AN OBSTRUCTION TO ISOMETRIC IMMERSIONS OF THE THREE DIMENSIONAL HEISENBERG GROUP INTO $\mathbb{R}^4$.

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ABSTRACT. It is known that three dimensional Riemannian spaces of negative curvature are not locally immersable into the Euclidean four-space $\mathbb{R}^4$. The Heisenberg group with an arbitrary G-invariant metric does not have negative curvature. In this paper we will prove that there are no local isometric immersions of the three dimensional Heisenberg group into the Euclidean space $\mathbb{R}^4$.

1. INTRODUCTION

A special case of a result, due to Otsuki [3] is that if the sectional curvature of a Riemannian manifold is strictly negative, then any local isometric immersion of the Riemannian manifold into an Euclidean space is of codimension at least one less that the dimension of the manifold. As a special case, this result gives obstructions on local isometric immersions to a wide class of Riemannian manifolds of dimension three into $\mathbb{R}^4$.

The Heisenberg group equipped with some G-invariant Riemannian metric does not have negative sectional curvature. The following theorem gives an obstruction to the existence of local isometric immersion for all G-invariant metrics of the Heisenberg group.

Let $*$ denote the Hodge star operator, and let $R : \bigwedge^2 TM \to \bigwedge^2 TM$ denote the curvature operator of a Riemannian manifold M. Let C denote the covariant derivative of R.

Theorem 1.1. *The following formula is an invariant of local codimension* 1 *isometric immersion of a three dimensional Riemannian manifold M into Euclidean space:*

$$f(R^{-1}, C) = \sum_{i=1}^{3} \langle * \circ R^{-1} \circ C_{\Xi}(e_i), e_i \rangle, \tag{1.1}$$

where $C_{\Xi}(X) = (\nabla_{\Xi(X)} R)(\Xi(X) \wedge X)$ and Ξ is an endomorphism of V which permuting two of the vectors and fixes the third vector in the basis $\{e_1, e_2, e_3\}$. This means that if R and C are the curvature tensor and its covariant derivative of an immersed manifold of codimension 1 *into Euclidean space, we have $f(R^{-1}, C) = 0$.*

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Remark 1.2. In (1.1) we can replace R^{-1} with the transposed cofactor matrix of R without violating theorem 1.1.

Theorem 1.3. *There is no isometric immersion of the three dimensional Heisenberg group equipped with a G-invariant metric into the Euclidean space $\mathbb{R}^4$.*

2. On local isometric immersions

Let M denote a Riemannian manifold and let p be a fixed point in M. For an eventual local isometric immersion, let α be the second fundamental form and let β be its normal covariant derivative. Recall that by the Codazzi equation, we have that β is symmetric. Define L and B by $\langle L(X), Y\rangle = \langle \alpha(X,Y), \xi\rangle$ and $\langle B(X,Y), Z\rangle = \langle \beta(X,Y,Z), \xi\rangle$, where $X, Y, Z \in T_eG$ are tangent vectors at e and $\xi \in N_eG$ is a normal vector of the immersion at e.

Proof of Theorem 1.1. The Gauss equation gives $R^{-1} = L^{-1} \wedge L^{-1}$ and the prolonged Gauss-Codazzi equation is $(\nabla_T R) = L \wedge B(T, \bullet) + B(T, \bullet) \wedge L$, (see [2].) Thus, $C_\Xi(X) = L(\Xi(X)) \wedge B(\Xi(X), X) + B(\Xi(X), \Xi(X)) \wedge L(X)$. Therefore, $R^{-1} \circ C_\Xi(X) = \Xi(X) \wedge L^{-1} \circ B(\Xi(X), X) + L^{-1} \circ B(\Xi(X), \Xi(X)) \wedge X$. Now, since $\langle *(X \wedge Y), Z\rangle = \det(X|Y|Z)$, we have

$$
\begin{aligned}
(2.1)\quad \langle * \circ R^{-1} \circ C_\Xi(X), X\rangle &= \det(\Xi(X)|L^{-1} \circ B(\Xi(X), X)|X) \\
&\quad + \det(L^{-1} \circ B(\Xi(X), \Xi(X))|X|X) \\
&= \det(L^{-1}) \det(L(\Xi(X))|B(\Xi(X), X)|L(X)) + 0 \\
&= \det(L^{-1})\langle * \circ R(\Xi(X) \wedge X), B(\Xi(X), X)\rangle.
\end{aligned}
$$

We can assume that Ξ permutes e_1 and e_2. Therefore

$$
\begin{aligned}
(2.2)\quad & f(R^{-1}, C) \\
&= \det(L^{-1})\langle R(e_2 \wedge e_1), B(e_2, e_1)\rangle + \det(L^{-1})\langle R(e_1 \wedge e_2), B(e_1, e_2)\rangle = 0.
\end{aligned}
$$

□

3. On the curvature of the Heisenberg group

3.1. **On the Levi-Civita connection of homogeneous spaces.** We recall some facts about homogeneous manifolds.

Definition 3.1. A Riemannian space M is called homogeneous if the isometry group acts transitively, i.e. for each pair of points of M there is an isometry which takes one of the points to the other.

A homogeneous space can be modeled by a quotient G/H of Lie groups $G \supset H$, where H is compact.

Definition 3.2. **A homogeneous space** is a reductive homogeneous space if there exists a vector space decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ of the Lie algebra $\mathfrak{g}$ of G where m is invariant under the action of H. Without the metric data we will denote G/H as an abstract homogeneous space.

Definition 3.3. If a Riemannian metric of G/H makes it to a homogeneous space, the metric is said to be a homogeneous metric.

Let $M = G/H$ be an n-dimensional abstract reductive homogeneous space, and let $\langle , \rangle$ be a H invariant metric on $\mathfrak{m}$. Then the metric induced by $\langle , \rangle$ is homogeneous. Let $\{Z_1^*, \dots, Z_n^*\}$ be Killing fields on M which have linearly independent values at the point $o = eH$ in M, where e is the identity element of G. This makes a local Killing frame in a neighborhood U_o of o. Define $g_{ij} = \langle Z_i^*, Z_j^* \rangle$. Let X^* and Y^* be Killing fields on M.

Lemma 3.4. *Let the data be as above. Then* $\nabla_{X^*} Y^* = \sum a^i Z_i^*$ *where*

$$a^i = \sum_j g^{ij} \frac{1}{2} \left\{ \langle [X^*, Y^*], Z_j^* \rangle + \langle [Y^*, Z_j^*], X^* \rangle + \langle [X^*, Z_j^*], Y^* \rangle \right\}.$$

Proof. Let $\nabla_{X^*} Y^* = \sum a^i Z_i^*$. It is well known (see e.g. [1, Lemma 7.27 on page 183]) that if X^*, Y^*, and Z^* are Killing fields, then

$$\langle \nabla_{X^*} Y^*, Z^* \rangle = \frac{1}{2} \left\{ \langle [X^*, Y^*], Z^* \rangle + \langle [Y^*, Z^*], X^* \rangle + \langle [X^*, Z^*], Y^* \rangle \right\}. \tag{3.1}$$

Therefore, we have

$$\begin{aligned} \sum g_{ij} a^i &= \langle \nabla_{X^*} Y^*, Z_j^* \rangle \\ &= \frac{1}{2} \left\{ \langle [X^*, Y^*], Z_j^* \rangle + \langle [Y^*, Z_j^*], X^* \rangle + \langle [X^*, Z_j^*], Y^* \rangle \right\}. \end{aligned}$$

□

3.2. **The Heisenberg Lie-algebra.** The Lie-algebra of the three dimensional Heisenberg group is known to be on the form shown in figure 1. It is not diffi-

[,]	e1	e2	e3
e1	0	0	0
e2	0	0	e1
e3	0	−e1	0

FIGURE 1. Multiplication table of the Heisenberg group

cult to see that for any inner product on the Heisenberg Lie-algebra, we can do a Gram-Schmidt process and obtain a new basis of perpendicular vectors satisfying the same multiplication table in figure 1.

By implementing lemma 3.4 into a computer by using the algebraic programming system Maple, we obtain the nonzero components of the curvature tensor and its covariant derivative:

$$\begin{aligned} R_{1212} &= -\tfrac{1}{4} g_{22} {g_{11}}^3 \\ R_{1313} &= -\tfrac{1}{4} g_{33} {g_{11}}^3 \\ R_{2323} &= \tfrac{3}{4} g_{11} \end{aligned}$$

$$\begin{aligned} R_{1223;2} &= \tfrac{1}{2} g_{22} {g_{11}}^3 \\ R_{1323;3} &= \tfrac{1}{2} g_{33} {g_{11}}^3 \end{aligned}$$

Remark 3.5. In the above expressions we have used as convention that the determinant of the metric tensor is 1 at the identity.

Let Ξ be the endomorphism of the Heisenberg Lie-algebra which permute e_1 and e_2 and fixes e_3. Now, $C_\Xi(\mathrm{e}_1) = R_{2123;2}\mathrm{e}_2 \wedge \mathrm{e}_3$, $C_\Xi(\mathrm{e}_2) = 0$ and $C_\Xi(\mathrm{e}_3) = 0$. Thus, we have $* \circ R^{-1} \circ C_\Xi(\mathrm{e}_1) = {R^{-1}}_{2323}R_{2123;2}\mathrm{e}_1$. Thus $f(R^{-1}, C) = -\frac{2}{3}g_{22}{g_{11}}^2 \neq 0$. This proves theorem 1.3.

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