Entropy in type I algebras

by

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Abstract

It is shown that if \((M, \phi, \alpha)\) is a W*-dynamical system with \(M\) a type I von Neumann algebra then the entropy of \(\alpha\) w.r.t. \(\phi\) equals the entropy of the restriction of \(\alpha\) to the center of \(M\). If furthermore \((N, \psi, \beta)\) is a W*-dynamical system with \(N\) injective then \(h_{\phi \otimes \psi}(\alpha \otimes \beta) = h_{\phi}(\alpha) + h_{\psi}(\beta)\).

1 Introduction

In the theory of non-commutative entropy the attention has almost exclusively been concentrated on non type I algebras. We shall in the present paper remedy this situation by proving the basic facts on entropy of automorphisms of type I C*- and von Neumann-algebras. The results are as nice as one can hope. The CNT-entropy of an automorphism of a von Neumann algebra of type I with respect to an invariant normal state is the classical entropy of the restriction of the automorphism to the center of the algebra. If one factor of a tensor product of two von Neumann algebras is of type I and the other injective, then the entropy of a tensor product automorphism with respect to an invariant product state is the sum of the entropies. The results have obvious corollaries to type I C*-algebras. The main idea behind our proofs is the use of conditional expectations of finite index, as employed in [GN].

We shall use the notation \(h_{\phi}(\alpha)\) for the CNT-entropy of a C*-dynamical system as defined by Connes, Narnhofer and Thirring in [CNT], and \(h'_{\phi}(\alpha)\) for the ST-entropy defined by Sauvageot and Thouvenot in [ST].

2 Main results

We first prove a general result for the Sauvageot-Thouvenot entropy for the restriction of an automorphism to a globally invariant C*-subalgebra of finite index. Recall the definition of ST-entropy and its connection with CNT-entropy.

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A stationary coupling of a C*-dynamical system \((A, \phi, \alpha)\) with a commutative system \((C, \mu, \beta)\) is an \(\alpha \otimes \beta\)-invariant state \(\lambda\) on \(A \otimes C\) such that \(\lambda|_A = \phi\) and \(\lambda|_C = \mu\). Given such a coupling and a finite-dimensional subalgebra \(P\) of \(C\) with atoms \(p_1, \ldots, p_n\), consider the quantity

\[
H_\mu(P|P^-) - H_\mu(P) + \sum_{i=1}^n \mu(p_i)S(\phi, \phi_i),
\]

where \(\phi_i(a) = \frac{1}{\mu(p_i)} \lambda(a \otimes p_i)\). By definition, the ST-entropy \(h_\phi'(\alpha)\) of the system \((A, \phi, \alpha)\) is the supremum of these quantities.

By [ST, Proposition 4.1], ST-entropy coincides with CNT-entropy for nuclear C*-algebras. In fact, the proof of the inequality \(h_\phi(\alpha) \leq h_\phi'(\alpha)\) does not use any assumptions on the algebra. On the other hand, given a coupling \(\lambda\) and an algebra \(P\) as above, for each \(m \in \mathbb{N}\) we can form the decomposition

\[
\phi = \sum_{i_1, \ldots, i_m=1}^n \phi_{i_1 \ldots i_m}(a) = \lambda(a \otimes p_1 \beta(p_{i_2}) \cdots k_{m-1}(p_{i_m})).
\]

If \(\gamma\) is a unital completely positive mapping of a finite-dimensional C*-algebra into \(A\), we can use these decompositions in computing the mutual entropy \(H_\phi(\gamma, \alpha \circ \gamma, \ldots, \alpha^{m-1} \circ \gamma)\) [CNT]. Indeed, since the atoms in \(\beta^j(P)\) are \(\beta^j(p_1), \ldots, \beta^j(p_n)\) we have by [CNT, III.3]

\[
H_\phi(\gamma, \alpha \circ \gamma, \ldots, \alpha^{m-1} \circ \gamma) \geq S\left(\mu \bigvee_{j=0}^{m-1} \beta^j(P)\right) - \sum_{j=0}^{m-1} S\left(\mu(\beta^j(P))\right)
\]

\[
+ \; \sum_{j}^{m-1} \sum_{i}^{m} \mu(\beta^j(p_i))S\left(\phi \circ \alpha^j \circ \gamma, \frac{\lambda((\alpha^j \circ \gamma)(\cdot) \otimes \beta^j(p_i))}{\mu(\beta^j(p_i))}\right).
\]

Hence by invariance of \(\phi, \mu\) and \(\lambda\) with respect to \(\alpha, \beta\) and \(\alpha \otimes \beta\) respectively

\[
\frac{1}{m} H_\phi(\gamma, \alpha \circ \gamma, \ldots, \alpha^{m-1} \circ \gamma) \geq \frac{1}{m} H_\mu\left(\bigvee_{j=0}^{m-1} \beta^j(P)\right) - H_\mu(P) + \sum_i \mu(p_i)S(\phi \circ \gamma, \phi_i \circ \gamma).\]

It follows that

\[
h_\phi(\alpha) \geq H_\mu(P|P^-) - H_\mu(P) + \sum_{i=1}^n \mu(p_i)S(\phi \circ \gamma, \phi_i \circ \gamma).\]

Thus what is really necessary for the coincidence of the entropies, is the existence of a net of unital completely positive mappings \(\gamma_i\) of finite-dimensional C*-algebras into \(A\) such that \(S(\phi, \psi) = \lim_i S(\phi \circ \gamma_i, \psi \circ \gamma_i)\) for any positive linear functional \(\psi\) on \(A\), \(\psi \leq \phi\). In particular, \(h_\phi(\alpha) = h_\phi'(\alpha)\) if \(A\) is an injective von Neumann algebra and \(\phi\) is a normal state on it.

**Proposition 1** Let \((A, \phi, \alpha)\) be a unital C*-dynamical system. Let \(B \subset A\) be an \(\alpha\)-invariant C*-subalgebra (with 1 \(\in B\)). Suppose there exists a conditional expectation \(E : A \rightarrow B\) such that \(E \circ \alpha = \alpha \circ E\), \(\phi \circ E = \phi\) and \(E(x) \geq cx\) for all \(x \in A^+\) for some \(c > 0\). Then \(h_\phi'(\alpha) = h_\phi'(\alpha|B)\).

**Proof.** Let \((C, \mu, \beta)\) be a C*-dynamical system with \(C\) abelian. Using \(E\) we can lift any stationary coupling on \(B \otimes C\) to a stationary coupling on \(A \otimes C\). This, together with the property of monotonicity of relative entropy, shows that \(h_\phi'(\alpha) \geq h_\phi'(\alpha|B)\).

Conversely, suppose \(\lambda\) is a stationary coupling of \((A, \phi, \alpha)\) with \((C, \mu, \beta), \) \(P\) a finite-dimensional subalgebra of \(C\) with atoms \(p_1, \ldots, p_n\), and \(\phi_i(a) = \frac{1}{\mu(p_i)} \lambda(a \otimes p_i)\) for \(a \in A\). Since
\[ \phi_i \leq \frac{1}{\mu(p)} \phi, \phi_i \text{ is normal in the GNS-representation of } \phi. \] Since \( E \) is \( \phi \)-invariant, it extends to a normal conditional expectation of the closure of \( A \) in the GNS-representation onto the closure of \( B \). Thus we can apply [OP, Theorem 5.15] to \( \phi \) and \( \phi_i \), and (as in the proof of Lemma 1.5 in [GN]) get

\[
\sum_{i=1}^{n} \mu(p_i) S(\phi, \phi_i) = \sum_{i=1}^{n} \mu(p_i) (S(\phi|B, \phi_i|B) + S(\phi_i \circ E, \phi_i)) \leq \sum_{i=1}^{n} \mu(p_i) S(\phi|B, \phi_i|B) - \log c.
\]

It follows that \( h'_\phi(\alpha) \leq h'_\phi(\alpha|B) - \log c \). Then for each \( m \in \mathbb{N} \)

\[
h'_\phi(\alpha) = \frac{1}{m} h'_\phi(\alpha^m) \leq \frac{1}{m} h'_\phi(\alpha^m|B) - \frac{1}{m} \log c = h'_\phi(\alpha|B) - \frac{1}{m} \log c.
\]

Thus \( h'_\phi(\alpha) \leq h'_\phi(\alpha|B) \).

**Corollary 2** If in the above proposition \( A \) and \( B \) are injective von Neumann algebras and \( \phi \) is normal then \( h_\phi(\alpha) = h_\phi(\alpha|B) \).

To prove our main result we need also two simple lemmas. The first lemma is more or less well-known.

**Lemma 3** Let \((M, \phi, \alpha)\) be a \( W^* \)-dynamical system. Then

(i) if \( p \) is an \( \alpha \)-invariant projection in \( M \) such that \( \text{supp } \phi \leq p \), then \( h_\phi(\alpha) = h_\phi(\alpha|\text{M}_p) \);

(ii) if \( \{p_i\}_{i \in I} \) is a set of mutually orthogonal \( \alpha \)-invariant central projections in \( M \), \( \sum_i p_i = 1 \), then

\[
h_\phi(\alpha) = \sum_i \phi(p_i) h_\phi(\alpha_i),
\]

where \( \phi_i = \frac{1}{\phi(p_i)} \phi \) is the normalized restriction of \( \phi \) to \( \text{M}_p \), and \( \alpha_i = \alpha|\text{M}_p_i \).

**Proof.** (i) easily follows from the definitions; (ii) follows from [CNT, VII.5(iii)], (i) and [SV, Lemma 3.3] applied to the subalgebras \( \text{M}(p_1 + \ldots + p_n) + C(1 - p_1 - \ldots - p_n) \).

The proof of the following lemma is left to the reader.

**Lemma 4** Let \( T \) be an automorphism of a probability space \((X, \mu), f \in L^\infty(X, \mu) \) a \( T \)-invariant function such that \( f \geq 0 \) and \( \int_X f \, d\mu = 1 \). Let \( \mu_f \) be the measure on \( X \) such that \( d\mu_f / d\mu = f \). Then \( h_{\mu_f}(T) \leq ||f||_\infty h_\mu(T) \).

**Theorem 5** Let \((M, \phi, \alpha)\) be a \( W^* \)-dynamical system with \( M \) a von Neumann algebra of type I. Let \( Z \) denote the center of \( M \). Then \( h_\phi(\alpha) = h_\phi(\alpha|Z) \).

**Proof.** By Lemma 3(i) we may suppose that \( \phi \) is faithful. Then \( M \) is a direct sum of homogeneous algebras of type \( I_n, n \in \mathbb{N} \cup \{\infty\} \). By Lemma 3(ii) we may assume that \( M \) is homogeneous of type \( I_n \). We first assume that \( n \in \mathbb{N} \). Then \( Z = L^\infty(X, \mu) \), where \((X, \mu)\) is a probability space and \( \phi|Z = \mu \). Thus

\[
M \cong Z \otimes \text{Mat}_n(\mathbb{C}) = L^\infty(X, \text{Mat}_n(\mathbb{C})), \quad \phi = \int_X \phi_x d\mu(x),
\]

where \( \phi_x = \text{Tr}(\cdot Q_x) \) is a state on \( \text{Mat}_n(\mathbb{C}) \), \( \text{Tr} \) the canonical trace on \( \text{Mat}_n(\mathbb{C}) \). We first assume \( Q_x \geq c > 0 \) for all \( x \).
If $s \in M^+$, $s$ is a function in $L^\infty(X, \text{Mat}_n(\mathbb{C}))$. Define the $\phi$-preserving conditional expectation $E: M \to Z$ by $E(s)(x) = \phi_x(s(x))$. Then

$$E(s)(x) = \text{Tr}(s(x)Q_x) \geq c\text{Tr}(s(x)) \geq cs(x),$$

so $E(s) \geq cs$, and it follows from Corollary 2 that $h_\phi(\alpha) = h_\phi(\alpha|Z)$.

If there is no $c > 0$ such that $Q_x \geq c$ for all $x$, let $X_c = \{x \in X \mid Q_x \geq c\}$, $(c > 0)$,

$$N_c = L^\infty(X_c, \text{Mat}_n(\mathbb{C})) \quad \text{and} \quad M_c = N_c + C\chi_{X \setminus X_c},$$

where $\chi_{X \setminus X_c}$ is the characteristic function of $X \setminus X_c$. Since $\phi$ is $\alpha$-invariant so is $M_c$, so by the above argument and Lemma 3, letting $\phi_c = \frac{1}{\mu(X_c)}\phi|_{N_c}$ and $\mu_c = \frac{1}{\mu(X_c)}\mu|_{X_c}$, we obtain

$$h_\phi(\alpha|M_c) = \mu(X_c)h_\phi(\alpha|N_c) = \mu(X_c)h_{\mu_c}(T|_{X_c}) \leq h_{\mu}(T),$$

where $T$ is the automorphism of $(X, \mu)$ induced by $\alpha$. Letting $c \to 0$ and using [SV, Lemma 3.3] we obtain the Theorem when $M$ is finite.

If $M$ is homogeneous of type I$_{\infty}$, we have $M \cong L^\infty(X, \mu) \otimes B(H)$, where $H$ is a separable Hilbert space. Let $\text{Tr}$ denotes the canonical trace on $B(H)$. Write again

$$\phi = \int_X \phi_x d\mu(x), \quad \phi_x = \text{Tr}(\cdot Q_x),$$

and let $E_x(U)$ denote the spectral projection of $Q_x$ corresponding to a Borel set $U$. Let $P_c \in M = L^\infty(X, B(H))$ be the projection defined by $P_c(x) = E_x([c, +\infty))$, where $c > 0$. Then $P_c$ is an $\alpha$-invariant finite projection. Let

$$M_c = P_cMP_c + C(1 - P_c).$$

Then $M_c$ is a finite type I von Neumann algebra. Its center is isomorphic to $L^\infty(X_c, \mu_c) \otimes C$, and the restriction of $\phi$ to it is $\phi(P_c)\mu_c \otimes \phi(1 - P_c)$, where $X_c = \{x \in X \mid P_c(x) \neq 0\}$ and

$$\int_{X_c} f(x) d\mu_c(x) = \frac{1}{\phi(P_c)} \int_{X_c} f(x)\phi_x(P_c(x)) d\mu(x).$$

So we can apply the first part of the proof to $M_c$. Since $d\mu_c/d\mu \leq \frac{1}{\phi(P_c)}$, applying Lemma 4 we get

$$h_\phi(\alpha|M_c) = \phi(P_c)h_{\mu_c}(T|_{X_c}) \leq h_{\mu}(T).$$

Now letting $c \to 0$ we conclude that $h_\phi(\alpha) = h_{\mu}(T)$.

It should be remarked that in a special case the above theorem was proved in [GS, Proposition 2.4].

If $A$ is a $C^*$-algebra and $\phi$ a state on $A$, the central measure $\mu_\phi$ of $\phi$ is the measure on the spectrum $\hat{A}$ of $A$ defined by $\mu_\phi(F) = \phi(\chi_F)$, where $\phi$ is regarded as a normal state on $A''$, see [P, 4.7.5]. Thus by Theorem 5 and [P, 4.7.6] we have the following

**Corollary 6** Let $(A, \phi, \alpha)$ be a $C^*$-dynamical system with $A$ a separable unital type I $C^*$-algebra. Then $h_\phi(\alpha) = h_{\mu_\phi}(\hat{\alpha})$, where $\hat{\alpha}$ is the automorphism of the measure space $(\hat{A}, \mu_\phi)$ induced by $\alpha$.

Since inner automorphisms act trivially on the center we have
Corollary 7 If \((M, \phi, \alpha)\) is a \(W^\ast\)-dynamical system with \(M\) of type I and \(\alpha\) an inner automorphism then \(h_\phi(\alpha) = 0\).

Note that in the finite case the above corollary also follows from a result of N. Brown [Br, Lemma 2.2].

The next result was shown in [S] when \(\phi\) is a trace.

**Corollary 8** Let \(R\) denote the hyperfinite II\(_1\) factor. Let \(A\) be a Cartan subalgebra of \(R\) and \(u\) a unitary operator in \(A\). If \(\phi\) is a normal state such that \(u\) belongs to the centralizer of \(\phi\) then \(h_\phi(\Ad u) = 0\).

**Proof.** As in [S], it follows from [CFW] that there exists an increasing sequence of full matrix algebras \(N_1 \subset N_2 \subset \ldots\) with union weakly dense in \(R\) such that \(A \cong A_n \otimes B_n\), where \(A_n = N_n \cap A\) and \(B_n = (N_n' \cap R) \cap A\) for all \(n \in \mathbb{N}\). Let \(M_n = N_n \otimes B_n\). Then \(M_n\) is of type I and contains \(u\). Hence \(h_\phi(\Ad u|_{M_n}) = 0\). Since \((\cup_n M_n)\sim = R\), \(h_\phi(\Ad u) = 0\) by [SV, Lemma 3.3].

If \((A, \phi, \alpha)\) and \((B, \psi, \beta)\) are \(C^\ast\)-dynamical systems we always have

\[h_{\phi \otimes \psi}(\alpha \otimes \beta) \geq h_\phi(\alpha) + h_\psi(\beta),\]

see [SV, Lemma 3.4]. The equality does not always hold, see [NST] or [Sa]. However, we have

**Theorem 9** Let \((A, \phi, \alpha)\) and \((B, \psi, \beta)\) be \(W^\ast\)-dynamical systems. Suppose that \(A\) is of type I, and \(B\) is injective. Then

\[h_{\phi \otimes \psi}(\alpha \otimes \beta) = h_\phi(\alpha) + h_\psi(\beta).\]

**Proof.** We shall rather prove that \(h_{\phi \otimes \psi}(\alpha \otimes \beta) = h_\phi(\alpha|_{\beta(A)}) + h_\psi(\beta)\). For this it suffices to consider the case when \(A\) is abelian; the general case will follow by the same arguments as in the proof of Theorem 5. (Note that the mapping \(x \mapsto \text{Tr}(x) - x\) on \(\text{Mat}_n(\mathbb{C})\) is not completely positive, but the mapping \(x \mapsto \text{Tr}(x) - \frac{1}{n}x\) is by the Pimsner-Popa inequality. Thus replacing \(M\) with \(M \otimes B\) and \(Z\) with \(Z \otimes B\) in the proof of Theorem 5 we have to replace the inequality \(E(s) \geq cs\) in the proof with \(E(s) \geq \frac{c}{s}\).

So suppose that \(A\) is abelian. It is clear that it suffices to prove that if \(A_1, \ldots, A_n\) are finite-dimensional subalgebras of \(A\), and \(B_1, \ldots, B_n\) are finite-dimensional subalgebras of \(B\), then

\[H_{\phi \otimes \psi}(A_1 \otimes B_1, \ldots, A_n \otimes B_n) = H_\phi(A_1, \ldots, A_n) + H_\psi(B_1, \ldots, B_n).\]

We always have the inequality "\(\geq\", [SV, Lemma 3.4]. To prove the opposite inequality consider a decomposition

\[
\phi \otimes \psi = \sum_{i_1, \ldots, i_n} \omega_{i_1 \ldots i_n}.
\]

Let \(H_{\phi \otimes \psi = \sum \omega_{i_1 \ldots i_n}}(A_1 \otimes B_1, \ldots, A_n \otimes B_n)\) be the entropy of the corresponding abelian model, so

\[H_{\phi \otimes \psi = \sum \omega_{i_1 \ldots i_n}}(A_1 \otimes B_1, \ldots, A_n \otimes B_n) = \sum_{i_1, \ldots, i_n} \eta \omega_{i_1 \ldots i_n}(1) + \sum_{k=1}^n S(\phi \otimes \psi|_{A_k \otimes B_k}, \sum_{i_k = i} \omega_{i_1 \ldots i_n}|A_k \otimes B_k).\]
Set $C = \sqrt{\sum_{k=1}^{n} A_k}$. Let $p_1, \ldots, p_r$ be those atoms $p$ of $C$ for which $\phi(p) > 0$. Define positive linear functionals $\psi_{m,i_1 \ldots i_n}$ on $B$, 

$$\psi_{m,i_1 \ldots i_n}(b) = \frac{\omega_{i_1 \ldots i_n}(p_m \otimes b)}{\phi(p_m)}.$$ 

Let also $\phi_m$ be the linear functional on $C$ defined by the equality $\phi_m(a) = \phi(ap_m)$. Then 

$$\omega_{i_1 \ldots i_n} = \sum_{m=1}^{r} \phi_m \otimes \psi_{m,i_1 \ldots i_n} \text{ on } C \otimes B,$$

and 

$$\psi = \sum_{i_1,\ldots,i_n} \psi_{m,i_1 \ldots i_n} \text{ for } m = 1, \ldots, r.$$ 

Since the supports of the states $\phi_m$ are mutually orthogonal minimal projections in $C$, we have 

$$\sum^n_{k=1} \sum_i S \left( \phi \otimes \psi|_{A_k \otimes B_k} \sum_{i_k=i} \omega_{i_1 \ldots i_n} |A_k \otimes B_k \right) \leq$$

$$\leq \sum^n_{k=1} \sum_i S \left( \phi \otimes \psi|_{C \otimes B_k} \sum_{i_k=i} \omega_{i_1 \ldots i_n} |C \otimes B_k \right)$$

$$= \sum^n_{k=1} \sum_i S \left( \phi \otimes \psi|_{C \otimes B_k} \sum_{m=1}^{r} \phi_m \otimes \left( \sum_{i_k=i} \psi_{m,i_1 \ldots i_n} \right) |C \otimes B_k \right)$$

$$= \sum^n_{k=1} \sum_i \sum_{m=1}^{r} \phi(p_m) S \left( \psi|_{B_k} \sum_{i_k=i} \psi_{m,i_1 \ldots i_n} |B_k \right).$$ 

If $a_i \geq 0$ and $\sum_i a_i \leq 1$ then $\eta(\sum_i a_i) \leq \sum_i \eta(a_i)$. Hence we have 

$$\sum_{i_1,\ldots,i_n} \eta \omega_{i_1 \ldots i_n}(1) \leq \sum_{m=1}^{r} \sum_{i_1,\ldots,i_n} \eta(\phi_m \otimes \psi_{m,i_1 \ldots i_n})(1)$$

$$= \sum_{m=1}^{r} \eta \phi(p_m) \sum_{i_1,\ldots,i_n} \psi_{m,i_1 \ldots i_n}(1) + \sum_{m=1}^{r} \phi(p_m) \sum_{i_1,\ldots,i_n} \eta \psi_{m,i_1 \ldots i_n}(1)$$

$$= \sum_{m=1}^{r} \eta \phi(p_m) + \sum_{m=1}^{r} \phi(p_m) \sum_{i_1,\ldots,i_n} \eta \psi_{m,i_1 \ldots i_n}(1).$$

Thus 

$$H_{\phi \otimes \psi = \sum \omega_{i_1 \ldots i_n}}(A_1 \otimes B_1, \ldots, A_n \otimes B_n) \leq$$

$$\leq \sum_{m=1}^{r} \eta \phi(p_m) + \sum_{m=1}^{r} \phi(p_m) H_{\psi = \sum \psi_{m,i_1 \ldots i_n}}(B_1, \ldots, B_n).$$

Since $\sum_m \eta \phi(p_m) = H_{\phi}(C) = H_{\phi}(A_1, \ldots, A_n)$, we conclude that 

$$H_{\phi \otimes \psi}(A_1 \otimes B_1, \ldots, A_n \otimes B_n) \leq H_{\phi}(A_1, \ldots, A_n) + H_{\psi}(B_1, \ldots, B_n),$$

completing the proof of the Theorem.
References


