A stochastic maximum principle for processes
driven by fractional Brownian motion

by

Y. Hu, B. Øksendal and A. Sulem
A STOCHASTIC MAXIMUM PRINCIPLE FOR
PROCESSES DRIVEN BY FRACTIONAL
BROWNIAN MOTION

Yaozhong Hu\(^1\), Bernt Øksendal\(^2,3\) and Agnès Sulem\(^4\)

October 18, 2000

1) Department of Mathematics, University of Kansas
405 Snow Hall, Lawrence, Kansas 66045-2142, USA
Email: hu@math.ukans.edu

2) Department of Mathematics, University of Oslo
Box 1053 Blindern, N-0316 Oslo, Norway
Email: oksendal@math.uio.no

3) Norwegian School of Economics and Business Administration,
Helleveien 30, N-5045 Bergen, Norway

4) INRIA, Domaine de Voluceau, Rocquencourt
B.P. 105, F-78153 Le Chesnay, Cedex, France
Email: agnes.sulem@inria.fr

Abstract

We prove a stochastic maximum principle for controlled processes \(X(t) = X^{(u)}(t)\) of the form

\[ dX(t) = b(t, X(t), u(t))dt + \sigma(t, X(t), u(t))dB^{(H)}(t) \]

where \(B^{(H)}(t)\) is \(n\)-dimensional fractional Brownian motion with Hurst parameter \(H = (H_1, \cdots, H_n) \in (1/2, 1)^n\). As an application we solve an optimal consumption problem with a terminal condition in an economy driven by a fractional Brownian motion.

1 INTRODUCTION

Let \(H = (H_1, \cdots, H_m)\) with \(1/2 < H_j < 1, j = 1, 2, \cdots, m\), and let \(B^{(H)}(t) = (B^{(H)}_1(t), \cdots, B^{(H)}_m(t))\), \(t \in \mathbb{R}\) be \(m\)-dimensional fractional Brownian motion, i.e. \(B^{(H)}(t) = B^{(H)}(t, \omega), (t, \omega) \in \mathbb{R} \times \Omega\) is a Gaussian process in \(\mathbb{R}^m\) such that

\[ \mathbb{E} \left[ B^{(H)}(t) \right] = B^{(H)}(0) = 0 \quad (1.1) \]

AMS 1991 subject classifications. Primary 60H05, 60H10; Secondary.
Key words and phrases: Stochastic maximum principle, stochastic control, fractional Brownian motion.

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and

\[ \mathbb{E} \left[ B_j^{(H)}(s)B_k^{(H)}(t) \right] = \frac{1}{2} \left\{ s^{2H_j} + t^{2H_j} - |t-s|^{2H_j} \right\} \delta_{jk}, \quad 1 \leq j, k \leq n, \quad s, t \in \mathbb{R}, \tag{1.2} \]

where

\[ \delta_{jk} = \begin{cases} 0 & \text{when } j \neq k \\ 1 & \text{when } j = k \end{cases} \]

Here \( \mathbb{E} = \mathbb{E}_\mu \) denotes the expectation with respect to the probability law \( \mu = \mu_\phi \) for \( B^{(H)}(\cdot) \). This means that the components \( B_1^{(H)}(\cdot), \ldots, B_m^{(H)}(\cdot) \) of \( B^{(H)}(\cdot) \) are \( m \) independent 1-dimensional fractional Brownian motions with Hurst parameter \( H_1, H_2, \ldots, H_m \), respectively. We refer to [MvN], [NVV] and [S] for more information about fractional Brownian motion. Because of its interesting properties (e.g. long range dependence and self-similarity of the components) \( B^{(H)}(t) \) has been suggested as a replacement of standard Brownian motion \( B(t) \) (corresponding to \( H_j = 1/2 \) for all \( j = 1, \ldots, m \)) in several stochastic models, including finance.

Unfortunately, \( B^{(H)}(\cdot) \) is neither a semimartingale nor a Markov process, so the powerful tools from the theories of such processes are not applicable when studying \( B^{(H)}(\cdot) \). Nevertheless, an efficient stochastic calculus of \( B^{(H)}(\cdot) \) can be developed. This calculus uses an Itô type of integration with respect to \( B^{(H)}(\cdot) \) and white noise theory. See [DHP] and [HØ2] for details. For applications to finance see [HØ2], [HØS1] [HØS2]. In [HØZ] and [OZ] the theory is extended to multi-parameter fractional Brownian fields \( B^{(H)}(x) ; x \in \mathbb{R}^d \) and applied to stochastic partial differential equations driven by such fractional white noise.

The purpose of this paper is to establish a stochastic maximum principle for stochastic control of processes driven by \( B^{(H)}(\cdot) \). We illustrate the result by applying it to a problem about optimal consumption in finance.

2 PRELIMINARIES

For the convenience of the reader we recall here some of the basic results of fractional Brownian motion calculus. Let \( B^{(H)}(t) \) be 1-dimensional in the following.

We let \( \int_{\mathbb{R}} \sigma(t, \omega) dB^{(H)}(t) \) denote the fractional Itô-integral of the process \( \sigma(t, \omega) \) with respect to \( B^{(H)}(t) \), as defined in [DHP]. In particular, this means that if \( \sigma \) belongs to the family \( S \) of step functions of the form

\[ \sigma(t, \omega) = \sum_{i=1}^N \sigma_i(\omega) \chi_{[t_i, t_{i+1})}(t), \quad (t, \omega) \in \mathbb{R} \times \Omega, \]

where \( 0 \leq t_1 < t_2 < \cdots < t_{N+1} \), then

\[ \int_{\mathbb{R}} \sigma(t, \omega) dB^{(H)}(t) = \sum_{i=1}^N \sigma_i(\omega) \circ \left( B^{(H)}(t_{i+1}) - B^{(H)}(t_i) \right), \tag{2.1} \]
where $\circ$ denotes the Wick product. For $\sigma(t) = \sigma(t, \omega) \in S$ we have
\[
E \left[ \int_{\mathbb{R}} \sigma(t, \omega) dB^{(H)}(t) \right]^2 = E \left[ \int_{\mathbb{R}_+^2} \sigma(s) \sigma(t) \phi(s, t) ds dt + \left( \int_{\mathbb{R}_+} D_s^\phi \sigma(s) ds \right)^2 \right], \tag{2.2}
\]
where $E = E_{\mu_H}$,
\[
\phi(s, t) = \phi_H(s, t) = H(2H - 1)|s - t|^{2H - 2} \tag{2.3}
\]
and $D_s^\phi$ denotes the Malliavin $\phi$-derivative at $s$ (see [DHP, Definition 3.1]). Using this we can extend the integral $\int_{\mathbb{R}} \sigma(t, \omega) dB^{(H)}(t)$ to the closure $\mathcal{L}_{\phi}^{1, 2} = \mathcal{L}_{\phi}^{1, 2}(\mathbb{R})$ of $S$ in the norm
\[
\|\sigma\|_{\mathcal{L}_{\phi}^{1, 2}}^2 = E \left[ \int_{\mathbb{R}_+^2} \sigma(s) \sigma(t) \phi(s, t) ds dt + \left( \int_{\mathbb{R}_+} D_s^\phi \sigma(s) ds \right)^2 \right]. \tag{2.4}
\]
This is in fact a Hilbert norm: If $\sigma, \theta \in \mathcal{L}_{\phi}^{1, 2}$, we have, by polarization,
\[
E \left[ \int_{\mathbb{R}} \sigma(t, \omega) dB^{(H)}(t) \int_{\mathbb{R}} \theta(t, \omega) dB^{(H)}(t) \right] = E \left[ \int_{\mathbb{R}_+^2} \sigma(s) \theta(t) \phi(s, t) ds dt + \left( \int_{\mathbb{R}_+} D_s^\phi \sigma(s) ds \int_{\mathbb{R}_+} D_s^\phi \theta(t) dt \right) \right]. \tag{2.5}
\]
We note that we need not assume that the integrand $\sigma \in \mathcal{L}_{\phi}^{1, 2}$ is adapted to the filtration $\mathcal{F}_t^{(H)}$ generated by $B^{(H)}(s, \cdot); s \leq t$.

An important property of this fractional Itô-integral is that
\[
E \left[ \int_{\mathbb{R}} \sigma(t, \omega) dB^{(H)}(t) \right] = 0 \quad \text{for all} \quad \sigma \in \mathcal{L}_{\phi}^{1, 2}. \tag{2.6}
\]
(see [DHP, Theorem 3.7]).

We give three versions of the fractional Itô formula, in increasing order of complexity.

**Theorem 2.1** ([DHP, Theorem 4.1]) Let $f \in C^2(\mathbb{R})$ with bounded derivatives. Then for $t \geq 0$
\[
f(B^{(H)}(t)) = f(B^{(H)}(0)) + \int_0^t f'(B^{(H)}(s)) dB^{(H)}(s) + H \int_0^t s^{2H - 1} f''(B^{(H)}(s)) ds. \tag{2.7}
\]

**Theorem 2.2** ([DHP, Theorem 4.3]) Let $X(t) = \int_0^t \sigma(s, \omega) dB^{(H)}(s)$, where $\sigma \in \mathcal{L}_{\phi}^{1, 2}$ and assume $f \in C^2(\mathbb{R}_+ \times \mathbb{R})$ with bounded derivatives. Then for $t \geq 0$
\[
f(t, X(t)) = f(0, 0) + \int_0^t \frac{\partial f}{\partial s}(s, X(s)) ds + \int_0^t \frac{\partial f}{\partial x}(s, X(s)) \sigma(s) dB^{(H)}(s) + \int_0^t \frac{\partial^2 f}{\partial x^2}(s, X(s)) \sigma(s) D_s^\phi X(s) ds. \tag{2.8}
\]

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Finally we give an $m$-dimensional version:

Let $B^{(H)}(t) = \left( B^{(H)}_1(t), \ldots, B^{(H)}_m(t) \right)$ be $m$-dimensional fractional Brownian motion with Hurst parameter $H = (H_1, \ldots, H_m) \in (1/2, 1)^m$, as in Section 1. Let $\sigma_{ij} \in \mathcal{L}^{1,2}_{\phi_{H_j}}$ for $1 \leq i \leq n, 1 \leq j \leq m$.

We can define $X(t) = (X_1(t), \ldots, X_n(t))$ where

$$X_i(t, \omega) = \sum_{j=1}^{m} \int_0^t \sigma_{ij}(s, \omega) dB^{(H)}_j(s) ; 1 \leq i \leq n.$$  \hspace{1cm} (2.9)

Then we have the following multi-dimensional fractional Itô formula:

**Theorem 2.3** Let $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n)$ with bounded derivatives. Then, for $t \geq 0$,

$$f(t, X(t)) = f(0, 0) + \int_0^t \frac{\partial f}{\partial s}(s, X(s)) ds + \int_0^t \sum_{i=1}^n \frac{\partial f}{\partial x_i}(s, X(s)) dX_i(s)$$

$$+ \int_0^t \left\{ \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(s, X(s)) \sum_{k=1}^m \sigma_{ik}(s) D_{k,i}^\phi(X_j(s)) \right\} ds$$ \hspace{1cm} (2.10)

$$= f(0, 0) + \int_0^t \frac{\partial f}{\partial s}(s, X(s)) ds + \sum_{j=1}^m \int_0^t \left[ \sum_{i=1}^n \frac{\partial f}{\partial x_i}(s, X(s)) \sigma_{ij}(s, \omega) \right] dB^{(H)}_j(s)$$

$$+ \int_0^t \text{Tr} \left[ \Lambda^T(s) f_{xx}(s, X(s)) \right] ds.$$ \hspace{1cm} (2.11)

Here $\Lambda = [\Lambda_{ij}] \in \mathbb{R}^{n \times m}$ with

$$\Lambda_{ij}(s) = \sum_{k=1}^m \sigma_{ik} D_{k,i}^\phi(X_j(s)) ; 1 \leq i \leq n, 1 \leq j \leq m,$$ \hspace{1cm} (2.12)

$$f_{xx} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_i \partial x_j} \end{bmatrix}_{1 \leq i, j \leq n}$$ \hspace{1cm} (2.13)

and $(\cdot)^T$ denotes matrix transposed, $\text{Tr}[:]$ denotes matrix trace.

Since we are here dealing with $m$ independent fractional Brownian motions we may regard $\Omega$ as the product of $m$ independent copies of $\Omega$ and write $\omega = (\omega_1, \cdots, \omega_m)$ for $\omega \in \Omega$. Then the notation $D_{k,s}^\phi Y$ in (2.10) and (2.12) means the Malliavin $\phi$-derivative with respect to $\omega_k$ and could also be written

$$D_{k,s}^\phi Y = \int_{\mathbb{R}} \phi_{H_k}(s, t) D_{k,t} Y dt = \int_{\mathbb{R}} \phi_{H_k}(s, t) \frac{\partial Y}{\partial \omega_k}(t, \omega) dt.$$  \hspace{1cm}

The following useful result is a multidimensional version of Theorem 4.2 in [DHP]:

**Theorem 2.4** Let

$$X(t) = \sum_{j=1}^m \int_0^t \sigma_j(r, \omega) dB^{(H)}_j(r) ; \sigma_j \in \mathcal{L}^{1,2}_{\phi_{H_j}} ; 1 \leq j \leq m.$$ \hspace{1cm} (2.14)
Then
\[ D^\phi_{k,s} X(t) = \sum_{j=1}^{m} \int_0^t D^\phi_{k,s} \sigma_j(r) dB_j^{(H)}(r) + \int_0^t \sigma_k(r) \phi_{H_k}(s,r) dr, \quad 1 \leq k \leq m. \] (2.15)

In particular, if \( \sigma_j(r) \) is deterministic for all \( j \in \{1, 2, \ldots, m\} \) then
\[ D^\phi_{k,s} X(t) = \int_0^t \sigma_k(r) \phi_{H_k}(s,r) dr. \] (2.16)

Now we have the following integration by parts formula.

**Corollary 2.5** Let \( X(t) \) and \( Y(t) \) be two processes of the form
\[ dX(t) = \mu(t, \omega) dt + \sigma(t, \omega) dB^{(H)}(t), \quad X(0) = x \in \mathbb{R}^n \]
and
\[ dY(t) = \nu(t, \omega) dt + \theta(t, \omega) dB^{(H)}(t), \quad Y(0) = y \in \mathbb{R}^n, \]
where \( \mu : \mathbb{R} \times \Omega \to \mathbb{R}^n, \nu : \mathbb{R} \times \Omega \to \mathbb{R}^n, \sigma : \mathbb{R} \times \Omega \to \mathbb{R}^{n \times m} \) and \( \theta : \mathbb{R} \times \Omega \to \mathbb{R}^{n \times m} \) are given processes with components \( \sigma_{ij}, \theta_{ij} \in L^1_{\phi_H} \) for \( 1 \leq i \leq n, 1 \leq j \leq m \) and \( B^{(H)}(\cdot) \) is \( m \)-dimensional.

Suppose that \( \sigma(\cdot) \) or \( \theta(\cdot) \) is deterministic. Then for \( T > 0, \)
\[ \mathbb{E} \left[ X(T) \cdot Y(T) \right] = x \cdot y + \mathbb{E} \left[ \int_0^T X(s) dY(s) \right] + \mathbb{E} \left[ \int_0^T Y(s) dX(s) \right] \]
\[ + \mathbb{E} \left[ \int_0^T \int_0^T \sum_{i=1}^n \sum_{k=1}^m \sigma_{ik}(s) \theta_{ik}(t) \phi_{H_k}(s,t) dsdt \right]. \] (2.17)

**Proof** This follows from Theorem 2.3 applied to the function \( f(t, x, y) = xy, \) combined with Theorem 2.4. \( \square \)

### 3 STOCHASTIC DIFFERENTIAL EQUATIONS

For given functions \( b : \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R} \) and \( \sigma : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) consider the stochastic differential equation
\[ dX(t) = b(t, X(t)) dt + \sigma(t, X(t)) dB^{(H)}(t), \quad t \in [0, T], \] (3.1)
where the initial value \( X(0) \in L^2(\mu_\phi) \) or the terminal value \( X(T) \in L^2(\mu_\phi) \) is given. The Itô isometry for the stochastic integral becomes
\[ \mathbb{E} \left( \int_0^T \sigma(t, X(t)) dB^{(H)}(t) \right)^2 = \mathbb{E} \left( \int_0^T \int_0^T \sigma(t, X(t)) \sigma(s, X(s)) \phi(s, t) dsdt \right) \]
\[ + \mathbb{E} \left\{ \left( \int_0^T \sigma_x(s, X(s)) D^\phi_{s,s} X(s) ds \right)^2 \right\}. \] (3.2)
Because of the appearance of the term $D_t X(s)$ on the right-hand-side of the above identity, we may not directly apply the Picard iteration to solve (3.1).

In this section, we will solve the following quasi-linear stochastic differential equations using the theory developed in [HÖ1], [HÖ2]:

$$dX(t) = b(t, X(t)) dt + (\sigma_t X(t) + a_t) dB^{(H)}(t),$$

(3.3)

where $\sigma_t$ and $a_t$ are given deterministic functions, $b(t, x) = b(t, x, \omega)$ is (almost surely) continuous with respect to $t$ and $x$ and globally Lipschitz continuous on $x$, the initial condition $X(0)$ or the terminal condition $X(T)$ is given. For simplicity we will discuss the case when $a_t = 0$ for all $t \in [0, T]$. Namely, we shall consider

$$dX(t) = b(t, X(t)) dt + \sigma_t X(t) dB^{(H)}(t).$$

(3.4)

We need the following result, which is a fractional version of Gjessing’s lemma (see e.g. Theorem 2.10.7 in [HÖUZ]).

**Lemma 3.1** Let

$$F = \exp^{\frac{1}{2}} \left( \int_{\mathbb{R}_+} f(t) dB^{(H)}(t) \right) = \exp \left( \int_{\mathbb{R}_+} f(t) dB^{(H)}(t) - \frac{1}{2} \|f\|_\phi^2 \right),$$

where $f$ is deterministic and such that

$$\|f\|_\phi^2 := \int_{\mathbb{R}_+^2} f(s) f(t) \phi(s, t) ds dt < \infty.$$

Then

$$F \circ G = F \tau_f G,$$

(3.5)

where $\circ$ is the Wick product defined in [HÖ2] and $\tau_f$ is given by

$$\int_{\mathbb{R}_+^2} f(s) g(t) \phi(s, t) ds dt = \int_{\mathbb{R}_+} \hat{f}(s) g(s) ds \quad \forall g \in C_0^\infty(\mathbb{R}_+)$$

(3.6)

and

$$\tau_f G(\omega) = G(\omega - \int_0^\cdot \hat{f}(s) ds).$$

**Proof** By [DHP, Theorem 3.1] it suffices to show the result in the case when

$$G(\omega) = \exp^{\frac{1}{2}} \left( \int_{\mathbb{R}_+} g(t) dB^{(H)}(t) \right) = \exp^{\frac{1}{2}} \langle \omega, g \rangle,$$

where $g$ is deterministic and $\|g\|_\phi < \infty$. In this case we have

$$F \circ G = \exp^{\frac{1}{2}} \left( \int_{\mathbb{R}_+} [f(t) + g(t)] dB^{(H)}(t) \right)$$

$$= \exp \left( \int_{\mathbb{R}_+} [f(t) + g(t)] dB^{(H)}(t) - \frac{1}{2} \|f\|_\phi^2 - \frac{1}{2} \|g\|_\phi^2 - \langle f, g \rangle_\phi \right),$$
where

$$(f, g)_\phi = \int_{\mathbb{R}^2_+} f(s)g(t)\phi(s,t)dsdt.$$  

But

$$\tau_f G = \exp \left( \int_{\mathbb{R}^2_+} g(t)dB^{(H)}(t) - \int_{\mathbb{R}^2_+} \hat{f}(t)g(t)dt \right)$$

$$= \exp \left( \int_{\mathbb{R}^2_+} g(t)dB^{(H)}(t) - (f, g)_\phi \right).$$  

Hence

$$F\tau_f G = \exp \left( \int_{\mathbb{R}^2_+} f(t)dB^{(H)}(t) - \frac{1}{2} \|f\|^2_\phi + \int_{\mathbb{R}^2_+} g(t)dB^{(H)}(t) - \frac{1}{2} \|g\|^2_\phi - (f, g)_\phi \right) = F \circ G.$$  

We now return to Equation (3.3). First let us solve the equation when $b = 0$ and with initial value $X(0)$ given. Namely, let us consider

$$dX(t) = -\sigma_t X(t)dB^{(H)}(t), \quad X(0) \text{ given}. \quad (3.7)$$

With the notion of Wick product, this equation can be written (see [HØ2, Def 3.11])

$$\dot{X}(t) = -\sigma_t X(t) \circ W^{(H)}(t), \quad (3.8)$$

where $W^{(H)} = \dot{B}^{(H)}$ is the fractional white noise. Using the Wick calculus, we obtain

$$X(t) = X(0) \circ J_\sigma(t)$$

$$= X(0) \circ \exp \left( -\int_0^t \sigma_s W^{(H)}(s)ds \right)$$

$$= X(0) \circ \exp \left( -\int_0^t \sigma_s dB^{(H)}(s) - \frac{1}{2} \|\sigma\|^2_{\phi,t} \right), \quad (3.9)$$

where

$$\|\sigma\|^2_{\phi,t} := \int_0^t \int_0^t \sigma_u \sigma_v \phi(u,v)du dv.$$  

To solve Equation (3.4) we let

$$Y_t = X(t) \circ J_\sigma(t). \quad (3.11)$$

This means

$$X(t) = Y_t \circ \dot{J}_\sigma(t), \quad (3.12)$$

where

$$\dot{J}_\sigma(t) = J_{-\sigma}(t) = \exp \left( \int_0^t \sigma_s dB^{(H)}(s) - \frac{1}{2} \|\sigma\|^2_{\phi,t} \right). \quad (3.13)$$

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Thus we have
\[
\frac{dY_t}{dt} = \frac{dX(t)}{dt} \circ J_\sigma(t) + X(t) \circ \frac{dJ_\sigma(t)}{dt}
\]
\[
= \frac{dX(t)}{dt} \circ J_\sigma(t) - \sigma_t J_\sigma(t) \circ X(t) \circ W^{(H)}(t)
\]
\[
= J_\sigma(t) \circ b(t, X(t), \omega)
\]
\[
= J_\sigma(t) b(t, \tau_{-\delta} X(t), \omega) + \int_0^\delta \dot{\sigma}(s) ds,
\]
where
\[
\int_{\mathbb{R}_+^2} \sigma_s g(t) \phi(s, t) ds dt = \int_{\mathbb{R}_+} \dot{\sigma}_s g(s) ds \quad \forall g \in C_0^\infty(\mathbb{R}_+)
\]  
(3.14)

We are going to relate \(\tau_{\delta} X(t)\) to \(Y_t\).
\[
\tau_{-\delta} X_t(t, \omega) = \tau_{-\delta} [J_{-\sigma}(t) \sigma \circ Y_t(t, \omega)]
\]
\[
= \tau_{-\delta} [J_{-\sigma}(t) \tau_{\delta} Y_t]
\]
\[
= \tau_{-\delta} J_{-\sigma}(t) Y_t.
\]
Since \(\tau_{-\delta} J_{-\sigma}(t) = [J_{-\delta}(t)]^{-1}\), we obtain the equivalent equation of \(Y_t\) for (3.4):
\[
\frac{dY_t}{dt} = J_{-\sigma}(t) b(t, [J_{-\sigma}(t)]^{-1} Y_t, \omega) + \int_0^\delta \dot{\sigma}(s) ds.
\]  
(3.15)

This is a deterministic equation. The initial value \(X(0)\) is equivalent to initial value \(Y_0 = X(0) \circ J_{-\sigma}(0) = X(0)\). Thus we can solve the quasilinear equation with given initial value.

The terminal value \(X(T)\) can also be transformed to the terminal value on \(Y(T) = X(T) \circ J_{-\sigma}(T)\). Thus the equation with given terminal value can be solved in a similar way. Note, however, that in this case the solution need not be \(\mathcal{F}^{(H)}\)-adapted. (But see the next section).

**Example 3.2** Let us consider the case \(b(t, x) = b_t x\) for some deterministic nice function \(b_t\) of \(t\).

This means that we are considering the linear stochastic differential equation:
\[
dx(t) = b_t X(t) dt + \sigma_t X(t) dB^{(H)}(t).
\]  
(3.16)

In this case it is easy to see that the equation satisfied by \(Y\) is
\[
\dot{Y}_t = b(t) Y_t.
\]

When the initial value is \(Y(0) = x\) (constant), \(x \in \mathbb{R}\), then
\[
Y_t = x e^{\int_0^t b(s) ds}.
\]

Thus we have the solution of (3.16) with \(X(0) = x\)
\[
X(t) = Y(t) \circ J_{-\sigma}(t)
\]
\[
= x \exp \left\{ \int_0^t b(s) ds + \int_0^t \sigma_s dB^{(H)}(s) - \frac{1}{2} \|\sigma\|^2_{\phi, t} \right\}.
\]  
(3.17)
If we assume the terminal value $X(T)$ given, then

$$ Y(t) = Y(T) e^{\int_t^T b(s) ds} $$

$$ = X(T) \cdot J_{-\sigma}(T) e^{\int_t^T b(s) ds} $$

Hence

$$ X(t) = Y(t) \cdot J_{-\sigma}(t) $$

$$ = X(T) \cdot \exp \left\{ \int_t^T b(s) ds - \int_t^T \sigma(s) dB^{(H)}(s) - \frac{1}{2} \int_t^T \int_t^T \sigma(u) \sigma(v) \phi(u,v) dudv \right\}. $$

(3.18)

4 FRACTIONAL BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS

Let $b : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a given function and let $F : \Omega \rightarrow \mathbb{R}$ be a given $\mathcal{F}_T^{(H)}$-measurable random variable, where $T > 0$ is a constant. Consider the problem of finding $\mathcal{F}^{(H)}$-adapted processes $p(t)$, $q(t)$ such that

$$ dp(t) = b(t, p(t), q(t)) dt + q(t) dB^{(H)}(t); \quad t \in [0, T] $$

(4.1)

$$ P(T) = F \quad \text{a.s.} $$

(4.2)

This is a fractional backward stochastic differential equation (FBSDE) in the two unknown processes $p(t)$ and $q(t)$. We will not discuss general theory for such equations here, but settle with a solution in a linear variant of (4.1)-(4.2), namely

$$ dp(t) = [\alpha(t) + b_t p(t) + c_t q(t)] dt + q(t) dB^{(H)}(t); \quad t \in [0, T] $$

(4.3)

$$ P(T) = F \quad \text{a.s.}, $$

(4.4)

where $b_t$ and $c_t$ are given continuous deterministic functions and $\alpha(t) = \alpha(t, \omega)$ is a given $\mathcal{F}^{(H)}$-adapted process s.t. $\int_0^T |\alpha(t, \omega)| dt < \infty$ a.s.

To solve (4.3)-(4.4) we proceed as follows: By the fractional Girsanov theorem (see e.g. [HØ2, Theorem 3.18]) we can rewrite (4.3) as

$$ dp(t) = [\alpha(t) + b_t p(t)] dt + q(t) d\hat{B}^{(H)}(t); \quad t \in [0, T] $$

(4.5)

where

$$ \hat{B}^{(H)}(t) = B^{(H)}(t) + \int_0^t c_s ds $$

(4.6)

is a fractional Brownian motion (with Hurst parameter $H$) under the new probability measure $\hat{\mu}$ on $\mathcal{F}_T^{(H)}$ defined by

$$ \frac{d\hat{\mu}(\omega)}{d\mu(\omega)} = \exp \{ -\langle \omega, \tilde{\xi} \rangle \} $$

(4.7)
where $\hat{c} = \hat{c}_t$ is the continuous function with supp $(\hat{c}) \subset [0, T]$ satisfying

$$
\int_0^T \hat{c}_s \phi(s, t) ds = \alpha_t; \quad 0 \leq t \leq T.
$$

(4.8)

If we multiply (4.5) with the integrating factor

$$
\beta_t := \exp\left(-\int_0^t b_s ds\right)
$$

we get

$$
d(\beta_t p(s)) = \beta_s \alpha(s) ds + \beta_s q(s) d\hat{B}^{(H)}(s)
$$

or, by integrating (4.9) from $s = t$ to $s = T$,

$$
\beta_T F = \beta_t p(t) + \int_t^T \beta_s \alpha(s) ds + \int_t^T \beta_s q(s) d\hat{B}^{(H)}(s).
$$

(4.10)

Assume from now on that

$$
\|\alpha\|_{L^2_{\phi}[0, T]}^2 := \mathbb{E}_{\hat{\mu}}\left[\int_{[0, T] \times [0, T]} \alpha(s)\alpha(t) \phi(s, t) ds dt + \left(\int_0^T \hat{D}_{s, \phi}^\alpha \alpha(s) ds\right)^2\right] < \infty.
$$

(4.11)

By the fractional Itô isometry (see [DHP, Theorem 3.7] or [HÖS2, (1.10)]) applied to $\hat{B}$, $\hat{\mu}$ we then have

$$
\mathbb{E}_{\hat{\mu}}\left[\left(\int_0^T \alpha(s) d\hat{B}^{(H)}(s)\right)^2\right] = \|\alpha\|_{L^2_{\phi}[0, T]}^2.
$$

(4.12)

From now on let us also assume that

$$
\mathbb{E}_{\hat{\mu}}[F^2] < \infty.
$$

(4.13)

We now apply the quasi-conditional expectation operator

$$
\mathbb{E}_{\hat{\mu}}[\cdot | \mathcal{F}_{t}^{(H)}]
$$

to both sides of (4.10) and get

$$
\beta_T \mathbb{E}_{\hat{\mu}}[F | \mathcal{F}_{t}^{(H)}] = \beta_t p(t) + \int_t^T \beta_s \mathbb{E}_{\hat{\mu}}[\alpha(s) | \mathcal{F}_{t}^{(H)}] ds.
$$

(4.14)

Here we have used that $p(t)$ is $\mathcal{F}_{t}^{(H)}$-measurable, that the filtration $\mathcal{F}_{t}^{(H)}$ generated by $\hat{B}^{(H)}(s); s \leq t$ is the same as $\mathcal{F}_{t}^{(H)}$, and that

$$
\mathbb{E}_{\hat{\mu}}\left[\int_t^T f(s, \omega) d\hat{B}^{(H)}(s) | \mathcal{F}_{t}^{(H)}\right] = 0, \quad \text{for all} \quad t \leq T
$$

(4.15)

for all $f \in \hat{L}^{1,2}_{\phi}[0, T]$. See [HÖ2, Def 4.9] and [HÖS2, Lemma 1.1].
From (4.14) we get the solution
\[
p(t) = \exp \left( - \int_t^T b_s \, ds \right) \tilde{E}_{\tilde{\mu}} \left[ F | \mathcal{F}_t^{(H)} \right] + \int_t^T \exp \left( - \int_t^s b_r \, dr \right) \tilde{E}_{\tilde{\mu}} \left[ \alpha(s) | \mathcal{F}_t^{(H)} \right] \, ds; \quad t \leq T.
\] (4.16)

In particular, choosing \( t = 0 \) we get
\[
p(0) = \exp \left( - \int_0^T b_s \, ds \right) \tilde{E}_{\tilde{\mu}} \left[ F \right] + \int_0^T \exp \left( - \int_0^s b_r \, dr \right) \tilde{E}_{\tilde{\mu}} \left[ \alpha(s) \right] \, ds.
\] (4.17)
Note that \( p(0) \) is \( \mathcal{F}_0^{(H)} \)-measurable and hence a constant. Choosing \( t = 0 \) in (4.10) we get
\[
G = \int_0^T \beta_s q(s) \, dB^{(H)}(s),
\] (4.18)
where
\[
G = G(\omega) = \beta_T F(\omega) - \int_0^T \beta_s \alpha(s, \omega) \, ds - p(0),
\] (4.19)
with \( p(0) \) given by (4.17).

By the fractional Clark-Ocone theorem [HØ1, Theorem 4.15 b)] applied to \( \tilde{B}^{(H)} \), \( \tilde{\mu} \) we have
\[
G = \tilde{E}_{\tilde{\mu}} \left[ G \right] + \int_0^T \tilde{E}_{\tilde{\mu}} \left[ \tilde{D}_s G | \mathcal{F}_s^{(H)} \right] \, dB^{(H)}(s),
\] (4.20)
where \( \tilde{D} \) denotes the stochastic gradient at \( s \) with respect to \( \tilde{B}^{(H)}(\cdot) \). Comparing (4.18) and (4.20) we see that we can choose
\[
q(t) = \exp \left( \int_0^t b_r \, dr \right) \tilde{E}_{\tilde{\mu}} \left[ \tilde{D}_t G | \mathcal{F}_t^{(H)} \right].
\] (4.21)
We have proved the first part of the following result:

**Theorem 4.1** Assume that (4.11) and (4.13) hold. Then a solution \( p(t), q(t) \) of (4.3)-(4.4) is given by (4.16) and (4.21) respectively. The solution is unique among all \( \mathcal{F}^{(H)} \)-adapted processes \( p(\cdot), q(\cdot) \in \tilde{L}_{\phi}^{1,2}[0, T] \).

**Proof** It remains to prove uniqueness. The uniqueness of \( p(\cdot) \) follows from the way we deduced formula (4.16) from (4.3)-(4.4). The uniqueness of \( q \) is deduced from (4.18) and (4.20) by the following argument: Substituting (4.20) from (4.18) and using that \( \tilde{E}_{\tilde{\mu}}(G) = 0 \) we get
\[
0 = \int_0^T \left( \beta_s q(s) - \tilde{E}_{\tilde{\mu}} \left[ \tilde{D}_s G | \mathcal{F}_s^{(H)} \right] \right) \, dB^{(H)}(s).
\]
Hence by the fractional Itô isometry (4.12)
\[
0 = \tilde{E}_{\tilde{\mu}} \left[ \left( \int_0^T \beta_s q(s) - \tilde{E}_{\tilde{\mu}} \left[ \tilde{D}_s G | \mathcal{F}_s^{(H)} \right] \right) \, dB^{(H)}(s) \right]^2
= \| \beta_s q(s) - \tilde{E}_{\tilde{\mu}} \left[ \tilde{D}_s G | \mathcal{F}_s^{(H)} \right] \|^2_{\tilde{L}_{\phi}^{1,2}[0, T]},
\]

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from which it follows that
\[ \beta_s q(s) - \mathbb{E}_\mu \left[ \hat{\mathcal{J}}_s \right] = 0 \quad \text{for } a.a. (s, \omega) \in [0, T] \times \Omega. \]

\[ \square \]

5 A STOCHASTIC MAXIMUM PRINCIPLE

We now apply the theory in the previous section to prove a maximum principle for systems driven by fractional Brownian motion. See e.g. \([H], [P]\) and \([YZ]\) and the references therein for more information about the maximum principle in the classical Brownian motion case.

Suppose \( X(t) = X^{(u)}(t) \) is a controlled system of the form
\[ dX(t) = b(t, X(t), u(t))dt + \sigma(t, X(t), u(t))dB^{(H)}(t); \quad X(0) = x \in \mathbb{R}^n \]
where \( b : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n \) and \( \sigma : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^{n \times m} \) are given \( C^1 \) functions. The control process \( u(\cdot) : [0, T] \times \Omega \rightarrow U \subset \mathbb{R}^k \) is assumed to be \( \mathcal{F}^{(H)} \)-adapted. \( U \) is a given closed convex set in \( \mathbb{R}^k \).

Let \( f : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R} \), \( g : \mathbb{R}^n \rightarrow \mathbb{R} \) and \( G : \mathbb{R}^n \rightarrow \mathbb{R}^N \) be given lower bounded \( C^1 \) functions and define the performance functional \( J(u) \) by
\[ J(u) = \mathbb{E} \left[ \int_0^T f(t, X(t), u(t))dt + g(X(T)) \right] \]
and the terminal condition by
\[ \mathbb{E} [G(X(T))] = 0. \]

Let \( \mathcal{A} \) denote the set of all \( \mathcal{F}^{(H)}_t \)-adapted processes \( u : [0, T] \times \Omega \rightarrow U \) such that \( X^{(u)}(t) \) does not explode in \( [0, T] \) and such that (5.3) holds. If \( u \in \mathcal{A} \) and \( X^{(u)}(t) \) is the corresponding state process we call \((u, X^{(u)})\) an admissible pair. Consider the problem to find \( \bar{J} \) and \( \bar{u} \in \mathcal{A} \) such that
\[ \bar{J} = \sup \{ J(u); u \in \mathcal{A} \} = J(\bar{u}). \]

If such \( \bar{u} \in \mathcal{A} \) exists, then \( \bar{u} \) is called an optimal control and \((\bar{u}, \bar{X})\), where \( \bar{X} = X^{\bar{u}} \), is called an optimal pair.

Define the Hamiltonian \( H : [0, T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \rightarrow \mathbb{R} \) by
\[ H(t, x, u, p, q) = f(t, x, u) + b(t, x, u)^T \sigma(t) + \sum_{i=1}^{m} \sum_{k=1}^{n} q_{ik}(t) \phi_{H_k}(s, t)ds. \]

Consider the following fractional stochastic backward differential equation in the pair of unknown \( \mathcal{F}^{(H)}_t \)-adapted processes \( p(t) \in \mathbb{R}^n \), \( q(t) \in \mathbb{R}^{n \times m} \), called the adjoint processes:
\[ \begin{cases} dp(t) = -H_x(t, X(t), u(t), p(t), q(t))dt + q(t)dB^{(H)}(t); & t \in [0, T] \\ p(T) = g_X(X(T)) + \lambda^T G_x(X(T)). & \end{cases} \]
where \( H_x = \nabla_x H = \left( \frac{\partial H}{\partial x_1}, \ldots, \frac{\partial H}{\partial x_n} \right)^T \) is the gradient of \( H \) with respect to \( x \) and similarly with \( g_x \) and \( G_x \). \( X(t) = X^{(u)}(t) \) is the process obtained by using the control \( u \in \mathcal{A} \) and \( \lambda \in \mathbb{R}_+^N \) is a constant. The equation (5.6) is called the adjoint equation and \( p(t) \) is sometimes interpreted as the shadow price (of a resource).

**Theorem 5.1 (The fractional stochastic maximum principle)** Suppose \( \bar{u} \in \mathcal{A} \) and put \( \bar{X} = X^{(u)} \). Let \( p(t), q(t) \) be a solution of the corresponding adjoint equation (5.6) for some \( \lambda \in \mathbb{R}_+^N \). Assume that the following, (5.7)-(5.9), hold:

\[
H(t, \cdot, p(t), q(t)), \ g(\cdot) \text{ and } G(\cdot) \text{ are concave, for all } t \in [0, T] \tag{5.7}
\]

\[
H(t, \bar{X}(t), \bar{u}(t), p(t), q(t)) = \max_{v \in \mathcal{U}} H(t, \bar{X}(t), v, p(t), q(t)) \tag{5.8}
\]

\[
q(\cdot) \text{ or } \sigma(\cdot, X(\cdot)) \text{ is deterministic.} \tag{5.9}
\]

Then if \( \lambda \in \mathbb{R}_+^N \) is such that \( (\bar{u}, \bar{X}) \) is admissible (i.e. (5.3) holds), the pair \( (\bar{u}, \bar{X}) \) is an optimal pair for problem (5.4).

**Proof** We first give a proof in the case when \( G(x) = 0 \), i.e. when there is no terminal condition.

With \( (\bar{u}, \bar{X}) \) as above consider

\[
\Delta := \mathbb{E} \left[ \int_0^T f(t, \bar{X}(t), \bar{u}(t))dt - \int_0^T f(t, X(t), u(t))dt \right]
\]

\[
= \mathbb{E} \left[ \int_0^T H(t, \bar{X}(t), \bar{u}(t), p(t), q(t))dt - \int_0^T H(t, X(t), u(t), p(t), q(t))dt \right]
\]

\[
- \mathbb{E} \left[ \int_0^T \{b(t, \bar{X}(t), \bar{u}(t))^T p(t)dt - \int_0^T b(t, X(t), u(t))^T p(t)dt \right]
\]

\[
- \mathbb{E} \left[ \int_0^T \int_0^T \sum_{i=1}^n \sum_{k=1}^m \{\sigma_{ik}(s, \bar{X}(s), \bar{u}(s)) - \sigma_{ik}(s, X(s), u(s))\} q_{ik}(t) \phi_{H_k}(s, t)dsdt \right]
\]

\[
=: \Delta_1 + \Delta_2 + \Delta_3. \tag{5.10}
\]

Since \( (x, u) \rightarrow H(x, u) = H(t, x, u, p, q) \) is concave we have

\[
H(x, u) - H(\bar{x}, \bar{u}) \leq H_x(\bar{x}, \bar{u}) \cdot (x - \bar{x}) + H_u(\bar{x}, \bar{u}) \cdot (u - \bar{u})
\]

for all \( (x, u), (\bar{x}, \bar{u}) \). Since \( v \rightarrow H(\bar{X}(t), v) \) is maximal at \( v = \bar{u}(t) \) we have

\[
H_u(\bar{x}, \bar{u}) \cdot (u(t) - \bar{u}(t)) \leq 0 \quad \forall t.
\]

Therefore

\[
\Delta_1 \geq \mathbb{E} \left[ \int_0^T -H_x(t, \bar{X}(t), \bar{u}(t), p(t), q(t)) \cdot (X(t) - \bar{X}(t))dt \right]
\]

\[
= \mathbb{E} \left[ \int_0^T (X(t) - \bar{X}(t))^T dp(t) - \int_0^T (X(t) - \bar{X}(t))^T q(t)dB^H(t) \right]
\]
Since $\mathbb{E} \left[ \int_0^T (X(t) - \bar{X}(t))^T q(t) dB^{(H)}(t) \right] = 0$ by (2.6), this gives
\begin{equation}
\Delta_1 \geq \mathbb{E} \left[ \int_0^T (X(t) - \bar{X}(t))^T dp(t) \right]. \tag{5.11}
\end{equation}

By (5.1) we have
\begin{align*}
\Delta_2 &= -\mathbb{E} \left[ \int_0^T \left\{ b(t, \bar{X}(t), \bar{u}(t)) - b(t, X(t), u(t)) \right\} \cdot p(t) dt \right] \\
&= -\mathbb{E} \left[ \int_0^T p(t) \left( d\bar{X}(t) - dX(t) \right) \right] - \mathbb{E} \left[ \int_0^T p(t)^T \left\{ \sigma(t, \bar{X}(t), \bar{u}(t)) - \sigma(t, X(t), u(t)) \right\} dB^{(H)}(t) \right] \\
&= \mathbb{E} \left[ \int_0^T p(t) \left( dX(t) - d\bar{X}(t) \right) \right]. \tag{5.12}
\end{align*}

Finally, since $g$ is concave we have
\begin{equation}
g(X(T)) - g(\bar{X}(T)) \leq g_{\bar{u}}(\bar{X}(T)) \cdot (X(T) - \bar{X}(T)) \tag{5.13}
\end{equation}

Combining (5.10)-(5.13) with Corollary 2.5 we get, using (5.2) and (5.6),
\begin{align*}
J(\bar{u}) - J(u) &= \Delta + \mathbb{E} \left[ g(\bar{X}(T)) - g(X(T)) \right] \\
&\geq \Delta + \mathbb{E} \left[ g_{\bar{u}}(\bar{X}(T)) \cdot (X(T) - \bar{X}(T)) \right] \\
&\geq \Delta - \mathbb{E} \left[ p(T) \cdot (X(T) - \bar{X}(T)) \right] \\
&= \Delta - \left\{ \mathbb{E} \left[ \int_0^T (X(t) - \bar{X}(t)) \cdot dp(t) \right] + \mathbb{E} \left[ \int_0^T p(t) \cdot (dX(t) - d\bar{X}(t)) \right] \\
&\quad + \mathbb{E} \left[ \int_0^T \int_0^T \sum_{i=1}^n \sum_{k=1}^m \left( \sigma_{ik}(s, X(s), u(s)) - \sigma_{ik}(s, \bar{X}(s), \bar{u}(s)) \right) q_{ik}(t) \phi_{H_k}(s, t) ds dt \right] \right\} \\
&\geq \Delta - (\Delta_1 + \Delta_2 + \Delta_3) = 0.
\end{align*}

This shows that $J(\bar{u})$ is maximal among all admissible pairs $(u(\cdot), X(\cdot))$.

This completes the proof in the case with no terminal conditions ($G = 0$). Finally consider the general case with $G \neq 0$. Suppose that for some $\lambda_0 \in \mathbb{R}_+^N$ there exists $\bar{u}_{\lambda_0}$ satisfying (5.7)-(5.9). Then by the above argument we know that if we put
\begin{equation}
J_{\lambda_0}(u) = \mathbb{E} \left[ \int_0^T f(t, X(t), u(t)) dt + g(X(T)) + \lambda_0^T G(X(T)) \right]
\end{equation}
then $J_{\lambda_0}(\bar{u}_{\lambda_0}) \geq J_{\lambda_0}(u)$ for all controls $u$ (without terminal condition). If $\lambda_0$ is such that $\bar{u}_{\lambda_0}$ satisfies the terminal condition (i.e. $\bar{u}_{\lambda_0} \in \mathcal{A}$) and $u$ is another control in $\mathcal{A}$ then
\begin{equation}
J(\bar{u}_{\lambda_0}) = J_{\lambda_0}(\bar{u}_{\lambda_0}) \geq J_{\lambda_0}(u) = J(u)
\end{equation}
and hence $\bar{u}_{\lambda_0} \in \mathcal{A}$ maximizes $J(u)$ over all $u \in \mathcal{A}$. \qed
6 APPLICATIONS: TWO OPTIMAL CONSUMPTION PROBLEMS

EXAMPLE 1 Suppose that the value of a firm at time $t$ is given by ($\mu, \alpha \neq 0$ are constants)

$$dX(t) = (\mu X(t) - u(t)) \, dt + \alpha X(t) dB^{(H)}(t) ; \ X(0) = x, \quad (6.1)$$

where $u(t) \geq 0$ is the consumption rate. The problem is to maximize the total discounted expected utility of the consumption, given by

$$J(u) = \mathbb{E} \left[ \int_0^T e^{-\delta t} \frac{u^\gamma(t)}{\gamma} \, dt \right], \quad (6.2)$$

where $\delta > 0, \gamma \in (0,1)$ are constants ($1 - \gamma$ is the relative risk aversion) under the terminal condition

$$\mathbb{E}[X(T)] = x_T \in \mathbb{R}. \quad (6.3)$$

We solve this problem by applying the fractional stochastic maximum principle.

In this case the Hamiltonian (5.5) is

$$H(t, x, u, p, q) = e^{-\delta t} \frac{u^\gamma}{\gamma} + (\mu x - u)p + q(t)\alpha x \int_0^T \phi(s, t)ds \quad (6.4)$$

and the adjoint equation (5.6) becomes

$$\begin{cases}
\dot{p}(t) = -\left\{ \mu p(t) + \alpha q(t) \int_0^T \phi(s, t)ds \right\} dt + q(t)dB^{(H)}(t) ; \quad t \in [0, T], \\
p(T) = \lambda
\end{cases} \quad (6.5)$$

We see immediately that this equation has the (unique) solution

$$p(t) = \lambda e^{\mu(T-t)}, \quad q(t) = 0. \quad (6.6)$$

To find $\bar{u}(t)$ we maximize

$$v \rightarrow H(t, x, v, p, 0) = e^{-\delta t} \frac{v^\gamma}{\gamma} + (\mu x - v)p$$

over all $v \geq 0$ and get

$$\bar{u}(t) = \left( e^{\delta t} p(t) \right)^{\frac{1}{\gamma-1}}. \quad (6.7)$$

To determine $\lambda$ (and hence $p(t)$) we put $u(t) = \bar{u}(t)$ in (6.1) and get, for $0 \leq t \leq T$,

$$\mathbb{E}[X(t)] = x + \mu \int_0^t \mathbb{E}[X(s)]ds - \int_0^t \bar{u}(s)ds.$$

This is a differential equation in $y(t) := \mathbb{E}[X(t)]$. Solving this equation and using the terminal condition (6.3) we get

$$x_T = x e^{\mu T} - e^{\mu T} \lambda^{\frac{1}{\gamma-1}} \exp\left\{ \frac{\mu T}{\gamma - 1} \right\} \int_0^T \exp\left\{ \frac{\mu(\gamma - 3)}{1 - \gamma} s \right\} ds$$
or

\[
\lambda = \begin{cases} 
(x e^{\mu T} - x_T)^{\gamma - 1} \left[ \frac{1 - \gamma}{\mu - \delta} \left\{ \exp \left( -\frac{\delta T}{1 - \gamma} \right) - \exp \left( -\frac{\mu T}{1 - \gamma} \right) \right\} \right]^{1 - \gamma} & ; \quad \mu \gamma \neq \delta \\
(x e^{\mu T} - x_T)^{\gamma - 1} \exp(-\delta T) & ; \quad \mu \gamma = \delta
\end{cases}
\] (6.8)

provided that

\[
x e^{\mu T} \geq x_T.
\] (6.9)

We have proved

**Theorem 6.1** Assume that (6.9) holds. Then the consumption rate \( \bar{u}(t) \) which maximizes (6.2) under the constraint (6.3) is given by

\[
\bar{u}(t) = \lambda^{\frac{1}{1 - \gamma}} \exp \left\{ \frac{1}{1 - \gamma} \{(\mu - \delta)t - \mu T\} \right\},
\] (6.10)

where \( \lambda \) is given by (6.8).

**Remark 6.2** Note that optimal consumption rate \( \bar{u}(t) \) in this model is independent of both the Hurst parameter \( H \) and the volatility \( \sigma \). So in fact \( \bar{u}(t) \) coincides with the optimal consumption rate in the deterministic case (\( \sigma = 0 \)) in this example.

**EXAMPLE 2** In the model (6.1) used in Example 1 we are assuming that if there is no consumption then the value \( X(t) \) at time \( t \) is the fractional geometric Brownian motion given by

\[
X(t) = x \exp \left\{ \alpha B^{(H)}(t) + \mu t - \frac{1}{2} \alpha^2 t^{2H} \right\},
\] (6.11)

which is the solution of (6.1) with \( u = 0 \) (see e.g. [HØ2, Example 3.14]). This is a natural choice of model from the point of stochastic differential equations because (6.1) is a natural fractional analogue of a well-known model in the standard Brownian motion case.

However, rather than taking the stochastic differential equation as the starting point we might choose a model where the value \( Y(t) \) at time \( t \) has the form

\[
Y(t) = x \exp \left\{ \alpha B^{(H)}(t) + \beta t \right\}
\] (6.12)

for some constants \( \alpha \) and \( \beta \). Such a choice is in agreement with claims that in finance the logarithmic returns

\[
h_n := \log \frac{Y(t_n)}{Y(t_{n-1})} = \alpha \left( B^{(H)}(t_n) - B^{(H)}(t_{n-1}) + \beta \Delta t \right)
\]

with \( \Delta t = t_n - t_{n-1} \) behave like fractional Brownian motions with Hurst coefficient \( H \in (1/2, 1) \). See e.g. [S, p.234].

If \( Y(t) \) is given by (6.12), then by Itô's formula (Theorem 2.2) \( Y(t) \) satisfies the stochastic differential equation

\[
dY(t) = Y(t) \left( \beta + H e^{2H-1} \alpha^2 \right) dt + \alpha Y(t) dB^{(H)}(t).
\] (6.13)
The corresponding value $Y(t) = Y^{(u)}(t)$ when the consumption rate is $u$ will hence satisfy the equation
\[
\begin{align*}
\begin{cases}
  dY(t) = \left[ Y(t) \left( \beta + H e^{2H-1} \alpha^2 \right) - u(t) \right] dt + \alpha Y(t) dB^{(H)}(t), \\
  Y(0) = y
\end{cases}
\end{align*}
\] (6.14)

With
\[
J(u) = E \left[ \int_0^T e^{-\delta t} Y(t) dt \right]
\] (6.15)
as in (6.2) and the terminal condition
\[
E[Y(T)] = y_T \in \mathbb{R},
\] (6.16)
we now consider the problem of maximizing $J(u)$.

In this case the Hamiltonian (5.5) gets the form
\[
H(t, y, u, p, q) = e^{-\delta t} \frac{u^\gamma}{\gamma} + \left[ y(\beta + H t^{2H-1} \alpha^2) - u \right] p + \alpha q \int_0^T \phi(s, t) ds
\] (6.17)
and the adjoint equation (5.6) becomes
\[
\begin{align*}
\begin{cases}
  dp(t) = - \left\{ \beta + H t^{2H-1} \alpha^2 + \alpha q(t) \int_0^T \phi(s, t) ds \right\} dt + q(t) dB^{(H)}(t), \\
  P(T) = \lambda
\end{cases}
\end{align*}
\] (6.18)

Again we see that we can choose $q = 0$ and this gives
\[
p(t) = \lambda \exp \left\{ \beta (T - t) + \frac{1}{2} \alpha^2 (T^{2H} - t^{2H}) \right\}.
\] (6.19)

To find the optimal consumption rate $u^*(t)$ we maximize
\[
v \rightarrow H(t, y, v, p, 0) = e^{-\delta t} \frac{v^\gamma}{\gamma} + \left[ y(\beta + H t^{2H-1} \alpha^2) - v \right] p
\]
over all $v \geq 0$ and get
\[
u^*(t) = \left( e^{\delta t} p(t) \right)^{\frac{1}{\gamma - 1}} = \lambda^{\frac{1}{\gamma - 1}} \exp \left\{ - \frac{\beta T + (\delta - \beta) t}{1 - \gamma} - \frac{\alpha^2 (T^{2H} - t^{2H})}{2(1 - \gamma)} \right\}.
\] (6.20)

Substituting this value for $u(t)$ is (6.14) we get, with $y(t) = E[Y(t)]$,
\[
y'(t) = (\beta + H t^{2H-1} \alpha^2) y(t) - E[u(t)]
\]
which gives
\[
y(t) = y(0) \exp \left\{ \beta t + \frac{1}{2} \alpha^2 t^{2H} \right\} \]
\[
- \int_0^t \exp \left\{ \beta (t - s) - \frac{1}{2} \alpha^2 (t^{2H} - s^{2H}) \right\} u^*(s) ds
\]
Combined with the terminal condition (6.16) this leads to

\[
y_T = y(0) \exp \left\{ \beta T + \frac{1}{2} \alpha^2 T^{2H} \right\} \\
- \lambda^{\frac{1}{\gamma - 1}} \int_0^T \exp \left\{ \beta(T - s) + \frac{1}{2} \alpha^2 (T^{2H} - s^{2H}) - \frac{\beta T + (\delta - \beta) s}{1 - \gamma} - \frac{\alpha^2 (T^{2H} - s^{2H})}{2(1 - \gamma)} \right\} \, ds
\]

or

\[
\lambda = (y \exp \left\{ \beta T + \frac{1}{2} \alpha^2 T^{2H} \right\} - y_T)^{\gamma - 1} \\
\cdot \exp \left\{ -\beta \gamma T - \frac{1}{2} \alpha^2 \gamma T^{2H} \right\} \left[ \int_0^T \exp \left\{ \frac{\beta \gamma - \delta}{1 - \gamma} s - \frac{\gamma \alpha^2 s^{2H}}{2(1 - \gamma)} \right\} \, ds \right]^{1 - \gamma},
\]

provided that

\[
y \exp \left\{ \beta T + \frac{1}{2} \alpha^2 T^{2H} \right\} \geq y_T. \tag{6.22}
\]

We summarize this in the following:

**Theorem 6.3** Assume that (6.22) holds. Then the consumption rate \( u^*(t) \) which maximizes (6.15) with the model (6.14) and the constraint (6.16) is given by

\[
u^*(t) = \lambda^{\frac{1}{\gamma - 1}} \exp \left\{ \frac{1}{1 - \gamma} \left[ (\beta - \delta) t - \beta T - \frac{1}{2} \alpha^2 (T^{2H} - t^{2H}) \right] \right\},
\]

where \( \lambda \) is given by (6.21).

**Acknowledgments.** This work is partially supported by the French-Norwegian cooperation project Stochastic Control and Applications, Aur 99-050. Y. Hu is partially supported by the National Science Foundation under Grant No. EPS-9874732 and matching support from the State of Kansas.

We are grateful to Fred Espen Benth and Nils Christian Framstad for helpful comments.

**References**


