ON ALGEBRAIC K–THEORY SPECTRA

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ABSTRACT. We identify the two–homotopy type of the algebraic K–theory spectra of rings of integers in non–exceptional two–regular number fields and of two–local number fields which contain a primitive fourth root of unity.

1. Results

Let $A$ be a unital ring. The homotopy groups of its algebraic K–theory spectrum $K(A)$ can be very hard to pin down, and an even harder task is to determine its homotopy type. The best results in this direction are due to Dwyer and Mitchell [DM]. In loc. cit. we can find an explicit description of the homotopy type of a certain localization of the algebraic K–theory spectrum at odd primes for rings of arithmetic type. The Quillen–Lichtenbaum conjecture predicts that their results give an identification of the homotopy type of the algebraic K–theory spectrum itself. Let $\ell$ be a prime number. Non–conjectural results on the homotopy type of the algebraic K–theory spectrum of the $\ell$–adic integers have been obtained by Bökstedt and Madsen for $\ell$ odd in [BM] and by Rognes for $\ell$ even in [R1]. The main results of this section address the two–homotopy type of the algebraic K–theory spectra of rings of integers in non–exceptional two–regular number fields and of two–local number fields which contain a primitive fourth root of unity.

A field $k$ is called non–exceptional if the Galois group $\text{Gal}(k \to k(\zeta_{2^n}))$ is cyclic for all $n$. No real number fields are non–exceptional. Let $F$ be a non–exceptional number field with ring of integers $O_F$ and ring of two–integers $R_F$. Let $r_2$ be the number of pairs of complex embeddings of $F$. In the following we adopt the notation from [DM]. Write $F_\infty$ for the direct limit of the fields $F(\mu_{2^n})$ obtained by adjoining two–primary roots of unity. We let $R_{F_n}$ signify the ring of two–integers in $F_n$. Let $\Gamma'_F$ denote the Galois group of the extension $F \to F_\infty$. The natural action of $\Gamma'_F$ on the group of two–primary roots of unity $\mu_{2^n}$ gives an embedding $c : \Gamma'_F \to \text{Aut}(\mu_{2^n})$. We fix a prime ideal $\wp$ of $R_F$ such that $\wp$ remains prime in $R_{n}$ for all $n \geq 0$. The ideal $\wp$ comes with a natural projection map $\pi_{\wp}$ from $R_F$ to its residue field $F[\wp]$. Čebotarev's density theorem guarantees the existence of an infinite number of such primes. The inertness property of $\wp$ is equivalent to the condition that the number of elements in $F[\wp]$ is a topological generator of $c(\Gamma'_F)$. The latter is topological cyclic since $F$ is assumed to be non–exceptional.

Recall from [GJ] that a number field is called two–regular if the 2–Sylow subgroup of its modified tame kernel is trivial. An equivalent characterization is that the


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rational prime ideal \( (2) \) does not split in the field, and that the narrow Picard group of the two-integers in the field has odd order. Finally we let \( ku \) denote the \((-1)\)-connected cover of the complex topological \( K \)-theory spectrum.

**Theorem 1.1.** Let \( F \) be a non-exceptional two-regular number field. Then the algebraic \( K \)-theory spectrum of its ring of two-integers has the two-homotopy type of the spectrum

\[
\bigvee \Sigma ku \vee K(F[\wp]).
\]

In the next result we use the standard notation \( \hat{U} \) for the infinite unitary group, and let \( \Omega^0_\infty X \) denote the basepoint component of the zeroth space of a spectrum \( X \). Theorem 1.1 is in fact a consequence of

**Proposition 1.2.** Let \( F \) be a non-exceptional two-regular number field. Then

\[
\prod_{\ell \not= 2} U \longrightarrow \Omega^\infty_0 K(R_F) \longrightarrow \Omega^\infty_0 K(F[\wp])
\]

is a split fibre sequence of spaces after two-adic completion. The map \( \Omega^\infty_0 K(R_F) \to \Omega^\infty_0 K(F[\wp]) \) is induced by the natural projection map.

Dwyer and Mitchell stated in [DM] an explicit conjectural calculation of the mod \( \ell \not= 2 \) group cohomology of the infinite general linear group \( GL(O_F) \), see also [Mi1] and [Mi2]. The following non-conjectural calculation which appears implicitly in the proof of Proposition 1.2 should be of interest in view of the comments on p.3 in [DM]. We write \( H^*(-) \) for the mod 2 cohomology group \( H^*(-; \mathbb{Z}/2) \), and \( SU \) for the special infinite unitary group.

**Corollary 1.3.** Let \( F \) be a non-exceptional two-regular number field. Then the mod 2 cohomology of \( BGL(O_F) \) is given by the formula:

\[
H^*(BGL(O_F)) \cong H^*(BGL(F[\wp])) \otimes H^*(SU) \otimes H^*(U)^{2-1}
\]

We call a degree \( n \) extension of the two-adic numbers a two-local number field of degree \( n \). Next we record the two-homotopy type of the algebraic \( K \)-spectrum of a two-local number field that contains a primitive fourth root of unity. We fix a prime number \( p \) such that its image in \( \text{Aut}(\mu_{2^n}) \) is a topological generator of \( c(\Gamma_E^\infty) \) (defined in the same way as for \( F \)). Let \( \mathbb{F}_p \) be the field of cardinality \( p \).

**Theorem 1.4.** Let \( E \) be a two-local number field of degree \( n \). Assume \( E \) contains a primitive fourth root of unity. Then the algebraic \( K \)-theory spectrum of \( E \) has the two-homotopy type of the spectrum

\[
\bigvee_{\ell} \Sigma ku \vee K(\mathbb{F}_p) \vee \Sigma K(\mathbb{F}_p).
\]

This result is not very surprising given Theorem 13.3 in [DF]. In fact, the proof of Theorem 1.4 is just a combination of Theorem 13.3 in loc. cit. and the following result which we may call étale descent for two-primary algebraic \( K \)-theory of two-local number fields.

Proof. A verbatim copy of the argument for étale descent for totally imaginary number fields in [RW2]. □

Method of proof. Next we explain in some detail how we derive Theorem 1.1. The assumption that $F$ is two–regular gives us control over the étale cohomology groups of $R_F$ with mod 2 coefficients. In result, the techniques developed by Dwyer and Friedlander in [DF1] give us a model for the two–adic étale topological $K$–theory space of $R_F$. Anton has carried out a similar analysis in [A1] and [A2]. By étale descent for $K$–theory of totally imaginary number fields from [RW2], we conclude that we have actually found a model for the two–adic algebraic $K$–theory space of $R_F$. To get the spectrum statement we employ the machinery of the $\mathcal{K}$–localization functor developed by Bousfield in [Bo]. Here we write $\mathcal{K}$ for the complex topological $K$–theory spectrum.

We mention that the two–primary algebraic $K$–groups of two–regular number fields were calculated in [RØ]. A typical example of such a field is $\mathbb{Q}(\zeta_{2^\nu})$. During the writing of this paper we learned that Hodgkin has also constructed a model for the two–adic algebraic $K$–theory space of $\mathbb{Z}[\zeta_{2^\nu}]$, and computed its topological $K$–theory. See reference [Ho] for details. I would like to thank Bill Dwyer and John Rognes for valuable help and comments.

2. Proofs

First we will introduce some notation, and explain briefly some constructions. Recall from [AM] that a pro–object in a category $C$ is a functor from a small left filtering category to $C$. The étale homotopy type functor as defined by Friedlander in [Fr] is a functor $(-)_{\text{ét}}$ from the category of locally Noetherian simplicial schemes to the category of pro–simplicial sets. In particular, given a Noetherian ring $R$ we may consider $\text{Spec}(R)$ as a simplicial scheme and hence also its étale homotopy type $\text{Spec}(R)_{\text{ét}}$. As examples of étale homotopy types we mention the following which we will need later. If $k$ is a field, then $\text{Spec}(k)_{\text{ét}}$ is of type $K(G_k,1)$ where $G_k$ denotes the absolute Galois group of $k$. Here $K(G_k,1)$ denotes the Eilenberg–Mac Lane space with its single homotopy group $G_k$ concentrated in degree one. If $A$ is a complete local ring with residue field $k$, then the naturally induced map $\text{Spec}(k)_{\text{ét}} \to \text{Spec}(A)_{\text{ét}}$ is an equivalence. In particular we have the homotopy equivalences:

$$* \to \text{Spec}(\mathbb{C})_{\text{ét}} \text{ and } \text{Spec}(\mathbb{F}_p)_{\text{ét}} \to \text{Spec}(\mathbb{Z}_p)_{\text{ét}}$$

Of utmost importance for us; the natural map

$$(2.1) \quad S^1 \to \text{Spec}(\mathbb{F})_{\text{ét}}$$

sending the generator of $\pi_1(S^1)$ to the Frobenius element of $\pi_1 \text{Spec}(\mathbb{F})_{\text{ét}}$ induces an isomorphism on mod $\ell$ cohomology if the finite field $\mathbb{F}$ has characteristic different from $\ell$.

Let $R = \mathbb{Z}[\frac{1}{\ell}]$ be the integers with $\ell$ inverted, and let $A$ be a finitely generated $R$–algebra. To construct the étale topological $K$–theory space of $A$ one can proceed
as follows, cf. [DF1] and [DF3]. Let $\text{GL}_{n,R}$ be the group scheme over $\text{Spec}(R)_{\text{et}}$ defined by $\text{GL}_{n,R} = \text{Spec} \frac{R[x_{i,j}, t]}{\det(x_{i,j}t) = 1}$ for $i, j = 1, \ldots, n$. Applying the bar construction determines a simplicial scheme $\text{BGL}_{n,R}$ with étale homotopy type $(\text{BGL}_{n,R})_{\text{et}}$. Let $\text{Hom}^0_{L}(\text{Spec}(A)_{\text{et}}, (\text{BGL}_{n,R})_{\text{et}})_{R_{\text{et}}}$ denote the connected component of the space of relative $\ell$-adic functions, i.e., the function complex of maps over $\text{Spec}(R)_{\text{et}}$ from $\text{Spec}(A)_{\text{et}}$ to the pro-space fibrewise mod $\ell$-completion over $R_{\text{et}}$ of $(\text{BGL}_{n,R})_{\text{et}}$, see Definition 2.2 in [DF1] for a precise statement. Next one defines the (connected) $\ell$-adic étale topological $K$-theory space $K^{\text{et}}(A)$ of $A$ as

$$\text{colim} \text{Hom}^0_{L}(\text{Spec}(A)_{\text{et}}, (\text{BGL}_{n,R})_{\text{et}})_{R_{\text{et}}}.$$ 

The $n$th $\ell$-adic étale topological $K$-group of $A$ is then defined as the $n$th homotopy group of $K^{\text{et}}(A)$.

Assume $X \to \text{Spec}(R)_{\text{et}}$ is a map of pro–spaces where $X$ is of finite mod $\ell$ cohomological dimension. Dwyer and Friedlander introduced in [DF2] the space $\tilde{K}^{\text{top}}(X)$ as the identity component of $\text{colim} \text{Hom}_{L}(X_{\text{ét}}, (\text{BGL}_{n,R})_{\text{et}})_{R_{\text{ét}}}$. By identity component one refers to the component which contains the composite map $X \to \text{Spec}(R)_{\text{et}} \to (\text{BGL}_{n,R})_{\text{et}}$ induced by the section $\text{Spec}(R) \to \text{BGL}_{n,R}$ corresponding to the identity matrix in $\text{GL}_n(R)$. Moreover, a map from $X$ to $\text{Spec}(A)_{\text{et}}$ induces a map $K^{\text{et}}(A) \to \tilde{K}^{\text{top}}(X)$.

**Proposition 2.2.** (Dwyer–Friedlander) Let $X$ be as above, and let $\ell = 2$. Assume $X \to A$ induces an isomorphism in mod 2 cohomology. Then the induced map $K^{\text{et}}(A) \to \tilde{K}^{\text{top}}(X)$ is a homotopy equivalence.

**Proof.** See Corollary 3.3 in [DF2] for a more general result. \qed

**Proof of Proposition 1.2.** The assumption that $F$ is totally imaginary and two–regular amounts to the calculation

$$H^{q}_{\text{ét}}(R_F; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & \text{for } q = 0, \\ (\mathbb{Z}/2)^{r_2+1} & \text{for } q = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

First we produce a pro–space $X$ along with a map to $\text{Spec}(R_F)_{\text{et}}$ that induces an isomorphism on $H^*(-; \mathbb{Z}/2)$. We claim the existence of a map $\beta^*$ with source $\bigvee^{r_2} S^1$ and target $\text{Spec}(R_F)_{\text{et}}$ which combined with the naturally induced map $\pi^{\ast}_{p} : \text{Spec}(F[p])_{\text{et}} \to \text{Spec}(R_F)_{\text{et}}$ delivers a mod 2 cohomology equivalence:

$$\text{Spec}(F[p])_{\text{et}} \bigvee^{r_2} S^1 \xrightarrow{\pi^{\ast}_{p} \vee \beta^{*}} \text{Spec}(R_F)_{\text{et}} \tag{2.4}$$

To verify this claim we detour the arguments in Propositions 4.2 and 4.5 in [DF2]. From (2.3) we can choose a map $\beta$ from the wedge of $r_2+1$ copies of a circle to $\text{Spec}(R_F)_{\text{et}}$ that induces an isomorphism on mod 2 cohomology. Recall that the two–integers $R_F$ comes with a natural structure map

$$\text{Spec}(R_F)_{\text{et}} \to K(\text{Aut}(\mu_{2^{\infty}}), 1)$$

which `classifies’ étale extensions of $R_F$. On $\pi_1$ the map is given by the action of the Galois group of the maximal unramified extension of $R_F$ on the group $\text{Aut}(\mu_{2^{\infty}})$.
of two–primary roots of unity. The prime ideal $\mathfrak{p}$ is chosen in such a way that the image (under the natural map) of its Frobenius in $\pi_1 \text{Spec}(R_F)_{\text{ét}}$ generates that action.

We claim the map $\beta$ can be chosen in such a way that under the composite map

$$\pi_1(\bigvee_{r_2+1} S^1) \to \pi_1 \text{Spec}(R_F)_{\text{ét}} \to \pi_1 K(\text{Aut}(\mu_{2^{2\infty}}), 1)$$

all of the circle generators except the first one map trivially, while the generator of the fundamental group of the first circle maps to a topological generator of $\mathcal{C}(\Gamma_F) \subseteq \text{Aut}(\mu_{2^{2\infty}})$. We may achieve this by the following argument due to Dwyer and Friedlander. Note that the two–profinite completion $\pi_1 \text{Spec}(R_F)_{\text{ét}}$ is a free pro–two–group on $r_2 + 1$ generators from (2.3). The image $f$ of the Frobenius of $\mathfrak{p}$ under the natural map $\pi_1 \text{Spec}(F[\mathfrak{p}])_{\text{ét}} \to \pi_1 \text{Spec}(R_F)_{\text{ét}}$ is one of the generators. Assume the other generators are $g_1, \ldots, g_{r_2}$. It is possible that one of the $g_i$'s do not have a trivial image in the image of $\pi_1 \text{Spec}(F[\mathfrak{p}])_{\text{ét}}$ in $\pi_1 \text{Spec}(R_F)_{\text{ét}}$. We may then multiply $g_i$ by some power of the Frobenius such that the resulting element $h_i$ does have a trivial image. The possibly new set of elements $f, h_1, \ldots, h_{r_2}$ is still a set of generators for $\pi_1 \text{Spec}(R_F)_{\text{ét}}$. Next we realize these generators by maps from the circle to $\text{Spec}(R_F)_{\text{ét}}$, and this defines the map $\beta$. Next we replace the first wedge summand in the domain of $\beta$ with a copy of the étale homotopy type of a finite field. This can be done in such a way that the induced map preserves the mod 2 cohomology equivalence, cf. (2.1). For this we employ the prime ideal $\mathfrak{p}$ which meets these requirements.\footnote{I thank Bill Dwyer for explaining this line of argument to me.}

By fixing an embedding of $F[\mathfrak{p}]$ into the complex numbers $\mathbb{C}$, we can think of (2.4) in terms of the diagram:

$$\begin{array}{ccc}
\text{Spec}(\mathbb{C})_{\text{ét}} & \longrightarrow & \bigvee_{r_2} S^1 \\
\downarrow & & \downarrow \beta^* \\
\text{Spec}(F[\mathfrak{p}])_{\text{ét}} & \xrightarrow{\pi_1^*} & \text{Spec}(R_F)_{\text{ét}}
\end{array}$$

(2.5)

Henceforth, we implicitly complete all spaces and spectra at the prime $\ell = 2$. For the circles mapping trivially under $\beta^*$ one can show that $\tilde{K}^{\text{top}}(S^1)$ is two–adically equivalent to the unpointed function space $BU^{S^1}$. The space $BU$ is an infinite loop space, so $BU^W$ where $W$ denotes a bouquet of $r_2$ circles is equivalent to the product $BU \times \bigvee_{r_2} U$. Further $K_{\text{ét}}(\mathbb{C})$ is two–adically equivalent as a $\mathbb{Z}/2$–space to $BU$ with the standard action of complex conjugation. Hence Proposition 2.2 and (2.5) give the homotopy Cartesian square:

$$\begin{array}{ccc}
K_{\text{ét}}(R_F) & \longrightarrow & BU \times \prod_{r_2} U \\
\downarrow \pi_1^* & & \downarrow \\
K_{\text{ét}}(F[\mathfrak{p}]) & \longrightarrow & BU
\end{array}$$

(2.6)

The Dwyer–Friedlander map from algebraic K–theory to étale topological K–theory is a two–adic equivalence for two–integers in totally imaginary number fields,
see [RW2]. Algebraic and étale topological K–theory coincide for finite fields by Corollary 8.6 of [DF1]. Thus we end up with the homotopy Cartesian square:

\[
\begin{array}{ccc}
\Omega^\infty_0 K(R_F) & \longrightarrow & BU \times \prod_{\mathbb{Z}} U \\
\pi_0^r & \downarrow & \\
\Omega^\infty_0 K(F[\varphi]) & \longrightarrow & BU
\end{array}
\]

(2.7)

The right hand side map in (2.7) is evaluation at the basepoint, cf. Proposition 4.5 in [DF2]. Moreover, it is a principal fibration and comes with an obvious section. In other words we have the claimed split fibre sequence

\[
\prod_{\mathbb{Z}} U \rightarrow \Omega^\infty_0 K(R_F) \rightarrow \Omega^\infty_0 K(F[\varphi])
\]

(2.8)

of spaces after two-adic completion. ⊓⊔

**Proof of Theorem 1.1.** Write \(L_K\) for the Bousfield localization functor associated to complex topological K–theory \(K\). Bousfield constructed in [Bo] a functor \(\Phi\) from the homotopy category of spaces to the homotopy category of spectra. One of the many outstanding properties of \(\Phi\) is a natural equivalence \(\Phi \Omega^\infty_0 X \cong L_K X\) for any CW–spectrum \(X\), cf. Theorem 2.3 of [Bo]. The Bousfield \(L_K\)–localization of \(U\) is the complex topological K–theory spectrum itself. Thus we obtain the split fibre sequence of spectra

\[
\bigvee_{\mathbb{Z}} \Sigma K \rightarrow L_K K(R_F) \rightarrow L_K K(F[\varphi])
\]

(2.9)

by applying the functor \(\Phi\) to (2.8).

Next we consider the square of spectra:

\[
\begin{array}{ccc}
K(R_F) & \longrightarrow & K^{\text{ét}}(R_F) \\
\downarrow & & \downarrow \\
L_K K(R_F) & \longrightarrow & L_K K^{\text{ét}}(R_F)
\end{array}
\]

Étale topological K–theory is \(K\)–local by construction, so the right hand vertical map is an equivalence. From étale descent for totally imaginary number fields in [RW2], the upper horizontal map is a two-adic equivalence on connected covers. Hence the lower horizontal map is an equivalence after two-adic completion, and the left vertical map is an equivalence in positive degrees. This remark combined with (2.9) finishes the proof. ⊓⊔

**Proof of Corollary 1.3.** This is immediate from (2.8), cf. Corollary 4.6 in [DF2]. ⊓⊔

**Proof of Theorem 1.4.** This is a consequence of Theorem 13.3 in [DM] and étale descent for local number fields. See Theorem 1.5. in this paper. ⊓⊔
REFERENCES


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