A semigroup approach to impulse control problems

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We state a verification theorem for a monotone control problem with fixed transaction costs. We then adapt the theory for semi group quasi variational inequalities to show that a solution exists. This will show that the dynamic programming principle holds and that an optimal control exists.

KEY WORDS Impulse control, semi group theory, quasi variational inequalities, transaction costs, dynamic programming principle.

1 Introduction

This paper focuses on the problem of investing in an uncertain market, when the investments are considered to be irreversible. This means that once an investment has been made and the market later drops to a less favorable state, we cannot undo the investment. The risk of overinvesting means that we should wait longer to invest, than if the investments were totally or partially reversible (see [1]). On the other hand we do not want to wait too long and miss out on any profits due to our lack of capacity. The problem then is to find a proper investment strategy taking the fluctuating marked into account. Numerous examples of irreversible investments exist, for example purchase of highly specified production machinery, educating staff members or spending money on advertising. See Dixit and Pindyck [4] for further economic discussions of the problem.

In order to model this problem we will use an n-dimensional Ito diffusion as the market process \( \Theta_t \). It takes values in the set \( E \subseteq \mathbb{R}^n \). We assume that our investments have two effects on our economy. The first is that the income will increase. In general the income will depend on the current state of the market and the current investment level, which will be denoted by \( \theta \) and \( k \), respectively. This is reasonable since a favourable market could for instance mean greater sales of a
product. On the other hand a high capacity could result in higher maintenance costs. The income function will be denoted by

\[ \Pi(\theta, k) : E \times [0, \infty)^m \rightarrow [0, \infty) \]

we assume that \( \Pi \) is bounded and uniformly continuous.

The second effect our investment has is obviously that it costs money. The cost of an investment increase by \( \delta \) when the market is in the state \((\theta, k)\) will be given by the function \( C \):

\[ C(\theta, k, \delta) : E \times [0, \infty)^m \times K \rightarrow [c, \infty) \]

Here \( K \) represents the set of all permitted investment increases. Also we assume that \( C \) takes values in the interval \([c, \infty)\) for some \( c > 0 \). This means that to each investment there is associated a minimum cost (transaction cost) independent of the size of our investment.

In addition we have a discount factor \( \lambda \) built into the model. This factor is considered to be strictly positive and constant.

We will show that the solution to the problem is to find a forbidden region \( \mathcal{F} \subseteq E \times [0, \infty)^m \) and an investment region \( \mathcal{I} \subseteq E \times [0, \infty)^m \) such that the optimal solution is to invest whenever \( (\Theta_t, k) \) hits \( \mathcal{F} \) and then invest until we are outside \( \mathcal{I} \). Suppose we start in the point \( A \) in the figure below. Then we should wait until \( (\Theta_t, k) \) hits \( B \) before we invest. The optimal strategy then is to invest our way out of \( \mathcal{I} \) (point \( C \)).

The object of this paper is to use semigroup methods to show that the optimal control exists and is of the form above. We also want to show that the value function satisfies the dynamic programming principle. In section 3 we give a verification theorem for the optimal value function. In section 4 we use theory for semi group quasi variational inequalities developed by Bensoussan and Lions [2] to show that there exist functions satisfying the conditions in the verification theorem. Note that for the results we only require boundedness from above of the diffusion term, and not boundedness from below or ellipticity as is required in some papers using ordinary quasi variational equations.
2 The control problem

We study the control of the n-dimensional process

\[ d\Theta_t = \mu(\Theta_t)dt + \sigma(\Theta_t)dB_t \]  

(1)

where \( B_t \) is n-dimensional Brownian motion and \( \mu : \mathbb{R}^n \to \mathbb{R}^n \) and \( \sigma : \mathbb{R}^n \to \mathbb{R}^n \) are bounded Lipshitz continuous functions. The controls are denoted by \( K_t(\omega) : [0, \infty) \times \Omega \to [0, \infty]^m \). The effect of the control on the market is given by

\[ d\Theta^K_t = \mu(\Theta^K_t)dt + \sigma(\Theta^K_t)dB_t - \gamma dK_t \]  

(2)

where \( \gamma \) is a \( n \times m \)-matrix.

Because of the fixed transaction costs, we must have a finite number of investments within each finite period of time. Therefore the controls can be written

\[ K_t = \sum_{i=0}^{\infty} k_i(\omega)\mathcal{X}_{\tau_i \leq t < \infty} \]

where we assume:

i. \( \tau_i \) is a \( \mathcal{F}_t \)-stopping time, where \( \mathcal{F}_t \) is the filtration generated by the driving Brownian motion.

ii. \( \tau_0 = 0 \)

iii. \( \tau_i \leq \tau_{i+1} \)

iv. \( \sum_{i=1}^{\infty} E\left[e^{-\lambda \tau_i}\right] < \infty. \)

v. \( k_i(\omega) \in \mathcal{K} \), where \( \mathcal{K} \subseteq [0, \infty)^m \) is a closed set.

vi. \( k_i(\omega) \) is \( \mathcal{F}_{\tau_i} \)-measurable.

If the fourth assumption does not hold, then the total costs of the investments will be infinite, therefore controls not satisfying (iv) are not optimal. Discarding such controls will simplify some of the proofs considerably.

The existence of the process \( \Theta_t \) satisfying (2) is established from ordinary theory of stochastic differential equations.

We want to maximize the expected profit after we have deducted the costs associated with each investment. In other words we want to find

\[
\sup_{K_t} \left\{ E^\theta \left[ \int_0^\infty e^{-\lambda t}\Pi(\Theta^K_t, K_t) \, dt \right] - \sum_{i=1}^{\infty} E^\theta \left[ e^{-\lambda \tau_i} C(\Theta^K_{\tau_i}, K_{\tau_i}, dK_{\tau_i}) \right] \right\}
\]

We also want to know if the optimal control exists.
3 SUFFICIENT CONDITIONS

We can now give sufficient conditions for a solution to the problem. In the following results \( \Theta_t \) denotes the uncontrolled process given in (1). The controlled process is denoted by \( \Theta^K_t \).

**Proposition 3.1.** Suppose we are given bounded continuous functions \( u \) and \( \{g_j\}_{j=1}^{\infty} \). Define

\[
    u_j \triangleq E^\theta \left[ \int_0^\infty e^{-\lambda t} g_j(\Theta_t, k) dt \right] \\
    Mu \triangleq \sup_{\delta \in K} \{ u(\theta - \gamma \delta, k + \delta) - C(\theta, k, \delta) \} \\
    D_j \triangleq \{ (\theta, k) : g_j = \Pi \} \\
    F \triangleq \{ (\theta, k) : u = Mu \}
\]

Suppose further that

i. \( u_j \to u \) uniformly

ii. \( g_j \geq \Pi \)

iii. \( u \geq Mu \)

iv. \( D_j \subseteq D_{j+1} \)

v. \( \bigcup_j D_j \supseteq F \)

Then for all stopping times \( \tau \)

\[
    u(\theta, k) = \sup_{K_t} \left\{ E^\theta \left[ \int_0^\tau e^{-\lambda \tau} \Pi(\Theta^K_t, K_t) d\tau - \sum_{\tau_i \leq \tau, \tau_i \geq 1} e^{-\lambda \tau_i} C(\Theta^K_{\tau_{i-1}}, K_{\tau_{i-1}}, dK_{\tau_i}) + e^{-\lambda \tau} u(\Theta^K_{\tau}, K_{\tau}) \right] \right\}
\]

And for all \( \tau \) the supremum holds for the control

\[
    K^*_t = \sum_{i=1}^{\infty} k_i(\omega) \chi_{\{ \tau_i^* \leq t < \infty \}}
\]

where \( k_0 = k \), \( \tau_0 = 0 \), \( \tau_{i+1}^* \) is the first hitting time after \( \tau_i^* \) of the process \( (\Theta^K_t, K_t(\omega)) \) to the set \( F \) and

\[
    k_{i+1}(\omega) = \inf \{ \delta > 0 : u(\Theta^K_{\tau_{i-1}}, \gamma \delta, k_i + \delta) - u(\Theta^K_{\tau_{i-1}}, k_i, \delta) = C(\Theta^K_{\tau_{i-1}}, k_i, \delta) \}
\]

Before we prove this we need some generalizations of Dynkin's theorem (see Dynkin [5] p.132). Here \( \Theta_t \) is the controlled process in (2) and \( \Theta_t^Z(\omega) \) denotes the Itô diffusion satisfying the equation

\[
    d\Theta_t^Z = Z(\omega) + \int_\tau^s \mu(\Theta_t^Z) dt + \int_\tau^s \sigma(\Theta_t^Z) dB_t \quad s \geq \tau
\]

for some \( \mathcal{F}_\tau \)-measurable \( Z(\omega) \). For simplicity of notation we will write:

\[
    R_\lambda f \triangleq E^\theta \left[ \int_0^\infty e^{-\lambda t} f(\Theta_t) dt \right]
\]

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Lemma 3.2. Let \( \tau \) denote a stopping time and suppose \( f \) is bounded. Then
\[
E \left[ \int_{\tau}^{\infty} e^{-\lambda s} f(\Theta_s^Z) ds \right] = E \left[ e^{-\lambda \tau} R_\lambda f(Z) \right]
\]

Proof.
\[
E \left[ \int_{\tau}^{\infty} e^{-\lambda s} f(\Theta_s^Z) ds \right] = E \left[ e^{-\lambda \tau} \int_{0}^{\infty} e^{-\lambda s} f(\Theta_{s+\tau}^Z) ds \right]
\]
\[
= E \left[ e^{-\lambda \tau} E \left[ \int_{0}^{\infty} e^{-\lambda s} f(\Theta_{s+\tau}^Z) ds \mid \mathcal{F}_\tau \right] \right]
\]

Moving the conditional expectation inside the integral gives
\[
= E \left[ e^{-\lambda \tau} \int_{0}^{\infty} e^{-\lambda s} E \left[ f(\Theta_{s+\tau}^Z) \mid \mathcal{F}_\tau \right] ds \right]
\]

From the Markov property
\[
= E \left[ e^{-\lambda \tau} \int_{0}^{\infty} e^{-\lambda s} E^Z \left[ f(\Theta_s) \right] ds \right] = E \left[ e^{-\lambda \tau} R_\lambda f(Z) \right]
\]

Corollary 3.3. Suppose \( \tilde{\tau} \geq \tau \) is a stopping time, then
\[
E \left[ \int_{\tilde{\tau}}^{\infty} e^{-\lambda s} f(\Theta_s^{\tilde{\tau} K}) ds \right] = E \left[ e^{-\lambda \tilde{\tau}} R_\lambda f(\Theta_{\tilde{\tau} K}) \right]
\]

Proof. Use \( Z = \Theta_{\tilde{\tau} K} \). Then \( \Theta_s^Z = \Theta_s^{\tilde{\tau} K} \) for \( t \geq \tilde{\tau} \). Thus
\[
E \left[ \int_{\tilde{\tau}}^{\infty} e^{-\lambda s} f(\Theta_s^{\tilde{\tau} K}) ds \right] = E \left[ \int_{\tilde{\tau}}^{\infty} e^{-\lambda s} f(\Theta_s^Z) ds \right]
\]
\[
= E \left[ e^{-\lambda \tilde{\tau}} f(Z) \right] = E \left[ e^{-\lambda \tilde{\tau}} R_\lambda f(\Theta_{\tilde{\tau} K}) \right]
\]

Corollary 3.4. Suppose \( P \) is a \( \mathcal{F}_{\tilde{\tau}, \tau} \)-measurable set and define
\[
\tilde{\tau}_i \triangleq \begin{cases} 
\tau_i \wedge \tau & \text{for } \omega \in P \\
\infty & \text{for } \omega \not\in P
\end{cases}
\]

Then \( \tilde{\tau}_i \) is a stopping time and
\[
E \left[ \int_{\tilde{\tau}_i}^{\tilde{\tau}_{i+1}} e^{-\lambda t} f(\Theta_t^K) dt \right] = E^\theta \left[ e^{-\lambda \tilde{\tau}_i} R_\lambda f(\Theta_{\tilde{\tau}_i}^K) \right] - E^\theta \left[ e^{-\lambda \tilde{\tau}_{i+1}} R_\lambda f(\Theta_{\tilde{\tau}_{i+1}}^K) \right] \tag{7}
\]
Proof. Since $P$ is $\mathcal{F}_{\tau_i}$-measurable we have
\[
\{\omega : \tau_i(\omega) \leq t\} = P \cap \{\omega : \tau_i \wedge t \leq t\} \in \mathcal{M}_t
\]
Thus $\tau_i$ is a stopping time.

From corollary 3.4 equation (7) holds for $\Theta_t^{\Theta_t^{\mathcal{F}_{\tau_i}}}$, but for $t \in [\tau_i \wedge \tau, \tau_{i+1} \wedge \tau)$ then $\Theta_t^K = \Theta_t^{\Theta_t^{\mathcal{F}_{\tau_i}}}$.

\begin{lemma}
Let $\tau$ be a stopping time and suppose $k(\omega)$ is $\mathcal{F}_{\tau_i \wedge \tau}$-measurable. Then for all bounded continuous $f$ we have
\[
E \left[ \int_{\tau_i \wedge \tau}^{\tau_{i+1} \wedge \tau} e^{-\lambda t} f(\Theta_t^K, k(\omega)) dt \right]
\]
\[
= E^\theta \left[ e^{-\lambda \tau_i \wedge \tau} R_{\lambda} f(\Theta_{\tau_i \wedge \tau}^{\mathcal{F}_{\tau_i}}, k(\omega)) \right] - E^\theta \left[ e^{-\lambda \tau_{i+1} \wedge \tau} R_{\lambda} f(\Theta_{\tau_{i+1} \wedge \tau}^{\mathcal{F}_{\tau_i}}, k(\omega)) \right]
\]
\end{lemma}

Proof. Assume that $k(\omega)$ only obtains the values $j \cdot 10^{-n} = (j_1, j_2, \ldots, j_m) \cdot 10^{-n}$, where $j \in \{0, 1, 2, \ldots, 10^2\}^m$. Define
\[
\tau_{i,j}^{n,j} = \begin{cases} 
\tau_i \wedge \tau & \text{if } k(\omega) = j \cdot 10^{-n} \\
\infty & \text{if } k(\omega) \neq j \cdot 10^{-n}
\end{cases}
\]
Then
\[
E^\theta \left[ \int_{\tau_i \wedge \tau}^{\tau_{i+1} \wedge \tau} e^{-\lambda t} f(\Theta_t^K, k(\omega)) dt \right] = \sum_j E^\theta \left[ \int_{\tau_i \wedge \tau}^{\tau_{i+1} \wedge \tau} e^{-\lambda t} f(\Theta_t^K, k(\omega)) \chi_{k(\omega) = j \cdot 10^{-n}} dt \right]
\]
But note that whenever $k(\omega) = j \cdot 10^{-n}$, then $\tau_{i,j}^{n,j} = \tau_i \wedge \tau$ and $\tau_{i+1,j}^{n,j} = \tau_{i+1} \wedge \tau$, and if $k(\omega) \neq j \cdot 10^{-n}$ then $\tau_{i,j}^{n,j} = \tau_{i+1}^{n,j} = \infty$. Therefore the expression equals
\[
= \sum_j E^\theta \left[ \int_{\tau_{i,j}^{n,j}}^{\tau_{i+1,j}^{n,j}} e^{-\lambda t} f(\Theta_t^K, j \cdot 10^{-n}) dt \right]
\]
Applying lemma 3.4 then gives
\[
= \sum_j \left( E^\theta \left[ e^{-\lambda \tau_{i,j}^{n,j}} R_{\lambda} f(\Theta_{\tau_{i,j}^{n,j}}^{\mathcal{F}_{\tau_i}}, j \cdot 10^{-n}) \right] - E^\theta \left[ e^{-\lambda \tau_{i+1,j}^{n,j}} R_{\lambda} f(\Theta_{\tau_{i+1,j}^{n,j}}^{\mathcal{F}_{\tau_i}}, j \cdot 10^{-n}) \right] \right)
\]
\[
= E^\theta \left[ e^{-\lambda \tau_i \wedge \tau} R_{\lambda} f(\Theta_{\tau_i \wedge \tau}^{\mathcal{F}_{\tau_i}}, k(\omega)) \right] - E^\theta \left[ e^{-\lambda \tau_{i+1} \wedge \tau} R_{\lambda} f(\Theta_{\tau_{i+1} \wedge \tau}^{\mathcal{F}_{\tau_i}}, k(\omega)) \right]
\]
The result holds if $k(\omega)$ only assumes finitely many values. The extension to general $k(\omega)$ functions is easy.

\begin{corollary}
\[
E^\theta \left[ \int_{\tau_i \wedge \tau}^{\tau_{i+1} \wedge \tau} e^{-\lambda t} f(\Theta_t^K, k_i(\omega)) dt \right]
\]
\[
= E^\theta \left[ e^{-\lambda \tau_i \wedge \tau} R_{\lambda} f(\Theta_{\tau_i \wedge \tau}^{\mathcal{F}_{\tau_i}}, k_i(\omega)) \right] - E^\theta \left[ e^{-\lambda \tau_{i+1} \wedge \tau} R_{\lambda} f(\Theta_{\tau_{i+1} \wedge \tau}^{\mathcal{F}_{\tau_i}}, k_i(\omega)) \right]
\]
\end{corollary}
Proof. $k_i(\omega)$ is not necessarily $F_{\tau_i,\tau}$-measurable. Choose

$$\tilde{k}_i(\omega) = k_i(\omega) \chi_{\tau_i \leq \tau}$$

Then $\tilde{k}_i(\omega)$ is $F_{\tau_i,\tau}$-measurable and the result holds for $\tilde{k}_i(\omega)$. The result follows since the equality above is unaffected by this alteration. \qed

**Lemma 3.7.**

$$E^\theta \left[ \int_{\tau_i,\tau}^{\tau_{i+1},\tau} e^{-\lambda t} \Pi(\Theta_t^K, k_i(\omega)) dt \right]$$

$$\leq E^\theta \left[ e^{-\lambda \tau_i,\tau} u(\Theta^K_{\tau_i,\tau}, k_i(\omega)) \right] - E^\theta \left[ e^{-\lambda \tau_{i+1},\tau} u(\Theta^K_{\tau_{i+1},\tau}, k_i(\omega)) \right]$$

Equality holds if $\{\tau_i\}_{i=1}^\infty$ and $\{k_i\}_{i=1}^\infty$ are chosen as in proposition 3.1.

**Proof.** The inequality follows easily since

$$E^\theta \left[ \int_{\tau_i,\tau}^{\tau_{i+1},\tau} e^{-\lambda t} \Pi(\Theta_t^K, k_i(\omega)) dt \right] \leq E^\theta \left[ \int_{\tau_i,\tau}^{\tau_{i+1},\tau} e^{-\lambda t} g_j(\Theta_t^K, k_i(\omega)) dt \right]$$

by corollary 3.6 this equals

$$= E^\theta \left[ e^{-\lambda \tau_i,\tau} u_j(\Theta^K_{\tau_i,\tau}, k_i(\omega)) \right] - E^\theta \left[ e^{-\lambda \tau_{i+1},\tau} u_j(\Theta^K_{\tau_{i+1},\tau}, k_i(\omega)) \right]$$

Letting $j \to \infty$ we get the inequality.

To see that the equality holds, let $\kappa_j$ denote the first exit time for the process $(\Theta_t, k_i)$ of the set $D_j$, where $D_j$ is as in proposition 3.1. Then since $g_j = \Pi$ in $D_j$ we have

$$E^\theta \left[ \int_{\tau_i,\tau \wedge \kappa_j}^{\tau_{i+1},\tau \wedge \kappa_j} e^{-\lambda t} \Pi(\Theta_t^K, k_i(\omega)) dt \right] = E^\theta \left[ \int_{\tau_i,\tau \wedge \kappa_j}^{\tau_{i+1},\tau \wedge \kappa_j} e^{-\lambda t} g_j(\Theta_t^K, k_i(\omega)) dt \right]$$

for all $j$. By using corollary 3.6 this equals

$$= E^\theta \left[ e^{-\lambda \tau_i,\tau \wedge \kappa_j} u_j(\Theta^K_{\tau_i,\tau \wedge \kappa_j}, k_i(\omega)) \right] - E^\theta \left[ e^{-\lambda \tau_{i+1},\tau \wedge \kappa_j} u_j(\Theta^K_{\tau_{i+1},\tau \wedge \kappa_j}, k_i(\omega)) \right]$$

Letting $j \to \infty$ we get the result. \qed

**Proof of Proposition 3.1.**

$$E^\theta \left[ \int_0^\tau e^{-\lambda t} \Pi(\Theta_t^K, K_i) dt \right] - E^\theta \left[ \sum_{\tau_i \leq \tau, i \geq 1} e^{-\lambda \tau_i} C(\Theta^K_{\tau_i}, K_{\tau_i}, dK_{\tau_i}) \right]$$

$$= \sum_{j=0}^\infty E^\theta \left[ \int_{\tau_i,\tau}^{\tau_{i+1},\tau} e^{-\lambda t} \Pi(\Theta_t^K, k_i) dt \right] - E^\theta \left[ \sum_{\tau_i \leq \tau, i \geq 1} e^{-\lambda \tau_i} C(\Theta^K_{\tau_i}, K_{\tau_i}, dK_{\tau_i}) \right]$$
for some constant $N$.

Proof. **Proof of 1)**

\[ E \left[ \left| \Theta_t^\theta - \Theta_t^{\tilde{\theta}} \right|^2 \right] = E \left[ \left( \theta - \tilde{\theta} + \int_0^t \mu(\Theta_s^\theta) - \mu(\Theta_s^{\tilde{\theta}}) ds + \int_0^t \sigma(\Theta_s^\theta) - \sigma(\Theta_s^{\tilde{\theta}}) dB_s \right)^2 \right] \]

By Hölder’s inequality and the Ito isometry

\[ \leq 3 |\theta - \tilde{\theta}|^2 + 3tE \left[ \int_0^t (\mu(\Theta_s^\theta) - \mu(\Theta_s^{\tilde{\theta}}))^2 ds \right] + E \left[ \int_0^t (\sigma(\Theta_s^\theta) - \sigma(\Theta_s^{\tilde{\theta}}))^2 ds \right] \]

Using the Lipschitz continuity we get

\[ \leq 3 |\theta - \tilde{\theta}|^2 + 3(1 + t)D^2 \int_0^t E \left[ \left| \Theta_s^\theta - \Theta_s^{\tilde{\theta}} \right|^2 \right] ds \]

By Gronwall’s inequality

\[ E \left[ \left| \Theta_t^\theta - \Theta_t^{\tilde{\theta}} \right|^2 \right] \leq |\theta - \tilde{\theta}|^2 Ne^{Nt} \]

for some constant $N$. Then applying Hölder’s inequality again we get the result.

**Proof of 2)**

\[ E \left[ \left| \Theta_t^\theta - \Theta_t^{\tilde{\theta}} \right|^2 \right] \leq E \left[ \left( \int_s^t \mu(\Theta_r^\theta) dr + \int_s^t \sigma(\Theta_r^\theta) dB_r \right)^2 \right] \]

Using the Ito isometry

\[ \leq E \left[ 2 \int_s^t \mu^2(\Theta_r^\theta) dr + 2 \int_s^t \sigma^2(\Theta_r^\theta) dr \right] \leq N(t - s) \]

for some $N \in \mathbb{R}$, since $\mu$ and $\sigma$ are bounded. Hölder’s inequality the gives the result. \[\square\]

**Proposition 4.4.** Suppose $\Pi \in \mathcal{C}$ and define

\[ \Phi(t)f \triangleq E^{\theta, k}[f(\Theta_t, k)] \]

Then $\Phi : \mathcal{B} \to \mathcal{B}$ is a semi-group and $\Phi(t)$ and $\Pi$ satisfy conditions 4.1.i-4.1.vii.

**Proof.** **Proof of 4.1.i)** This is the Markov property.

The proof of 4.1.ii)-4.1.iv) is omitted.

**Proof of 4.1.v)** Define $r(\delta)$ to be

\[ r(\delta) = \sup_{|\theta - \tilde{\theta}| \leq \delta} \{|f(\theta) - f(\tilde{\theta})|\} \quad (10) \]

then

\[ |f(\theta) - f(\tilde{\theta})| \leq r(\delta) + \frac{r(\delta)}{\delta} |\theta - \tilde{\theta}| \]

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Then
\[ |E \left[ f(\Theta_t^\delta) - f(\Theta_t^\bar{\delta}) \right] | \leq r(\delta) + \frac{r(\delta)}{\delta} E \left[ |\Theta_t^\delta - \Theta_t^{\bar{\delta}}| \right] \]

Lemma 4.3.1 gives:
\[ \leq r(\delta) + \frac{r(\delta)|\theta - \bar{\theta}|}{\delta} N e^{N t} \]  
(11)

\( f \) is uniformly continuous, therefore \( \lim_{\delta \to 0} r(\delta) = 0 \). Then (11) gives uniform continuity of \( \Phi(t)f \) also.

**Proof of 4.1.vi)** Define \( r(\delta) \) as in (10). Then
\[ |E^{\delta} [f(\Theta_t)] - f(\theta)| \leq r(\delta) + \frac{r(\delta)}{\delta} E^{\delta} \left[ |\Theta_t - \theta| \right] \]

Lemma 4.3.2 then shows the assertion.

**Proof of 4.1.vii)** Since we assumed \( \Pi \in C \) we have by the same methods as before.
\[ E \left[ |\Pi(\Theta_t^\delta) - \Pi(\Theta_t^{\bar{\delta}})| \right] \leq r(\delta) + \frac{r(\delta)}{\delta} E \left[ |\Theta_t^\delta - \Theta_t^{\bar{\delta}}| \right] \]

But by lemma 4.3.2
\[ \leq \sqrt{N(t - s)} \]
for some \( N \in R \). Therefore
\[ \sup_{\theta \in E} |E \left[ \Pi(\Theta_t^\delta) \right] - E \left[ \Pi(\Theta_t^{\bar{\delta}}) \right] | \to 0 \text{, as } t \to s \]

Then \( \Phi(t)\Pi \) is in fact continuous as a function of \( t \) into \( C \).

We have now proved all the assertions for each \( k \in [0, \infty)^m \). Generalizing the results to functions of \( (\theta, k) \) is then easy.

\( \square \)

From Bensoussan and Lions [2] we know that there exists a solution of the semigroup quasivariational inequality (SGQVI):

**Theorem 4.5.** Suppose \( \Pi \in C \), then the problem
\[ u \in C \]  
(12)
\[ u \geq Mu \]  
(13)
\[ u \geq E^{\theta} \left[ \int_0^t e^{-\lambda s} \Pi(\Theta_s, k) ds \right] + E^{\theta} \left[ e^{\lambda t} u(\Theta_t, k) \right] \text{ for all } t \geq 0 \]  
(14)

Has a minimum solution.

**Proof.** See Bensoussan and Lions [2] Theorem 5.2 pg.436 for the proof in a semigroup setting. By proposition 4.4 the result holds for \( \Phi(t)f \triangleq E^{\theta,k} \left[ f(\Theta_t) \right] \).

\( \square \)
We also present the semi group variational inequality (SGVI). Here $\psi$ is a fixed function in $C$

\[ u \in C \]  \hspace{1cm} (15)

\[ u \geq \psi \]  \hspace{1cm} (16)

\[ u \geq E^\theta \left[ \int_0^t e^{-\lambda s} \Pi(\Theta_s, k) ds \right] + E^\theta \left[ e^{-\lambda t} u(\Theta_t, k) \right] \quad \text{for all } t \geq 0 \]  \hspace{1cm} (17)

Then there exists a minimum element solving the semi group variational inequalities also (See [2], Theorem 5.1 on page 425.). For the SGVI we have the following result:

**Lemma 4.6.** Suppose $u$ is the minimum element that solves the SGVI. Then there exist functions $u_j \in C$ satisfying:

\[ u_j = E^\theta \left[ \int_0^\infty e^{-\lambda t} (\Pi + j(\psi - u_j^+)) \Theta_t, k dt \right] \]  \hspace{1cm} (18)

such that

\[ u_j \leq u_{j+1} \]

and

\[ \lim_{j \to \infty} |u_j - u|_{\text{sup}} = 0 \]

**Proof.** The proof can be found in Bensoussan and Lions [2]. The first part is Lemma 5.3 on page 426. The second is Lemma 5.7 on page 429. \hfill \square

From this result we deduce the following:

**Proposition 4.7.** Suppose $u$ is the minimum element of the SGQVI. Then $u$ is also the minimum element of the SGVI with $\psi = Mu$.

**Proof.** We know that the minimum solution of the SGQVI exists and belongs to $C$. We let this solution be denoted by $u$. Choose $\psi = Mu$ (which belongs to $C$) and let $\tilde{u}$ be the minimum solution of the SGVI with $Mu$ as the fixed function $\psi$. Then $u$ also satisfies this SGVI. Since $\tilde{u}$ is the minimum solution of this problem, then $\tilde{u} \leq u$. $M$ is an increasing operator, therefore $M\tilde{u} \leq Mu$. But $\tilde{u}$ solves the SGVI with $\psi = Mu$, in particular $\tilde{u} \geq Mu$. Hence $\tilde{u} \geq M\tilde{u}$ and $\tilde{u}$ solves the SGQVI also, but since $u$ is the minimum element satisfying the SGQVI, we get $u = \tilde{u}$. Therefore $u$ is the minimum element satisfying the SGVI with $\psi = Mu$. \hfill \square

**Theorem 4.8.** Let $u$ denote the minimum element solving the SGQVI. Let $u_j$ denote the solution of (18) with $\psi = Mu$ and define

\[ g_j \triangleq \Pi + j(Mu - u_j)^+ \]

Then $u$ and $\{g_j\}_{j=1}^\infty$ satisfy the conditions of proposition 3.1.
Proof. By lemma 4.6 and proposition 4.7 there exist $u_j \leq u_{j+1}$ such that

$$u_j = E^\theta \left[ \int_0^\infty e^{-\lambda t}(\Pi + j(Mu - u_j)^+)(\Theta_t, k)dt \right]$$

and $|u_j - u|_{sup} \to 0$ as $j \to \infty$. Easily $g_j \geq \Pi$. Furthermore

$$D_j \triangleq \{(\theta, k) : g_j = \Pi \} = \{(\theta, k) : u_j \geq Mu\}$$

Then easily $D_j \subseteq D_{j+1}$, since $u_j \leq u_{j+1}$. Whenever $u > Mu$ then $u_j \geq Mu$ for all large enough $j$. Thus

$$\bigcup_{j=1}^\infty D_j \supseteq \{(\theta, k) : u > Mu\} \triangleq \mathcal{F}$$

$\Box$
References


