PORTFOLIO SEPARATION IN MARKETS DRIVEN BY INDEPENDENT INCREMENT PROCESSES

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Abstract

This paper generalizes the mutual fund representation theorem obtained by Khanna & Kulldorff [KK] to the case of the driving noise being stable-like processes with independent increments, and allowing for more general income, preferences and portfolio constraints; in particular, the classical case of no risk-free asset is treated. Furthermore, we solve the Merton problem for a modified version of the HARA utility family in a geometric Lévy motion market with bankruptcy, to obtain weaker separation results.

Introduction

This paper concerns the portfolio (and in Section 2 consumption) optimization problem for a small agent in a frictionless market where the assets are driven by stochastic processes with independent increments. We consider in particular $m$ fund separation results for the optimal portfolio.

The separation into two mutual funds in the discrete one period model was due to Tobin [T]. The classical conditions were that either utility functions were quadratic, or returns jointly normal. Later works have usually approached the problem either from the direction of restricting the distributions and let preferences be arbitrary (though sufficiently regular), or the converse. In the former case, the canonical assumption is the normal distribution. An agent maximizing a concave increasing utility function will then minimize variance given the mean. A more complete characterization of the separating distributions were given by Ross [R]. In the continuous time setting with assets being geometric Brownian, two fund separation was proven by Merton [M] by means of dynamic programming. Instead of minimize variance given mean, Khanna & Kulldorff [KK] choose to maximize mean given the variance, and are by remarkably simple methods able to remove the risk aversion assumption and also allow for “no short sale” conditions on a subset of the portfolios, as well as incomplete markets; they do however assume the existence of a risk free asset. The starting point of this paper was originally to remove that assumption, but further generalizations are also obtained in Section 1:

- As we are still doing mean-dispersion trade-off, it is in fact not necessary to assume expected utility maximization. We have kept a weak greed assumption, but as [KK] laconically point out, even that assumption may be dropped if the agent is allowed to give away or dispose of unwanted wealth; however, risk aversion will turn out to be a valuable tool for reducing the number of funds.
• We allow for more general linear constraints than short sale bans or absence of risk free assets, usually at the cost of increasing the number of funds necessary.

• We allow for non-traded income not independent of the market, and for variable interest rate (in particular, different interest rates for lending and borrowing is covered), again at the cost of more funds.

• Stable distributions are treated as well.

Mutual fund theorems under the assumption of (symmetric) stable distributions (for $\alpha > 1$) are briefly mentioned in Cass & Stiglitz [CS], Section 9, in the single period setting. A more thorough treatment was given earlier by Fama [F], who also (Section IV) gives valuable insight into the peculiar diversification problem for $\alpha \leq 1$ (where expectation does in fact not exist), briefly treated in this paper, see Remark 5 to Proposition 1 and paragraph 2.1. [F] is confined to symmetric laws, but remark that this assumption is not really necessary as long as the agent is only doing trade-off between location and the scale parameter, and disregarding skewness - which is quite unreasonable, as it assumes that the agent will be indifferent between two particular distributions concentrated on $(-\infty, \mu)$ and $(\mu, \infty)$ respectively (for totally skewed law with $\alpha < 1$; For laws symmetric (around the location parameter), it does however make sense.) We shall therefore (essentially) assume that all noise sources have the same skewness, and as this parameter changes sign when the random variable is multiplied by a negative factor, we will see that the existence “no short sale” constraints will frequently be crucial.

A note is appropriate on the stable distributions: In discrete time, normal and (other) stable laws are in some sense “equally bad” at modeling stock markets, as both could attain arbitrarily negative values. In the continuous-time, however, things are different: The geometric Brownian motion given by $dS = S \cdot (\mu dt + \sigma dW)$ does never change sign, but replacing the Brownian motion $W$ by a (not totally skewed) stable Lévy motion will give a positive probability of changing sign in any positive time interval (unless, of course, $\sigma \neq 0$). This is an imperfection if the the theory is intended to model limited liability assets such as stocks.

The other approach to separation results is to restrict the preferences. The class of quadratic utility is an archetypical example of one allowing for separation theorems in discrete time finance. In the single period model, the aforementioned [CS] article gives a complete characterization of the sufficiently smooth utility functions admitting such separation. A special case is the result that the only (sufficiently regular) utility functions admitting separation into risk free asset and one fund (the weak form of separation, where the fund may be individual), are the ones admitting hyperbolic or absolute risk aversion (their Section 10, point C.) In Section 2, we consider the continuous time analog in a geometric Lévy motion market with risk free asset. For completeness, we show the expected counterexample to the aforementioned two fund separation. In the case of HARA utility, the consumption-investment problem is solved under quite general intertemporal trade-off, where the assets are allowed to go bankrupt and disappear. The possibility of an investment opportunity disappearing is often overlooked, as it is of no interest in a single period discrete model and will never occur in a lognormal market. Actually, we choose to modify the HARA utility into everywhere nondecreasing, concave functions which do void the regularity conditions of [CS]. Section 2 generalizes earlier works by Aase [A] as well as one co-written by this author ([FØS]), where it was assumed that the Lévy measure was finite. Benth et al. [BKR] have criticized this assumption on the grounds
that this assumption is void by the normal-inverse Gaussian laws used in recent stock models (as the stable laws have infinite Lévy measure as well). That assumption is now removed.

1 Separation theorems, stable case

Recall that an \( \alpha \)-stable real random variable \( Z \) is one satisfying

\[
\log \mathbb{E}[e^{i\theta Z}] = i\mu \theta - Q^\alpha |\theta|^\alpha \cdot \begin{cases} 
(1 - i\beta_0 \tan \frac{\alpha}{2} \text{sign} \, \theta) & \text{if } \alpha \neq 1 \\
(1 + i\beta_0 \frac{\alpha}{2} \log |\theta| \text{sign} \, \theta) & \text{if } \alpha = 1
\end{cases}
\]

for an index of stability \( \alpha \in (0, 2] \), a scale parameter \( Q \geq 0 \), a skewness parameter \( \beta_0 \in [-1, 1] \) and a location parameter \( \mu_0 \in \mathbb{R} \); We write \( Z \sim S_\alpha(Q, \beta_0, \mu_0) \). The case \( \alpha = 2 \) is the Gaussian which is independent of \( \beta_0 \); we shall always use skewness parameter zero if \( \alpha = 2 \). For the purposes of this paper, it will suffice to consider only identically distributed laws located at zero — i.e., we’ll let the location parameter enter as an additive constant in (4). Also, we shall assume nonnegative skewness (if not, consider \(-Z\) instead); For \( Z_1, Z_2 \) i.i.d. \( S_\alpha(1, \beta, 0) \), \( \beta \geq \beta_0 \geq 0 \) and \( \beta \neq 0 \), we can write

\[
Z = Q\left(\frac{\beta + \beta_0}{2\beta}\right)^{\frac{1}{\alpha}} \cdot Z_1 - Q\left(\frac{\beta - \beta_0}{2\beta}\right)^{\frac{1}{\alpha}} \cdot Z_2 + \mu_0 + \frac{2}{\pi} \beta (a_1 \log a_1 - a_2 \log a_2) \chi_{\{\alpha=1\}}
\]

For our purposes, it will be most convenient to choose \( \beta \geq 0 \) less than one if possible, and equal to zero if possible. For \( \beta = 0 \), we shall instead of (2) write \( Z = QZ_1 + \mu_0 \).

Throughout this section let \( X = (X_1, \ldots, X_d)^T \) where \( X_i \) are real-valued, independent and right-continuous with \( \alpha \)-stable independent increments; by (2) above, we shall assume that the increment \( X(t+1) - X(t) \) is distributed \( S_\alpha(1, \beta, 0) \) with \( \beta \geq 0 \). Then for \( v = (v_1, \ldots, v_d)^T \in \mathbb{R}^d \), we have (with abuse of notation),

\[
v^T \, dX_i \sim S_\alpha(Q \cdot (dt)^{\frac{1}{\alpha}}, B, D \cdot (dt)^{\frac{1}{\alpha}})
\]

where

\[
\begin{align*}
Q &= \|v\| = \|v\|_{\alpha} := \left( \sum |v_i|^\alpha \right)^{\frac{1}{\alpha}}, \\
B &= \beta \sum \frac{v_i^{<\alpha>}}{Q^\alpha} \quad \text{in } [-\beta, \beta], \\
D &= -\frac{2}{\pi} \beta \sum v_i \log |v_i| \chi_{\{\alpha=1\}};
\end{align*}
\]

where the signed power function \( v^{<\alpha>} \) equals \( |v|^\alpha \cdot \text{sign} \, v \). If \( \alpha \neq 1 \) or \( \beta = 0 \), the law is strictly stable, while the skew 1-stable case is the only one which cannot be made strictly stable by the translation we have performed. Note that \( \| \cdot \|_{\alpha} \) is only a quasi-norm for \( \alpha < 1 \). Note also that \( B = \beta \) iff all \( v_i \) are nonnegative or \( \beta = 0 \).

1.1 The market. No short sale-alike conditions.

Consider a frictionless market with optimization in
• a risk-free asset $S_0$ following
  \[ dS_0(t) = rS_0(t) \, dt. \]  

• $n$ risky assets \( \{S_i\}_{i=1,\ldots,n} \) following
  \[ dS_i(t) = S_i(t^-)[\tilde{\mu}_i \, dt + \sigma_i \, dX(t)] \]  
  such that $S_i$ can be expressed as a combination of the others (if so, leave it out of the model).

• non-traded income $I$ following
  \[ dI(t) = d\tilde{I}(t) + \sigma_0 \, dX(t) \]  
  with $d\tilde{I}$ independent of $dX$,

• and furthermore, consumption $\tilde{C}(t)$.

The coefficients $r = r(t, \omega)$, $\tilde{\mu}_i = \tilde{\mu}_i(t, \omega)$ and $\sigma_i = \sigma_i(t, \omega)$ are assumed to be $\mathcal{F}_t$-predictable. Furthermore, the agent is supposed to be "small" in the sense that the market dynamics does not depend on the agent’s choice of (consumption and) portfolio vector $u$ in the following:

From the assets, we form a self-financed portfolio which when discounted with the interest rate $r$ has the dynamics
  \[ dY(t) = Y(t^-)u^T[\mu \, dt + \Sigma \, dX(t)] + \sigma_0 \, dX(t) - dC(t) \]  
where $\mu = \tilde{\mu} - r1$. Putting $v^T := \sigma_0^T + Y(t^-)u^T\Sigma$, we have
  \[ dY(t) = Y(t^-)u^T\mu \, dt + v^T \, dX(t) - dC(t) \]  

Below, for restrictions of the form $u \in U$, then $V := \{v = \sigma_0 + Y\Sigma^T u\}_{u \in U}$. We shall only consider $U$ generated by linear constraints; they will in various cases give rise to separation theorems. Furthermore, we shall always assume $U$ to be closed.

Remarks.

1. If the dynamics of the assets in addition have a term identically distributed for all assets, and independent of $X$, then it may be included in $\tilde{I}$.

2. The assumptions made on the asset dynamics do not ensure that $S_i \neq 0$; it does, however, imply that 0 is a trap for the asset price. Should that happen, we can leave that asset out of the model, as we need not assume the number of assets to be constant in time. Alternatively, we could impose the constraint $u_i = 0$, which is a special case of the constraints allowed for below.

3. As noted in the introduction, most stable laws may have arbitrarily large jumps, both positive and negative, meaning that the asset $S_i$ may change sign, unlike the real life stock market. The exceptions are $\alpha = 2$ (the Gaussian case, where $\beta$ is arbitrary), and the totally skewed laws with $\beta \text{sign } \sigma_i = \pm 1$. Totally skewed processes with $\alpha < 1$ may admit arbitrage, depending on the sign of the drift terms.
4. The condition that all noise sources have the same $\beta$, will turn out to be quite essential. However, we have to remark that the construction (2) may disguise the problem with different $\beta$'s; introducing more noise sources will make the market incomplete, and with the dimension of $\mathbf{u}$ being strictly smaller than the dimension of $\mathbf{v}$, (8) will only hold in special cases.

We make the following assumptions on preferences, portfolio and (discounted) consumption:

**ASSUMPTION**

1. Portfolio optimization is taken with respect to $\mathcal{F}_T$-predictable portfolios $\mathbf{u}(t, \omega)$ with $\| \cdot \|_\alpha$ a.s. locally integrable; in addition, we may impose the restrictions $\mathbf{u} \in U$ as below, and furthermore, any restriction on $\|\mathbf{v}\|$.

2. The class of admissible consumption processes is such that if $\hat{C}$ is admissible, then any $\hat{C}^*$ with

$$d\hat{C}^* = d\hat{C} + c \, dt \quad \text{for some } c \geq 0, \text{ for a.a. } (t, \omega),$$

is admissible as well.

3. Preferences form a partial ordering on the (wealth, consumption) pairs such that if (6) holds and $Y^* \sim Y$, then $(Y^*, \hat{C}^*)$ is (weakly) preferred to $(Y, \hat{C})$; We shall say that a $\mathbf{u}^*$ is preferred (say, to $\mathbf{u}$) if for each admissible $\hat{C}$, there exists an admissible $\hat{C}^*$ satisfying (6) and such that the corresponding wealth processes have the same law.

4. (5a) has a unique weak solution, and an admissible (consumption, portfolio) pair exists.

Note that preferences are more general than expected utility maximization; for example, we allow for infinite expected utility. The assumptions made are exactly what is required for the following argument (the skew 1-stable case is not covered, but follows likewise): Given two vectors $\mathbf{u}^*$ and $\mathbf{u}$ with $Y \mathbf{u}^T \mu \geq Y \mathbf{u}^T \mu$, and both $\|\mathbf{v}\|_\alpha = \|\mathbf{v}\|_\alpha = Q$ and $\beta \sum (v_i^*)^{\sigma \alpha} = \beta \sum v_i^{\sigma \alpha} > B Q^\alpha$, then $\mathbf{u}^*$ is preferred. Consider the consumption-investment pairs $(C, \mathbf{u})$ and $(C^*, \mathbf{u}^*)$ with $C^*$ given by

$$dC^* = dC + Y \cdot (\mathbf{u}^* - \mathbf{u})^T \mu \, dt \quad (\geq dC),$$

i.e. (6) holds; for the corresponding wealth processes, we have

$$dY^* = Y \mathbf{u}^T \mu \, dt + \mathbf{v}^T d\mathbf{X} - dC^*$$

$$= Y (\mathbf{u}^T \mu \, dt - (\mathbf{u}^* - \mathbf{u})^T \mu \, dt) + \mathbf{v}^T d\mathbf{X} - dC$$

$$= Y \mathbf{u}^T \mu \, dt + Q \, dX^* - dC$$

$$\sim Y \mathbf{u}^T \mu \, dt + Q \, dX - dC$$

$$= dY$$
since $X^* \sim X$ (see (3)).

If

$$\beta = 0 \quad \text{or both} \quad (\alpha \neq 1 \quad \text{and} \quad V \subseteq \{v; v_i \geq 0 \forall i\}),$$

then an optimal $u^*$ can therefore be chosen so that it solves the static optimization problem

$$\sup_u Y^{\top} \mu \quad \text{subject to} \quad \begin{cases} \|v\|_\alpha = Q \\ u \in U \end{cases}$$

We will call $Q \geq 0$ attainable if the constraints of (9) form a nonempty set.

To recover and generalize the mutual fund separation result of [KK], Lemma 3 (though not the explicit formulae), we state the following easy result:

**Proposition 1.**

Suppose (8). Consider problem (9) with $U$ of the form $U = U_0 + \tilde{U}$ where all $u_0 \in U_0$ satisfies $\sigma_0 + Y^{\top} u = 0$, and $\tilde{U}$ is a closed family of half-lines from 0 such that $\Sigma^{\top} \tilde{u} \neq 0$ for all $0 \neq \tilde{u} \in \tilde{U}$. If $u^*_0$ maximizes $u^{\top}_0 \mu$ for $u_0 \in U_0$, then for all attainable $Q$, there exists an optimal $u^* \in U$ of the form $u^* = u^*_0 + Q \tilde{u}^*$, with $\tilde{u}^*$ independent of $Q$.

**Proof.** Under the assumptions, $v = QY^{\top} \tilde{u}$. Problem (9) therefore takes the form

$$\max_u (u_0 + Q\tilde{u})^{\top} \mu \quad \text{subject to} \quad \begin{cases} Q\|Y^{\top} \tilde{u}\|_\alpha = Q \\ Q\tilde{u} \in \tilde{U} \end{cases}$$

which is the same as

$$u^*_0^{\top} \mu + Q \max_{\tilde{u} \in \tilde{U}} \tilde{u}^{\top} \mu \quad \text{subject to} \quad \|Y^{\top} \tilde{u}\|_\alpha = 1.$$ 

By our assumptions, a solution exists.

**Remarks.**

1. This gives $m_0 + 2$ fund separation if $u^*_0$ separates into $m_0$ funds. (The last fund is the bank.)

2. The conditions are a bit ad hoc, but the idea is the following: À priori, the market may admit an arbitrage – which could be eliminated by the constraints on the portfolio. For example, if two assets have the same noise but one has higher drift, then the arbitrage is eliminated by forbidding short sales. Note that in this case, $\Sigma$ is not invertible. Assuming $\alpha = 0$, the conditions are set to ensure that one cannot exploit any arbitrage. With general $\sigma_0$, there could be some opportunity to cancel the uncertainty from the income, which is the interpretation of maximizing $u^*_0^{\top} \mu$ first. Certainly, the presence of possibly individual $\sigma_0$ weakens the realism of the conditions.
3. The condition that if $\beta \neq 0$ then all $v_i$ are nonnegative (implying $B = \beta$ unless $Q = 0$), is essential. It can be replaced by nonpositivity, but let us consider the case where $V$ admits both $v$ with only nonnegative components and $v$ with only nonpositive components: We'll need an extra fund; splitting into two separate cases, with solutions, say $v = Qt^{+}$ and $v = Qt^{-}$ respectively, we see that we cannot have $t^{+} = -t^{-}$ as the one maximizing one problem will minimize the other; so the no short sale condition is necessary except in degenerate cases.

4. The interest rate (typically) affects the representation.

5. It is worth remarking some consequences for $\alpha \leq 1$: Assume for simplicity that $\Sigma$ is the identity. First, if $\alpha = 1$ and $V$ is a union of orthants, then one shall only invest in one asset (the one with highest drift, or highest negative drift if negative position allowed.) If $\alpha < 1$ then the same holds if $V$ is the entire space, or if $V$ is the first orthant and at least one of the $\theta_i$ are nonnegative. Such a result indicates that in many cases, there is no bounded optimal control, see paragraph 2.1.

6. Note that strict stability is crucial for $\alpha = 1$. We shall treat the skew 1-stable case separately in the next subsection.

1.2 Alternative constraints for $\alpha \geq 1$; risk aversion and absence of arbitrage.

Throughout this paragraph, we shall assume the market to be arbitrage-free in the usual sense, i.e. regardless of $U$. If the market is incomplete, we augment it with fictional "dummy" assets in which only the zero position is allowed. Therefore, we can assume $\Sigma$ to be invertible (which implies absence of arbitrage if $\alpha \geq 1$, but not for $\alpha < 1$ in the totally skew case). Then we have $Y u^T = (v - \sigma_0)^T \Sigma^{-1}$ and in particular, there is an $u$ such that $v = 0$. The dynamics then takes the form

$$dY = v^T [\theta \, dt + dX] - dC$$

with $\theta = \Sigma^{-1} \mu$ and $dC = dC + \sigma_0^T \theta \, dt$. We then consider linear conditions of the form

$$V = \{v; \quad v^T z_i \leq \zeta_i, \quad i = -m, \ldots, n\}$$

Conditions on the sign of $Yu_k$ (including the forbidding of the dummy assets) are numbered from 1 to $n$.

The equivalence-in-law argument (7) may then be repeated. Let us first do away with the 1-stable case: If $\beta = 0$, then we (essentially) have a linear programming problem, which may separate into fewer vectors (again, see paragraph 2.1.) For the skew 1-stable case, we have the following:

**Proposition 2.**

Assume $\alpha = 1$, $\beta \neq 0$ and $V$ of the form $\{v; \ v_i \geq 0 \ \forall i, \ v_i = 0$ for some $i$'s}. Then we have $m_0 + 2$ fund separation if $\sigma_0$ separates into $m_0$ vectors.
Proof. The drift term to be maximized is

\[ D = v^T \theta - \frac{2\beta}{d} \sum (v_t \log v_t). \]  

(12)

For \( \beta < 0 \), this is a convex function of \( v \), and the optimal is to invest all in the asset with highest \( \theta_i \). Assume \( \beta > 0 \). Then \( v \mapsto D \) is concave. The first order condition of the Lagrangian \( D - \lambda v^T 1 \), is satisfied by

\[ v = \exp\{ -1 - \frac{\pi}{2\beta} \lambda \} \cdot f = \frac{Q}{f^T 1} f \]  

(13)

where \( f_i = \exp\{ \frac{\pi}{2\beta} \theta_i \} \).

The case \( \beta < 1 \) yields the same result as the symmetric case (see Remark 5 to Proposition 1), and in many problems we will have no bounded optimal control in this case as well. Again, see paragraph 2.1.

From now on, we shall assume \( \alpha > 1 \), implying that \( v_t \mapsto \|v\|_\alpha \) is Lipschitz, and Kuhn-Tucker will be utilized below. One of the key features of [KK] is the removal of risk aversion. It turns out that for \( \alpha \geq 1 \) where \( \cdot \|_\alpha \) is a norm, some (narrow) concept of risk aversion is a useful tool nevertheless:

**DEFINITION**

An agent is called risk averse if for an independent increment process \( X_0 \sim X \), independent of \( X \) and \( \tilde{I} \), then \( v_0 = 0 \) is preferred among the wealth processes

\[ dY = D \, dt + Q \, dX + v_0 \, dX_0 - dC \]

(14)  

(ref. (7b).)

A risk averse agent will therefore, given drift and skewness, minimize the scale parameter; if \( \|v\| > \|v^*\| \) (everything else equal), then (14) holds with \( v_0 = \|v^*\| - \|v\| \).

À priori, we would choose \( v^* \) to solve

\[
\begin{align*}
\text{max } v^T \theta & \quad \text{subject to} \\
\|v\|_\alpha &= Q \quad \text{(Q)} \\
\beta \sum v^{<\alpha>} &= B Q^\alpha \quad \text{(B)} \\
v &\in V \quad \text{(V)}
\end{align*}
\]

(15a)

For a risk averse agent, we will instead choose \( v_* \) to solve

\[
\begin{align*}
\text{min } \|v\|_\alpha & \quad \text{subject to} \\
\|v\|_\alpha &= D \quad \text{(D)} \\
\sum (\beta v^{<\alpha>} - B |v_i|^\alpha) &= 0 \quad \text{(B)} \\
v &\in V \quad \text{(V)}
\end{align*}
\]

(15b)

As condition (D) is linear, we shall assume it to be one of the constraints in (11), namely number zero. We will call \( Q \geq 0 \), \( D \in \mathbb{R} \) and/or a \( B \in [-1, 1] \) attainable if the constraints of (15) form a nonempty set. For simplicity, we shall assume that condition (D) in (15b), is
constraint number 0 in (11).

Solving (15b) for \( v_* = v_*(D) \) for all attainable \( D \) (even though a risk averse agent will never choose \( D < 0 \)), we find that the solution \( v^* = v^*(Q) \) of (15a) can be chosen as \( v^*(Q) = v_*(D) + p(Q) \) with \( p(Q) \perp \theta \) being "pure uncertainty". Under our assumptions, \( p(Q) \) can be written as a finite sum \( \sum \lambda_i p_i \) with \( p_i \perp \theta \) independent of \( Q \): There will only be a finite number of \( D_i \) for which there is a need to invoke \( p(Q) \). For each \( D_i \), only one \( p_i \) is needed, by convexity of \( V \).

The Kuhn-Tucker condition now takes the form

\[
v^*_k < \sigma_0^{-1} > \cdot (1 + \lambda (\beta \text{sign } v_k - B)) = \sum \lambda_i z_{i,k}
\]  

(16)

In the presence of inequality constraints, we therefore have the following:

**Proposition 3.**

Let \( \beta = 0 \) and let \( a := \frac{1}{a-1} \) be an odd integer. Let \( V \) be given by (11), and assume that \( \sigma_0 \) separates into \( m_0 \) vectors (common to all agents). Then we have \( \tilde{m} \) fund separation for all risk averse agents, with \( \tilde{m} = m_0 + \bar{m} + 1 \), where \( \tilde{m} \) is the number of independent vectors in expanding the power

\[
v^*_k = \left( \sum_{i=-m}^{n} \lambda_i z_{i,k} \right)^a
\]

(17a)

and solving with respect to \( v \):

\[
v^* = \sum_{i=1}^{\tilde{m}} \bar{\lambda}_i f_i.
\]

(17b)

**Proof.** Equation (17) holds because \( \beta = 0 \) and \( a \) is odd. We have

\[
Y u^* T = (v^* - \sigma_0) T \Sigma^{-1} = \sum_{i=1}^{\tilde{m}} \bar{\lambda}_i (f_i^T \Sigma^{-1} - \sigma_0 \Sigma^{-1})
\]

(18)

So \( u^* \) separates into \( m_0 + \tilde{m} \) funds. The last fund is the bank.

\[\Delta\]

**Remarks.**

1. If \( a \) is even, then (16) cannot determine the sign of the coordinates \( v_k \). *Imposing* nonnegativity or nonpositivity will require at least as many funds as assets in (17a) — a number possibly larger than the original number of assets due to the augmentation with the "dummy" assets.

2. We may want to write the dependence of the interest rate explicitly. The drift constraint is \( D = Y u^T \mu = Y u^T \tilde{\mu} - r Y u^T 1 \). Impose instead the two constraints

\[
\begin{align*}
Y u^T 1 &= K \\
Y u^T \tilde{\mu} &= D + r K
\end{align*}
\]

(19)
at the cost of requiring extra funds. Note that \( r \) may depend on \( K \), which is economically quite plausible; for example, one may have \( r = r^+ \) for \( K \leq Y \) and \( r = r^- \) for \( K > Y \), i.e. different rates for lending and borrowing – but interest rate may also depend on the degree of leverage.

3. In real world market, the commission fraction charged for managing the portfolio, is not the same for all agents. This can be handled by the same extra fund as individual interest rate assuming that the vector \( \delta \) of commissions is given by \( \delta_0 \mathbf{1} \) for individual \( \delta_0 \). More generally, one may allow \( \delta \) to be spanned by more vector at the cost of increasing the number of funds.

1.3 Further reduction if \( \alpha = 2 \)

As Remark 1 to Proposition 3 shows, the result is quite weak for \( \alpha > 1 \). For the Gaussian case, we can use \( L^2 \) orthogonality to achieve even fewer funds. Let us first rewrite the process into

\[
dY = Y u^T [\mu \, dt + dW] + \rho^T dW - dC
\]

where \( W \) is a Wiener process with covariance matrix \( R \) (assumed invertible; no "dummy assets" are required). The problem (15b) for the risk averse agent, then becomes

\[
\rho^T R \rho + \min_{u \in U} (Y^2 u^T R u + 2 Y u^T R \rho)
\]

where the drift condition is the zeroth constraint of \( U \), which is given by:

\[
U = \{ u; \quad Y u^T w_i \leq \omega_i, \quad i = -m, \ldots, n \}
\]

By Kuhn-Tucker, we immediately have \( \bar{m} \) fund separation if \( \rho \) separates into \( m_0 \) vectors:

\[
R(Y u + \rho) = \sum \lambda_i w_i
\]

A stronger result is then obtained if some of the equality constraints have \( \omega_i \) common to all agents:

**PROPOSITION 4.**

Assume that constraints \(-\bar{m} - 1, \ldots, -m\) are equalities with \( \omega_i \) common to all agents. Then if \( \rho \) separates into \( m_0 \) vectors, we can take \( \bar{m} \) to be \( 2 + m_0 \) plus the number of independent vectors in \( \{ i \geq -\bar{m} \} \).

*Proof.* The latter \( m - \bar{m} \) constraints form an affine subspace \( qu_0 + \bar{U} \) with \( qu_0 \) minimizing \( u^T R u \) (and choose \( \| u \|_0 = 1 \)); then \( u_0^T R \bar{u} = 0 \) for \( \bar{u} \in \bar{U} \), and (21) reduces to the minimization

\[
\min_{\bar{u} \in \bar{U}} \bar{u}^T R \bar{u} + \frac{2}{Y} \bar{u}^T R \rho
\]

and by a rotation we can assume \( \bar{U} \) to be Euclidean \( \bar{n} \)-space and use (23).

△
We then obtain the following classical result for the case where there is no risk free asset:

**Corollary**

Assume that \( \rho = 0 \) and that the only constraint is \( Y u^\top 1 = Y \). Then

\[
u^* = \lambda_0(R^{-1} \mu) + \lambda(R^{-1} 1).
\]

and we have two fund separation.

Note that a non-risk averse agent will need to invoke a “pure risk fund” \( p \perp \mu \) iff \( \mu \) is a scalar multiple of \( 1 \) (in which case one fund suffices for the risk averse agent.)

**Concluding remarks to Section 1**

We have seen that the Gaussian case apparently admits better separation results than the stable case, in which the most interesting results are found if risk is diversifiable and the dependence of the skewness parameter degenerates. In fact, “no short sale” conditions may even be essential for the representation.

We make a remark to the Assumption on the preferences: In the Gaussian case, it may be replaced by the assumption that given \( Y^* \geq \bar{Y} \) a.s. with \( \bar{Y} \sim Y \), then given the consumption, \( Y^* \) is preferred. Then we can still maximize drift given variance, and apply to well-known comparison theorems (see [IW], Theorem 1.1 for the result, and section 2 for an application to stochastic control.) However, in the discontinuous case \( \alpha < 2 \), this breaks down, as wealth can with positive probability jump by a factor less than \(-1\) (unless we choose the zero portfolio).

It is mathematically a curious point that that the elegant setup of [KK] is not as crucial to their continuous setting as to the stable case.

We have assumed that the portfolio vector belongs to some closed set; if not, an optimal vector need not exist. [KK] point out that if there is no optimal portfolio, then for each admissible strategy there is one which is at least as good and uses the mutual funds. A careful note is appropriate, as we cannot a\( \alpha \) priori achieve \( \epsilon \)-optimality with \( \epsilon \) independent of desired uncertainty level \( Q \) (or drift \( D \)). Our assumptions on the preferences do not imply any ordering between two strategies where one is better (in the static optimization problem) for some \( Q \) and the other is better for other \( Q \).

If the stability index \( \alpha \) is constant, then \( X \) will be a Lévy process, but we may allow for \( X \) to be stable-like, i.e. \( \alpha = \alpha(t) \) time-dependent (or even stochastic if independent of everything). However, as Proposition 3 only admits discrete values \( \alpha = 1 + \frac{1}{\omega} \), this generalization is most interesting in the setting of Proposition 1. Note however that all \( X_i \) are still supposed to have the same \( \alpha \); We can adapt the theory to the \( X_i \) having different stability indices if \( \Sigma \) is invertible, by separating the market into independent groups of assets, each with common \( \alpha \).

[KK] note that also the number of stocks may vary (stochastically) in time; this is of course correct if the investor is allowed to sell a stock which is about to disappear. One may want to ask: What if the stock goes bankrupt and not only does the investment opportunity disappear, but the money invested in the stock is lost? We will address this question in the next section, in the more general setting of jump processes.
2 Dynamic programming and geometric Lévy motion markets.

Throughout this section, we shall assume that a risk free asset exists, and we shall assume that the stock prices follow the geometric Lévy motion (in fact, we allow for time-dependent coefficients). We shall first (paragraph 2.1) briefly revisit the stable laws in a dynamic programming setting. Dropping the assumptions on the distribution of the noise sources, we no longer in general have separation. In paragraph 2.2, we shall go into further detail: Most of the paragraph will be devoted to solving the well-known Merton problem when direct utility is a modification of the HARA type. The solution will admit two fund separation with money as one fund, and one will see that the fund will be common to some agents; in the usual setup where investment opportunities are constant, all agents with the same γ will have the same fund. However, a main point of this section is to allow for investment opportunities to disappear, for example through bankruptcy; this is a case which is not covered by the naïve generalization of the wealth process, where the value invested in each stock is taken as our control (which is only possible as long as the stock has nonzero price). We will address this problem by defining a market state process Θ driven by Poisson jumps only:

\[
d\Theta(s) = \int (\nu(s, \Theta(s^-), \xi) - \Theta(s^-)) J(ds, d\xi),
\]

i.e. that a jump will bring the market dynamics from state Θ to a new state ν. Assume that the wealth process – which is assumed to have no income apart from the traded assets – has the dynamics

\[
dY(s) = w^T(s^-) d\Lambda(s) - c(s) ds
\]

where \( d\Lambda \) has the Lévy-Khintchine representation

\[
d\Lambda = b dt + dW + \int p(\xi) N(ds, d\xi)
\]

where \( p_t \geq -1 \),

\[
d\bar{N} = dN - \bar{\chi} dq ds = dN - \bar{\chi} E[N(1, d\xi)] ds
\]

and \( \bar{\chi} = \chi|p| \leq 1 \). We allow for \( b, p \) and the covariance matrix \( R \) to depend on both time and \( \Theta \) (actually, it may be useful to take \( \Theta \) to be the triple \( (b, p, R) \)). For convenience, we may assume that the number of assets do not change (a disappearing asset will get zero dynamics). For simplicity, we assume that the intensity of jumps in \( \Theta \), namely \( q(\{\xi; \nu(t, \theta, \xi) \neq \theta\}) \), is (uniformly) bounded – a condition which should admit generalizations; however, as the main purpose is to allow for some of the (finite number of) assets to disappear, the assumption is not too restrictive.

2.1 Stable laws revisited.

Let us first recapitulate some results from Section 1 assuming that we are maximizing expected utility under "no short sale" constraints or the unconstrained case, and the value function \( \Phi \) (assumed strictly increasing) satisfies the Hamilton-Jacobi-Bellman equation. Let for simplicity all assets be independent with α-stable noise, and suppress the index \( i \) in the notation.
For $\alpha < 2$, the Lévy measure of $X_t$ has the form

$$
(\kappa^- \chi_{\{z<0\}} + \kappa^+ \chi_{\{z>0\}}) \frac{dz}{|z|^{1+\alpha}}
$$

(28)

and $\beta = \frac{\kappa^-}{\kappa^+ + \kappa^-}$; for the correspondences between the Poisson measure representation (27b) and the parameters of the stable law, we refer to [ST], Theorem 3.12.2.

In the Gaussian case, the optimal $w^*$ maximizes

$$
w b \Phi' + \frac{1}{2} \sigma^2 |w|^2 \Phi''
$$

(29)

(here $\sigma^2$ is the variance, equaling twice the scale parameter) and if $\Phi'' < 0$, an interior solution exists with the mutual fund property. For $\alpha \in (1, 2)$, we may work with $d\tilde{N} := dN - dq ds$ instead of $d\tilde{N}$, adjusting the drift term into $\tilde{b}$. An optimal $w^*$ must maximize

$$
w b \Phi'(y) + \int (\Phi(y + wz) - \Phi(y) - wz \Phi'(y)) (\kappa^- \chi_{\{z<0\}} + \kappa^+ \chi_{\{z>0\}}) \frac{dz}{|z|^{1+\alpha}}
$$

(30a)

Substituting for $wz$, the expression transforms into

$$
w b \Phi'(y) + |w|^\alpha \cdot \begin{cases} 
[\kappa^+ \int_0^\infty + \kappa^- \int_0^0] (\Phi(y + z) - \Phi(y) - z \Phi'(y)) \frac{dz}{|z|^{1+\alpha}} & \text{if } w \geq 0, \\
[\kappa^- \int_0^\infty + \kappa^+ \int_0^0] (\Phi(y + z) - \Phi(y) - z \Phi'(y)) \frac{dz}{|z|^{1+\alpha}} & \text{if } w < 0
\end{cases}
$$

(30b)

(30b)

where $\tilde{I}^\pm$ are the integrals over $\mathbb{R}^\pm$, respectively. Analogous to the Gaussian case, if $\Phi$ is concave and not affine, then $\tilde{I}^\pm$ are both negative and we have an interior solution (valid also for $\kappa^- = 0$) of the form

$$
w^*_i = \left( \frac{\Phi'}{\tilde{I}^+ + \tilde{I}^- \pm \beta_i (\tilde{I}^+ - \tilde{I}^-)} \right)^{\frac{1}{\alpha-1}} \cdot f_i
$$

(31a)

where "$\pm$" equals $\text{sign} w$ and

$$
f_i = \left( \frac{2\tilde{b}_i}{\alpha_i (\kappa^+_i + \kappa^-_i)} \right)^{\frac{1}{\alpha-1}}.
$$

(31b)

We see that if all assets have common $\alpha_i$ and $\beta_i$, and if nonnegativity constraints apply unless $\beta = 0$, then $\mathbf{f}$ is a mutual fund. For $\alpha < 1$, we may work with $dN$ instead of $d\tilde{N}$, adjusting the drift term into $\tilde{b}$. An optimal $w^*$ must maximize

$$
w b \Phi'(y) + \int (\Phi(y + wz) - \Phi(y)) (\kappa^- \chi_{\{z<0\}} + \kappa^+ \chi_{\{z>0\}}) \frac{dz}{|z|^{1+\alpha}},
$$

(32a)

or performing the same substitution as above,

$$
w b \Phi'(y) + \frac{\kappa^-}{1 - \beta} |w|^\alpha \cdot \left[ \tilde{I}^+ + \tilde{I}^- + \beta (\tilde{I}^+ - \tilde{I}^-) \text{sign } w \right]
$$

(32b)
where the latter factor may have either sign. Since \( \alpha < 1 \), then it will be optimal to hold an infinite position in the unconstrained case (except possibly if \( \beta = 0 \)). So let us assume that short sale is forbidden, and that \( \beta < 0 \) in order to have a finite solution, which takes the form

\[
w^*_i = \max \left( 0, \left( \frac{\Phi'}{f^* + \tilde{I}^- + \beta_i(f^* - \tilde{I}^-)} \right)^{\frac{1}{\alpha-1}} \right) \cdot f_i
\]

where

\[
f_i = \left( \frac{2\beta_i}{\alpha_i(\kappa^+_i + \kappa^-_i)} \right)^{\frac{1}{\alpha-1}}.
\]

is a mutual fund if the \( \alpha_i \)'s and the \( \beta_i \)'s are common and all \( \beta_i < 0 \).

Let us then consider the case \( \alpha = 1 \). The optimal \( w \) maximizes

\[
w w \Phi'(y) + \int \left( \Phi(y + wz) - \Phi(y) - wz \Phi'(y) \chi(|z| \leq 1) \left( \kappa^- \chi_{|z| < 0} + \kappa^+ \chi_{|z| > 0} \right) \right) \frac{dz}{z^2}
\]

\[
= w w \Phi'(y) - |w| \cdot (\kappa^- - \kappa^+) \Phi'(y) \log |w|
\]

\[
+ |w| \cdot \left[ \kappa^+ \int_{-\infty}^{0} + \kappa^- \int_{0}^{\infty} \right] \left( \Phi(y + z) - \Phi(y) - z \chi(|z| \leq 1) \Phi'(y) \right) \frac{dz}{z^2}
\]

\[
= w w \Phi'(y) - |w| \cdot (\kappa^- - \kappa^+) \Phi'(y) \log |w| + |w| \cdot [\kappa^+ I^+ + \kappa^- I^-]
\]

if \( w \geq 0 \), and

\[
w w \Phi'(y) - |w| \cdot (\kappa^- - \kappa^+) \Phi'(y) \log |w| + |w| \cdot [\kappa^- I^+ + \kappa^+ I^-]
\]

if \( w < 0 \). The strictly stable case of course yields an extremal |w| (possibly 0), also if \( w \)

is constrained to a compact – which is not surprising, as the method of Section 1 leads to linear programming. If \( \kappa^+ \neq \kappa^- \) then it is still optimal to choose \( w \) infinite, if unconstrained. Disallowing short sale, only the positive \( \beta \) (\( \iff \kappa^+ > \kappa^- \)) yield interior solution, which is

\[
w = \exp \left\{ \frac{b}{\kappa^+ - \kappa^-} + \frac{\kappa^+ I^+ + \kappa^- I^-}{(\kappa^+ - \kappa^-) \Phi'} - 1 \right\}
\]

Again, assuming all \( \beta_i \) equal, we have a mutual fund

\[
f_i = \exp \left\{ \frac{b_i}{\kappa^+ - \kappa^-} \right\}.
\]

We conclude this subsection by noting that like in the Gaussian case, we are to maximize a linear term plus a |w|^\alpha-term, and that the case \( \alpha > 1 \) behaves very much like the Gaussian, while the \( \alpha \leq 1 \) cases are more extreme and do require restrictions on the portfolio on order to have interior solutions, except in degenerate cases.

### 2.2 Separation within classes of utility functions.

Let us first give a brief example of a utility function for which there is no two fund separation:
PROPOSITION 5.
There is a finite horizon problem of the form
\[ \Phi(y) = \sup_w J^{(w)}(y) = \sup_w E^\nu[\Upsilon(Y(1))] \]
where \( \Upsilon \) is concave (and there is no consumption) such that the optimal \( w^* \) does not separate into two funds independent of wealth.

Proof. Consider a strictly increasing, concave \( C^1 \) modification \( \Upsilon \) of the square root utility \( \hat{\Upsilon}(y) = 2\sqrt{y} \), such that
\[
\Upsilon(y) = \begin{cases} 
\hat{\Upsilon}, & \text{if } y \leq 1 \\
< \hat{\Upsilon}, & \text{if } y > 1.
\end{cases}
\]

Let the market consist of two (nonnegative) assets,
\[ dX_i(t) = bX_i(t^-)(dt - dN_i(t)) \]
where \( E[N_i(1)] = q_i \in (0, 1) \) such that \( X_i \) are submartingales. We assume the assets are not identical in law, i.e. \( q_1 \neq q_2 \). (The common constant \( b \) could for convenience be put equal to 1 if we accept that the asset jumps to zero without disappearing, i.e. it is immediately reborn with a positive value and the same dynamics.) Let \( \hat{\Phi} \) and \( \hat{J}^{(w)} \) correspond to the limiting case \( \Upsilon \nearrow \hat{\Upsilon} \). Then it is an easy exercise (and it follows from Proposition 6) that the solution is of the form \( \hat{\Phi}(y) = K \cdot 2\sqrt{y} \) and the (unique) optimal control is given by
\[
b\hat{w}_i = (1 - q_i^2)y.
\]

and if there is two fund separation, the funds are determined by \( \hat{w} \) and the origin; in particular, the bank can be taken as one of the funds. Now if
\[ y \leq y_0 := \exp\left\{-2 - q_1^2 - q_2^2\right\}, \]
we will almost surely have \( Y(1) \leq 1 \) and therefore \( \Phi(y) = \hat{\Phi}(y) \) with optimal control \( w^* = \hat{w} \) for all choices of \( \Upsilon \) as above. On the other hand, for \( y > y_0 \) we have
\[
J^{(w)}(y) < \hat{J}^{(w)}(y) \quad \text{and thus} \quad \Phi(y) < \hat{\Phi}(y).
\]

We want to prove that \( \Phi \) is differentiable at \( y_0 \), which by concavity will imply continuous differentiability up to some \( y_1 > y_0 \). Consider the control \( \hat{w} \), and let \( y = y_0 + \epsilon \) with \( \epsilon > 0 \) so small be so small that if there is at least one jump, \( Y(1) \leq 1 \) almost surely. Let \( \sigma \) be the probability that no jump occurs before time 1. Then
\[
J^{(w)}(y_0 + \epsilon) - \hat{\Phi}(y_0 + \epsilon) = \sigma \cdot \left( \Upsilon((y_0 + \epsilon)e^{2-q_1^2-q_2^2}) - \hat{\Upsilon}((y_0 + \epsilon)e^{2-q_1^2-q_2^2}) \right)
\]
Divide by \( \epsilon \) and let \( \epsilon \searrow 0 \), we see that the right hand side tends to \( \sigma \cdot (\Upsilon'(1) - \hat{\Upsilon}'(1)) = 0 \) by (40). So \( \Phi \) is differentiable at \( y_0 \) as claimed. Therefore, since there is no restriction to assume nonnegative positions, the HJB equation holds for \( y > y_0 \) small enough; \( w_i \) then satisfies the first order condition
\[
\Phi'(y) = q_i \Phi'(y - bw_i) = q_i K \cdot (y - bw_i)^{-\frac{1}{2}}
\]
for small enough \( y \). Assume for contradiction that there is a separation into (the bank and) \( a(y)f \), in which case \( w_2/w_1 = (1 - q_2^2)/(1 - q_1^2) \) by (39). Inserting and rearranging terms, we find that this is the same as

\[
y \cdot (\Phi'(y))^2 = K^2. \tag{43b}
\]

By continuity, \( \Phi(y) = K \cdot 2\sqrt{y} = \hat{\Phi}(y) \) on some nonempty interval \((y_0, y_2)\), a contradiction. \( \triangle \)

We shall now confine ourselves to a class of utility functions admitting two fund separation. We will choose to consider a modification of the HARA utility family; in the finite horizon setting (36), we would see similar properties with exponential utility, just as in the single period setting of [CS]. We shall however skip the exponential utility class, as more interesting properties are found by studying the HARA family, given by

\[
\frac{1 - \gamma}{\gamma} \left( \frac{y + \eta}{1 - \gamma} \right)^\gamma \tag{44a}
\]

where \( 0 \neq \gamma \neq 1 \); in addition, we let the class include

\[
\log(y + \eta) = \lim_{\gamma \to 0} \frac{1 - \gamma}{\gamma} \left( \frac{y + \eta}{1 - \gamma} \right)^\gamma - 1 \tag{44b}
\]

to correspond to \( \gamma = 0 \). Strictly speaking, the HARA class has an additional positive scaling parameter – which without loss of generality can be put equal to one in our setting, as we are soon to introduce a discounting factor.

We may work with the HARA function as is, but for \( \frac{\gamma + \eta}{1 - \gamma} < 0 \), the utility function is not very interesting (\( \gamma > 1 \)) or not possibly not even defined (\( \gamma < 1 \)). To cope with this, we can modify the utility function: The computationally easiest modifications are as follows:

- use \( \frac{\gamma + \eta}{1 - \gamma} \) instead of \( \frac{\gamma + \eta}{1 - \gamma} \). Then the utility function is not monotone.
- putting the utility function equal to a constant when \( \frac{\gamma + \eta}{1 - \gamma} < 0 \).

We shall use the latter modification. Notice that if \( \gamma < 1 \), then this constant has to be minus infinity for the utility function to be concave. Otherwise, it is intuitively obvious that some infinite position is optimal, and it is not difficult to show that there will be such an optimal strategy to the problem we are about to pose (the case where we stop the process the moment of bankruptcy corresponds to zero utility). We skip the details. If \( \gamma > 1 \), the only constant extension which makes utility nondecreasing and concave, is to put utility equal to zero when \( \frac{\gamma + \eta}{1 - \gamma} < 0 \). We shall therefore choose to work with the utility functions

\[
\Upsilon_{\gamma}(y) = \begin{cases} 
-\infty & \text{if } \gamma < 1 \text{ and } \frac{\gamma + \eta}{1 - \gamma} < 0 \\
0 & \text{if } \gamma > 1 \text{ and } \frac{\gamma + \eta}{1 - \gamma} < 0 \\
\frac{1 - \gamma}{\gamma} \left( \frac{\gamma + \eta}{1 - \gamma} \right)^\gamma & \text{if } 0 \neq \gamma \neq 1 \text{ and } \frac{\gamma + \eta}{1 - \gamma} \geq 0 \\
\log(y + \eta) & \text{if } \gamma = 0 \text{ and } \frac{\gamma + \eta}{1 - \gamma} \geq 0
\end{cases} \tag{45}
\]

with \( \log 0 = -0^\gamma = -\infty \) if \( \gamma < 0 \).
Let $\Delta \geq 0$ be the discount term. We will consider the finite horizon problem

$$\Phi(t, y, \theta) := \sup_{\mathcal{F}^{t,Y,\theta}} \left[ \int_t^T \Delta(s, \Theta(s)) \cdot \mathcal{Y}_{\eta(s)}^\gamma (c(s)) \, ds + \tilde{\Delta}(\Theta(T)) \cdot \mathcal{Y}_{\tilde{\eta}}^\gamma (Y(T)) \right]$$

(46)

where $\mathcal{F}^{t,Y,\theta}$ denotes expectation with respect to the probability law $P^{t,Y,\theta}$ of the processes starting at $\Theta(t) = \theta$, $Y(t) = y$. We impose the restriction that $c(s) = 0$ whenever $\Delta(s) = 0$; without loss of generality, we can therefore assume that

$$\eta(s) = 0 \text{ whenever } \Delta(s) = 0$$

(47)

We will have to assume further conditions which will be stated in Proposition 6.

The $\Theta$-dependence in the bequest function seems natural, as the bequest function may represent future investment optimization problems (beyond time $T$). The $\Theta$-dependence in running utility has not the same intuitive interpretation, though.

If $\gamma > 1$, then for large enough $y$, namely $y \geq -h$ with $h$ given by

$$-\tilde{\eta} + h(t) = \int_t^T -c(s) \, ds = \int_t^T \eta(s) \, ds$$

(48)

one can use the zero portfolio, consume $c = -\eta$ (which yields maximum direct utility) and still end up with $Y(T) \geq -\tilde{\eta}$ (and maximum terminal utility). On the other hand, if $Y(t) < -h(t)$, then there is no portfolio (satisfying the tameness conditions in Proposition 6) for which $Y(T) \geq -\tilde{\eta}$ a.s. Therefore, $\Phi(t, y, \theta) = 0$ if $y \geq -h(t)$ with $h$ as in (48). Arguing similarly for $\gamma < 1$, we arrive at the following property:

$$\Phi(t, -h, \theta) = \begin{cases} 0 & \text{if } \gamma > 0 \\ -\infty & \text{if } \gamma < 0 \end{cases}$$

(49a)

$$\Phi(t, y, \theta) = \begin{cases} 0 & \text{if } \gamma > 1 \\ -\infty & \text{if } \gamma < 1 \text{ if } \frac{y + h}{1 - \gamma} < 0. \end{cases}$$

(49b)

Therefore, it is no restriction to assume

$$w = \left(\frac{y + h}{1 - \gamma}\right)^+ \cdot f.$$ 

(50)

It will turn out that the optimal $f^*$ does not depend on $h$ nor $y$. We will consider candidates $\phi = \phi_\epsilon$ for the value function:

$$\phi = \phi_\epsilon(t, y, \theta) = \begin{cases} D(t, \theta) \mathcal{Y}_{\theta(t) + \epsilon}(y) + g(t, \theta) & \text{if } \frac{y + h}{1 - \gamma} > 0 \\ 0 \text{ or } \infty \text{ as above} & \text{if } \frac{y + h}{1 - \gamma} \leq 0 \end{cases}$$

(51)

where $D$ is supposed to be positive. The case $\frac{y + h}{1 - \gamma} \leq 0$ is already solved, so assume $\frac{y + h}{1 - \gamma} > 0$. It will turn out that $g = 0$ for $\gamma \neq 0$. 
In the following, \( \hat{D} \) will denote \( \frac{\partial D(t, \theta)}{\partial \theta} \) and so forth. Put \( Z = Z_c(s) = \phi_c(s, \Theta(s), Y(s)) \). Using the feedback control \( (c, (\frac{2\gamma + h + \epsilon}{1 - \gamma}) + f) \), we get

\[
\Delta(s, \Theta) \cdot \nabla_c \gamma(c) \, ds + dZ = \left[ \Delta \frac{1 - \gamma}{\gamma} \left( \frac{c + \eta}{1 - \gamma} \right)^\gamma - cZ \frac{\gamma}{y + h + \epsilon} \right] \, ds \\
+ Z \left[ \frac{\hat{D}(s, \Theta)}{D(s, \Theta)} + \frac{\gamma}{y + h + \epsilon} \right] \, ds \\
+ Z \left[ \frac{\gamma}{1 - \gamma} (f^\top b - \frac{1}{2} f^\top Rf) \right. \\
\left. + \int \left( \frac{D(s, \nu(\Theta(s^-)))}{D(s, \Theta(s^-))} \right) \frac{(1 + \frac{f^\top p}{1 - \gamma})^\gamma - 1}{1 - \gamma} \frac{\gamma}{1 - \gamma} \, d\tilde{N} \right] \\
\right]
\]

where \( d\tilde{N} = dN - dq \, ds \) and where if \( \gamma < 1 \), we only consider \( f \) such that

\[
1 - \gamma + f^\top p \geq 0 \quad \text{q-a.e.}
\]

(The others yield minus infinity; note that if \( \gamma < 1 \), then arbitrary large positive or negative direction jumps will forbid negative or positive position, respectively. In particular, the zero position is the only allowed for an asset having not totally skewed \( \alpha \)-stable noise for \( \alpha < 2 \), if \( \gamma < 1 \).) The \( c^* \) maximizing the right hand side of (52) with respect to \( c \), satisfies

\[
c^* = \left( \frac{D}{\Delta} \right)^{\frac{1}{1-\gamma}} (y + h + \epsilon) \chi_\Delta - \eta
\]

where \( \chi_\Delta(s) = \chi_{\Delta(s > 0)} \), and thus

\[
\Delta \frac{1 - \gamma}{\gamma} \left( (\frac{c^* + \eta}{1 - \gamma})^\gamma - (h - c^*)Z \frac{\gamma}{y + h + \epsilon} \right) = Z (1 - \gamma) \left( \frac{D}{\Delta} \right)^{\frac{1}{1-\gamma}} \chi_\Delta
\]

Therefore, if \( D \) satisfies the inequality

\[
0 \geq \hat{D}(t, \vartheta) \frac{1 - \gamma}{\gamma} + D(t, \vartheta) \frac{(1 - \gamma)^2}{\gamma} \left( \frac{D(t, \vartheta)}{\Delta(t, \vartheta)} \right)^{\frac{1}{1-\gamma}} \cdot \chi_\Delta + D(t, \vartheta) (f^\top b - \frac{1}{2} f^\top Rf) \\
+ \frac{1 - \gamma}{\gamma} \int \left( D(t, \nu(\vartheta)) \left( (1 + \frac{f^\top p}{1 - \gamma})^\gamma - D(t, \vartheta) - \frac{\gamma}{1 - \gamma} D(t, \vartheta) f^\top p \right) \right) \, dq
\]

for all \( f \), then

\[
\Delta(s, \Theta) \cdot \nabla_c \gamma(c) \, ds + dZ \leq Z \left[ \frac{\gamma}{1 - \gamma} f^\top dW + \int \left( \frac{D(s, \nu(\Theta(s^-)))}{D(s, \Theta(s^-))} \right) \frac{(1 + \frac{f^\top p}{1 - \gamma})^\gamma - 1}{1 - \gamma} \, d\tilde{N} \right]
\]

with equality if both \( c = c^* \) and (55) holds with equality for \( f^* \) maximizing its right hand side.
Similarly, for log utility we get an optimal consumption $c^*$ given by (54a) with $\gamma = 0$, and the following for $D,$ $g$:

\[
0 = \dot{D}(t, \vartheta) + \int \left( D(t, \nu(\vartheta)) - D(t, \vartheta) \right) dq + \Delta(t, \vartheta) \tag{57a}
\]

\[
0 \geq \dot{g}(t, \vartheta) + \int \left( g(t, \nu(\vartheta)) - g(t, \vartheta) \right) dq + \Delta(t, \vartheta) \left( \log(\Delta(t, \vartheta)) - \log(D(t, \vartheta)) - 1 \right)
+ D(t, \vartheta) \left( \bar{f}^T \vartheta - \frac{1}{2} \bar{f}^T R_f \right) + \int \left( D(t, \nu(\vartheta)) \log(1 + \bar{f}^T p) - D(t, \vartheta) \tilde{c} \bar{f}^T p \right) dq \tag{57b}
\]

for all $f,$ $\bar{f}^T p \geq -1$ (here, $0 \log 0 := 0$). If (57) holds, then

\[
\Delta(s, \Theta) \log(c + \eta) ds + dZ \leq D(s, \Theta(s)) \bar{f}^T dW
+ \int \left( D(s, \nu(\Theta(s^-))) \log(1 + \bar{f}^T p) + g(s, \nu(\Theta(s^-))) - g(s, \Theta(s^-)) \right) d\tilde{N} \tag{58}
\]

with equality if both $c = c^*$ and (57b) holds with equality for $f^*$ maximizing its right hand side.

We may now proceed to prove that the solution suggested is optimal under certain conditions:

**Proposition 6.**
Consider problem (46), where the supremum is taken among all consumption-investment strategies $(c, w) = (c(s), (\frac{f(s) + h(s)}{1-\gamma})^+ \cdot f)$ satisfying:

- a). (27) has a unique weak solution
- b). $c(s) = 0$ whenever $\Delta(s) = 0$
- c). $f$ is bounded, and $f = 0$ whenever $Y(s)$ is less than some lower bound.
- d). For all $f$ such that $1 - \gamma + \bar{f}^T p \geq 0$ $q$-a.s., and for all $f$ if $\gamma > 1$,

\[
\int \left( \left( \frac{1}{\gamma} \right) \left( \log(1 + \bar{f}^T p) \right)^2 dq < \infty \quad \text{for} \quad \gamma \neq 0 \tag{59a}
\]

\[
\int \left( \log(1 + \bar{f}^T p) \right)^2 dq < \infty \quad \text{for} \quad \gamma = 0. \tag{59b}
\]

Assume furthermore that there exist bounded $D$ satisfying (55) (if $\gamma \neq 0$) or $D$, $g$ satisfying (57) (if $\gamma = 0$), with $D > 0$ and

\[
D(T, \cdot) \geq \bar{\Delta}, \quad g(T, \cdot) \geq 0. \tag{60}
\]

Then $\Phi \leq \phi_0$ given by (51) with $h$ given by (48).

Suppose in addition that $c^*$ is given by (54a), and that (55) (if $\gamma \neq 0$) or (57b) (if $\gamma = 0$) holds with equality for the $f^* = f^*(\vartheta)$ maximizing the right hand side, (60) holds with equality and

\[
\mathbb{E}^{t, \vartheta} \left[ \int_t^T f^*(\Theta(s))^T R_f f^*(\Theta(s)) ds \right] < \infty. \tag{61}
\]

Then $\Phi = \phi_0$ and $(c^*, (\frac{Y(s) + h(s)}{1-\gamma})^+ \cdot f^*)$ is optimal.
Proof. Consider the process $Z_t(s)$. For $\gamma \leq 0$ let $\epsilon > 0$ be given and note that for $y < -h$, then $c$ and $f$ are both zero; For $\gamma > 0$, use $\epsilon = 0$. Thus if we start at $y > -h$, $Z_t(s)$ is lower bounded if $\gamma < 1$. Define

$$
\tau_M = T \wedge \inf \{ s \geq t_1 | Z_t(s) \geq M \}.
$$

(62)

By the boundedness of $f$ and $D$,

$$
Z_t(t) \geq E[Z_t(\tau_M) + \int_t^{\tau_M} \Delta \cdot \mathcal{T}(c) \, ds]
$$

(63)

Let $M \to \infty$. By lower boundedness and Fatou’s lemma, we get $Z_t(t) \geq \Phi$ and thus $Z_0(t) \geq \Phi$ if $\gamma < 1$. The same conclusion holds for $\gamma > 1$; though $Z$ is not necessarily lower bounded, $c$ above and (59a) suffice.

To prove optimality, consider $(c^*, D_1(s) + h(s) + f^*)$. The case of log utility is easy, so consider the case $\gamma \neq 0$. Notice that

$$
\Delta \cdot \mathcal{T}(c^*) = \chi_\Delta \left( \frac{D}{\Delta} \right)^{\frac{1}{1-\gamma}} Z
$$

(64)

and therefore, $Z$ is a geometric process, since $f^*$ depends only on $Z$ through $D$. Solving the equation (55) for $Z$, we get

$$
Z(T) \cdot \exp \left\{ \int_t^T \chi_\Delta \left( \frac{D(s, \Theta(s))}{\Delta(s, \Theta(s))} \right)^{\frac{1}{1-\gamma}} \, ds \right\} = Z(t) \cdot X(T)
$$

(65a)

where $X(t) = 1$ and

$$
dX(t) = X(t^-) \left( \int_t^T \frac{\gamma}{1-\gamma} f^T \, dW + \int_t^T \left( \frac{D(s, \nu(\Theta(s)))}{D(s, \Theta(s))} \right) \left( (1 + f^T p)^+ \gamma \right) \, d\tilde{N} \right)
$$

(65b)

has zero expectation by (59a), (61). So the right hand side is a martingale. Differentiating, we get that the right hand side of (56) — which now holds with equality — has zero expectation. It follows that $Z_0(t) = \Phi$.

\[ \square \]

Remark. Obviously the boundedness assumptions on $D$, $g$ and $f$ may be weakened. Note also that the optimal $f^*$ does not satisfy condition $c$ of Proposition 6 for $\gamma > 1$. These “tameness conditions” on the portfolio have the economic interpretation of excluding “doubling strategies” which would be optimal if allowed.

\[ \triangle \]

COROLLARY

The fund $f^*$ does not depend on $h$, and is thus a mutual fund for all agents with $(\Delta, \tilde{\Delta})$ common (up to the obvious multiplicative constant). If $\nu p = 0$ g-a.e., i.e. $\Theta$ does not jump simultaneously as the assets do, then $f^*$ does not depend on $\Delta$, i.e. it is a mutual fund for all agents with the same $\gamma$.

\[ \triangle \]

We remark that if $\nu p$ is not necessarily 0, but all agents have the same $D(\cdot, \nu(\cdot))/D(\cdot, \cdot)$, then $f^*$ is still common for all with common $\gamma$. $f^*$ does (typically) depend on $\gamma$, though.
This far we have only considered finite time horizon. For $T = \infty$, consider the following criterion with $\eta = 0$, $\gamma < 1$:

$$
\Phi(t, y, \theta) := \sup_{T \to \infty} \lim E^{t, y, \theta} \left[ \int_t^T \Delta(s, \Theta(s)) \cdot \gamma_0(c(s)) \, ds \right],
$$

(66)

the supremum being taken over all strategies such that almost surely, $Y(s) \geq 0$ for all $s \geq t$. Then we have the following:

**Proposition 7.**

Consider the problem (66) subject to $Y(s) \geq 0\forall s$, a.s., with $\gamma < 1$, $\eta = 0$, optimizing with respect to consumption-investment strategies satisfying a) - d) of Proposition 6. Assume that there exist bounded $D$ satisfying (55) (if $\gamma \neq 0$) or $D$, $g$ satisfying (57) (if $\gamma = 0$), with $D > 0$. Define $\phi_0$ by (51) with $h = 0$, and suppose that for $y > 0$,

$$
\limsup_{M \to \infty} \phi_0(\tau_M, Y(\tau_M), \Theta(\tau_M)) \geq 0
$$

(67)

with $\tau_M$ given by (62). Then $\Phi \leq \phi_0$.

Suppose in addition that $c^*$ is given by (54a), and that (55) (if $\gamma \neq 0$) or (57b) (if $\gamma = 0$) holds with equality for the $f^* = f^*(\theta)$ maximizing the right hand side and that

$$
\limsup_{M \to \infty} E[\phi_0(\tau_M, Y^*(\tau_M), \Theta(\tau_M))] = 0
$$

(68)

with $Y^*$ being the process obtained by using $(c^*, \frac{Y^*}{1-\gamma}f^*)$.

Then $\Phi = \phi_0$ and $(c^*, \frac{Y^*}{1-\gamma}f^*)$ is optimal. If in addition equation (68) holds with $\lim$ instead of $\limsup$, then we can have $\lim$ instead of $\liminf$ in the definition of the value function.

**Proof.** As in the proof of Proposition 6, we have

$$
Z(\epsilon) = E[Z(\tau_M)] \geq E\left[ \int_0^{\tau_M} \Delta \cdot \gamma_t(c) \, ds \right]
$$

with equality for $\epsilon = 0$, $c^*$, $f^*$. Take $\liminf$ on both sides. If (68) holds with $\lim$ instead of $\limsup$, we can take $\liminf$ for superoptimality and then $\lim$ for optimality.

Again, we remark that if $D(\cdot, \nu(\cdot))/D(\cdot, \cdot)$ is common to all agents, we have a mutual fund within the CRRA class (i.e. HARA with $\eta = 0$), for fixed $\gamma < 1$. In the special case

$$
\Delta(t, \theta) = H(\theta)e^{-\delta t},
$$

(70a)

we often have a finite solution of the form

$$
D(t, \theta) = K(\theta)e^{-\delta t},
$$

(70b)

and then $f$ will not depend on $\delta$. We shall consider an example where this holds, and where (55) (with equality) can be solved explicitly as inductively given Bernoulli equations:
Example. Assume that there is a total of $n$ assets; assets $1, \ldots, n_0$ are always accessible, and assets $n_0 + 1, \ldots, n$ are bonds, each being driven by one Poisson jump source $N_i$ of which everything else is independent, and each bond will become inaccessible at first jump in $N_i$. By scaling the drift term, we can without loss of generality assume that a bond jumps to 0; also, it is more convenient working with $dN_i$ instead of $d\tilde{N}_i = dN_i - \tilde{x}_i dt dq$. The bond then gets an adjusted drift term $\tilde{b}_i$ which has to be $\geq 1$ in order to avoid arbitrage. The market state can now be represented as $\Theta(s) = (\Theta_{n_0+1}(s), \ldots, \Theta_n(s))$, where $\Theta_i$ is one if the bond is still accessible, zero if not:

$$d\Theta_i(s) = -\Theta_i(s^-) dN_i$$

(71)

where each $N_i$ has Lévy measure $\gamma > 0$. Consider (55) with equality. For the first $n_0$ assets, the maximization with respect to the vector $\hat{f}$ consisting of the $n_0$ first $f_i$'s, is independent of $D$:

$$\sup_{\hat{f}} \left\{ \hat{f}^T \mathbf{b} - \frac{1}{2} \hat{f}^T \hat{R} \hat{f} + \frac{1 - \gamma}{\gamma} \int \left( ((1 + \frac{\hat{f}^T \hat{p}}{1 - \gamma})^+ \right)^\gamma - 1 - \hat{x}_i \frac{\gamma}{1 - \gamma} \hat{f}^T \hat{p} \right) dq \right\} =: F(0)^{1 - \gamma}$$

(72)

where the $^+$ accents on the vectors and matrix denote the obvious truncation of the dimension.

Again, if $\gamma < 1$ the supremum is only taken among the $f$ which satisfy $\frac{f_i}{1 - \gamma} \geq -1$ $q$-a.e. The last $n - n_0$ assets are each independent of everything, so the portfolio optimization can be carried out for each $i > n_0$. We write $\vartheta \equiv \epsilon$ if $\epsilon$ and $\vartheta - \epsilon$ are also vectors of zeros and ones (i.e., the latter has no negative components and all assets accessible with $\vartheta$ are accessible with $\epsilon$.) Then $\nu(\vartheta) = \vartheta - \epsilon_i$ $q_i$-a.e. and the maximization with respect to $f_i$ becomes:

$$\sup_{f_i} \left\{ D(t, \vartheta) f_i \tilde{b}_i + \frac{1 - \gamma}{\gamma} q_i D(t, \vartheta - \epsilon_i)((1 - \frac{f_i}{1 - \gamma})^+)^\gamma - D(t, \vartheta) \right\}$$

(73)

and the optimal $f_i$ is given by

$$f_i^* = (1 - \gamma) \left(1 - \left(\frac{q_i D(t, \vartheta - \epsilon_i)}{\tilde{b}_i D(t, \vartheta)}\right)^{1 - \gamma}\right) \vartheta_i$$

(74)

Inserting into (55), we get the following Bernoulli equation for $D(t, \vartheta)$:

$$0 = \frac{\dot{D}(t, \vartheta)}{D(t, \vartheta)} + \frac{F(\vartheta)}{1 - \gamma}$$

$$+ (1 - \gamma) \left[ (\Delta(t, \vartheta))^{1 - \gamma} \chi_{\Delta} + \sum_{i > n_0} \left( \frac{q_i}{\tilde{b}_i} D(t, \vartheta - \epsilon_i) \right)^{1 - \gamma} \cdot \tilde{b}_i \vartheta_i \right] (D(t, \vartheta))^{-1 - \gamma}$$

(75a)

with

$$F(\vartheta) := F(0) + \sum_{i > n_0} (\gamma \tilde{b}_i - q_i) \vartheta_i.$$ 

(75b)

The solution is

$$D(t, \vartheta) = e^{-F(\vartheta) t}$$

$$\cdot \left( A(\vartheta) - \int^t e^{-F(\vartheta) s} \cdot \left( (\Delta(s, \vartheta))^{1 - \gamma} \chi_{\Delta} + \sum_{i > n_0} \left( \frac{q_i}{\tilde{b}_i} D(s, \vartheta - \epsilon_i) \right)^{1 - \gamma} \cdot \tilde{b}_i \vartheta_i \right) ds \right)^{1 - \gamma}$$

(76)
(with abuse of notation if $\theta_i = 0$) to be solved inductively starting from $\theta = 0$. In the special infinite horizon case where $\Delta$ is given by (70a), we find an easy generalization of the version of the Merton problem treated in [FOS]. We can take $A = 0$, and by induction, it follows that if $\delta > \max_{\varepsilon \in \Theta} F(\varepsilon)$, then $D$ satisfies (70b) with $K$ inductively given by

$$
(K(\theta))^{1-\gamma} = \frac{1 - \gamma}{\delta - F(\theta)} \left[ (H(\theta))^{1-\gamma} + \sum_{i > n_0} \left( \frac{q_i}{b_i} K(\theta - e_i) \right)^{1-\gamma} \cdot \bar{b}_i \theta_i \right].
$$

(77)

To prove optimality, note that by (64) and (58) with equality, it suffices that $\Delta/D$ is bounded away from 0 to conclude that $E[Z(\tau_M)]$ tends to 0. However, in our case the fraction is constant. For $\gamma \in (0, 1)$ it also follows, by solving the problem for a sequence $\delta_n \nearrow F(\theta)$ that the value function is $+\infty$ for $\delta \leq F$ (which is impossible for $\gamma < 0$, as $F < 0$ as well). Finally, we remark that if $H$ is common (up to a multiplicative constant) to all agents, then $\varphi$ is a mutual fund for all agents.

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\section*{References}


