The strong Euler scheme for stochastic differential equations driven by Lévy processes

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Abstract

The strong Euler scheme for stochastic differential equations is the stochastic analog of the Euler scheme for ordinary differential equations. If τ is a partition with mesh(τ)$\leq \delta$ and Y^{δ} denotes the approximation of X, where X denotes the solution of the SDE, then under some moment conditions we show that $E\left[|X(T)-Y^{\delta}(T)|\right] \leq C\cdot\delta^{\frac{1}{2}}$. I.e the strong Euler scheme converges with order $\frac{1}{2}$.

1 Introduction

Lévy processes arises in a wide variety of different areas such as laser physics, mathematical finance and turbulence. This type of processes include familiar processes such as the Brownian motion, the Poisson process and the stable processes. Lévy processes posess a lot of properties which are desirable both from a modelling and a mathematical point of view.

In the case where the Lévy process is the Brownian motion the stochastic Euler scheme and higher order schemes are treated extensively in [2]. The weak Euler scheme for Lévy processes has been treated by Taqqu and Protter in [3]. This paper gives a treatment of the strong Euler scheme for Lévy process.

We will consider stochastic differential equations of the form:

$$dX_t = a(X_t)dt + b(X_t)dL_t \quad , \qquad X_0 = x_0 \tag{1}$$

where a and b are some deterministic functions and L is a Lévy process. An explicit solution to differential equations of this type are hard to find, if possible. Instead of finding an explicit solution one can try to find an approximate solution. One common way to make such an approximation is to discretesize the equation according to the so called Euler scheme. The stochastic Euler scheme is the stochastic analog of the classic Euler method for stochastic differential equations.

We say that Y^n converges to X strongly if $Y^n(T)$ converges to $X(T)$ in $L^1(P)$. The Euler scheme algorithm for SDEs of the form (1) is given by $Y^\delta(t_0) = x_0$ and

$$Y^\delta(t_k) = Y^\delta(t_{k-1}) + a(Y^\delta(t_{k-1}))\Delta t_{k-1} + b(Y^\delta(t_{k-1}))\Delta L_{k-1} \tag{2}$$

where $\Delta t_k = t_{k+1} - t_k$ and $\Delta L_k = L(t_{k+1}) - L(t_k)$. Since L is a Lévy process ΔL_k is mutually independent of ΔL_m for all $k \neq m$ and all the ΔL_ks have the same distribution. Often this distribution is known, for instance when the Lévy process is a Brownian motion we know that ΔB_k is normal distributed.

2 Error bounds for truncated Ito-Taylor expansions

The rate of convergence proof for the strong Euler scheme has several steps. The first step is to obtain error bounds for truncated first order Ito-Taylor approximation. The way we obtain these bounds are similar to the deterministic case where one can obtain error bounds using Taylor expansion. In the stochastic case the method is much the same, but istead of using Taylor's formula we use Ito's formula. In this way we obtain an expression for the approximation error we make for each time discretization.

For notational convenience we make the convention $R_0 = \mathbb{R} \setminus \{0\}$. Let L be a square integrable Lévy process. Then the process can be written $L_t = \sigma B_t + \int_{R_0} z(\mu - \pi)(t, dz)$, where μ is a Poisson random measure and π its compensator. In the Lévy process case this compensator is equal to $\pi(dt, dz) = \nu(dz)dt$ where ν is the Lévy measure. The problem in this section consists of attaining some error bound on the truncated Ito-Taylor expansion of X. We start with a little lemma,

Lemma 1. *Assume X_t is any process adapted to the filtration generated by the Lévy process $L_t = \sigma B_t + \int_{R_0} z(\mu - \pi)(t, dz)$. Then*

$$E\big[\sup_{0\le t\le T} |\int_0^t f(X_s)ds|^2\big] \le T^2 E\big[\sup_{0\le t\le T} |f(X_t)|^2\big] \tag{3}$$

$$E\big[\sup_{0\le t\le T} |\int_0^t f(X_s)dB_s|^2\big] \le 4TE\big[\sup_{0\le t\le T} |f(X_t)|^2\big] \tag{4}$$

$$E\big[\sup_{0\le t\le T} |\int_0^t \int_{R_0} f(X_s, z)(\mu - \pi)(ds, dz)|^2\big] \le 4T \int_{R_0} E\big[\sup_{0\le t\le T} |f(X_t, z)|^2\big]\nu(dz) \tag{5}$$

Proof. Inequality (3) is obtained by using the Cauchy-Schwarz inequality. The second inequality (4) is obtained by first using Doob's maximal inequality, and then the Ito isometry the following way:

$$E\big[\sup_{0\le t\le T} |\int_0^t f(X_s)dB_s|^2\big] \le 4E\big[|\int_0^T f(X_s)dB_s|^2\big] = 4E\big[\int_0^T |f(X_s)|^2 ds\big] \le 4TE\big[\sup_{0\le t\le T} |f(X_t)|^2\big]$$

Inequality (5) is obtained by using first Doob's maximal inequality, then the Ito isometry and finally Tonelli's theorem the following way:

$$
\begin{aligned}
&E\big[\sup_{0\le t\le T}|\int_0^t\int_{R_0} f(X_s,z)(\mu-\pi)(ds,dz)|^2\big] \\
&\quad\le 4E\big[|\int_0^T\int_{R_0} f(X_s,z)(\mu-\pi)(ds,dz)|^2\big] \\
&\quad= 4E\big[\int_0^T\int_{R_0} |f(X_s,z)|^2\nu(dz)ds\big] \\
&\quad\le 4T\int_{R_0} E\big[\sup_{0\le t\le T}|f(X_t,z)|^2\big]\nu(dz)
\end{aligned}
$$

□

Theorem 2. *Assume $a(\cdot)$ and $b(\cdot)$ satisfy the Lipschitz conditions*

$$|a(x)|+|b(x)|\le C_1(1+|x|) \tag{6}$$

$$|a(x)-a(y)|+|b(x)-b(y)|\le C_2|x-y| \tag{7}$$

that a', a'', b', b'' all are totally bounded, and that $\int_{|z|\ge 1} z^4\nu(dz)<\infty$. Let X denote the solution to equation (1) and assume that X_T has finite fourth order moment. Then

$$E\big[\sup_{0\le t\le T}|X(t)-T_x(t)|^2\big]\le CT^2 \tag{8}$$

for some C not depending on T, where $T_x(t)=x+a(x)t+b(x)L_t$ is the first non-trivial truncated Ito-Taylor expansion of X.

Proof. Let $S_{a'}=\sup_x|a'(x)|$, $S_{a''}=\sup_x|a''(x)|$, $S_{b'}=\sup_x|b'(x)|$ and $S_{b''}=\sup_x|b''(x)|$. The differential equation can be written,

$$X_t = x_0+\int_0^t a(X_s)ds+\int_0^t \sigma b(X_s)dB_s+\int_0^t\int_{R_0} b(X_s)z(\mu-\pi)(dt,dz) \tag{9}$$

Ito's formula gives us the following formulas for $a(X_t)$ and $b(X_t)$:

$$
\begin{aligned}
a(X_t) = a(x_0) &+ \int_0^t (a(X_s)a'(X_s) + \frac{1}{2}\sigma^2 b^2(X_s)a''(X_s))ds \\
&+ \int_0^t \sigma b(X_s)a'(X_s)dB_s \\
&+ \int_0^t \int_{R_0} (a(X_s + b(X_s)z) - a(X_s))(\mu - \pi)(ds, dz) \\
&+ \int_0^t \int_{R_0} (a(X_s + b(X_s)z) - a(X_s) - a'(X_s)b(X_s)z)\pi(ds, dz) \quad (10)
\end{aligned}
$$

and

$$
\begin{aligned}
b(X_t) = b(x_0) &+ \int_0^t (a(X_s)b'(X_s) + \frac{1}{2}\sigma^2 b^2(X_s)b''(X_s))ds \\
&+ \int_0^t \sigma b(X_s)b'(X_s)dB_s + \int_0^t \int_{R_0} b(X_s + b(X_s)z) - b(X_s)(\mu - \pi)(ds, dz) \\
&+ \int_0^t \int_{R_0} (b(X_s + b(X_s)z) - b(X_s) - b'(X_s)b(X_s))\tilde{z}\pi(ds, d\tilde{z}) \quad (11)
\end{aligned}
$$

Applying (10) and (11) to $a(X_t)$ and $b(X_t)$ in equation 9 we get:

$$
X_t = x_0 + a(x_0)dt + b(x_0)L_t + \sum_{i=1}^{12} R_i \quad (12)
$$

where:

$$
\begin{aligned}
R_1 &= \int_0^t \int_0^s (a(X_l)a'(X_l) + \frac{1}{2}\sigma^2 b^2(X_l)a''(X_l))dlds \\
R_2 &= \int_0^t \int_0^s \sigma b(X_l)a'(X_l)dB_l ds \\
R_3 &= \int_0^t \int_0^s \int_{R_0} (a(X_l + b(X_l)z) - a(X_l))(\mu - \pi)(dl, dz)ds \\
R_4 &= \int_0^t \int_0^s \int_{R_0} (a(X_l + b(X_l)z) - a(X_l) - a'(X_l)z)\pi(dl, dz)ds \\
R_5 &= \sigma \int_0^t \int_0^s a(X_l)b'(X_l) + \frac{1}{2}\sigma^2 b^2(X_l)b''(X_l)dldB_s \\
R_6 &= \sigma^2 \int_0^t \int_0^s b(X_l)b'(X_l)dB_l dB_s
\end{aligned}
$$

$$R_7 = \sigma \int_0^t \int_0^s \int_{R_0} (b(X_l + b(X_l)z) - b(X_l))(\mu - \pi)(dl, dz)dB_s$$

$$R_8 = \sigma \int_0^t \int_0^s \int_{R_0} (b(X_l + b(X_l)z) - b(X_l) - b'(X_l)b(X_l)z)\pi(dl, dz)dB_s$$

$$R_9 = \int_0^t \int_{R_0} \int_0^s (a(X_l)b'(X_l) + \frac{1}{2}\sigma^2 b^2(X_l)b''(X_l))dl z(\mu - \pi)(ds, dz)$$

$$R_{10} = \int_0^t \int_{R_0} \int_0^s \sigma b(X_l)b'(X_l)dB_l z(\mu - \pi)(ds, dz)$$

$$R_{11} = \int_0^t \int_{R_0} \int_0^s \int_{R_0} (b(X_l + b(X_l)z) - b(X_l)(\mu - \pi)(dl, dz))\tilde{z}(\mu - \pi)(ds, d\tilde{z})$$

$$R_{12} = \int_0^t \int_{R_0} \int_0^s \int_{R_0} \big(b(X_l + b(X_l)z) - b(X_l) - b'(X_l)b(X_l)z\big) \times \pi(dl, dz)\tilde{z}(\mu - \pi)(ds, d\tilde{z}) \qquad (13)$$

Let $r_i(\cdot, z)$ denote the kernel of R_i. We now want to obtain bounds on the r_i's. Starting with r_1:

$$\begin{aligned} E\big[\sup_{0\le t\le T} |r_1(X_t, z)|^2\big] &= E\big[\sup_{0\le t\le T} |a(X_t)a'(X_t) + \frac{1}{2}\sigma^2 b^2(X_t)a''(X_t)|^2\big] \\ &\le E\big[\sup_{0\le t\le T} \{2(a(X_t)a'(X_t))^2 + \frac{1}{2}\sigma^4(b^2(X_t)a''(X_t))^2\}\big] \\ &\le \big(2(S_{a'})^2 E\big[\sup_{0\le t\le T} |a(X_t)|^2\big] + \frac{1}{2}\sigma^4(S_{a''})^2 E\big[\sup_{0\le t\le T} |b(X_t)|^4\big]\big) \qquad (14) \end{aligned}$$

using first the Lipschitz condition, eq.(6), and then the assumption that X has finite fourth moment,

$$\begin{aligned} &\le 4(S_{a'})^2(C_1)^2 E\big[\sup_{0\le t\le T}(1 + |X_t|^2)\big] + 2\sigma^4(S_{a''})^2(C_1)^4 E\big[\sup_{0\le t\le T}(1 + |X_t|^2)^2\big] \\ &\le C_{R_1} \end{aligned}$$

A bound for r_2 is obtained by first using the Lipschitz condition eq.(6), and then the assumptions about the moments of X,

$$\begin{aligned} E\big[\sup_{0\le t\le T} |r_2|^2\big] &= E\big[\sup_{0\le t\le T} |\sigma b(X_t)a'(X_t)|^2\big] \\ &\le \sigma^2(S_{a'})^2 2(C_1)^2 E\big[\sup_{0\le t\le T}(1 + |X_t|^2)\big] \\ &\le C_{R_2} \end{aligned}$$

Similarily we get a bound for r_3 by first using the Lipschitz condition, eq. (7), then the Lipschitz condition eq.(6) and finally the moment assumption on X in the following way:

$$\begin{aligned} E\big[\sup_{0\leq t\leq T}|r_3(X_s,z)|^2\big] &= E\big[\sup_{0\leq t\leq T}|a(X_s+b(X_s)z)-a(X_s)|^2\big] \\ &\leq C_2E\big[\sup_{0\leq t\leq T}|b(X_t)|^2z^2\big] \\ &\leq z^2C_22(C_1)^2E\big[\sup_{0\leq t\leq T}(1+|X_t|^2)\big] \\ &\leq z^2C_{R_3} \end{aligned}$$

We can by using the preceeding techniques obtain similar bounds on the following r_i terms:

$$\begin{aligned} &E\big[\sup_{0\leq t\leq T}|r_5(X_t,z)|^2\big] \leq C_{R_5} && E\big[\sup_{0\leq t\leq T}|r_6(X_t,z)|^2\big] \leq C_{R_6} \\ &E\big[\sup_{0\leq t\leq T}|r_7(X_t,z)|^2\big] \leq z^2C_{R_7} && E\big[\sup_{0\leq t\leq T}|r_9(X_t,z)|^2\big] \leq C_{R_9} \\ &E\big[\sup_{0\leq t\leq T}|r_{10}(X_t,z)|^2\big] \leq C_{R_{10}} && E\big[\sup_{0\leq t\leq T}|r_{11}(X_t,z)|^2\big] \leq z^2C_{R_{11}} \end{aligned}$$

Using Lemma 1 twice, we obtain the following bounds:

$$\begin{aligned} &E\big[\sup_{0\leq t\leq T}|R_1|^2\big] \leq C_{R_1}T^4 \\ &E\big[\sup_{0\leq t\leq T}|R_2|^2\big] \leq C_{R_2}T^3 \\ &E\big[\sup_{0\leq t\leq T}|R_3|^2\big] \leq C_{R_3}\int_R z^2\nu(dz)T^3 \leq \tilde{C}_{R_3}T^3 \\ &E\big[\sup_{0\leq t\leq T}|R_5|^2\big] \leq C_{R_5}T^3 \\ &E\big[\sup_{0\leq t\leq T}|R_6|^2\big] \leq C_{R_6}T^2 \\ &E\big[\sup_{0\leq t\leq T}|R_7|^2\big] \leq C_{R_7}\int_R z^2\nu(dz)T^2 \leq \tilde{C}_{R_7}T^2 \\ &E\big[\sup_{0\leq t\leq T}|R_9|^2\big] \leq C_{R_9}T^3 \\ &E\big[\sup_{0\leq t\leq T}|R_{10}|^2\big] \leq C_{R_{10}}T^2 \\ &E\big[\sup_{0\leq t\leq T}|R_{11}|^2\big] \leq C_{R_{11}}(\int_R z^2\nu(dz))^2T^2 \leq \tilde{C}_{R_{11}}T^2 \qquad (15) \end{aligned}$$

We want similar bounds on R_4, R_8 and R_{12}. These terms are treated separately since the bounds on these terms are obtained with a different method. Since the bounds for R_8 and R_{12} are obtained in almost the same way as the bound for R_4, we will only treat R_4. First we use Taylor's formula to expand $a(\cdot)$ around X_t,

$$a(X_t + b(X_t)z) = a(X_t) + a'(X_t)b(X_t)z + \frac{a''(y)}{2}b(X_t)^2z^2$$

for some y between X_t and $X_t + b(X_t)z$. Hence

$$r_4(X_t, z) = a(X_t + b(X_t)z) - a(X_t) - a'(X_t)b(X_t)z = \frac{a''(y)}{2}b(X_t)^2z^2 \quad (16)$$

Then we use the Schwarz inequality and (16),

$$\begin{aligned} &E\big[\sup_{0\le t\le T} \big|\int_0^t \int_{R_0} r_4(X_s, z)\nu(dz)ds\big|^2\big] \\ &\le T^2 E\big[\sup_{0\le t\le T} \big|\int_R r_4(X_s, z)\nu(dz)\big|^2\big] \\ &\le T^2 E\big[\sup_{0\le t\le T} \big|\int_{R_0} \frac{a''(y)}{2} b(X_t)^2 z^2 \nu(dz)\big|^2\big] \\ &\le \frac{1}{4}T^2 S_{a''}\big(\int_R z^2\nu(dz)\big)^2 E\big[\sup_{0\le t\le T} b(X_t)^2\big] \end{aligned} \quad (17)$$

Now using first the Lipschitz condition eq.(6), and then the moment assumption on X we obtain

$$\le \frac{1}{4}(S_{a''})^2\big(\int_R z^2\nu(dz)\big)^2 T^2 4(C_1)^4 E\big[\sup_{0\le t\le T}(1 + |X_t|^2)^2\big] \le C_{R_4}T^4 \quad (18)$$

Lemma 1 now yields the desired result, namely

$$E\big[\sup_{0\le t\le T} |R_4|^2\big] \le C_{R_4}T^4$$

The bounds for R_8 and R_{12} are similarily given by:

$$E\big[\sup_{0\le t\le T} |R_8|^2\big] \le C_{R_8}T^3 \qquad E\big[\sup_{0\le t\le T} |R_{12}|^2\big] \le C_{R_{12}}T^3$$

Finally by using the triangel inequality,

$$E\big[\sup_{0\le t\le T} |X(t) - T_X(t)|^2\big] = E\big[\sup_{0\le t\le T} \big|\sum_{i=1}^{12} R_i\big|^2\big] \le \sum_{i=1}^{12} E\big[\sup_{0\le t\le T} |R_i|^2\big] \le CT^2$$

□

One of the assumptions in Theorem 2 is that the solution of the SDE (1) has finite fourth order moment. Conditions concerning this can be found in [1, pp. 144].

Corollary 3. *Let R_i, i=1,..,12 be the remainder terms in Theorem 2, then $\tilde{R} = \sum_{i=1}^{12} R_i$ admits the following representation:*

$$\tilde{R}(t, t+\delta) = \int_t^{t+\delta} g_1(t,s)ds + \int_t^{t+\delta} g_2(t,s)dB_s + \int_t^{t+\delta} \int_{R_0} g_3(t,s)z(\mu - \pi)(ds, dz)$$

Where $E\big[\sup_{t\le s\le t+\delta} |g_i(t,s)|^2\big] \le C_i\delta$ for i=1,2 and 3, and some C_i not depending on δ.

Proof. From the proof of the theorem we have that the kernels r_i of the remainder terms R_i (equations (13)) satisfy

$$E\big[\sup_{0\le t\le T} |r_i(X_t, z)|^2\big] \le K_1 \tag{19}$$

for $i \in \{1, 2, 5, 6, 9, 10\}$ and some K_1 not depending on T. And

$$E\big[\sup_{0\le t\le T} |r_i(X_t, z)|^2\big] \le z^2 K_2 \tag{20}$$

for $i \in \{3, 4, 7, 8, 11, 12\}$ and some K_2 not depending on T. The result then follows by applying Lemma 1 to the remainder terms R_i. □

3 The strong Euler scheme

Theorem 4. *Let X denote the solution of eq. (1) and assume that the conditions in Theorem 2 is satisfied. Let $\{\tau_n\}$ be a random partition of the interval $[0,T]$, where $P(|\tau_{n+1}-\tau_n| \leq \delta) = 1$ for all n. Define*

$$Y_{n+1} = Y_n + \int_{\tau_n}^{\tau_{n+1}} a(Y_n)ds + \int_{\tau_n}^{\tau_{n+1}} b(Y_n)dL_s, \quad Y_0 = x_0 \tag{21}$$

and set

$$Y(t) = Y_n + \int_{\tau_n}^{t} a(Y_s)ds + \int_{\tau_n}^{t} b(Y_n)dL_s \tag{22}$$

Then

$$E\big[|X(T)-Y(T)|\big] \leq E\big[\sup_{0\leq t\leq T} |X(t)-Y(t)|^2\big]^{\frac{1}{2}} \leq C\delta^{\frac{1}{2}}$$

for some C not depending on δ.

Proof. Define a stochastic process $\{X_n\}$ by

$$X_{n+1} = X_n + \int_{\tau_n}^{\tau_{n+1}} a(X_n)ds + \int_{\tau_n}^{\tau_{n+1}} b(X_n)dL_s + \tilde{R}(\tau_n, \tau_{n+1}) \tag{23}$$

where $\tilde{R} = \sum R_i$ is as in Corollary 2. By Corollary 3 $\tilde{R}$ admits a representation,

$$\begin{aligned} R(\tau_n, \tau_{n+1}) = &\int_{\tau_n}^{\tau_{n+1}} r_1(\tau_n, s)ds + \int_{\tau_n}^{\tau_{n+1}} r_2(\tau_n, s)dB_s \\ &+ \int_{\tau_n}^{\tau_{n+1}} \int_{R_0} r_3(\tau_n, s)z(\mu-\pi)(ds,dz) \end{aligned} \tag{24}$$

where

$$E\big(\sup_{\tau_n\leq s\leq\tau_{n+1}} |r_1(\tau_n, s)|^2\big) \leq C(\tau_{n+1}-\tau_n) \tag{25}$$

$$E\big(\sup_{\tau_n\leq s\leq\tau_{n+1}} |r_2(\tau_n, s)|^2\big) \leq C(\tau_{n+1}-\tau_n) \tag{26}$$

$$E\big(\sup_{\tau_n\leq s\leq\tau_{n+1}} |r_3(\tau_n, s)|^2\big)\} \leq C(\tau_{n+1}-\tau_n) \tag{27}$$

for some C not depending on δ. We have the following expressions for X:

$$X(t) = X_n + \int_{\tau_n}^{\tau_{n+1}} a(X_n)ds + \int_{\tau_n}^{\tau_{n+1}} b(X_n)dL_s + R(\tau_n, t) \tag{28}$$

$$
\begin{aligned}
X_n - X_0 =& \sum_{i=1}^{n}(X_i - X_{i-1}) \\
=& \sum_{i=1}^{n}(\int_{\tau_{i-1}}^{\tau_i} a(X_{i-1})ds + \int_{\tau_{i-1}}^{\tau_i} b(X_{i-1})dL_s + R(\tau_{i-1}, \tau_i)) \quad (29)
\end{aligned}
$$

From which we obtain the following representation for X and Y respectively:

$$
\begin{aligned}
X(t) =& x_0 + \sum_{i=1}^{n}(\int_{\tau_{i-1}}^{\tau_i} a(X_{i-1})ds + \int_{\tau_{i-1}}^{\tau_i} b(X_{i-1})dL_s) + \int_{\tau_n}^{t} a(X_n)ds \\
&+ \int_{\tau_n}^{t} b(X_n)dL_s + \sum_{i=1}^{n} R(\tau_{i-1}, \tau_i) + R(\tau_n, t) \quad (30)
\end{aligned}
$$

$$
\begin{aligned}
Y(t) =& Y_0 + \sum_{i=1}^{n}(\int_{\tau_{i-1}}^{\tau_i} a(Y_{i-1})ds + \int_{\tau_{i-1}}^{\tau_i} b(Y_{i-1})dL_s) \\
&+ \int_{\tau_n}^{t} a(Y_n)ds + \int_{\tau_n}^{t} b(Y_n)dL_s \quad (31)
\end{aligned}
$$

where $n = n(t, \omega)$. Set $Z(t) = E\big[\sup_{0 \leq s \leq t} |X(s) - Y(s)|^2\big]$ and define a function $p(t) = max\{n : \tau_n \leq t\}$. The next step is to find a bound on Z,

$$
\begin{aligned}
Z(t) =& E\big[\sup_{0 \leq s \leq t} |\sum_{i=1}^{n}(\int_{\tau_{i-1}}^{\tau_i} a(X_{i-1}) - a(Y_{i-1})dl) \\
&+ \int_{\tau_n}^{s} a(X_n) - a(Y_n)dl + \sum_{i=1}^{n}(\int_{\tau_{i-1}}^{\tau_i} b(X_{i-1}) - b(Y_{i-1})dL_l) \\
&+ \int_{\tau_n}^{s} b(X_n) - b(Y_n)dL_l + \sum_{i=1}^{n} R(\tau_{i-1}, \tau_i) + R(\tau_n, s)|^2\big] \quad (32)
\end{aligned}
$$

Then using the inequality $(a + b + c)^2 \leq 3a^2 + 3b^2 + 3c^2$,

$$
\begin{aligned}
&\leq 3E\big[\sup_{0 \leq s \leq t} |\int_0^s a(X_{p(l)}) - a(Y_{p(l)})dl|^2\big] \\
&+ 3E\big[\sup_{0 \leq s \leq t} |\int_0^s b(X_{p(l)}) - b(Y_{p(l)})dL_l|^2\big] \\
&+ 3E\big[\sup_{0 \leq s \leq t} |\int_0^s r_1(p(l), l)dl \\
&+ \int_0^s r_2(p(l), l)dB_l + \int_0^s \int_{R_0} r_3(p(l), l)z(\mu - \pi)(dl, dz)|^2\big] \quad (33)
\end{aligned}
$$

again using the inequality $(a+b+c)^2 \leq 3a^2+3b^2+3c^2$ then Doob's maximal inequality and the Ito isometries,

$$
\begin{aligned}
&\leq 3TE\big[\int_0^t |a(X_{p(l)}) - a(Y_{p(l)})|^2 dl\big] \\
&\quad + 24\sigma^2 E\big[(\int_0^t b(X_{p(l)}) - b(Y_{p(l)}) dB_l)^2\big] \\
&\quad + 24E\big[(\int_0^t \int_{R_0} (b(X_{p(l)}) - b(Y_{p(l)}))z(\mu - \pi)(dl, dz))^2\big] \\
&\quad + 9TE\big[\int_0^t |r_1(p(l), l)|^2 dl\big] + 36E\big[\int_0^t |r_2(p(l), l)|^2 dl\big] \\
&\quad + 36 \int_{R_0} z^2 \nu(dz) E\big[\int_0^t |r_3(p(l), l)|^2 dl\big]
\end{aligned}
\tag{34}
$$

Then we proceed by again using the Ito isometries

$$
\begin{aligned}
&\leq 3T(C_2)^2 E\big[\int_0^t |X_{p(l)} - Y_{p(l)}|^2 dl\big] \\
&\quad + 24\sigma^2(C_2)^2 E\big[\int_0^t |X_{p(l)} - Y_{p(l)}|^2 dl\big] \\
&\quad + 24(C_2)^2 \int_{R_0} z^2 \nu(dz) E\big[\int_0^t |X_{p(l)} - Y_{p(l)}|^2\big] \\
&\quad + (9T + 72) \int_0^t E\big[\max_{1 \leq i \leq 3} |r_i(p(l), l)|^2\big] dl \\
&\leq d_1 \int_0^t E\big[\sup_{0 \leq l \leq s} |X(l) - Y(l)|^2\big] ds + (9T + 72)TC\delta \\
&= d_1 \int_0^t Z(s) ds + d_2\delta
\end{aligned}
\tag{35}
$$

where d_1 and d_2 are constants not depending on δ. By Gronwall's inequality we then obtain the following bound on $Z(t)$:

$$
Z(t) \leq d_2\delta + d_1 \int_0^t e^{d_1(t-s)} d_2 \delta ds \leq d_2(1 + d_1 T e^{d_1 T})\delta \leq C\delta \tag{36}
$$

Or, equivalently: $E\big(\sup_{0 \leq t \leq T} |X(t) - Y(t)|^2\big) \leq C\delta$. The proof is now completed by using the Cauchy-Schwarz inequality,

$$
E\big[\sup_{0 \leq t \leq T} |X(t) - Y(t)|\big] \leq E\big[\sup_{0 \leq t \leq T} |X(t) - Y(t)|^2\big]^{\frac{1}{2}} \leq C^{\frac{1}{2}} \delta^{\frac{1}{2}}
$$

□

References

[1] I.I. Gihman and A.V. Skorohod, *The Theory of Stochastic Processes 3*, Springer-Verlag, 1979.

[2] Kloeden and Platen, *Numerical Methods for Stochastic Differential Equations*, Springer-Verlag, 1992.

[3] P. Protter and D. Talay, *The Euler scheme for Lévy driven stochastic differential equations*, The Annals of Probability. 25(1), 393-423, 1997.