The strong Euler scheme for stochastic differential equations driven by Lévy processes

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Abstract

The strong Euler scheme for stochastic differential equations is the stochastic analog of the Euler scheme for ordinary differential equations. If $\tau$ is a partition with mesh($\tau$) $\leq \delta$ and $Y^\delta$ denotes the approximation of $X$, where $X$ denotes the solution of the SDE, then under some moment conditions we show that $E[|X(T) - Y^\delta(T)|] \leq C \cdot \delta^{1/2}$. I.e. the strong Euler scheme converges with order $\frac{1}{2}$. 
1 Introduction

Lévy processes arise in a wide variety of different areas such as laser physics, mathematical finance and turbulence. This type of processes include familiar processes such as the Brownian motion, the Poisson process and the stable processes. Lévy processes possess a lot of properties which are desirable both from a modelling and a mathematical point of view.

In the case where the Lévy process is the Brownian motion the stochastic Euler scheme and higher order schemes are treated extensively in [2]. The weak Euler scheme for Lévy processes has been treated by Taqqu and Protter in [3]. This paper gives a treatment of the strong Euler scheme for Lévy process.

We will consider stochastic differential equations of the form:

\[ dX_t = a(X_t)dt + b(X_t)dL_t \quad , \quad X_0 = x_0 \]  

(1)

where \(a\) and \(b\) are some deterministic functions and \(L\) is a Lévy process. An explicit solution to differential equations of this type are hard to find, if possible. Instead of finding an explicit solution one can try to find an approximate solution. One common way to make such an approximation is to discretize the equation according to the so called Euler scheme. The stochastic Euler scheme is the stochastic analog of the classic Euler method for stochastic differential equations.

We say that \(Y^n\) converges to \(X\) strongly if \(Y^n(T)\) converges to \(X(T)\) in \(L^1(P)\). The Euler scheme algorithm for SDEs of the form (1) is given by \(Y^\delta(t_0) = x_0\) and

\[ Y^\delta(t_k) = Y^\delta(t_{k-1}) + a(Y^\delta(t_{k-1}))\Delta t_{k-1} + b(Y^\delta(t_{k-1}))\Delta L_{k-1} \]  

(2)

where \(\Delta t_k = t_{k+1} - t_k\) and \(\Delta L_k = L(t_{k+1}) - L(t_k)\). Since \(L\) is a Lévy process \(\Delta L_k\) is mutually independent of \(\Delta L_m\) for all \(k \neq m\) and all the \(\Delta L_k\)s have the same distribution. Often this distribution is known, for instance when the Lévy process is a Brownian motion we know that \(\Delta B_k\) is normal distributed.
2 Error bounds for truncated Ito-Taylor expansions

The rate of convergence proof for the strong Euler scheme has several steps. The first step is to obtain error bounds for truncated first order Ito-Taylor approximation. The way we obtain these bounds are similar to the deterministic case where one can obtain error bounds using Taylor expansion. In the stochastic case the method is much the same, but instead of using Taylor’s formula we use Ito’s formula. In this way we obtain an expression for the approximation error we make for each time discretization.

For notational convenience we make the convention $R_0 = \mathbb{R} \setminus \{0\}$. Let $L$ be a square integrable Lévy process. Then the process can be written $L_t = \sigma B_t + \int_{R_0} z(\mu - \pi)(t, dz)$, where $\mu$ is a Poisson random measure and $\pi$ its compensator. In the Lévy process case this compensator is equal to $\pi(dt, dz) = \nu(dz)dt$ where $\nu$ is the Lévy measure. The problem in this section consists of attaining some error bound on the truncated Ito-Taylor expansion of $X$. We start with a little lemma,

**Lemma 1.** Assume $X_t$ is any process adapted to the filtration generated by the Lévy process $L_t = \sigma B_t + \int_{R_0} z(\mu - \pi)(t, dz)$. Then

\[
E\left[ \sup_{0 \leq t \leq T} \int_0^t f(X_s) ds \right]^2 \leq T^2 E\left[ \sup_{0 \leq t \leq T} |f(X_t)|^2 \right] \tag{3}
\]

\[
E\left[ \sup_{0 \leq t \leq T} \int_0^t f(X_s) dB_s \right]^2 \leq 4TE\left[ \sup_{0 \leq t \leq T} |f(X_t)|^2 \right] \tag{4}
\]

\[
E\left[ \sup_{0 \leq t \leq T} \int_0^t \int_{R_0} f(X_s, z)(\mu - \pi)(ds, dz) \right]^2 \leq 4T \int_{R_0} E\left[ \sup_{0 \leq t \leq T} |f(X_t, z)|^2 \right] \nu(dz) \tag{5}
\]

**Proof.** Inequality (3) is obtained by using the Cauchy-Schwarz inequality. The second inequality (4) is obtained by first using Doob’s maximal inequality, and then the Ito isometry the following way:

\[
E\left[ \sup_{0 \leq t \leq T} \int_0^t f(X_s) dB_s \right]^2 \leq 4E\left[ \int_0^T f(X_s) dB_s \right]^2 \leq 4E\left[ \int_0^T |f(X_s)|^2 ds \right] \leq 4TE\left[ \sup_{0 \leq t \leq T} |f(X_t)|^2 \right]
\]
Inequality (5) is obtained by using first Doob’s maximal inequality, then the Ito isometry and finally Tonelli’s theorem the following way:

\[
E \left[ \sup_{0 \leq t \leq T} \left| \int_0^T \int_{R_0} f(X_s, z)(\mu - \pi)(ds, dz) \right|^2 \right]
\leq 4E \left[ \left( \int_0^T \int_{R_0} f(X_s, z)(\mu - \pi)(ds, dz) \right)^2 \right]
= 4E \left[ \int_0^T \int_{R_0} |f(X_s, z)|^2 \nu(dz)ds \right]
\leq 4T \int_{R_0} E \left[ \sup_{0 \leq t \leq T} |f(X_t, z)|^2 \right] \nu(dz)
\]

\[\Box\]

Theorem 2. Assume \(a(\cdot)\) and \(b(\cdot)\) satisfy the Lipschitz conditions

\[|a(x)| + |b(x)| \leq C_1(1 + |x|)\]  \hspace{1cm} (6)

\[|a(x) - a(y)| + |b(x) - b(y)| \leq C_2|x - y|\]  \hspace{1cm} (7)

that \(a', a'', b', b''\) all are totally bounded, and that \(\int_{|z| \geq 1} z^4 \nu(dz) < \infty\). Let \(X\) denote the solution to equation (1) and assume that \(X_T\) has finite fourth order moment. Then

\[E \left[ \sup_{0 \leq t \leq T} |X(t) - T_x(t)|^2 \right] \leq CT^2\]  \hspace{1cm} (8)

for some \(C\) not depending on \(T\), where \(T_x(t) = x + a(x)t + b(x)L_t\) is the first non-trivial truncated Ito-Taylor expansion of \(X\).

Proof. Let \(S_{a'} = \sup_x |a'(x)|\), \(S_{a''} = \sup_x |a''(x)|\), \(S_{b'} = \sup_x |b'(x)|\) and \(S_{b''} = \sup_x |b''(x)|\). The differential equation can be written,

\[X_t = x_0 + \int_0^t a(X_s)ds + \int_0^t \sigma b(X_s)dB_s + \int_0^t \int_{R_0} b(X_s)z(\mu - \pi)(dz, dt)\]  \hspace{1cm} (9)
Ito's formula gives us the following formulas for $a(X_t)$ and $b(X_t)$:

$$a(X_t) = a(x_0) + \int_0^t (a(X_s)a'(X_s) + \frac{1}{2}\sigma^2 b^2(X_s)a''(X_s))ds$$

$$+ \int_0^t \sigma b(X_s)a'(X_s)dB_s$$

$$+ \int_0^t \int_{R_0}^t (a(X_s + b(X_s)z) - a(X_s))(\mu - \pi)(ds, dz)$$

$$+ \int_0^t \int_{R_0}^t (a(X_s + b(X_s)z) - a(X_s) - a'(X_s)b(X_s)z)\pi(ds, dz)$$

(10)

and

$$b(X_t) = b(x_0) + \int_0^t (a(X_s)b'(X_s) + \frac{1}{2}\sigma^2 b^2(X_s)b''(X_s))ds$$

$$+ \int_0^t \sigma b(X_s)b'(X_s)dB_s + \int_0^t \int_{R_0}^t b(X_s + b(X_s)z) - b(X_s)(\mu - \pi)(ds, dz)$$

$$+ \int_0^t \int_{R_0}^t (b(X_s + b(X_s)z) - b(X_s) - b'(X_s)b(X_s))\tilde{z}\pi(ds, d\tilde{z})$$

(11)

Applying (10) and (11) to $a(X_t)$ and $b(X_t)$ in equation 9 we get:

$$X_t = x_0 + a(x_0)dt + b(x_0)L_t + \sum_{i=1}^{12} R_i$$

(12)

where:

$$R_1 = \int_0^t \int_0^s (a(X_i)a'(X_i) + \frac{1}{2}\sigma^2 b^2(X_i)a''(X_i))dlds$$

$$R_2 = \int_0^t \int_0^s \sigma b(X_i)a'(X_i)dBlds$$

$$R_3 = \int_0^t \int_0^s \int_{R_0}^t (a(X_i + b(X_i)z) - a(X_i))(\mu - \pi)(dl, dz)ds$$

$$R_4 = \int_0^t \int_0^s \int_{R_0}^t (a(X_i + b(X_i)z) - a(X_i) - a'(X_i)z)\pi(dl, dz)ds$$

$$R_5 = \sigma \int_0^t \int_0^s a(X_i)b'(X_i) + \frac{1}{2}\sigma^2 b^2(X_i)b''(X_i)dldB_s$$

$$R_6 = \sigma^2 \int_0^t \int_0^s b(X_i)b'(X_i)dBlds$$
\[
R_7 = \sigma \int_0^t \int_0^s \int_{R_0} (b(X_t + b(X_i)z) - b(X_i))(\mu - \pi)(dl, dz)dB_s \\
R_8 = \sigma \int_0^t \int_0^s \int_{R_0} (b(X_t + b(X_i)z) - b(X_i) - b'(X_i)b(X_i)z)\pi(dl, dz)dB_s \\
R_9 = \int_0^t \int_0^s \int_{R_0} (a(X_t)b'(X_i) + \frac{1}{2}\sigma^2 b^2(X_i)b''(X_i))dlz(\mu - \pi)(ds, dz) \\
R_{10} = \int_0^t \int_0^s \int_{R_0} \sigma b(X_i)b'(X_i)dB_lz(\mu - \pi)(ds, dz) \\
R_{11} = \int_0^t \int_0^s \int_{R_0} (b(X_t + b(X_i)z) - b(X_i)(\mu - \pi)(dl, dz))\tilde{z}(\mu - \pi)(ds, d\tilde{z}) \\
R_{12} = \int_0^t \int_0^s \int_{R_0} (b(X_t + b(X_i)z) - b(X_i) - b'(X_i)b(X_i)z) \\
\times \pi(dl, dz)\tilde{z}(\mu - \pi)(ds, d\tilde{z}) \tag{13}
\]

Let \( r_i(\cdot, z) \) denote the kernel of \( R_i \). We now want to obtain bounds on the \( r_i \)'s. Starting with \( r_1 \):

\[
E\left[ \sup_{0 \leq t \leq T} |r_1(X_t, z)|^2 \right] \\
= E\left[ \sup_{0 \leq t \leq T} |a(X_t)a'(X_t) + \frac{1}{2}\sigma^2 b^2(X_t)a''(X_t)|^2 \right] \\
\leq E\left[ \sup_{0 \leq t \leq T} \left\{ 2(a(X_t)a'(X_t))^2 + \frac{1}{2}\sigma^4 (b^2(X_t)a''(X_t))^2 \right\} \right] \\
\leq (2S^2)E\left[ \sup_{0 \leq t \leq T} |a(X_t)|^2 \right] + \frac{1}{2}\sigma^4 (S^4)E\left[ \sup_{0 \leq t \leq T} |b(X_t)|^4 \right] \tag{14}
\]

using first the Lipschitz condition, eq.(6), and then the assumption that \( X \) has finite fourth moment,

\[
\leq 4(S^2)(C^2)(C_1)^2 E\left[ \sup_{0 \leq t \leq T} (1 + |X_t|^2) \right] + 2\sigma^4 (S^4)(C_1)^4 E\left[ \sup_{0 \leq t \leq T} (1 + |X_t|^2) \right] \\
\leq C_{R_1}
\]

A bound for \( r_2 \) is obtained by first using the Lipschitz condition eq.(6), and then the assumptions about the moments of \( X \),

\[
E\left[ \sup_{0 \leq t \leq T} |r_2|^2 \right] = E\left[ \sup_{0 \leq t \leq T} |\sigma b(X_t)a'(X_t)|^2 \right] \\
\leq \sigma^2 (S^2)2(C_1)^2 E\left[ \sup_{0 \leq t \leq T} (1 + |X_t|^2) \right] \\
\leq C_{R_2}
\]
Similarly we get a bound for $r_3$ by first using the Lipschitz condition, eq. (7), then the Lipschitz condition eq.(6) and finally the moment assumption on $X$ in the following way:

$$
E\left[ \sup_{0 \leq t \leq T} |r_3(X_t, z)|^2 \right] = E\left[ \sup_{0 \leq t \leq T} |a(X_t + b(X_t)z) - a(X_t)|^2 \right] \\
\leq C_3 E\left[ \sup_{0 \leq t \leq T} |b(X_t)|^2 z^2 \right] \\
\leq z^2 C_3 2(C_1)^3 E\left[ \sup_{0 \leq t \leq T} (1 + |X_t|^2) \right] \\
\leq z^2 C_{R_3}
$$

We can by using the preceding techniques obtain similar bounds on the following $r_i$ terms:

$$
E\left[ \sup_{0 \leq t \leq T} |r_5(X_t, z)|^2 \right] \leq C_{R_5} \\
E\left[ \sup_{0 \leq t \leq T} |r_6(X_t, z)|^2 \right] \leq C_{R_6} \\
E\left[ \sup_{0 \leq t \leq T} |r_7(X_t, z)|^2 \right] \leq z^2 C_{R_7} \\
E\left[ \sup_{0 \leq t \leq T} |r_9(X_t, z)|^2 \right] \leq C_{R_9} \\
E\left[ \sup_{0 \leq t \leq T} |r_10(X_t, z)|^2 \right] \leq C_{R_{10}} \\
E\left[ \sup_{0 \leq t \leq T} |r_{11}(X_t, z)|^2 \right] \leq z^2 C_{R_{11}}
$$

Using Lemma 1 twice, we obtain the following bounds:

$$
E\left[ \sup_{0 \leq t \leq T} |R_1|^2 \right] \leq C_{R_1} T^4 \\
E\left[ \sup_{0 \leq t \leq T} |R_2|^2 \right] \leq C_{R_2} T^3 \\
E\left[ \sup_{0 \leq t \leq T} |R_3|^2 \right] \leq C_{R_3} \int_R z^2 \nu(dz) T^3 \leq C_{R_3} T^3 \\
E\left[ \sup_{0 \leq t \leq T} |R_6|^2 \right] \leq C_{R_6} T^2 \\
E\left[ \sup_{0 \leq t \leq T} |R_7|^2 \right] \leq C_{R_7} \int_R z^2 \nu(dz) T^2 \leq C_{R_7} T^2 \\
E\left[ \sup_{0 \leq t \leq T} |R_9|^2 \right] \leq C_{R_9} T^3 \\
E\left[ \sup_{0 \leq t \leq T} |R_{10}|^2 \right] \leq C_{R_{10}} T^2 \\
E\left[ \sup_{0 \leq t \leq T} |R_{11}|^2 \right] \leq C_{R_{11}} \left( \int_R z^2 \nu(dz) \right)^2 T^2 \leq C_{R_{11}} T^2
$$

\[ (15) \]
We want similar bounds on $R_4$, $R_8$ and $R_{12}$. These terms are treated separately since the bounds on these terms are obtained with a different method. Since the bounds for $R_8$ and $R_{12}$ are obtained in almost the same way as the bound for $R_4$, we will only treat $R_4$. First we use Taylor’s formula to expand $a(·)$ around $X_t$,

$$a(X_t + b(X_t)z) = a(X_t) + a'(X_t)b(X_t)z + \frac{a''(y)}{2}b(X_t)^2z^2$$

for some $y$ between $X_t$ and $X_t + b(X_t)z$. Hence

$$r_4(X_t, z) = a(X_t + b(X_t)z) - a(X_t) - a'(X_t)b(X_t)z = \frac{a''(y)}{2}b(X_t)^2z^2 \quad (16)$$

Then we use the Schwarz inequality and (16),

$$E\left[ \sup_{0 \leq t \leq T} | \int_0^t \int_{R_0} r_4(X_s, z)\nu(dz)ds|^2 \right]$$

$$\leq T^2 E\left[ \sup_{0 \leq t \leq T} \left| \int_{R_0} r_4(X_s, z)\nu(dz) \right|^2 \right]$$

$$\leq T^2 E\left[ \sup_{0 \leq t \leq T} \left| \int_{R_0} \frac{a''(y)}{2}b(X_t)^2z^2\nu(dz) \right|^2 \right]$$

$$\leq \frac{1}{4}T^2 S_0 a'' \left( \int_{R_0} z^2\nu(dz) \right)^2 E\left[ \sup_{0 \leq t \leq T} b(X_t)^2 \right] \quad (17)$$

Now using first the Lipschitz condition eq.(6), and then the moment assumption on $X$ we obtain

$$\leq \frac{1}{4}(S_0 a'')^2 \left( \int_{R_0} z^2\nu(dz) \right)^2 T^2 4(C_1)^4 E\left[ \sup_{0 \leq t \leq T} (1 + |X_t|^2)^2 \right] \leq C R_4 T^4 \quad (18)$$

Lemma 1 now yields the desired result, namely

$$E\left[ \sup_{0 \leq t \leq T} |R_4|^2 \right] \leq C R_4 T^4$$

The bounds for $R_8$ and $R_{12}$ are similarly given by:

$$E\left[ \sup_{0 \leq t \leq T} |R_8|^2 \right] \leq C R_8 T^3 \quad E\left[ \sup_{0 \leq t \leq T} |R_{12}|^2 \right] \leq C R_{12} T^3$$

Finally by using the triangular inequality,

$$E\left[ \sup_{0 \leq t \leq T} |X(t) - T_X(t)|^2 \right] = E\left[ \sup_{0 \leq t \leq T} \left| \sum_{i=1}^{12} R_i \right|^2 \right] \leq \sum_{i=1}^{12} E\left[ \sup_{0 \leq t \leq T} |R_i|^2 \right] \leq C T^2$$

\[ \square \]
One of the assumptions in Theorem 2 is that the solution of the SDE (1) has finite fourth order moment. Conditions concerning this can be found in [1, pp. 144].

**Corollary 3.** Let \( R_i, i=1,\ldots,12 \) be the remainder terms in Theorem 2, then \( \tilde{R} = \sum_{i=1}^{12} R_i \) admits the following representation:

\[
\tilde{R}(t, t + \delta) = \int_t^{t+\delta} g_1(t, s)ds + \int_t^{t+\delta} g_2(t, s)dB_s \\
+ \int_t^{t+\delta} \int_{\mathbb{R}_0} g_3(t, s)z(\mu - \pi)(ds, dz)
\]

Where \( E[\sup_{t \leq s \leq t+\delta} |g_i(t, s)|^2] \leq C_i\delta \) for \( i=1,2 \) and \( 3 \), and some \( C_i \) not depending on \( \delta \).

**Proof.** From the proof of the theorem we have that the kernels \( r_i \) of the remainder terms \( R_i \) (equations (13)) satisfy

\[
E[\sup_{0 \leq t \leq T} |r_i(X_t, z)|^2] \leq K_i \tag{19}
\]

for \( i \in \{1, 2, 5, 6, 9, 10\} \) and some \( K_i \) not depending on \( T \). And

\[
E[\sup_{0 \leq t \leq T} |r_i(X_t, z)|^2] \leq z^2 K_i \tag{20}
\]

for \( i \in \{3, 4, 7, 8, 11, 12\} \) and some \( K_i \) not depending on \( T \). The result then follows by applying Lemma 1 to the remainder terms \( R_i \).

\( \square \)
3 The strong Euler scheme

Theorem 4. Let $X$ denote the solution of eq. (1) and assume that the conditions in Theorem 2 is satisfied. Let $\{\tau_n\}$ be a random partition of the interval $[0, T]$, where $P(|\tau_{n+1} - \tau_n| \leq \delta) = 1$ for all $n$. Define

$$Y_{n+1} = Y_n + \int_{\tau_n}^{\tau_{n+1}} a(Y_n) ds + \int_{\tau_n}^{\tau_{n+1}} b(Y_n) dL_s, \quad Y_0 = x_0$$ (21)

and set

$$Y(t) = Y_n + \int_{\tau_n}^{t} a(Y_s) ds + \int_{\tau_n}^{t} b(Y_s) dL_s$$ (22)

Then

$$E[\|X(T) - Y(T)\|] \leq E[\sup_{0 \leq t \leq T} |X(t) - Y(t)|^2]^{\frac{1}{2}} \leq C\delta^{\frac{1}{2}}$$

for some $C$ not depending on $\delta$.

Proof. Define a stochastic process $\{X_n\}$ by

$$X_{n+1} = X_n + \int_{\tau_n}^{\tau_{n+1}} a(X_n) ds + \int_{\tau_n}^{\tau_{n+1}} b(X_n) dL_s + \tilde{R}(\tau_n, \tau_{n+1})$$ (23)

where $\tilde{R} = \sum R_i$ is as in Corollary 2. By Corollary 3 $\tilde{R}$ admits a representation,

$$R(\tau_n, \tau_{n+1}) = \int_{\tau_n}^{\tau_{n+1}} r_1(\tau_n, s) ds + \int_{\tau_n}^{\tau_{n+1}} r_2(\tau_n, s) dB_s$$

$$+ \int_{\tau_n}^{\tau_{n+1}} \int_{R_0} r_3(\tau_n, s) \varphi(\mu - \pi)(ds, dz)$$ (24)

where

$$E\left( \sup_{\tau_n \leq s \leq \tau_{n+1}} |r_1(\tau_n, s)|^2 \right) \leq C(\tau_{n+1} - \tau_n)$$ (25)

$$E\left( \sup_{\tau_n \leq s \leq \tau_{n+1}} |r_2(\tau_n, s)|^2 \right) \leq C(\tau_{n+1} - \tau_n)$$ (26)

$$E\left( \sup_{\tau_n \leq s \leq \tau_{n+1}} |r_3(\tau_n, s)|^2 \right) \leq C(\tau_{n+1} - \tau_n)$$ (27)

for some $C$ not depending on $\delta$. We have the following expressions for $X$:

$$X(t) = X_n + \int_{\tau_n}^{t} a(X_n) ds + \int_{\tau_n}^{t} b(X_n) dL_s + R(\tau_n, t)$$ (28)
\[ X_n - X_0 = \sum_{i=1}^{n} (X_i - X_{i-1}) \]
\[ = \sum_{i=1}^{n} \left( \int_{\tau_{i-1}}^{\tau_i} a(X_{i-1}) ds + \int_{\tau_{i-1}}^{\tau_i} b(X_{i-1}) dL_s + R(\tau_{i-1}, \tau_i) \right) \]  \hspace{1cm} (29)

From which we obtain the following representation for \( X \) and \( Y \) respectively:

\[ X(t) = x_0 + \sum_{i=1}^{n} \left( \int_{\tau_{i-1}}^{\tau_i} a(X_{i-1}) ds + \int_{\tau_{i-1}}^{\tau_i} b(X_{i-1}) dL_s \right) + \int_{\tau_n}^{t} a(X_n) ds \]
\[ + \int_{\tau_n}^{t} b(X_n) dL_s + \sum_{i=1}^{n} R(\tau_{i-1}, \tau_i) + R(\tau_n, t) \]  \hspace{1cm} (30)

\[ Y(t) = y_0 + \sum_{i=1}^{n} \left( \int_{\tau_{i-1}}^{\tau_i} a(Y_{i-1}) ds + \int_{\tau_{i-1}}^{\tau_i} b(Y_{i-1}) dL_s \right) \]
\[ + \int_{\tau_n}^{t} a(Y_n) ds + \int_{\tau_n}^{t} b(Y_n) dL_s \]  \hspace{1cm} (31)

where \( n = n(t, \omega) \). Set \( Z(t) = E \left[ \sup_{0 \leq s \leq t} |X(s) - Y(s)|^2 \right] \) and define a function \( p(t) = \max \{ n : \tau_n \leq t \} \). The next step is to find a bound on \( Z \),

\[ Z(t) = E \left[ \sup_{0 \leq s \leq t} \left| \sum_{i=1}^{n} \left( \int_{\tau_{i-1}}^{\tau_i} a(X_{i-1}) - a(Y_{i-1}) dl \right) \right. \]
\[ + \int_{\tau_n}^{s} a(X_n) - a(Y_n) dl + \sum_{i=1}^{n} \left( \int_{\tau_{i-1}}^{\tau_i} b(X_{i-1}) - b(Y_{i-1}) dL_l \right) \]
\[ + \int_{\tau_n}^{s} b(X_n) - b(Y_n) dL_l + \sum_{i=1}^{n} R(\tau_{i-1}, \tau_i) + R(\tau_n, s)^2 \right] \]  \hspace{1cm} (32)

Then using the inequality \( (a + b + c)^2 \leq 3a^2 + 3b^2 + 3c^2 \),

\[ \leq 3E \left[ \sup_{0 \leq s \leq t} \left| \int_{0}^{s} a(X_{p(l)}) - a(Y_{p(l)}) dl \right|^2 \right] \]
\[ + 3E \left[ \sup_{0 \leq s \leq t} \left| \int_{0}^{s} b(X_{p(l)}) - b(Y_{p(l)}) dL_l \right|^2 \right] \]
\[ + 3E \left[ \sup_{0 \leq s \leq t} \left| \int_{0}^{s} r_1(p(l), l) dl \right| \right] \]
\[ + \int_{0}^{s} r_2(p(l), l) dB_l + \int_{0}^{s} \int_{R_0} \left( r_3(p(l), l) (\mu - \pi)(dl, dz) \right)^2 \]  \hspace{1cm} (33)
again using the inequality \((a+b+c)^2 \leq 3a^2 + 3b^2 + 3c^2\) then Doob's maximal inequality and the Ito isometries,

\[
\leq 3TE\left[\int_0^t |a(X_{p(l)}) - a(Y_{p(l)})|^2 \, dl\right]
+ 24\sigma^2 E\left[\int_0^t b(X_{p(l)}) - b(Y_{p(l)}) dB_l\right]^2
+ 24E\left[\int_0^t \int_{R_0} (b(X_{p(l)}) - b(Y_{p(l)}))z(\mu - \pi)(dl, dz)\right]^2
+ 9TE\left[\int_0^t |r_1(p(l), l)|^2 \, dl\right] + 36E\left[\int_0^t |r_2(p(l), l)|^2 \, dl\right]
+ 36 \int_{R_0} z^2 \nu(dz) E\left[\int_0^t |r_3(p(l), l)|^2 \, dl\right]
\]

\[
(34)
\]

Then we proceed by again using the Ito isometries

\[
\leq 3T(C_2)^2 E\left[\int_0^t |X_{p(l)} - Y_{p(l)}|^2 \, dl\right]
+ 24\sigma^2(C_2)^2 E\left[\int_0^t |X_{p(l)} - Y_{p(l)}|^2 \, dl\right]
+ 24(C_2)^2 \int_{R_0} z^2 \nu(dz) E\left[\int_0^t |X_{p(l)} - Y_{p(l)}|^2 \right]
+ (9T + 72) \int_0^T E\left[\max_{1 \leq i \leq 3} |r_i(p(l), l)|^2 \right] \, dl
\leq d_1 \int_0^T E\left[\sup_{0 \leq l \leq s} |X(l) - Y(l)|^2 \right] \, ds + (9T + 72)TC\delta
= d_1 \int_0^T Z(s) \, ds + d_2 \delta
\]

\[
(35)
\]

where \(d_1\) and \(d_2\) are constants not depending on \(\delta\). By Gronwall's inequality we then obtain the following bound on \(Z(t)\):

\[
Z(t) \leq d_2 \delta + d_1 \int_0^t e^{d_1(t-s)} d_2 \delta \, ds \leq d_2(1 + d_1 T e^{d_1 T}) \delta \leq C \delta
\]

Or, equivalently: \(E(\sup_{0 \leq t \leq T} |X(t) - Y(t)|^2) \leq C \delta\). The proof is now completed by using the Cauchy-Schwarz inequality,

\[
E\left[\sup_{0 \leq t \leq T} |X(t) - Y(t)|\right] \leq E\left[\sup_{0 \leq t \leq T} |X(t) - Y(t)|^2\right]^{\frac{1}{2}} \leq C^{\frac{1}{2}} \delta^{\frac{1}{2}}
\]

\[
\square
\]

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References

