\mathrm{u}} \to A_{\mathrm{c}}$ and $\theta_{\mathrm{c}}\colon A_{\mathrm{c}} \to A_{\mathrm{r}}$ extending, in each case, the identity automorphism on $\mathcal{A}$. Both maps are quantum group morphisms.*

Proof. Given the maps ψ_{c} and θ_{c} exist, it follows trivially from density of $\mathcal{A}$ in A_{u} and A_{c}, respectively, that they are unique and are quantum group morphisms.

The norm inequality $||\cdot||_{\rm c} \le ||\cdot||_{\rm u}$ is already known and the existence of the map $\psi_{\rm c}$ is obvious. If we show $||\cdot||_{\rm r} \le ||\cdot||_{\rm c}$, the existence of $\theta_{\rm c}$ follows trivially. We turn now to showing this inequality. Before proceeding, let us first observe that $|h(a)| \le ||a||_{\rm c}$, for all $a \in \mathcal{A}$. Let $h_{\rm c}$ denote the Haar integral of $(A_{\rm c}, \Delta_{\rm c})$. When we regard $\mathcal{A}$ as a Hopf $*$-subalgebra of $(A_{\rm c}, \Delta_{\rm c})$ and of (A, Δ), as we do here, we have $h_{\rm c}(a) = h(a)$ for all $a \in \mathcal{A}$, by uniqueness of Haar integrals. Consequently, $|h(a)| = |h_{\rm c}(a)| \le ||a||_{\rm c}$, as claimed.

Again suppose $a \in \mathcal{A}$. Since the Haar integral $h_{\rm r}$ of $(A_{\rm r}, \Delta_{\rm r})$ is a faithful state of $A_{\rm r}$, it follows from [14, Theorem 10.1] that

$$||a^*a||_{\rm r} = ||\theta(a)^*\theta(a)|| = \lim[h_{\rm r}((\theta(a)^*\theta(a))^n)]^{1/n}.$$

Using the fact that $h = h_{\rm r}\theta$, we get $||a^*a||_{\rm r} = \lim[h(a^*a)^n)]^{1/n}$. By our observations in the preceding paragraph, $h((a^*a)^n) \le ||(a^*a)^n||_{\rm c}$. Therefore, $||a^*a||_{\rm r} \le \lim ||(a^*a)^n||_{\rm c}^{1/n} = ||a^*a||_{\rm c}$ and hence $||a||_{\rm r} \le ||a||_{\rm c}$, as required. □

Corollary 3.9 *Let (A, Δ) be a compact quantum group with associated Hopf $*$-algebra $\mathcal{A}$. Then (A, Δ) is co-amenable if, and only if, $\mathcal{A}$ admits only one quantum group completion (up to isomorphism).*

Proof. This follows immediately from the theorem and the observation that (A, Δ) is co-amenable if, and only if, $||\cdot||_{\rm u} = ||\cdot||_{\rm r}$, as norms on $\mathcal{A}$; this observation is an immediate consequence of Theorem 3.6. □

The qualifying word *regular* may not be dropped in the statement of Theorem 3.8. This may be seen as follows: Let Γ denote a discrete group. Set $(A, \Delta) = (C^*(\Gamma), \Delta)$, and recall that $\mathcal{A}$ is the group algebra of Γ. Let W be a unitary representation of Γ on a Hilbert space H and denote by π the associated representation of $C^*(\Gamma)$ on H, so that $\pi(C^*(\Gamma)) = C^*(W)$, where $C^*(W)$ denotes the C^*-algebra generated by all W_x ($x \in \Gamma$). Then define a C^*-seminorm $||\cdot||_\pi$ on $\mathcal{A}$ by setting $||a||_\pi = ||\pi(a)||$. Assume that $||\cdot||_\pi$ is a C^*-norm on $\mathcal{A}$; that is, π is faithful on $\mathcal{A}$. Then the completion of $\mathcal{A}$ with respect to $||\cdot||_\pi$ may be identified with $C^*(W)$.

If we now assume that Theorem 3.8 holds without the qualifying word *regular*, the regular representation L of Γ is clearly weakly contained in W; that is, there exists a $*$-homomorphism ϕ from $C^*(W)$ onto $C^*(L)$ satisfying $\phi(W_x) = L_x$, for all $x \in \Gamma$. If we also assume that Γ is amenable, then φ is a $*$-isomorphism (since it clearly admits an inverse in this case). Now set $\Gamma = \mathbf{Z}$. Then $C^*(L) = C(\mathbf{T})$ and L_1 has spectrum $\mathbf{T}$. This forces W_1 to have spectrum $\mathbf{T}$ also. To get a contradiction we need now only show W_1 does not have to have spectrum $\mathbf{T}$. To do this, choose a unitary V on a Hilbert space with infinite spectrum not equal to $\mathbf{T}$. This induces a representation W of $\mathbf{Z}$ and the corresponding homomorphism π is injective on $\mathbf{C}(\mathbf{Z})$, since ${\rm sp}(V)$ is infinite (this implies all the powers $1, V, V^2, \ldots$ are linearly independent). Thus, this representation W satisfies the required conditions and the spectrum of $W_1 = V$ is not equal to $\mathbf{T}$.

An open question in this setting is whether $||\cdot||_\pi$ is necessarily regular whenever L is weakly contained in W. We doubt that the answer is positive. It is worth mentioning here that Woronowicz shows in [19, Theorem 1.6] that if Γ is finitely generated and W is a faithful representation of Γ such that $W \otimes W$ is (strongly) contained in a multiple of W, then $||\cdot||_\pi$ is regular. However, the only known representations satisfying these assumptions seem to be the universal and the regular ones, and the external tensor product of such representations.

4 Quantum semigroups and co-amenability

In this short section we give a sufficient condition ensuring that a compact quantum semigroup is a compact quantum group. Recall that a *compact quantum semigroup* is a pair (A, Δ) consisting of a unital C*-algebra A and a co-multiplication $\Delta: A \to A \otimes A$. Of course, if, in addition, the linear spans of the spaces $(A \otimes 1)\Delta A$ and $(1 \otimes A)\Delta A$ are each equal to $A \otimes A$, then (A, Δ) is a compact quantum group. A *Haar integral* on a compact quantum semigroup (A, Δ) is defined in the usual way as a state on h on A for which we have $(\mathrm{id} \otimes h)\Delta(a) = (h \otimes \mathrm{id})\Delta(a) = h(a)1$, for all $a \in A$. It is trivial to verify that at most one Haar integral can exist. Not every compact quantum semigroup admits a Haar integral, nor does the existence of a Haar integral imply that a compact quantum semigroup is a compact quantum group [14].

A *bounded co-unit* for a compact quantum semigroup (A, Δ) is defined as a unital $*$-homomorphism ε from A to $\mathbf{C}$ such that, for all $a \in A$, $(\varepsilon \otimes \mathrm{id})\Delta(a) = (\mathrm{id} \otimes \varepsilon)\Delta(a) = a$. The example given in [14] of a compact quantum semigroup having no Haar integral has got a bounded co-unit. Thus, the existence of a bounded co-unit does not ensure that a compact quantum semigroup is a compact quantum group.

We shall need some notation for the following two results. If $a, b \in A$, we write $a * (hb)$ for the element $(h \otimes \mathrm{id})((b \otimes 1)\Delta(a))$ and $(ha) * b$ for the element $(\mathrm{id} \otimes h)((1 \otimes a)\Delta(b))$.

Lemma 4.1 *Let (A, Δ) be a compact quantum semigroup admitting a Haar integral h. Then, for all $a, b \in A$, the element $1 \otimes a * hb$ belongs to the closed linear span of $(A \otimes 1)\Delta A$. Likewise, $(ha) * b \otimes 1$ belongs to the closed linear span of $(1 \otimes A)\Delta A$.*

Proof. If $F: A \otimes A \to A \otimes A$ is the flip automorphism, then the *opposite* compact quantum semigroup $(A, F\Delta)$ also has the state h as its Haar integral and ε as a bounded co-unit. It follows that if we show that $1 \otimes a * hb$ belongs to the closed linear span of $(A \otimes 1)\Delta A$, then we can deduce from this result applied to $(A, F\Delta)$ that $(ha) * b \otimes 1$ belongs to the closed linear span of $(1 \otimes A)\Delta A$. The demonstration that $1 \otimes a * hb$ belongs to the closed linear span of $(A \otimes 1)\Delta A$ is given in the proof of Theorem 3.3 of [14]. The strong hypotheses of Theorem 3.3 are not needed for our result, which only needs the fact that (A, Δ) is a compact quantum semigroup admitting a Haar integral, as can be verified by a careful reading of the proof in [14]. □

Theorem 4.2 *Let (A, Δ) be a compact quantum semigroup admitting a faithful Haar integral and a bounded co-unit. Then (A, Δ) is a co-amenable compact quantum group.*

Proof. If we show that (A, Δ) is a compact quantum group, its co-amenability follows from Theorem 2.2. By the preceding lemma, we need then only show that the closed linear span L of the elements $a * hb$, where $a, b \in A$, and the closed linear span R of the elements $(ha) * b$, are both equal to A. For, in this case, $1 \otimes A$ and $A \otimes 1$ are subsets of the closed linear spans of $(A \otimes 1)\Delta A$ and $(1 \otimes A)\Delta A$, respectively and therefore each of these closed linear spans is equal to $A \otimes A$, thereby ensuring (A, Δ) is a compact quantum group. Co-amenability is then immediate. We shall show only that $L = A$; the proof that $R = A$ is similar. Arguing by contradiction, suppose that $L \neq A$, so that there exists a non-zero element $\tau \in A^*$ that vanishes on L. Then $\tau(a * hb) = 0$, for all $a, b \in A$. Thus, $(\tau \otimes hb)\Delta(a) = 0$; that is, $h(b((\tau \otimes \mathrm{id})\Delta(a))) = 0$. By faithfulness of h we deduce that $(\tau \otimes \mathrm{id})\Delta(a) = 0$. Applying ε now we get $0 = \varepsilon((\tau \otimes \mathrm{id})\Delta(a)) = \tau((\mathrm{id} \otimes \varepsilon)\Delta(a)) = \tau(a)$. Hence, $\tau = 0$, a contradiction. Therefore, $L = A$, as required. □

The question arises as to whether one can drop the faithfulness requirement on the Haar integral h in the preceding theorem. The answer is no. To see this let $A = C(\mathbf{D})$, the C*-algebra of continuous complex-valued functions on the closed unit disc $\mathbf{D}$. A co-multiplication Δ on A is given by setting $\Delta(f)(s, t) = f(st)$, for all $s, t \in \mathbf{C}$. The linear functional δ_0 on A defined by evaluation at the origin, $\delta_0(f) = f(0)$, is a Haar integral for (A, Δ) and the functional δ_1 is a bounded co-unit. But (A, Δ) is not a compact quantum group, by [14, Proposition 2.2].

5 Appendix

For the convenience of the reader we gather here some basic facts about compact quantum groups (see [11, 14, 20] for more information).

A compact quantum group (A, Δ) consists of a unital $C*$-algebra A and a unital $*$-homomorphism $\Delta : A \to A \otimes A$ (called the co-multiplication) satisfying

$$(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta$$

and such that the linear spans of $(1 \otimes A)\Delta A$ and $(A \otimes 1)\Delta A$ are each dense in $A \otimes A$. A *morphism* from (A, Δ) to a compact quantum group (B, Δ') is a unital $*$-homomorphism $\pi: A \to B$ satisfying $\Delta'\pi = (\pi \otimes \pi)\Delta$.

There exists a unique state h on A called the Haar integral of (A, Δ) which satisfies

$$(h \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes h)\Delta = h(\cdot)1.$$

By a Hopf $*$-subalgebra $\mathcal{A}$ of a compact quantum group (A, Δ) we mean a Hopf $*$-algebra such that $\mathcal{A}$ is a $*$-subalgebra of A with co-multiplication given by restricting the co-multiplication Δ from A to $\mathcal{A}$. The co-unit $\varepsilon: \mathcal{A} \to \mathbf{C}$ and the co-inverse $\kappa: \mathcal{A} \to \mathcal{A}$ of $\mathcal{A}$ are linear maps satisfying

$$(\varepsilon \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \varepsilon)\Delta = \mathrm{id},$$

$$m(\kappa \otimes \mathrm{id})\Delta = m(\mathrm{id} \otimes \kappa)\Delta = \varepsilon(\cdot)1,$$

where $m: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ denotes the multiplication map. The co-unit ε is known to be a $*$-homomorphism.

Any compact quantum group (A, Δ) has a canonical dense Hopf $*$-subalgebra $\mathcal{A}$ consisting of the linear span of the matrix entries of all finite dimensional co-representations of (A, Δ). By abuse of language ε and κ are also refered to as the co-unit and the co-inverse of (A, Δ). We call $\mathcal{A}$ the associated Hopf $*$-algebra of (A, Δ).

The associated Hopf $*$-algebra of a compact quantum group has the following uniqueness property (which is stated without proof in [11]).

Theorem 5.1 *The associated Hopf $*$-algebra $\mathcal{A}$ of a compact quantum group (A, Δ) is the unique dense Hopf $*$-subalgebra of (A, Δ).*

Proof. Let $\mathcal{B}$ be another dense Hopf $*$-subalgebra of (A, Δ). We must show that $\mathcal{A} = \mathcal{B}$. First we show that $\mathcal{B}$ is the linear span of the matrix entries of those finite-dimensional co-representations which have matrix entries belonging to $\mathcal{B}$. This will immediately imply that $\mathcal{B} \subset \mathcal{A}$. Thus let $x \in \mathcal{B}$. Then we may write $\Delta(x) = \sum_i x_i \otimes y_i$, for finitely many $x_i, y_i \in \mathcal{B}$ with $\{y_i\}$ linearly independent. Pick linear functionals $\{\xi_i\}$ on $\mathcal{B}$ such that $\xi_i(y_j) = \delta_{ij}$, for all i, j. Then

$$\Delta(x_i) = (\mathrm{id} \otimes \mathrm{id} \otimes \xi_i) \sum_j \Delta(x_j) \otimes y_j = (\mathrm{id} \otimes \mathrm{id} \otimes \xi_i)(\Delta \otimes \mathrm{id})\Delta(x)$$

$$= (\mathrm{id} \otimes \mathrm{id} \otimes \xi_i)(\mathrm{id} \otimes \Delta)\Delta(x) = \sum_j x_j \otimes (\mathrm{id} \otimes \xi_i)\Delta(y_j),$$

for all i. Thus, if we let $\{e_i\}$ denote a linear basis for the vector subspace of $\mathcal{B}$ spanned by $\{x_i\}$, there exist finitely many elements z_i, w_{kl} in $\mathcal{B}$ such that

$$\Delta(x) = \sum_i e_i \otimes z_i \qquad \text{and} \qquad \Delta(e_j) = \sum_k e_k \otimes w_{kj} \ ,$$

for all j. Now

$$\sum_{k,l} e_l \otimes w_{lk} \otimes w_{kj} = \sum_k \Delta(e_k) \otimes w_{kj} = (\Delta \otimes \mathrm{id})\Delta(e_j)$$

$$= (\mathrm{id} \otimes \Delta)\Delta(e_j) = \sum_l e_l \otimes \Delta(w_{lj}),$$

so by linear independence of $\{e_i\}$, we get

$$\Delta(w_{lj}) = \sum_k w_{lk} \otimes w_{kj} \ ,$$

for all j, l. It follows that $w = (w_{ij})$ is a finite-dimensional co-representation of (A, Δ) with matrix entries belonging to $\mathcal{B}$. Furthermore, the element x is a linear combination of the matrix entries of w because

$$x = (\mathrm{id} \otimes \varepsilon)\Delta(x) = \sum_i \varepsilon(z_i)e_i = \sum_i \varepsilon(z_i)(\varepsilon \otimes \mathrm{id})\Delta(e_i) = \sum_{ij} \varepsilon(z_i e_j)w_{ji} \ ,$$

where ε is the co-unit of $\mathcal{B}$. This proves that $\mathcal{B} \subset \mathcal{A}$.

To prove the converse inclusion, first observe that $\mathcal{B}$ is the linear span of the matrix entries of those finite-dimensional irreducible unitary co-representations of (A, Δ) with matrix entries belonging to $\mathcal{B}$. To see this, consider the co-representation w constructed above, and define elements $v_{ij} = w_{ij} + (\delta_{ij} - \varepsilon(w_{ij}))I \in \mathcal{B}$, for all i, j, where ε now denotes the co-unit of $\mathcal{A}$. It is easily checked that $v = (v_{ij})$ is a co-representation of (A, Δ). Since $\varepsilon(v_{ij}) = \delta_{ij}$, for all i, j, the co-representation v is invertible with inverse $v^{-1} = (\kappa(v_{ij}))$, where κ is the co-inverse of $\mathcal{A}$. Now it is known [11, 19] that any invertible co-representation is equivalent to a direct sum of irreducible unitary ones. Since the invertible co-representation v has matrix entries in $\mathcal{B}$, its irreducible components are easily seen to also have matrix entries belonging to $\mathcal{B}$. It then follows that $\mathcal{B}$ is a linear span of the required sort.

To conclude that $\mathcal{A} \subset \mathcal{B}$, we now show that every finite-dimensional irreducible unitary co-representation of (A, Δ) is equivalent to one with matrix entries belonging to $\mathcal{B}$. Assume, for contradiction, that $v = (v_{ij})$ is a finite-dimensional irreducicble unitary co-representation not equivalent to any finite-dimensional irreducicble unitary co-representation $u = (u_{ij})$ with matrix entries u_{ij} belonging to $\mathcal{B}$. From [14, Theorem 7.4], we get that $h(u_{ij}v_{kl}) = 0$, for all i, j, k, l. Since $\mathcal{B}$ is linearly spanned by elements of the type u_{ij}, as observed above, and $\mathcal{B}$ is dense in A, this implies that $h(av_{kl})$, for all k, l and $a \in A$. In particular, we get $h(v_{kl}^* v_{kl}) = 0$, and therefore $v_{kl} = 0$, for all k, l, since h is faithful on $\mathcal{A}$. This is impossible as v is unitary. □

Note that the first part of the proof shows that $\mathcal{A}$ is maximal among all Hopf $*$-subalgebras of (A, Δ).

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