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A STOCHASTIC MAXIMUM PRINCIPLE FOR PROCESSES DRIVEN BY FRACTIONAL BROWNIAN MOTION

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Abstract

We prove a stochastic maximum principle for controlled processes $X(t) = X^{(u)}(t)$ of the form

$$dX(t) = b(t, X(t), u(t))dt + \sigma(t, X(t), u(t))dB^{(H)}(t)$$

where $B^{(H)}(t)$ is n -dimensional fractional Brownian motion with Hurst parameter $H = (H_1, \dots, H_n) \in (1/2, 1)^n$. As an application we solve an optimal consumption problem with a terminal condition in an economy driven by a fractional Brownian motion.

1 INTRODUCTION

Let $H = (H_1, \dots, H_m)$ with $1/2 < H_j < 1$, $j = 1, 2, \dots, m$, and let $B^{(H)}(t) = (B_1^{(H)}(t), \dots, B_m^{(H)}(t))$, $t \in \mathbb{R}$ be m -dimensional fractional Brownian motion, *i.e.* $B^{(H)}(t) = B^{(H)}(t, \omega)$, $(t, \omega) \in \mathbb{R} \times \Omega$ is a Gaussian process in \mathbb{R}^m such that

$$\mathbb{E} [B^{(H)}(t)] = B^{(H)}(0) = 0 \tag{1.1}$$

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and

$$\mathbb{E} \left[B_j^{(H)}(s) B_k^{(H)}(t) \right] = \frac{1}{2} \left\{ s^{2H_j} + t^{2H_j} - |t - s|^{2H_j} \right\} \delta_{jk}; 1 \leq j, k \leq n, \quad s, t \in \mathbb{R}, \quad (1.2)$$

where

$$\delta_{jk} = \begin{cases} 0 & \text{when } j \neq k \\ 1 & \text{when } j = k \end{cases}$$

Here $\mathbb{E} = \mathbb{E}_\mu$ denotes the expectation with respect to the probability law $\mu = \mu_\phi$ for $B^{(H)}(\cdot)$. This means that the components $B_1^{(H)}(\cdot), \dots, B_m^{(H)}(\cdot)$ of $B^{(H)}(\cdot)$ are m independent 1-dimensional fractional Brownian motions with Hurst parameter H_1, H_2, \dots, H_m , respectively. We refer to [MvN], [NVV] and [S] for more information about fractional Brownian motion. Because of its interesting properties (e.g. long range dependence and self-similarity of the components) $B^{(H)}(t)$ has been suggested as a replacement of *standard Brownian motion* $B(t)$ (corresponding to $H_j = 1/2$ for all $j = 1, \dots, m$) in several stochastic models, including finance.

Unfortunately, $B^{(H)}(\cdot)$ is neither a semimartingale nor a Markov process, so the powerful tools from the theories of such processes are not applicable when studying $B^{(H)}(\cdot)$. Nevertheless, an efficient stochastic calculus of $B^{(H)}(\cdot)$ can be developed. This calculus uses an Itô type of integration with respect to $B^{(H)}(\cdot)$ and white noise theory. See [DHP] and [HØ2] for details. For applications to finance see [HØ2], [HØS1] [HØS2]. In [HØZ] and [ØZ] the theory is extended to multi-parameter fractional Brownian fields $B^{(H)}(x); x \in \mathbb{R}^d$ and applied to stochastic partial differential equations driven by such fractional white noise.

The purpose of this paper is to establish a stochastic maximum principle for stochastic control of processes driven by $B^{(H)}(\cdot)$. We illustrate the result by applying it to a problem about optimal consumption in finance.

2 PRELIMINARIES

For the convenience of the reader we recall here some of the basic results of fractional Brownian motion calculus. Let $B^{(H)}(t)$ be 1-dimensional in the following.

We let $\int_{\mathbb{R}} \sigma(t, \omega) dB^{(H)}(t)$ denote the *fractional Itô-integral* of the process $\sigma(t, \omega)$ with respect to $B^{(H)}(t)$, as defined in [DHP]. In particular, this means that if σ belongs to the family \mathcal{S} of step functions of the form

$$\sigma(t, \omega) = \sum_{i=1}^N \sigma_i(\omega) \chi_{[t_i, t_{i+1})}(t), \quad (t, \omega) \in \mathbb{R} \times \Omega,$$

where $0 \leq t_1 < t_2 < \dots < t_{N+1}$, then

$$\int_{\mathbb{R}} \sigma(t, \omega) dB^{(H)}(t) = \sum_{i=1}^N \sigma_i(\omega) \diamond \left(B^{(H)}(t_{i+1}) - B^{(H)}(t_i) \right), \quad (2.1)$$

where \diamond denotes the Wick product. For $\sigma(t) = \sigma(t, \omega) \in \mathbb{S}$ we have

$$\mathbb{E} \left[\int_{\mathbb{R}} \sigma(t, \omega) dB^{(H)}(t) \right]^2 = \mathbb{E} \left[\int_{\mathbb{R}_+^2} \sigma(s) \sigma(t) \phi(s, t) ds dt + \left(\int_{\mathbb{R}_+} D_s^\phi \sigma(s) ds \right)^2 \right], \quad (2.2)$$

where $\mathbb{E} = \mathbb{E}_{\mu_H}$,

$$\phi(s, t) = \phi_H(s, t) = H(2H - 1)|s - t|^{2H-2} \quad (2.3)$$

and D_s^ϕ denotes the Malliavin ϕ -derivative at s (see [DHP, Definition 3.1]). Using this we can extend the integral $\int_{\mathbb{R}} \sigma(t, \omega) dB^{(H)}(t)$ to the closure $\mathcal{L}_\phi^{1,2} = \mathcal{L}_\phi^{1,2}(\mathbb{R})$ of \mathbb{S} in the norm

$$\|\sigma\|_{\mathcal{L}_\phi^{1,2}}^2 = \mathbb{E} \left[\int_{\mathbb{R}_+^2} \sigma(s) \sigma(t) \phi(s, t) ds dt + \left(\int_{\mathbb{R}_+} D_s^\phi \sigma(s) ds \right)^2 \right]. \quad (2.4)$$

This is in fact a Hilbert norm: If $\sigma, \theta \in \mathcal{L}_\phi^{1,2}$, we have, by polarization,

$$\begin{aligned} & \mathbb{E} \left[\int_{\mathbb{R}} \sigma(t, \omega) dB^{(H)}(t) \int_{\mathbb{R}} \theta(t, \omega) dB^{(H)}(t) \right] \\ &= \mathbb{E} \left[\int_{\mathbb{R}_+^2} \sigma(s) \theta(t) \phi(s, t) ds dt + \left(\int_{\mathbb{R}_+} D_s^\phi \sigma(s) ds \int_{\mathbb{R}_+} D_t^\phi \theta(t) dt \right) \right]. \end{aligned} \quad (2.5)$$

We note that we need not assume that the integrand $\sigma \in \mathcal{L}_\phi^{1,2}$ is adapted to the filtration $\mathcal{F}_t^{(H)}$ generated by $B^{(H)}(s, \cdot); s \leq t$.

An important property of this fractional Itô-integral is that

$$\mathbb{E} \left[\int_{\mathbb{R}} \sigma(t, \omega) dB^{(H)}(t) \right] = 0 \quad \text{for all } \sigma \in \mathcal{L}_\phi^{1,2}. \quad (2.6)$$

(see [DHP, Theorem 3.7]).

We give three versions of the fractional Itô formula, in increasing order of complexity.

Theorem 2.1 ([DHP, Theorem 4.1]) *Let $f \in C^2(\mathbb{R})$ with bounded derivatives. Then for $t \geq 0$*

$$f(B^{(H)}(t)) = f(B^{(H)}(0)) + \int_0^t f'(B^{(H)}(s)) dB^{(H)}(s) + H \int_0^t s^{2H-1} f''(B^{(H)}(s)) ds. \quad (2.7)$$

Theorem 2.2 ([DHP, Theorem 4.3]) *Let $X(t) = \int_0^t \sigma(s, \omega) dB^{(H)}(s)$, where $\sigma \in \mathcal{L}_\phi^{1,2}$ and assume $f \in C^2(\mathbb{R}_+ \times \mathbb{R})$ with bounded derivatives. Then for $t \geq 0$*

$$\begin{aligned} f(t, X(t)) &= f(0, 0) + \int_0^t \frac{\partial f}{\partial s}(s, X(s)) ds \\ &\quad + \int_0^t \frac{\partial f}{\partial x}(s, X(s)) \sigma(s) dB^{(H)}(s) + \int_0^t \frac{\partial^2 f}{\partial x^2}(s, X(s)) \sigma(s) D_s^\phi X(s) ds. \end{aligned} \quad (2.8)$$

Finally we give an m -dimensional version:

Let $B^{(H)}(t) = (B_1^{(H)}(t), \dots, B_m^{(H)}(t))$ be m -dimensional fractional Brownian motion with Hurst parameter $H = (H_1, \dots, H_m) \in (1/2, 1)^m$, as in Section 1. Let $\sigma_{ij} \in \mathcal{L}_{\phi_{H_j}}^{1,2}$ for $1 \leq i \leq n, 1 \leq j \leq m$. We can define $X(t) = (X_1(t), \dots, X_n(t))$ where

$$X_i(t, \omega) = \sum_{j=1}^m \int_0^t \sigma_{ij}(s, \omega) dB_j^{(H)}(s); 1 \leq i \leq n. \quad (2.9)$$

Then we have the following multi-dimensional fractional Itô formula:

Theorem 2.3 *Let $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n)$ with bounded derivatives. Then, for $t \geq 0$,*

$$\begin{aligned} f(t, X(t)) &= f(0, 0) + \int_0^t \frac{\partial f}{\partial s}(s, X(s)) ds + \int_0^t \sum_{i=1}^n \frac{\partial f}{\partial x_i}(s, X(s)) dX_i(s) \\ &\quad + \int_0^t \left\{ \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(s, X(s)) \sum_{k=1}^m \sigma_{ik}(s) D_{k,s}^\phi(X_j(s)) \right\} ds \end{aligned} \quad (2.10)$$

$$\begin{aligned} &= f(0, 0) + \int_0^t \frac{\partial f}{\partial s}(s, X(s)) ds + \sum_{j=1}^m \int_0^t \left[\sum_{i=1}^n \frac{\partial f}{\partial x_i}(s, X(s)) \sigma_{ij}(s, \omega) \right] dB_j^{(H)}(s) \\ &\quad + \int_0^t \text{Tr} \left[\Lambda^T(s) f_{xx}(s, X(s)) \right] ds. \end{aligned} \quad (2.11)$$

Here $\Lambda = [\Lambda_{ij}] \in \mathbb{R}^{n \times m}$ with

$$\Lambda_{ij}(s) = \sum_{k=1}^m \sigma_{ik} D_{k,s}^\phi(X_j(s)); \quad 1 \leq i \leq n, \quad 1 \leq j \leq m, \quad (2.12)$$

$$f_{xx} = \left[\frac{\partial^2 f}{\partial x_i \partial x_j} \right]_{1 \leq i, j \leq n} \quad (2.13)$$

and $(\cdot)^T$ denotes matrix transposed, $\text{Tr}[\cdot]$ denotes matrix trace.

Since we are here dealing with m independent fractional Brownian motions we may regard Ω as the product of m independent copies of $\bar{\Omega}$ and write $\omega = (\omega_1, \dots, \omega_m)$ for $\omega \in \Omega$. Then the notation $D_{k,s}^\phi Y$ in (2.10) and (2.12) means the Malliavin ϕ -derivative with respect to ω_k and could also be written

$$D_{k,s}^\phi Y = \int_{\mathbb{R}} \phi_{H_k}(s, t) D_{k,t} Y dt = \int_{\mathbb{R}} \phi_{H_k}(s, t) \frac{\partial Y}{\partial \omega_k}(t, \omega) dt.$$

The following useful result is a multidimensional version of Theorem 4.2 in [DHP]:

Theorem 2.4 *Let*

$$X(t) = \sum_{j=1}^m \int_0^t \sigma_j(r, \omega) dB_j^{(H)}(r); \quad \sigma_j \in \mathcal{L}_{\phi_{H_j}}^{1,2}; \quad 1 \leq j \leq m. \quad (2.14)$$

Then

$$D_{k,s}^\phi X(t) = \sum_{j=1}^m \int_0^t D_{k,s}^\phi \sigma_j(r) dB_j^{(H)}(r) + \int_0^t \sigma_k(r) \phi_{H_k}(s, r) dr, \quad 1 \leq k \leq m. \quad (2.15)$$

In particular, if $\sigma_j(r)$ is deterministic for all $j \in \{1, 2, \dots, m\}$ then

$$D_{k,s}^\phi X(t) = \int_0^t \sigma_k(r) \phi_{H_k}(s, r) dr. \quad (2.16)$$

Now we have the following integration by parts formula.

Corollary 2.5 *Let $X(t)$ and $Y(t)$ be two processes of the form*

$$dX(t) = \mu(t, \omega)dt + \sigma(t, \omega)dB^{(H)}(t), \quad X(0) = x \in \mathbb{R}^n$$

and

$$dY(t) = \nu(t, \omega)dt + \theta(t, \omega)dB^{(H)}(t), \quad Y(0) = y \in \mathbb{R}^n,$$

where $\mu : \mathbb{R} \times \Omega \rightarrow \mathbb{R}^n$, $\nu : \mathbb{R} \times \Omega \rightarrow \mathbb{R}^n$, $\sigma : \mathbb{R} \times \Omega \rightarrow \mathbb{R}^{n \times m}$ and $\theta : \mathbb{R} \times \Omega \rightarrow \mathbb{R}^{n \times m}$ are given processes with components $\sigma_{ij}, \theta_{ij} \in \mathcal{L}_{\phi_{H_j}}^{1,2}$ for $1 \leq i \leq n$, $1 \leq j \leq m$ and $B^H(\cdot)$ is m -dimensional.

Suppose that $\sigma(\cdot)$ or $\theta(\cdot)$ is deterministic. Then for $T > 0$,

$$\begin{aligned} \mathbb{E}[X(T) \cdot Y(T)] &= x \cdot y + \mathbb{E} \left[\int_0^T X(s) dY(s) \right] + \mathbb{E} \left[\int_0^T Y(s) dX(s) \right] \\ &\quad + \mathbb{E} \left[\int_0^T \int_0^T \sum_{i=1}^n \sum_{k=1}^m \sigma_{ik}(s) \theta_{ik}(t) \phi_{H_k}(s, t) ds dt \right]. \end{aligned} \quad (2.17)$$

Proof This follows from Theorem 2.3 applied to the function $f(t, x, y) = xy$, combined with Theorem 2.4. \square

3 STOCHASTIC DIFFERENTIAL EQUATIONS

For given functions $b : \mathbb{R} \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ and $\sigma : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ consider the stochastic differential equation

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dB^{(H)}(t), \quad t \in [0, T], \quad (3.1)$$

where the initial value $X(0) \in L^2(\mu_\phi)$ or the terminal value $X(T) \in L^2(\mu_\phi)$ is given. The Itô isometry for the stochastic integral becomes

$$\begin{aligned} \mathbb{E} \left(\int_0^T \sigma(t, X(t)) dB^{(H)}(t) \right)^2 &= \mathbb{E} \left(\int_0^T \int_0^T \sigma(t, X(t)) \sigma(s, X(s)) \phi(s, t) ds dt \right) \\ &\quad + \mathbb{E} \left\{ \left(\int_0^T \sigma'_x(s, X(s)) D_s^\phi X(s) ds \right)^2 \right\}. \end{aligned} \quad (3.2)$$

Because of the appearance of the term $D_s X(s)$ on the right-hand-side of the above identity, we may not directly apply the Picard iteration to solve (3.1).

In this section, we will solve the following quasi-linear stochastic differential equations using the theory developed in [HØ1], [HØ2]:

$$dX(t) = b(t, X(t))dt + (\sigma_t X(t) + a_t) dB^{(H)}(t), \quad (3.3)$$

where σ_t and a_t are given deterministic functions, $b(t, x) = b(t, x, \omega)$ is (almost surely) continuous with respect to t and x and globally Lipschitz continuous on x , the initial condition $X(0)$ or the terminal condition $X(T)$ is given. For simplicity we will discuss the case when $a_t = 0$ for all $t \in [0, T]$. Namely, we shall consider

$$dX(t) = b(t, X(t))dt + \sigma_t X(t)dB^{(H)}(t). \quad (3.4)$$

We need the following result, which is a fractional version of Gjessing's lemma (see e.g. Theorem 2.10.7 in [HØUZ]).

Lemma 3.1 *Let*

$$F = \exp^\diamond \left(\int_{\mathbb{R}_+} f(t) dB^{(H)}(t) \right) = \exp \left(\int_{\mathbb{R}_+} f(t) dB^{(H)}(t) - \frac{1}{2} \|f\|_\phi^2 \right),$$

where f is deterministic and such that

$$\|f\|_\phi^2 := \int_{\mathbb{R}_+^2} f(s)f(t)\phi(s,t)dsdt < \infty.$$

Then

$$F \diamond G = F \tau_{\hat{f}} G, \quad (3.5)$$

where \diamond is the Wick product defined in [HØ2] and \hat{f} is given by

$$\int_{\mathbb{R}_+^2} f(s)g(t)\phi(s,t)dsdt = \int_{\mathbb{R}_+} \hat{f}(s)g(s)ds \quad \forall g \in C_0^\infty(\mathbb{R}_+) \quad (3.6)$$

and

$$\tau_{\hat{f}} G(\omega) = G(\omega - \int_0^\cdot \hat{f}(s)ds)$$

Proof By [DHP, Theorem 3.1] it suffices to show the result in the case when

$$G(\omega) = \exp^\diamond \left(\int_{\mathbb{R}_+} g(t) dB^{(H)}(t) \right) = \exp^\diamond \langle \omega, g \rangle,$$

where g is deterministic and $\|g\|_\phi < \infty$. In this case we have

$$\begin{aligned} F \diamond G &= \exp^\diamond \left(\int_{\mathbb{R}_+} [f(t) + g(t)] dB^{(H)}(t) \right) \\ &= \exp \left(\int_{\mathbb{R}_+} [f(t) + g(t)] dB^{(H)}(t) - \frac{1}{2} \|f\|_\phi^2 - \frac{1}{2} \|g\|_\phi^2 - (f, g)_\phi \right), \end{aligned}$$

where

$$(f, g)_\phi = \int_{\mathbb{R}_+^2} f(s)g(t)\phi(s, t)dsdt.$$

But

$$\begin{aligned}\tau_{\hat{f}}G &= \exp^\diamond \left(\int_{\mathbb{R}_+} g(t)dB^{(H)}(t) - \int_{\mathbb{R}_+} \hat{f}(t)g(t)dt \right) \\ &= \exp^\diamond \left(\int_{\mathbb{R}_+} g(t)dB^{(H)}(t) - (f, g)_\phi \right).\end{aligned}$$

Hence

$$F\tau_{\hat{f}}G = \exp \left(\int_{\mathbb{R}_+} f(t)dB^{(H)}(t) - \frac{1}{2}\|f\|_\phi^2 + \int_{\mathbb{R}_+} g(t)dB^{(H)}(t) - \frac{1}{2}\|g\|_\phi^2 - (f, g)_\phi \right) = F \diamond G.$$

□

We now return to Equation (3.3). First let us solve the equation when $b = 0$ and with initial value $X(0)$ given. Namely, let us consider

$$dX(t) = -\sigma_t X(t)dB^{(H)}(t), \quad X(0) \text{ given.} \quad (3.7)$$

With the notion of Wick product, this equation can be written (see [HØ2, Def 3.11])

$$\dot{X}(t) = -\sigma_t X(t) \diamond W^{(H)}(t), \quad (3.8)$$

where $W^{(H)} = \dot{B}^{(H)}$ is the fractional white noise. Using the Wick calculus, we obtain

$$\begin{aligned}X(t) &= X(0) \diamond J_\sigma(t) \\ &:= X(0) \diamond \exp^\diamond \left(- \int_0^t \sigma_s W^{(H)}(s)ds \right) \\ &= X(0) \diamond \exp \left(- \int_0^t \sigma_s dB^{(H)}(s) - \frac{1}{2}\|\sigma\|_{\phi, t}^2 \right),\end{aligned} \quad (3.9)$$

where

$$\|\sigma\|_{\phi, t}^2 := \int_0^t \int_0^t \sigma_u \sigma_v \phi(u, v)du dv. \quad (3.10)$$

To solve Equation (3.4) we let

$$Y_t = X(t) \diamond J_\sigma(t). \quad (3.11)$$

This means

$$X(t) = Y_t \diamond \hat{J}_\sigma(t), \quad (3.12)$$

where

$$\hat{J}_\sigma(t) = J_{-\sigma}(t) = \exp \left(\int_0^t \sigma_s dB^{(H)}(s) - \frac{1}{2}\|\sigma\|_{\phi, t}^2 \right). \quad (3.13)$$

Thus we have

$$\begin{aligned}
\frac{dY_t}{dt} &= \frac{dX(t)}{dt} \diamond J_\sigma(t) + X(t) \diamond \frac{dJ_\sigma(t)}{dt} \\
&= \frac{dX(t)}{dt} \diamond J_\sigma(t) - \sigma_t J_\sigma(t) \diamond X(t) \diamond W^{(H)}(t) \\
&= J_\sigma(t) \diamond b(t, X(t), \omega) \\
&= J_\sigma(t) b(t, \tau_{-\hat{\sigma}} X(t), \omega + \int_0^\cdot \hat{\sigma}(s) ds),
\end{aligned}$$

where

$$\int_{\mathbb{R}_+^2} \sigma_s g(t) \phi(s, t) ds dt = \int_{\mathbb{R}_+} \hat{\sigma}_s g(s) ds \quad \forall g \in C_0^\infty(\mathbb{R}_+) \quad (3.14)$$

We are going to relate $\tau_{\hat{\sigma}} X(t)$ to Y_t .

$$\begin{aligned}
\tau_{-\hat{\sigma}} X_t(t, \omega) &= \tau_{-\hat{\sigma}} [J_{-\sigma}(t) \sigma \diamond Y_t(t, \omega)] \\
&= \tau_{-\hat{\sigma}} [J_{-\sigma}(t) \tau_{\hat{\sigma}} Y_t] \\
&= \tau_{-\hat{\sigma}} J_{-\sigma}(t) Y_t.
\end{aligned}$$

Since $\tau_{-\hat{\sigma}} J_{-\sigma}(t) = [J_{-\hat{\sigma}}(t)]^{-1}$, we obtain the equivalent equation of Y_t for (3.4):

$$\frac{dY_t}{dt} = J_{-\sigma}(t) b(t, [J_{-\sigma}(t)]^{-1} Y_t, \omega + \int_0^\cdot \hat{\sigma}(s) ds). \quad (3.15)$$

This is a deterministic equation. The initial value $X(0)$ is equivalent to initial value $Y_0 = X(0) \diamond J_{-\sigma}(0) = X(0)$. Thus we can solve the quasilinear equation with given initial value.

The terminal value $X(T)$ can also be transformed to the terminal value on $Y(T) = X(T) \diamond J_{-\sigma}(T)$. Thus the equation with given terminal value can be solved in a similar way. Note, however, that in this case the solution need not be $\mathcal{F}^{(H)}$ -adapted. (But see the next section).

Example 3.2 *Let us consider the case $b(t, x) = b_t x$ for some deterministic nice function b_t of t . This means that we are considering the linear stochastic differential equation:*

$$dX(t) = b_t X(t) dt + \sigma_t X(t) dB^{(H)}(t). \quad (3.16)$$

In this case it is easy to see that the equation satisfied by Y is

$$\dot{Y}_t = b(t) Y_t.$$

When the initial value is $Y(0) = x$ (constant), $x \in \mathbb{R}$, then

$$Y_t = x e^{\int_0^t b(s) ds}.$$

Thus we have the solution of (3.16) with $X(0) = x$

$$\begin{aligned}
X(t) &= Y(t) \diamond J_{-\sigma}(t) \\
&= x \exp \left\{ \int_0^t b(s) ds + \int_0^t \sigma_s dB^{(H)}(s) - \frac{1}{2} \|\sigma\|_{\phi, t}^2 \right\}.
\end{aligned} \quad (3.17)$$

If we assume the terminal value $X(T)$ given, then

$$\begin{aligned} Y(t) &= Y(T) e^{\int_t^T b(s) ds} \\ &= X(T) \diamond J_\sigma(T) e^{\int_t^T b(s) ds} \end{aligned}$$

Hence

$$\begin{aligned} X(t) &= Y(t) \diamond J_{-\sigma}(t) \\ &= X(T) \diamond \exp \left\{ \int_t^T b(s) ds - \int_t^T \sigma_s dB^{(H)}(s) - \frac{1}{2} \int_t^T \int_t^T \sigma(u) \sigma(v) \phi(u, v) du dv \right\}. \end{aligned} \quad (3.18)$$

4 FRACTIONAL BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS

Let $b : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a given function and let $F : \Omega \rightarrow \mathbb{R}$ be a given $\mathcal{F}_T^{(H)}$ -measurable random variable, where $T > 0$ is a constant. Consider the problem of finding $\mathcal{F}^{(H)}$ -adapted processes $p(t)$, $q(t)$ such that

$$dp(t) = b(t, p(t), q(t)) dt + q(t) dB^{(H)}(t); \quad t \in [0, T] \quad (4.1)$$

$$P(T) = F \quad \text{a.s.} \quad (4.2)$$

This is a *fractional backward stochastic differential equation* (FBSDE) in the two unknown processes $p(t)$ and $q(t)$. We will not discuss general theory for such equations here, but settle with a solution in a linear variant of (4.1)-(4.2), namely

$$dp(t) = [\alpha(t) + b_t p(t) + c_t q(t)] dt + q(t) dB^{(H)}(t); \quad t \in [0, T] \quad (4.3)$$

$$P(T) = F \quad \text{a.s.}, \quad (4.4)$$

where b_t and c_t are given continuous deterministic functions and $\alpha(t) = \alpha(t, \omega)$ is a given $\mathcal{F}^{(H)}$ -adapted process s.t. $\int_0^T |\alpha(t, \omega)| dt < \infty$ a.s.

To solve (4.3)-(4.4) we proceed as follows: By the fractional Girsanov theorem (see e.g. [HØ2, Theorem 3.18]) we can rewrite (4.3) as

$$dp(t) = [\alpha(t) + b_t p(t)] dt + q(t) d\hat{B}^{(H)}(t); \quad t \in [0, T] \quad (4.5)$$

where

$$\hat{B}^{(H)}(t) = B^{(H)}(t) + \int_0^t c_s ds \quad (4.6)$$

is a fractional Brownian motion (with Hurst parameter H) under the new probability measure $\hat{\mu}$ on $\mathcal{F}_T^{(H)}$ defined by

$$\frac{d\hat{\mu}(\omega)}{d\mu(\omega)} = \exp^\diamond \{ -\langle \omega, \hat{c} \rangle \} \quad (4.7)$$

where $\hat{c} = \hat{c}_t$ is the continuous function with $\text{supp } (\hat{c}) \subset [0, T]$ satisfying

$$\int_0^T \hat{c}_s \phi(s, t) ds = c_t; \quad 0 \leq t \leq T. \quad (4.8)$$

If we multiply (4.5) with the integrating factor

$$\beta_t := \exp\left(-\int_0^t b_s ds\right)$$

we get

$$d(\beta_s p(s)) = \beta_s \alpha(s) ds + \beta_s q(s) d\hat{B}^{(H)}(s) \quad (4.9)$$

or, by integrating (4.9) from $s = t$ to $s = T$,

$$\beta_T F = \beta_t p(t) + \int_t^T \beta_s \alpha(s) ds + \int_t^T \beta_s q(s) d\hat{B}^{(H)}(s). \quad (4.10)$$

Assume from now on that

$$\|\alpha\|_{\hat{\mathcal{L}}_\phi^{1,2}[0,T]} := \mathbb{E}_{\hat{\mu}} \left[\int_{[0,T] \times [0,T]} \alpha(s) \alpha(t) \phi(s, t) ds dt + \left(\int_0^T \hat{D}_s^\phi \alpha(s) ds \right)^2 \right] < \infty. \quad (4.11)$$

By the fractional Itô isometry (see [DHP, Theorem 3.7] or [HØS2, (1.10)]) applied to \hat{B} , $\hat{\mu}$ we then have

$$\mathbb{E}_{\hat{\mu}} \left[\left(\int_0^T \alpha(s) d\hat{B}^{(H)}(s) \right)^2 \right] = \|\alpha\|_{\hat{\mathcal{L}}_\phi^{1,2}[0,T]}^2. \quad (4.12)$$

From now on let us also assume that

$$\mathbb{E}_{\hat{\mu}} [F^2] < \infty. \quad (4.13)$$

We now apply the quasi-conditional expectation operator

$$\tilde{\mathbb{E}}_{\hat{\mu}} [\cdot | \mathcal{F}_t^{(H)}]$$

to both sides of (4.10) and get

$$\beta_T \tilde{\mathbb{E}}_{\hat{\mu}} [F | \mathcal{F}_t^{(H)}] = \beta_t p(t) + \int_t^T \beta_s \tilde{\mathbb{E}}_{\hat{\mu}} [\alpha(s) | \mathcal{F}_t^{(H)}] ds. \quad (4.14)$$

Here we have used that $p(t)$ is $\mathcal{F}_t^{(H)}$ -measurable, that the filtration $\hat{\mathcal{F}}_t^{(H)}$ generated by $\hat{B}^{(H)}(s); s \leq t$ is the same as $\mathcal{F}_t^{(H)}$, and that

$$\tilde{\mathbb{E}}_{\hat{\mu}} \left[\int_t^T f(s, \omega) d\hat{B}^{(H)}(s) | \hat{\mathcal{F}}_t^{(H)} \right] = 0, \quad \text{for all } t \leq T \quad (4.15)$$

for all $f \in \hat{\mathcal{L}}_\phi^{1,2}[0, T]$. See [HØ2, Def 4.9] and [HØS2, Lemma 1.1].

From (4.14) we get the solution

$$\begin{aligned} p(t) &= \exp \left(- \int_t^T b_s ds \right) \tilde{\mathbb{E}}_{\hat{\mu}} \left[F | \mathcal{F}_t^{(H)} \right] \\ &\quad + \int_t^T \exp \left(- \int_t^s b_r dr \right) \tilde{\mathbb{E}}_{\hat{\mu}} \left[\alpha(s) | \mathcal{F}_t^{(H)} \right] ds; \quad t \leq T. \end{aligned} \quad (4.16)$$

In particular, choosing $t = 0$ we get

$$p(0) = \exp \left(- \int_0^T b_s ds \right) \tilde{\mathbb{E}}_{\hat{\mu}} [F] + \int_0^T \exp \left(- \int_0^s b_r dr \right) \tilde{\mathbb{E}}_{\hat{\mu}} [\alpha(s)] ds. \quad (4.17)$$

Note that $p(0)$ is $\mathcal{F}_0^{(H)}$ -measurable and hence a constant. Choosing $t = 0$ in (4.10) we get

$$G = \int_0^T \beta_s q(s) d\hat{B}^{(H)}(s), \quad (4.18)$$

where

$$G = G(\omega) = \beta_T F(\omega) - \int_0^T \beta_s \alpha(s, \omega) ds - p(0), \quad (4.19)$$

with $p(0)$ given by (4.17).

By the fractional Clark-Ocone theorem [HØ1, Theorem 4.15 b)] applied to $\hat{B}^{(H)}$, $\hat{\mu}$ we have

$$G = \mathbb{E}_{\hat{\mu}}[G] + \int_0^T \tilde{\mathbb{E}}_{\hat{\mu}} \left[\hat{D}_s G | \hat{\mathcal{F}}_s^{(H)} \right] d\hat{B}^{(H)}(s), \quad (4.20)$$

where \hat{D} denotes the stochastic gradient at s with respect to $\hat{B}^{(H)}(\cdot)$. Comparing (4.18) and (4.20) we see that we can choose

$$q(t) = \exp \left(\int_0^t b_r dr \right) \tilde{\mathbb{E}}_{\hat{\mu}} \left[\hat{D}_t G | \mathcal{F}_t^{(H)} \right]. \quad (4.21)$$

We have proved the first part of the following result:

Theorem 4.1 *Assume that (4.11) and (4.13) hold. Then a solution $p(t)$, $q(t)$ of (4.3)-(4.4) is given by (4.16) and (4.21) respectively. The solution is unique among all $\mathcal{F}^{(H)}$ -adapted processes $p(\cdot)$, $q(\cdot) \in \hat{\mathcal{L}}_\phi^{1,2}[0, T]$.*

Proof It remains to prove uniqueness. The uniqueness of $p(\cdot)$ follows from the way we deduced formula (4.16) from (4.3)-(4.4). The uniqueness of q is deduced from (4.18) and (4.20) by the following argument: Substituting (4.20) from (4.18) and using that $\mathbb{E}_{\hat{\mu}}(G) = 0$ we get

$$0 = \int_0^T \left(\beta_s q(s) - \tilde{\mathbb{E}}_{\hat{\mu}} \left[\hat{D}_s G | \hat{\mathcal{F}}_s^{(H)} \right] \right) d\hat{B}^{(H)}(s).$$

Hence by the fractional Itô isometry (4.12)

$$\begin{aligned} 0 &= \mathbb{E}_{\hat{\mu}} \left[\left\{ \int_0^T \left(\beta_s q(s) - \tilde{\mathbb{E}}_{\hat{\mu}} \left[\hat{D}_s G | \hat{\mathcal{F}}_s^{(H)} \right] \right) d\hat{B}^{(H)}(s) \right\}^2 \right] \\ &= \|\beta_s q(s) - \tilde{\mathbb{E}}_{\hat{\mu}} [\hat{D}_s G | \hat{\mathcal{F}}_s^{(H)}]\|_{\hat{\mathcal{L}}_\phi^{1,2}[0, T]}^2, \end{aligned}$$

from which it follows that

$$\beta_s q(s) - \tilde{\mathbb{E}}_{\hat{\mu}} \left[\hat{D}_s G | \hat{\mathcal{F}}_s^{(H)} \right] = 0 \quad \text{for} \quad a.a.(s, \omega) \in [0, T] \times \Omega.$$

□

5 A STOCHASTIC MAXIMUM PRINCIPLE

We now apply the theory in the previous section to prove a maximum principle for systems driven by fractional Brownian motion. See e.g. [H], [P] and [YZ] and the references therein for more information about the maximum principle in the classical Brownian motion case.

Suppose $X(t) = X^{(u)}(t)$ is a controlled system of the form

$$dX(t) = b(t, X(t), u(t))dt + \sigma(t, X(t), u(t))dB^{(H)}(t); \quad X(0) = x \in \mathbb{R}^n \quad (5.1)$$

where $b : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$ and $\sigma : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^{n \times m}$ are given C^1 functions. The control process $u(\cdot) : [0, T] \times \Omega \rightarrow U \subset \mathbb{R}^k$ is assumed to be $\mathcal{F}^{(H)}$ -adapted. U is a given closed convex set in \mathbb{R}^k .

Let $f : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}$, $g : \mathbb{R}^n \rightarrow \mathbb{R}$ and $G : \mathbb{R}^n \rightarrow \mathbb{R}^N$ be given lower bounded C^1 functions and define the *performance functional* $J(u)$ by

$$J(u) = \mathbb{E} \left[\int_0^T f(t, X(t), u(t))dt + g(X(T)) \right] \quad (5.2)$$

and the *terminal condition* by

$$\mathbb{E} [G(X(T))] = 0. \quad (5.3)$$

Let \mathcal{A} denote the set of all $\mathcal{F}_t^{(H)}$ -adapted processes $u : [0, T] \times \Omega \rightarrow U$ such that $X^{(u)}(t)$ does not explode in $[0, T]$ and such that (5.3) holds. If $u \in \mathcal{A}$ and $X^{(u)}(t)$ is the corresponding state process we call $(u, X^{(u)})$ an *admissible pair*. Consider the problem to find \bar{J} and $\bar{u} \in \mathcal{A}$ such that

$$\bar{J} = \sup \{ J(u) ; u \in \mathcal{A} \} = J(\bar{u}). \quad (5.4)$$

If such $\bar{u} \in \mathcal{A}$ exists, then \bar{u} is called an *optimal control* and (\bar{u}, \bar{X}) , where $\bar{X} = X^{\bar{u}}$, is called an *optimal pair*.

Define the *Hamiltonian* $H : [0, T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$H(t, x, u, p, q) = f(t, x, u) + b(t, x, u)^T p + \sum_{i=1}^n \sum_{k=1}^m q_{ik}(t) \int_0^T \sigma_{ik}(s, x, u) \phi_{H_k}(s, t) ds. \quad (5.5)$$

Consider the following *fractional stochastic backward differential equation* in the pair of unknown $\mathcal{F}_t^{(H)}$ -adapted processes $p(t) \in \mathbb{R}^n$, $q(t) \in \mathbb{R}^{n \times m}$, called the *adjoint processes*:

$$\begin{cases} dp(t) = -H_x(t, X(t), u(t), p(t), q(t))dt + q(t)dB^{(H)}(t); & t \in [0, T] \\ p(T) = g_x(X(T)) + \lambda^T G_x(X(T)). \end{cases} \quad (5.6)$$

where $H_x = \nabla_x H = \left(\frac{\partial H}{\partial x_1}, \dots, \frac{\partial H}{\partial x_n} \right)^T$ is the gradient of H with respect to x and similarly with g_x and G_x . $X(t) = X^{(u)}(t)$ is the process obtained by using the control $u \in \mathcal{A}$ and $\lambda \in \mathbb{R}_+^N$ is a constant. The equation (5.6) is called the adjoint equation and $p(t)$ is sometimes interpreted as the *shadow price* (of a resource).

Theorem 5.1 (The fractional stochastic maximum principle) Suppose $\bar{u} \in \mathcal{A}$ and put $\bar{X} = X^{(\bar{u})}$. Let $p(t), q(t)$ be a solution of the corresponding adjoint equation (5.6) for some $\lambda \in \mathbb{R}_+^N$. Assume that the following, (5.7)-(5.9), hold:

$$H(t, \cdot, \cdot, p(t), q(t)), \quad g(\cdot) \quad \text{and} \quad G(\cdot) \quad \text{are concave, for all } t \in [0, T] \quad (5.7)$$

$$H(t, \bar{X}(t), \bar{u}(t), p(t), q(t)) = \max_{v \in U} H(t, \bar{X}(t), v, p(t), q(t)) \quad (5.8)$$

$$q(\cdot) \quad \text{or} \quad \sigma(\cdot, X(\cdot)) \quad \text{is deterministic.} \quad (5.9)$$

Then if $\lambda \in \mathbb{R}_+^N$ is such that (\bar{u}, \bar{X}) is admissible (i.e. (5.3) holds), the pair (\bar{u}, \bar{X}) is an optimal pair for problem (5.4).

Proof We first give a proof in the case when $G(x) = 0$, i.e. when there is no terminal condition.

With (\bar{u}, \bar{X}) as above consider

$$\begin{aligned} \Delta &:= \mathbb{E} \left[\int_0^T f(t, \bar{X}(t), \bar{u}(t)) dt - \int_0^T f(t, X(t), u(t)) dt \right] \\ &= \mathbb{E} \left[\int_0^T H(t, \bar{X}(t), \bar{u}(t), p(t), q(t)) dt - \int_0^T H(t, X(t), u(t), p(t), q(t)) dt \right] \\ &\quad - \mathbb{E} \left[\int_0^T \{b(t, \bar{X}(t), \bar{u}(t))\}^T p(t) dt - \int_0^T b(t, X(t), u(t))^T p(t) dt \right] \\ &\quad - \mathbb{E} \left[\int_0^T \int_0^T \sum_{i=1}^n \sum_{k=1}^m \{ \sigma_{ik}(s, \bar{X}(s), \bar{u}(s)) - \sigma_{ik}(s, X(s), u(s)) \} q_{ik}(t) \phi_{H_k}(s, t) ds dt \right] \\ &=: \Delta_1 + \Delta_2 + \Delta_3. \end{aligned} \quad (5.10)$$

Since $(x, u) \rightarrow H(x, u) = H(t, x, u, p, q)$ is concave we have

$$H(x, u) - H(\bar{x}, \bar{u}) \leq H_x(\bar{x}, \bar{u}) \cdot (x - \bar{x}) + H_u(\bar{x}, \bar{u}) \cdot (u - \bar{u})$$

for all $(x, u), (\bar{x}, \bar{u})$. Since $v \rightarrow H(\bar{X}(t), v)$ is maximal at $v = \bar{u}(t)$ we have

$$H_u(\bar{x}, \bar{u}) \cdot (u(t) - \bar{u}(t)) \leq 0 \quad \forall t.$$

Therefore

$$\begin{aligned} \Delta_1 &\geq \mathbb{E} \left[\int_0^T -H_x(t, \bar{X}(t), \bar{u}(t), p(t), q(t)) \cdot (X(t) - \bar{X}(t)) dt \right] \\ &= \mathbb{E} \left[\int_0^T (X(t) - \bar{X}(t))^T dp(t) - \int_0^T (X(t) - \bar{X}(t))^T q(t) dB^{(H)}(t) \right] \end{aligned}$$

Since $\mathbb{E} \left[\int_0^T (X(t) - \bar{X}(t))^T q(t) dB^{(H)}(t) \right] = 0$ by (2.6), this gives

$$\Delta_1 \geq \mathbb{E} \left[\int_0^T (X(t) - \bar{X}(t))^T dp(t) \right]. \quad (5.11)$$

By (5.1) we have

$$\begin{aligned} \Delta_2 &= -\mathbb{E} \left[\int_0^T \{b(t, \bar{X}(t), \bar{u}(t)) - b(t, X(t), u(t))\} \cdot p(t) dt \right] \\ &= -\mathbb{E} \left[\int_0^T p(t) (d\bar{X}(t) - dX(t)) \right] - \mathbb{E} \left[\int_0^T p(t)^T \{ \sigma(t, \bar{X}(t), \bar{u}(t)) - \sigma(t, X(t), u(t)) \} dB^{(H)}(t) \right] \\ &= \mathbb{E} \left[\int_0^T p(t) (dX(t) - d\bar{X}(t)) \right]. \end{aligned} \quad (5.12)$$

Finally, since g is concave we have

$$g(X(T)) - g(\bar{X}(T)) \leq g_x(\bar{X}(T)) \cdot (X(T) - \bar{X}(T)) \quad (5.13)$$

Combining (5.10)-(5.13) with Corollary 2.5 we get, using (5.2) and (5.6),

$$\begin{aligned} J(\bar{u}) - J(u) &= \Delta + \mathbb{E} [g(\bar{X}(T)) - g(X(T))] \\ &\geq \Delta + \mathbb{E} [g_x(\bar{X}(T)) \cdot (\bar{X}(T) - X(T))] \\ &\geq \Delta - \mathbb{E} [p(T) \cdot (X(T) - \bar{X}(T))] \\ &= \Delta - \left\{ \mathbb{E} \left[\int_0^T (X(t) - \bar{X}(t)) \cdot dp(t) \right] + \mathbb{E} \left[\int_0^T p(t) \cdot (dX(t) - d\bar{X}(t)) \right] \right. \\ &\quad \left. + \mathbb{E} \left[\int_0^T \int_0^T \sum_{i=1}^n \sum_{k=1}^m \{ \sigma_{ik}(s, X(s), u(s)) - \sigma_{ik}(s, \bar{X}(s), \bar{u}(s)) \} q_{ik}(t) \phi_{H_k}(s, t) ds dt \right] \right\} \\ &\geq \Delta - (\Delta_1 + \Delta_2 + \Delta_3) = 0. \end{aligned}$$

This shows that $J(\bar{u})$ is maximal among all admissible pairs $(u(\cdot), X(\cdot))$.

This completes the proof in the case with no terminal conditions ($G = 0$). Finally consider the general case with $G \neq 0$. Suppose that for some $\lambda_0 \in \mathbb{R}_+^N$ there exists \bar{u}_{λ_0} satisfying (5.7)-(5.9). Then by the above argument we know that if we put

$$J_{\lambda_0}(u) = \mathbb{E} \left[\int_0^T f(t, X(t), u(t)) dt + g(X(T)) + \lambda_0^T G(X(T)) \right]$$

then $J_{\lambda_0}(\bar{u}_0) \geq J_{\lambda_0}(u)$ for all controls u (without terminal condition). If λ_0 is such that \bar{u}_{λ_0} satisfies the terminal condition (i.e. $\bar{u}_{\lambda_0} \in \mathcal{A}$) and u is another control in \mathcal{A} then

$$J(\bar{u}_{\lambda_0}) = J_{\lambda_0}(\bar{u}_{\lambda_0}) \geq J_{\lambda_0}(u) = J(u)$$

and hence $\bar{u}_{\lambda_0} \in \mathcal{A}$ maximizes $J(u)$ over all $u \in \mathcal{A}$. □

6 APPLICATIONS: TWO OPTIMAL CONSUMPTION PROBLEMS

EXAMPLE 1 Suppose that the value of a firm at time t is given by ($\mu, \alpha \neq 0$ are constants)

$$dX(t) = (\mu X(t) - u(t)) dt + \alpha X(t) dB^{(H)}(t); X(0) = x, \quad (6.1)$$

where $u(t) \geq 0$ is the *consumption rate*. The problem is to maximize the total discounted expected utility of the consumption, given by

$$J(u) = \mathbb{E} \left[\int_0^T e^{-\delta t} \frac{u^\gamma(t)}{\gamma} dt \right], \quad (6.2)$$

where $\delta > 0$, $\gamma \in (0, 1)$ are constants ($1 - \gamma$ is the relative risk aversion) under the terminal condition

$$\mathbb{E}[X(T)] = x_T \in \mathbb{R}. \quad (6.3)$$

We solve this problem by applying the fractional stochastic maximum principle.

In this case the Hamiltonian (5.5) is

$$H(t, x, u, p, q) = e^{-\delta t} \frac{u^\gamma}{\gamma} + (\mu x - u)p + q(t)\alpha x \int_0^T \phi(s, t) ds \quad (6.4)$$

and the adjoint equation (5.6) becomes

$$\begin{cases} dp(t) = - \left\{ \mu p(t) + \alpha q(t) \int_0^T \phi(s, t) ds \right\} dt + q(t) dB^{(H)}(t); & t \in [0, T] \\ p(T) = \lambda \end{cases} \quad (6.5)$$

We see immediately that this equation has the (unique) solution

$$p(t) = \lambda e^{\mu(T-t)}, \quad q(t) = 0. \quad (6.6)$$

To find $\bar{u}(t)$ we maximize

$$v \rightarrow H(t, x, v, p, 0) = e^{-\delta t} \frac{v^\gamma}{\gamma} + (\mu x - v)p$$

over all $v \geq 0$ and get

$$\bar{u}(t) = \left(e^{\delta t} p(t) \right)^{\frac{1}{\gamma-1}}. \quad (6.7)$$

To determine λ (and hence $p(t)$) we put $u(t) = \bar{u}(t)$ in (6.1) and get, for $0 \leq t \leq T$,

$$\mathbb{E}[\bar{X}(t)] = x + \mu \int_0^t \mathbb{E}[\bar{X}(s)] ds - \int_0^t \bar{u}(s) ds.$$

This is a differential equation in $y(t) := \mathbb{E}[\bar{X}(t)]$. Solving this equation and using the terminal condition (6.3) we get

$$x_T = x e^{\mu T} - e^{\mu T} \lambda^{\frac{1}{\gamma-1}} \exp\left\{ \frac{\mu T}{\gamma-1} \right\} \int_0^T \exp\left\{ \frac{(\mu\gamma - \delta)s}{1-\gamma} \right\} ds$$

or

$$\lambda = \begin{cases} (xe^{\mu T} - x_T)^{\gamma-1} \left[\frac{1-\gamma}{\mu\gamma-\delta} \left\{ \exp\left(-\frac{\delta T}{1-\gamma}\right) - \exp\left(-\frac{\mu\gamma T}{1-\gamma}\right) \right\} \right]^{1-\gamma}; & \mu\gamma \neq \delta \\ (xe^{\mu T} - x_T)^{\gamma-1} T^{1-\gamma} \exp(-\delta T); & \mu\gamma = \delta \end{cases} \quad (6.8)$$

provided that

$$xe^{\mu T} \geq x_T. \quad (6.9)$$

We have proved

Theorem 6.1 *Assume that (6.9) holds. Then the consumption rate $\bar{u}(t)$ which maximizes (6.2) under the constraint (6.3) is given by*

$$\bar{u}(t) = \lambda^{\frac{1}{\gamma-1}} \exp \left\{ \frac{1}{1-\gamma} [(\mu - \delta)t - \mu T] \right\}, \quad (6.10)$$

where λ is given by (6.8).

Remark 6.2 *Note that optimal consumption rate $\bar{u}(t)$ in this model is independent of both the Hurst parameter H and the volatility σ . So in fact $\bar{u}(t)$ coincides with the optimal consumption rate in the deterministic case ($\sigma = 0$) in this example.*

EXAMPLE 2 In the model (6.1) used in Example 1 we are assuming that if there is no consumption then the value $X(t)$ at time t is the *fractional geometric Brownian motion* given by

$$X(t) = x \exp \left\{ \alpha B^{(H)}(t) + \mu t - \frac{1}{2} \alpha^2 t^{2H} \right\}, \quad (6.11)$$

which is the solution of (6.1) with $u = 0$ (see e.g. [HØ2, Example 3.14]). This is a natural choice of model from the point of stochastic differential equations because (6.1) is a natural fractional analogue of a well-known model in the standard Brownian motion case.

However, rather than taking the stochastic differential equation as the starting point we might choose a model where the value $Y(t)$ at time t has the form

$$Y(t) = x \exp \left\{ \alpha B^{(H)}(t) + \beta t \right\} \quad (6.12)$$

for some constants α and β . Such a choice is in agreement with claims that in finance the *logarithmic returns*

$$h_n := \log \frac{Y(t_n)}{Y(t_{n-1})} = \alpha \left(B^{(H)}(t_n) - B^{(H)}(t_{n-1}) \right) + \beta \Delta t$$

with $\Delta t = t_n - t_{n-1}$ behave like fractional Brownian motions with Hurst coefficient $H \in (1/2, 1)$. See e.g. [S, p.234].

If $Y(t)$ is given by (6.12), then by Itô's formula (Theorem 2.2) $Y(t)$ satisfies the stochastic differential equation

$$dY(t) = Y(t) \left(\beta + H e^{2H-1} \alpha^2 \right) dt + \alpha Y(t) dB^{(H)}(t). \quad (6.13)$$

The corresponding value $Y(t) = Y^{(u)}(t)$ when the consumption rate is u will hence satisfy the equation

$$\begin{cases} dY(t) = [Y(t) (\beta + H e^{2H-1} \alpha^2) - u(t)] dt + \alpha Y(t) dB^{(H)}(t), \\ Y(0) = y \end{cases} \quad (6.14)$$

With

$$J(u) = \mathbb{E} \left[\int_0^T e^{-\delta t} \frac{u^\gamma(t)}{\gamma} dt \right] \quad (6.15)$$

as in (6.2) and the terminal condition

$$\mathbb{E}[Y(T)] = y_T \in \mathbb{R}, \quad (6.16)$$

we now consider the problem of maximizing $J(u)$.

In this case the Hamiltonian (5.5) gets the form

$$H(t, y, u, p, q) = e^{-\delta t} \frac{u^\gamma}{\gamma} + [y(\beta + H t^{2H-1} \alpha^2) - u] p + \alpha y q \int_0^T \phi(s, t) ds \quad (6.17)$$

and the adjoint equation (5.6) becomes

$$\begin{cases} dp(t) = - \left\{ \beta + H t^{2H-1} \alpha^2 + \alpha q(t) \int_0^T \phi(s, t) ds \right\} dt + q(t) dB^{(H)}(t) \\ P(T) = \lambda \end{cases} \quad (6.18)$$

Again we see that we can choose $q = 0$ and this gives

$$p(t) = \lambda \exp \left\{ \beta(T - t) + \frac{1}{2} \alpha^2 (T^{2H} - t^{2H}) \right\}. \quad (6.19)$$

To find the optimal consumption rate $u^*(t)$ we maximize

$$v \rightarrow H(t, y, v, p, 0) = e^{-\delta t} \frac{v^\gamma}{\gamma} + [y(\beta + H t^{2H-1} \alpha^2) - v] p$$

over all $v \geq 0$ and get

$$u^*(t) = \left(e^{\delta t} p(t) \right)^{\frac{1}{\gamma-1}} = \lambda^{\frac{1}{\gamma-1}} \exp \left\{ -\frac{\beta T + (\delta - \beta)t}{1 - \gamma} - \frac{\alpha^2 (T^{2H} - t^{2H})}{2(1 - \gamma)} \right\}. \quad (6.20)$$

Substituting this value for $u(t)$ in (6.14) we get, with $y(t) = \mathbb{E}[Y(t)]$,

$$y'(t) = (\beta + H t^{2H-1} \alpha^2) y(t) - \mathbb{E}[u(t)]$$

which gives

$$\begin{aligned} y(t) &= y(0) \exp \left\{ \beta t + \frac{1}{2} \alpha^2 t^{2H} \right\} \\ &\quad - \int_0^t \exp \left\{ \beta(t - s) - \frac{1}{2} \alpha^2 (t^{2H} - s^{2H}) \right\} u^*(s) ds \end{aligned}$$

Combined with the terminal condition (6.16) this leads to

$$y_T = y(0) \exp \left\{ \beta T + \frac{1}{2} \alpha^2 T^{2H} \right\} - \lambda^{\frac{1}{\gamma-1}} \int_0^T \exp \left\{ \beta(T-s) + \frac{1}{2} \alpha^2 (T^{2H} - s^{2H}) - \frac{\beta T + (\delta - \beta)s}{1-\gamma} - \frac{\alpha^2 (T^{2H} - s^{2H})}{2(1-\gamma)} \right\} ds$$

or

$$\lambda = (y \exp \left\{ \beta T + \frac{1}{2} \alpha^2 T^{2H} \right\} - y_T)^{\gamma-1} \cdot \exp \left\{ -\beta \gamma T - \frac{1}{2} \alpha^2 \gamma T^{2H} \right\} \left[\int_0^T \exp \left\{ \frac{\beta \gamma - \delta}{1-\gamma} s - \frac{\gamma \alpha^2 s^{2H}}{2(1-\gamma)} \right\} ds \right]^{1-\gamma}, \quad (6.21)$$

provided that

$$y \exp \left\{ \beta T + \frac{1}{2} \alpha^2 T^{2H} \right\} \geq y_T. \quad (6.22)$$

We summarize this in the following:

Theorem 6.3 *Assume that (6.22) holds. Then the consumption rate $u^*(t)$ which maximizes (6.15) with the model (6.14) and the constraint (6.16) is given by*

$$u^*(t) = \lambda^{\frac{1}{\gamma-1}} \exp \left\{ \frac{1}{1-\gamma} \left[(\beta - \delta)t - \beta T - \frac{1}{2} \alpha^2 (T^{2H} - t^{2H}) \right] \right\},$$

where λ is given by (6.21).

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