3-DIMENSIONAL SURGERY THEORY,
UNil–GROUPS AND THE BOREL CONJECTURE

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1. Introduction.

One of the most important and powerful techniques of higher dimensional geometric topology is surgery. After it was invented in the 1960s by Browder and Novikov and further developed by Wall, it has become an indispensable tool in any attempt to solve classification problems in higher dimensional manifolds. Moreover, surgery theory provides a natural framework for some of the most important unsolved problems in topology of manifolds, as, for example, the Borel conjecture. In addition, like all profound theories, it introduces its own set of new, interesting problems; the most famous being the Novikov conjecture.

Classically, surgery theory only applies to manifolds of dimensions \( \geq 5 \), with some partial (though very deep) results due to Freedman in the topological case of dimension 4. In dimension 3 completely different techniques, which, on the other hand, in general are not available in higher dimensions, are used. Thus dimension 3 and higher dimensions seem to be two completely different fields. It may be argued that most of the techniques may be combined under a general heading of “cutting and pasting”, and 3–manifold topologists also use surgery (on knots and links) as a tool, but this does not lend itself to the kind of algebraization that makes surgery such a powerful tool in higher dimensions.

This paper may be seen as an attempt to bridge the gap between dimension 3 and higher, by studying classification problems in dimension 3 from the point of view of higher dimensional surgery theory. This is achieved by a suitable modification of what “classification” should mean, which in an appropriate sense makes surgery “work” in this dimension (See also [KT]). Then we show how one can use classical 3–manifold techniques to do computations. In particular we prove that the integral Novikov conjecture is true in dimension 3.

This 3–dimensional theory is interesting enough by itself, but perhaps even more interesting are applications to the theory of manifolds of higher dimensions. The applications we give here are to questions about the Borel conjecture, the vanishing of Cappell’s exotic UNil–groups, and relations between these groups and the Borel conjecture. The known positive results on the Borel conjecture require some extra geometrical structure; the most general being Farrell and Jones’ proof for nonpositively curved manifolds [FJ]. Using results and techniques of the present paper, we
can e.g. in every dimension $n \geq 5$ construct infinitely many smooth, aspherical manifolds $M^n$ which do not admit any metric of nonpositive curvature, but for which the the Borel conjecture still holds.

In section 2 we set up the theory and state all the results. The proofs are in section 3. Section 4 contains the applications to the Borel conjecture and Cappell's UNil–groups, and a fairly detailed description of our approach to a construction of a possible counterexample to the Borel conjecture via the UNil–groups.

2. Statements of results.

From the point of view of classification of topological (or smooth or PL) manifolds, the central result of surgery theory is the exact sequence of Sullivan–Wall, cf. [W1], chapter 10: Let $M^n$, $n \geq 5$ be a closed, connected, orientable topological manifold. Then, in its simplest version, the Sullivan–Wall sequence is (continuing to the left):

$$
\cdots \longrightarrow L_{n+1}(\pi_1(M)) \xrightarrow{\gamma(M)} S^s_{TOP}(M) \xrightarrow{\eta(M)} [M, G/TOP] \xrightarrow{\theta(M)} L_n^s(\pi_1(M))
$$

Here $L_n^s(\pi_1(M))$ are the surgery groups of Wall, $[M, G/TOP]$ is the set (group) of normal invariants and $S^s_{TOP}(M)$ is the structure set (see [W1]). The broken arrow represents an action of the group $L_{n+1}(\pi_1(M))$ on $S^s_{TOP}(M)$ such that the orbits are exactly identified by $\eta(M)$. The decorations $\varepsilon = h, s$ refer to the case of homotopy equivalences or simple homotopy equivalences in $S_{TOP}(M)$.

There are also analogous sequences for

(a) nonorientable manifolds, where the surgery groups also depend on the orientation homomorphism $\omega_M : \pi_1(M) \to \mathbb{Z}/2$,

(b) manifolds with boundary, keeping the boundary fixed,

(c) smooth and PL manifolds, where the $L$–groups are the same, but $G/TOP$ is replaced by $G/O$ or $G/PL$,

(d) 4–dimensional topological manifolds with “good” fundamental groups. (See [FQ].)

In this paper we study of a corresponding sequence in dimension 3. Before we describe this sequence, recall the definition of $S^s_{TOP}(M^n)$ for $n \geq 5$ (in the appropriate category). The elements of $S^s_{TOP}(M^n)$ are equivalence classes of homotopy equivalences ($\varepsilon = h$) or simple homotopy equivalences ($\varepsilon = s$) $f : N \to M$, $f_1 : N_1 \to M$ and $f_2 : N_2 \to M$ are equivalent if there is an $\varepsilon$–cobordism $W$ between $N_1$ and $N_2$ and a map $F : W \to M$ extending $f_1$ and $f_2$. (If $\varepsilon = s$, this means that there is a homeomorphism (diffeomorphism, PL homeomorphism) $g : N_1 \to N_2$ such that $f_2 \circ g \simeq f_1$.) The modifications necessary in dimension 3 are clear from the following two observations:

1. Surgery can be performed on any 1–cycle, but we change the fundamental group. Therefore we can only hope to achieve homology equivalences (over $\mathbb{Z}[\pi_1(M)]$).

2. The action of $L_{n+1}(\pi_1(M))$ on $S^s_{TOP}(M)$ in higher dimensions is well defined because a normal map $F : (W; N_1, N_2) \to (M \times I; M \times 0, M \times 1)$ with trivial surgery obstruction can be modified to the relation in $S^s_{TOP}(M)$ described above.
It is now not hard to see that we get a Sullivan–Wall sequence in dimension three if $S_{TOP}^e(M)$ is defined as the set of equivalence classes of $e$–homology equivalences (with $\mathbb{Z}[\pi_1(M)]$, hence all coefficients) $f : N \to M$, where $f_1 : N_1 \to M$ and $f_2 : N_2 \to M$ are equivalent if there is a normal map with trivial surgery obstruction in $\Omega_3^e(\pi_1(M), \omega_M)$

$$F : (W; N_1, N_2) \to (M \times I; M \times 0, M \times 1)$$

extending $f_1$ and $f_2$. If $M$ has boundary, we require that $f|\partial M$ is the identity.

With this definition there is an exact sequence

$$\begin{align*}
\Sigma(M/\partial M), G/TOP & \xrightarrow{\theta^e(M \times I)} L_4^e(\pi_1(M), \omega_M) \xrightarrow{\gamma(M)} S_{TOP}^e(M) \\
& \xrightarrow{\eta(M)} [M/\partial M, G/TOP] \xrightarrow{\theta^e(M)} L_3^e(\pi_1(M), \omega_M)
\end{align*}$$

and the purpose of this paper is to study this sequence.

Remark. This definition can be found in [KT]. For “good” fundamental groups, surgery can be performed on $F$ to change it into a $\mathbb{Z}[\pi_1(M)]$–homology equivalence, and we obtain the sequence in [FQ].

In higher dimensions it is very difficult to say much about the maps in the sequence for an arbitrary manifold. For example, it is still a conjecture (a strong version of the Novikov conjecture) that

$$\theta^e(M) : [M, G/TOP] \to L_4^e(\pi_1(M), \omega_M)$$

is injective for an aspherical manifold $M^n$. We shall see in this paper that in dimension 3, however, the situation is completely different, and we shall achieve an almost complete computation of $\theta^e(M)$ for arbitrary 3–manifolds $M$. In particular, we show that the Novikov conjecture is always true in this dimension (Corollary 3).

Before stating our main results, we make an important observation. Recall that Whitney sum defines an $H$–space structure on $G/TOP$ and hence a group structure on $[M, G/TOP]$, but $\theta^e(M)$ is not in general a homomorphism. However, there is a different $H$–space structure on $G/TOP$ such that it is, and for 3–manifolds the two group structures on $[M, G/TOP]$ coincide. The reason for this is that in this range $G/TOP \simeq K(\mathbb{Z}/2, 2)$, and $K(\mathbb{Z}/2, 2)$ has a unique $H$–space structure up to homotopy. This means that for a 3–manifold with (possibly empty) boundary there is an isomorphism of groups

$$[M/\partial M, G/TOP] \simeq H^2(M/\partial M; \mathbb{Z}/2) \simeq H_1(M; \mathbb{Z}/2),$$

and the surgery obstruction map

$$\theta^e(M) : H_1(M; \mathbb{Z}/2) \to L_3^e(\pi_1(M), \omega_M)$$

is a homomorphism.

The difference between dimension three and higher can be seen already in our first result:
Theorem 1. Let $M$ be a compact, connected, orientable 3–manifold. Then the surgery map $\theta^s(M)$ is a split monomorphism.

The simple example $M = \mathbb{R}P^2 \times S^1$ is easily seen to have $[M, G/TOP] \approx \mathbb{Z}/2 \times \mathbb{Z}/2$ and $L^2_\ast (\pi_1(M), \omega_M) \approx \mathbb{Z}/2$, hence Theorem 1 cannot be extended to arbitrary 3–manifolds. However, it turns out that the crucial geometric property of $\mathbb{R}P^2 \times S^1$ is that it contains a two–sided (i.e. with trivial normal bundle) $\mathbb{R}P^2$. Our next result implies a weaker form of Theorem 1, but depends on it for the proof. It also follows from Theorem 5 below, but is stated separately because it gives the precise conditions for $\theta^s(M)$ to be injective.

Theorem 2. Let $M$ be a compact, connected 3–manifold. Then $\theta^s(M)$ is injective if and only if $M$ does not contain a two–sided $\mathbb{R}P^2$.

Now let $M^3$ be a $K(\pi, 1)$–manifold. Since $\pi_1(M) \approx \pi$ is torsion free, there is no two–sided $\mathbb{R}P^2$ in $M$. As a consequence we get:

Corollary 3. (Integral Novikov conjecture in dimension 3) Let $M^3$ be a compact, aspherical manifold. Then the surgery map $\theta^s(M)$ is a monomorphism.

Remark. It should be pointed out that this 3-dimensional version of the Novikov conjecture is proved without any geometric restriction on the manifolds. In particular, there is no appeal to Thurston’s geometrisation conjecture. This is in sharp contrast with most higher dimensional results, where always some geometric input is needed (e.g. existence of metric of nonpositive curvature or some restriction on $\pi_1(M)$, cf. [FH], [C4]).

Our next result identifies the kernel of $\theta^s(M)$ in the presence of two–sided $\mathbb{R}P^2$s. First we need the following definition:

Definition. An irreducible 3–manifold $N$ is called special if the following conditions are satisfied:

1. The boundary of $N$ is nonempty and incompressible in $N$
2. $N$ is not sufficiently large

These conditions are surprisingly restrictive:

Lemma 4. If $N^3$ is special, then $H_1(M; \mathbb{Z})$ is finite, $\partial N$ consists of two copies of $\mathbb{R}P^2$ and any embedded $\mathbb{R}P^2$ in the interior of $N$ is boundary–parallel.

(Recall that $\mathbb{R}P^2 \subset N$ is called boundary–parallel if it bounds a cylinder $\mathbb{R}P^2 \times I$ together with one of the boundary components. $M$ is sufficiently large [Wa1] if it contains a two–sided incompressible surface. If it is orientable and sufficiently large, it is called Haken.)

Before we state Theorem 5, recall that the isomorphisms (1.2) and the surgery maps $\theta^s(M)$ are natural with respect to codimension 0 inclusions.

Theorem 5.

a) Let $M$ be a compact 3–manifold. The kernel of $\theta^s(M)$ is generated by all $\ker \theta^s(N)$, where $N$ runs through all special submanifolds $N$ of $M$.

b) For each $N \approx \mathbb{R}P^2 \times I \subset M$, $\text{im}(\ker \theta^s(N)) \approx \mathbb{Z}/2$, generated by a simple self–homotopy equivalence of $M$. 
Remark. By Lemma 4, \( M \) can only have special submanifolds if it contains two-sided \( \mathbb{RP}^2 \)'s. The simplest special submanifolds are just neighborhoods of two-sided \( \mathbb{RP}^2 \)'s, and a possible conjecture is that we can replace special submanifolds by such neighborhoods in theorem 5. In fact, we observe below that \( \theta^\varepsilon(\mathbb{RP}^2 \times I) = 0 \), hence (using (1.2)) the conjecture would be that \( \ker \theta^\varepsilon(M) \subset H_1(M; \mathbb{Z}/2) \) is the subgroup generated by \( H^1 \) of all the two-sided \( \mathbb{RP}^2 \)'s. This, by a result of Epstein [E], is precisely the subgroup generated by the orientation reversing elements of order two in \( \pi_1(M) \). Note that if this conjecture is true, \( \ker \theta^\varepsilon(M) \) is independent of \( \varepsilon \)!

Observe that by the Sullivan–Wall sequence, Theorem 5 already gives some information about \( S^*_\text{TOP}(M) \). Our next results concern this set.

First we show that for some manifolds the structure set decomposes according to the prime decomposition of \( M \). This result once again points out the special role of two-sided projective planes in 3–manifolds (cf. Theorem 2).

**Theorem 6.** Let \( M \) be a compact, connected 3–manifold which is either a connected sum or boundary connected sum of two non–simply connected manifolds \( M_1 \) and \( M_2 \). Then

\[
S^*_{\text{TOP}}(M) \cong S^*_{\text{TOP}}(M_1) \times S^*_{\text{TOP}}(M_2)
\]

if and only if \( M \) does not contain a two–sided \( \mathbb{RP}^2 \).

**Remarks.** (6.1) \( M \) contains a two-sided \( \mathbb{RP}^2 \) if and only if either \( M_1 \) or \( M_2 \) does. (6.2) If \( M_1 \), say, is simply connected, then \( S^*_{\text{TOP}}(M_1) = 0 \), and we trivially have the decomposition in Theorem 6, even when \( M_2 \) does contain a two–sided \( \mathbb{RP}^2 \). It follows that if \( M \) is aspherical, we can replace all homotopy spheres and disks in its prime decomposition by real spheres and disks, and obtain an irreducible manifold with the same structure set.

Using Theorem 6 we can often reduce to computing \( S^*_{\text{TOP}}(M) \) for prime 3-manifolds. The next theorem analyzes homology equivalences in this case.

**Theorem 7.** Suppose \( M \) is a closed, prime 3-manifold and assume \( N_1 \# N_2 \xrightarrow{f} M \) is a homology equivalence over \( \mathbb{Z}[\pi_1(M)] \). Then \( N_1 \) or \( N_2 \) is a homology sphere, which we may assume is mapped trivially by \( f \).

Since \( S^*_{\text{TOP}}(S^3) = 0 \), this means, in particular, that if \( M \) is prime, then every element in \( S^*_{\text{TOP}}(M) \) may be represented by homology equivalences \( N \to M \) with \( N \) also prime.

When considering manifolds with boundary, it is often of substantial advantage to know that the boundary is incompressible (for example the so-called JSJ splitting theorem for Haken manifolds, cf. [JS] [J]). Let \( M \) be a compact, connected, irreducible 3–manifold with nonempty boundary \( \partial M \). Then there is a standard procedure for converting \( M \) into a manifold \( \overline{M} \) with incompressible boundary \( \partial \overline{M} \), by cutting along properly embedded 2-disks. In the case of manifolds without two–sided projective planes, it turns out that this operation does nothing to the structure sets. Namely, despite the fact that \( \pi_1(M) \neq \pi_1(\overline{M}) \) we have:

**Theorem 8.** If \( M \) contains no two–sided \( \mathbb{RP}^2 \)'s, then \( S^*_{\text{TOP}}(M) \cong S^*_{\text{TOP}}(\overline{M}) \).

Any further information about \( S^*_{\text{TOP}}(M) \) depends on knowledge of \( L^*_\delta(\pi_1(M), \omega_M) \). In the finite fundamental group case, computation of these groups is essentially complete (cf. [LM], [W2]). (Note that a closed manifold with finite fundamental group
is orientable, so then $\theta^s(M)$ is injective.) In particular, the structure set can be quite large in this case, as illustrated by the following example. Note that although all our theorems have the same formulation for $\varepsilon = s$ or $h$, the computations are different in the two cases.

**Example 9.** Let $M = X \# L$, where $X = S^3/Q2^k$, $k \geq 3$, is the quaternionic space form, and $L = S^3/(Z/2^n)$, $n \geq 1$, is a lens space. Then

$$S^s_{TOP}(M) \cong S^s_{TOP}(X) \times S^s_{TOP}(L) \cong \mathbb{Z}^r,$$

where $r = 2^{k-2} + 2^{n-1} + 2(k + 1)$, and

$$S^h_{TOP}(M) \cong S^h_{TOP}(X) \times S^h_{TOP}(L) \cong \mathbb{Z}^s \oplus (Z/2)^t,$$

where $s = 2^{k-2} + 2^{n-1} + 2$ and $t = \lfloor 2(2^{n-2} + 2)/3 \rfloor - \lfloor n/2 \rfloor - 1$. Moreover, one can show that no nontrivial element in $S^s_{TOP}(M)$ can be represented by a homology self equivalence of $M$. However, for some $M$ there are nontrivial elements in $S^h_{TOP}(M)$ that can be represented even by homotopy self equivalences; the simplest example is $X \# L(8; 1)$.

In the infinite fundamental group case, e.g. for a $K(\pi, 1)$–manifolds, however, the situation is completely different:

**Theorem 10.**

a) Let $M^3$ be a compact aspherical manifold with nonempty boundary. Then $S^s_{TOP}(M) = 0$.

b) Let $M^3$ is a closed aspherical manifold which is either Seifert fibered, hyperbolic or Haken with at least one hyperbolic piece in the torus decomposition. Then $S^h_{TOP}(M) = 0$.

**Remark.** Thurston’s famous geometrization conjecture [T] says that a closed, orientable, irreducible 3–manifold is either hyperbolic, Seifert fibered or Haken. If this conjecture is true, Theorems 6 and 10 go a long way towards computing $S^s_{TOP}(M)$ for all compact 3–manifolds.

3. Proofs.

**Notation.** It will be convenient to use the standard notation $\mathcal{N}(M)$ for the set of topological normal cobordism classes of normal maps to $M$, homeomorphisms on the boundary if $\partial M \neq \emptyset$. Thus, for a 3-manifold with boundary

$$\mathcal{N}(M) \cong [M/\partial M, G/TOP] \cong H_1(M; \mathbb{Z}/2).$$

Also, if $M = \cup_i M_i$ is the manifold $M$ written as the union of its connected components, we shall write

$$L_n^s(M) = \oplus_i L_n^s(\pi_1(M_i), \omega_{M_i}).$$

($n$ does not have to be the dimension of $M$!) In the following there will usually be no need to distinguish between $L^s$ and $L^h$, because the proofs are the same in the two cases. If so, we just write $L$. Similarly for $S$. 

The orientable case: Theorem 1.

Let $M^3$ be orientable, and let $\gamma \subset M$ be an imbedded loop. Then $\gamma$ has a trivial normal bundle $T_\gamma \approx S^1 \times D^2$, and the natural map

$$\mathbb{Z}/2 \approx N(T_\gamma) \xrightarrow{\cdot i} N(M) \approx H_1(M; \mathbb{Z}/2)$$

is onto the summand represented by $\gamma$. Since $\pi_1(M)$ is finitely generated, there is an isomorphism $\beta : H_1(M; \mathbb{Z}/2) \approx (\mathbb{Z}/2)^l$ for some $l$, and we choose embedded circles $\gamma_1, \ldots, \gamma_l$ which represent a basis. Let $\beta_i : \pi_1(M) \to \mathbb{Z}/2, i = 1 \ldots l$ be the corresponding projections. (The Hurewicz map $\pi_1(M) \to H_1(M; \mathbb{Z}/2)$ is surjective.) Consider now the diagram

$$\begin{array}{c}
\oplus_i N(T_{\gamma_i}) \xrightarrow{\oplus_i \cdot i} N(M) \\
\approx \oplus_i \theta(T_{\gamma_i}) \\
\oplus_i L_3(\mathbb{Z}) \xrightarrow{\oplus_i \cdot i} L_3(M) \\
\Pi_i \beta_i.
\end{array}$$

Since $L_3(\mathbb{Z}) \xrightarrow{\sim} L_3(\mathbb{Z}/2) \xrightarrow{\sim} \mathbb{Z}/2$ ([W1], ch. 13), the result follows. \square

Lemma 4.

The assertion is a standard fact, cf. [Hp], Lemma 6.6. Since $N$ is not sufficiently large, the boundary must contain at least one $RP^2$ (see [Hp], Lemma 6.6 & 6.7). If there is an $RP^2$ in the interior of $N$, then it has to be two–sided, since $N$ is irreducible. Then $N$ must be nonorientable, and if $F$ is such $RP^2$, the restriction $H^1(M; \mathbb{Z}/2) \to H^1(F; \mathbb{Z}/2)$ is surjective, since $H^1(F; \mathbb{Z}/2)$ is generated by $w_1(F)$. Therefore the inclusion induces an injection $H_1(F; \mathbb{Z}/2) \to H_1(M; \mathbb{Z}/2)$ and hence also $\pi_1(F) \to \pi_1(M)$, so $F$ is necessarily incompressible. Since $N$ is not sufficiently large, $F$ must be boundary–parallel. But Theorem 4.1 in [Sw] then says that unless $\partial N$ consist consists of exactly two copies of $RP^2$, $N$ must be sufficiently large. \square

$M^3$ non–orientable. Theorems 2 and 5.

Theorem 2 clearly follows from Theorem 5, so we prove this. Our strategy is to try to cut $M$ along two–sided surfaces and use splitting theorems for $L$–groups to inductively reduce to simpler situations. If $F \subset M$ is a connected, properly embedded (i.e. $F \cap \partial M = \partial F$), two–sided surface in $M$, let $M'$ be the manifold obtained by cutting $M$ along $F$ (e.g. $M' = M$ – open neighborhood of $F$). Assume now that $\pi_1(F) \to \pi_1(M')$ is injective. From work of Cappell [C1–4] (cf. Ranicki [R3]) it then follows that there is a commutative diagram (note that $Wh(\pi_1(F)) = 0$):

$$\begin{array}{cccccccc}
\cdots & \to & N(F \times I, ) & \to & N(M') & \to & N(M) & \to & N(F) & \to & \cdots \\
\downarrow \approx & & \downarrow \approx & & \downarrow \approx & & \downarrow \approx & & \downarrow \approx & & \downarrow \approx \\
\cdots & \to & H_1(F; \mathbb{Z}/2) & \to & H_1(M'; \mathbb{Z}/2) & \to & H_1(M; \mathbb{Z}/2) & \to & H_0(F; \mathbb{Z}/2) & \to & \cdots \\
\downarrow \theta(F \times I) & & \downarrow \theta(M') & & \downarrow \theta(M) & & \downarrow \theta(F) & & \downarrow \theta'(F) & & \downarrow \theta'(F) \\
\cdots & \to & L_3(F) & \to & L_3(M') & \to & L_3(M) & \to & L_3'(F) & \to & \cdots
\end{array}$$
with exact rows. Here $L_2^2(F)$ is the direct sum of $L_2(F)$ and a certain \textit{UNil}-group, see [R3], Theorem 8.3. It is known [Cl] that $\vartheta(F \times I)$ is surjective (in fact an isomorphism if $F \not\cong P^2$), and that $\vartheta(F)$ is always injective. Let $K(M)$ denote the kernel of $\vartheta(M)$. Then half of the argument for the 5-lemma proves

\textbf{Lemma.} The induced homomorphism $K(M') \to K(M)$ is surjective.

Suppose now that we cut $M$ repeatedly along such surfaces, ending up with a disjoint collection $\{M_i\}$ of codimension 0 submanifolds $M_i \subset M$ and a gluing map $\prod_i M_i \to M$. Then the Lemma implies that $K(M)$ is generated by the images of all the $K(M_i)$. Hence, to prove the first part of Theorem 5, it suffices to show that we can cut $M$ into pieces such that $K(M_i) = 0$ unless $M_i$ is a special submanifold.

We start by cutting along $S^3$'s and $D^3$'s until all properly imbedded spheres and disks in any of the pieces are isotopic to the boundary. If any of the pieces thus obtained are orientable, we know already by Theorem 1 that they do not contribute anything to the kernel. Hence it suffices to consider the case $M_i$ prime, non-orientable and such that all the components of $\partial M_i$ are incompressible. In fact, we may even reduce to the \textit{irreducible} case: Any nontrivial embedded sphere is parallel to the boundary, and if $\tilde{M}_i = M_i \cup [D^3]$ attached to $S^3$-components of $\partial M_i$, then $N(\tilde{M}_i) \approx N(M_i)$ and $L_* (\tilde{M}_i) \approx L_*(M_i)$, hence also $K(\tilde{M}_i) \approx K(M_i)$. Thus we have a surjection $\oplus_i K(\tilde{M}_i) \to K(M)$, where each $\tilde{M}_i$ is irreducible, nonorientable and has incompressible boundary (although there is no map $\prod_i \tilde{M}_i \to M$ inducing it).

If $M_i$ contains no two-sided $RP^2$, neither does $\tilde{M}_i$. Therefore it is sufficiently large [HI], and we can reduce (by repeated cutting along incompressible surfaces) to the trivial case $M_i = D^3$. But then $K(M_i) = 0$.

The more interesting case is when $M_i$ (and hence also $\tilde{M}_i$) does contain two-sided $RP^2$'s. Then we can cut $\tilde{M}_i$ along a minimal (finite) number of two-sided $RP^2$'s into pieces where all two-sided $RP^2$'s are boundary-parallel (see e.g. [Hp], Lemma 13.2). Consider the resulting pieces. They all have nonempty, incompressible boundary, so they are either \textit{special} or sufficiently large. But the pieces that are sufficiently large, admit \textit{generalized hierarchies} [Sw], which means that we can inductively cut them along incompressible surfaces such that the resulting components are either 3-disk or homeomorphic to $RP^2 \times I$. The $RP^2 \times I$'s are, of course, also special submanifolds, so we conclude that only special submanifolds of $\tilde{M}_i$ contribute to $K(M_i)$. But it is obvious that special submanifolds of $\tilde{M}_i$ are isotopic to special submanifolds of $M_i$ and hence also of $M$. This completes the first part of Theorem 5.

For the second part, let $P \times I \subset M$ be a product neighborhood of a two-sided $RP^2$. We then observed above that

$$\mathcal{N}(P \times I) \approx H_1(P;\mathbb{Z}/2) \to H_1(M;\mathbb{Z}/2) \approx \mathcal{N}(M)$$

is injective, hence $\mathcal{N}(P \times I)$ gives rise to a $\mathbb{Z}/2 \subset \mathcal{N}(M)$. That this lies in the kernel of $\vartheta(M)$, follows from the diagram
and the calculation $L_3(\mathbb{Z}/2^-) = 0$ ([W1], p. 162). But the nontrivial element of $\mathcal{N}(P \times I)$ can be represented by a simple homotopy equivalence which is the identity on the boundary. To be more specific, let $f : P \times I \to P \times I$ be the composition

$$P \times I \to P \times I \vee S^3 \xrightarrow{id \cup g} P \times I,$$

where the first map is a standard pinch and $g$ is the Hopf map $S^3 \to S^2$ composed with the covering $S^2 \to P \times \{1/2\} \subset P \times I$. It is then not difficult to see (cf. [Rul]) that $f$ is a simple homotopy equivalence which represents the nontrivial element in $\mathcal{N}(P \times I)$. But then $f$ extends to a simple homotopy equivalence on all of $M$. 

**Theorem 6.**

We first prove the Theorem in the connected sum case. Let $M'_i$, $i = 1, 2$, be $M_1$ — an open disk. Then it is easy to see that $S_{TOP}(M'_i) \approx S_{TOP}(M_i)$, and gluing along the boundary defines a map

$$S_{TOP}(M'_1) \times S_{TOP}(M'_2) \xrightarrow{\rho} S_{TOP}(M)$$

which is part of an obvious map from the product of the surgery sequences for $M'_1$ and $M'_2$ to the one for $M$. Consider now the diagram

$$\begin{array}{cccccc}
\mathcal{N}(S^2 \times I^2) & \xrightarrow{\rho} & L_4(S^2) \\
\downarrow & & \downarrow \\
\mathcal{N}(M'_1 \times I) \times \mathcal{N}(M'_2 \times I) & \longrightarrow & L_4(M'_1) \times L_4(M'_2) & \longrightarrow & S_{TOP}(M'_1) \times S_{TOP}(M'_2) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \rho \\
\mathcal{N}(M \times I) & \longrightarrow & L_4(M) & \longrightarrow & S_{TOP}(M) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0
\end{array}$$

By Theorem 2, the rows are exact if and only if there are no two-sided $RP^2$'s. The columns are Mayer–Vietoris sequences of the type considered in the proof of Theorem 5 above. The first column is exact because $\mathcal{N}(S^2 \times I) = 0$. The second is exact modulo Cappell’s $UNil_4$-group, which by [C2] is trivial if there are no orientation reversing elements of order two in $\pi_1(M_1)$ or $\pi_1(M_2)$. But this is equivalent to the nonexistence of a two-sided $RP^2$. Hence both the rows and columns are exact in this case, and standard diagram chasing shows that $\rho$ is an isomorphism. (Note that the structure sets admit natural abelian group structures in this case.)

Conversely, if the fundamental groups both are nontrivial and there exists an orientation reversing element of order two in one of them, Cappell’s calculations
[C2] show that coker \((L_4(M_1) \times L_4(M_2) \to L_4(M))\) will be infinitely generated. Hence coker \(\rho\) will also be infinitely generated.

The only modification necessary in the boundary connected sum case is that \(S^2\) in the argument above should be replaced by \(D^2\). \(\Box\)

**Theorem 7.**

Let \(p : \tilde{M} \to M\) be the universal cover of \(M\), and let

\[
\begin{array}{c}
\tilde{N}_1 \# \tilde{N}_2 \xrightarrow{f} \tilde{M} \\
\downarrow p' \quad \downarrow p \\
N_1 \# N_2 \xrightarrow{f} M
\end{array}
\]

be the pullback along \(f\). Then \(\tilde{f}\) is a homotopy equivalence over \(\mathbb{Z}\).

Let \(\Sigma\) be the connecting sphere in \(N_1 \# N_2\), and write \(N_1 \# N_2 = N_1^0 \cup_\Sigma N_2^0\). Then \(p'^{-1}(\Sigma) = \bigsqcup \gamma S_\gamma\), a disjoint union of spheres indexed by elements \(\gamma \in \pi_1(M)\). We claim that each \(S_\gamma\) is mapped trivially (up to homotopy) by \(\tilde{f}\).

Since \(M\) is prime, there are three cases to consider: (i) \(\tilde{M}\) contractible, (ii) \(\tilde{M} \simeq S^3\), and (iii) \(\tilde{M} \simeq \mathbb{R} \times S^2\). In the first two cases, the claim is obvious. In case (iii), \(M\) is either an \(S^2\)- or \(RP^2\)-bundle over \(S^1\), and we can reduce to the case of \(S^1 \times S^2\) by (in the other cases) considering a double cover. But for \(M = S^1 \times S^2\) we have

\[
\pi_2(\tilde{M}) \approx \pi_2(M) \approx H_2(M),
\]

with \([\tilde{f}]|S_\gamma^2\) \(\in \pi_2(\tilde{M})\) corresponding to \([f]|\Sigma\) \(\in H_2(M)\). Since \(\Sigma\) bounds in \(N_1 \# N_2\), the latter represents the trivial element in homology.

Thus, in all cases, each \(S_\gamma\) is homologous to zero in \(H_2(N_1 \# \bar{N}_2)\), hence bounds a compact submanifold \(V_\gamma \subset N_1 \# \bar{N}_2\). Then int \(V_\gamma\) can only contain finitely many \(S_\gamma\)'s, each of which splits \(V_\gamma\) into two components. Therefore we can find an “innermost” \(V_\gamma\); one that does not contain any \(S_\gamma\)'s in its interior. But this \(V_\gamma\) must map homeomorphically onto either \(N_1^0\) or \(N_2^0\), say \(N_2\). Moreover, we can write \(N_1 \# \bar{N}_2 = W_\gamma \cup S_\gamma V_\gamma\) and simple Mayer-Vietoris arguments show that \(V_\gamma\) must be a homology disk.

It now only remains to observe that \(f|N_2\) can be identified with \((p\tilde{f})|V_\gamma\), which clearly is null-homotopic. \(\Box\)

**Theorem 8.**

It suffices to prove that \(S^*_\text{TOP}(M) \approx S^*_\text{TOP}(\tilde{M})\) if \(M\) is \(\tilde{M}\) with a 1-handle attached. Then we have a diagram

\[
\begin{array}{c}
\mathcal{N}(\tilde{M} \times I) \to L_4(\tilde{M}) \to S^*_\text{TOP}(\tilde{M}) \to 0 \\
\downarrow \quad \downarrow \quad \downarrow \\
\mathcal{N}(M \times I) \to L_4(M) \to S^*_\text{TOP}(M) \to 0,
\end{array}
\]
where the vertical maps are induced by codimension 0 inclusions. Since $M \simeq \overline{M} \setminus S^1$, it is easy to see that $\mathcal{N}(\overline{M} \times I) \to \mathcal{N}(M \times I)$ is surjective. (Note that $M$ may be nonorientable even if $\overline{M}$ is orientable, so the map may not be an isomorphism.)

Furthermore, $\pi_1 M \approx \pi_1 \overline{M} \ast \mathbb{Z}$ and $\pi_1 M$ has no element of order two ($M$ being acyclic), so $L(\overline{M}) \to L(M)$ is an isomorphism by the results of [C2]. It then follows that the last vertical map is an isomorphism.

**Example 9.**
We need to determine the surgery obstruction map

$$\mathcal{N}((X \# L) \times I) \to L^s_4(X \# L).$$

This splits into three maps

$$H_0(M \times I, \partial(M \times I); \mathbb{Z}) \approx \mathbb{Z} \xrightarrow{\sim} L_4(1),$$

$$H_2(L; \mathbb{Z}/2) \approx \mathbb{Z}/2 \to \tilde{L}_4^s(L)$$

and $H_2(L; \mathbb{Z}/2) \approx \mathbb{Z}/2 \to \tilde{L}_4^s(L)$. The surgery groups are computed in [W2] and [CS] (cf. also [HM]), and they are all free, except $\tilde{L}_4^s(L)$ which has two-torsion which might conceivably be hit by the two-torsion in $\mathcal{N}(L \times I)$. However, the $\mathbb{Z}/2$ in $\mathcal{N}(L \times I)$ is represented by a self homotopy equivalence [KwS], p.531, hence has surgery obstruction zero. Therefore $\mathcal{S}^s_{TOP}(M) \approx \tilde{L}_4^s(X) \oplus \tilde{L}_4^s(L)$.

The assertion concerning simple homology equivalences of $X \# L$ is based on two facts:

1. Every (simple) self homotopy equivalence (in fact, every degree one map) of $X$ and $L$ is a (simple) homotopy equivalence.
2. Every simple self homotopy equivalence of $X$ and $L$ is homotopic to a homeomorphism [Co], p. 100, [KwS], Proposition 1.4.

On the other hand, it is not hard to see that $L(8;1)$ has four (homotopy classes of) self homotopy equivalences, but up to isotopy only two self homeomorphisms (cf. [B], Theorem 3). Hence there is a self homotopy equivalence $f$ of $L(8;1)$ which is not simple. If $f$ is trivial in $\mathcal{S}^s_{TOP}(L)$, there is a normal cobordism with trivial surgery obstruction between $f$ and the identity. But finite groups are "good," so surgery can be performed to give an $h$-cobordism between $f$ and the identity. By [KwS], Theorem 2.1, such $h$-cobordisms are trivial, so $f$ must be homotopic to a homeomorphism. This is a contradiction. □

**Theorem 10.**
The proof of this theorem is based on the following idea: Recall from [KS], p. 277 (cf. also [N]) that for any $M^3$ there is a natural periodicity diagram

\[
\begin{array}{c}
\mathcal{N}(M \times I) \xrightarrow{\theta^s(M \times I)} L^s_4(M) \\
\Gamma \downarrow^\approx \quad \downarrow^\approx \times CP^2 \\
\mathcal{N}(M \times I^5) \xrightarrow{\theta^s(M \times I^5)} L^s_8(M)
\end{array}
\]

If $M$ is a closed manifold which admits a metric of nonpositive sectional curvature, then $\theta^s(M \times I^5)$ is an isomorphism, by [FJ], Theorem 2.1 and Addendum 2.4.
Thus \( \theta^e(M \times I) \) is also an isomorphism. But such manifolds are aspherical, hence can not contain any \( RP^2 \)s. Hence \( \theta^e(M) \) is injective by Theorem 2, and it follows from the surgery exact sequence that \( S^e_{\text{TOP}}(M) = 0 \).

In case b) of the theorem, we now only have to observe that all these manifolds have metrics of nonpositive curvature. The hyperbolic case is obvious, and the Seifert fibered case follows from the classification in e.g. [Sc1]. The remaining cases are covered by Theorem 3.3 in [L].

a) requires more work. First we observe that by Remark (6.2) following Theorem 6 we may assume that \( M \) is irreducible, and by Theorem 8 we may assume that \( M \) also has incompressible boundary. Now recall the homotopy functor \( S^e_{\ast} \) of Ranicki [R1]. This has the property that if \( X \) is an \( n \)-manifold (possibly with boundary) with \( n \geq 5 \), then

\[
S^e_{\text{TOP}}(X) \approx S^e_{n+1}(X).
\]

Our idea is to embed \( M \) as a codimension zero retract of a closed manifold \( W \) which admits a metric of nonpositive curvature. By functoriality and (3.2) we then have

\[
S^e_{\text{TOP}}(M \times I^k) \approx S^e_{k+4}(M) \subseteq S^e_{k+4}(W) \approx S^e_{\text{TOP}}(W \times I^k),
\]

and the last group is trivial for all \( k \geq 2 \), by [FJ]. In particular, \( S^e_{\text{TOP}}(M \times I^4) = S^e_{\text{TOP}}(M \times I^5) = 0 \), so by the exactness of the surgery sequence \( N(M \times I^5) \approx L^5_5(M) \). The result then follows from diagram (3.1) again.

It remains to construct such manifold \( W \). If all the boundary components of \( M \) are tori or Klein Bottles, it follows from [L], Theorem 3.2 and Theorem 3.3 that \( M \) has a metric of nonpositive curvature, and such that the boundary components are totally geodesic and the metric is a product near the boundary. (Let us call a metric with these properties good near the boundary.) Then we simply let \( W \) be the double of \( M \). Then the metric extends to a metric of nonpositive curvature on all of \( W \), and \( W \) has an obvious retraction down to (each copy of) \( M \).

If there are other boundary components, none of these can be spheres of projective planes, since \( M \) is aspherical. However, in the presence of boundary components of genus \( \geq 2 \), \( M \) may not have a metric which is good near the boundary, so the simple doubling above may not work, in general. For example, even if \( M \) is atoroidal, hence has a hyperbolic structure, its double is not, if there are annuli in the JSJ decomposition of \( M \). However, we can get around this problem by reducing to the case of genus 1 boundaries as follows:

In each of the components of genus \( \geq 2 \) we choose an incompressible annulus. Now we double \( M \) along the complements of these annuli. That is, if \( A_1, \ldots, A_k \) are the chosen annuli, let \( F = \partial M - \bigcup_i \text{int} A_i \). Then define \( V = M \bigsqcup_F M \). \( V \) has only tori and Klein bottles as boundary components, and it contains \( M \) as a retract.

Claim: \( V \) has incompressible boundary.

Let \( D \subset V \) be a properly embedded 2-disk, i.e. such that \( D \cap \partial V = \partial D \). We may assume that \( D \) intersects \( F \subset V \) transversally in a finite union of circles and arcs. An “innermost” circle \( C \) bounds a disk in one of the copies of \( M \), hence also in \( \partial M \) since \( \partial M \) is incompressible. By the irreducibility of \( M \) these two disks bound a ball, which may be used to eliminate \( C \) (and possibly other circles). Repeating
this we may assume that $D \cap F$ only consists of arcs. All arcs bound two disks with $\partial D$, and we may choose one such disk $E$ which contains no arc in its interior. The argument above now allows us to eliminate the arc on the boundary of $E$ (and possibly other arcs). Continuing this way, we may eliminate all of $\text{int}D \cap F$. But this means that $D$ lies completely in one of the copies of $M$, with its boundary in one of the annuli $A_j$. Since $A_j$ is incompressible in $M$, $\partial D$ must bound a disk in $A_j$, hence also in the boundary component of $V$ which contains it.

By the results of Leeb cited above, $V$ now has the required metric, and we let $W$ be the double of $V$. Then $W$ retracts onto $V$, hence also onto $M$. □

Remark. The proof of part a) actually shows more, namely: $S^n_\epsilon(K(\pi_1(M), 1)) = 0$ for all $k$, where $M^3$ is a compact, sufficiently large 3-manifold with nonempty, incompressible boundary. This observation will be important in section 4 below.

4. Applications to Cappell’s $UNil$–groups and the Borel Conjecture.

We start with the following observation:

**Theorem 11.** Let $(M^3, \partial M)$ be a compact, connected and aspherical 3-manifold with nonempty boundary. Set $\pi = \pi_1(M)$. Assume that $\Phi$ is a diagram describing $\pi$ as a free product with amalgamation $\pi = A \ast_C B$ or an HNN extension $\pi = A \ast_C \{t\}$ of finitely generated subgroups:

\[
\begin{array}{ccc}
C & \rightarrow & B \\
\Phi & \downarrow & \downarrow \\
A & \rightarrow & A \ast_C B
\end{array}
\text{ or }
\begin{array}{ccc}
A & \rightarrow & A \ast_C \{t\}
\end{array}
\]

where the homomorphisms $C \rightarrow A$ and $C \rightarrow B$ are injective. Then Cappell’s unitary nilpotent groups $UNil_\ast(\Phi)$ vanish for $\ast = 0, 1, 2, 3$.

**Proof.** First we observe again that we may replace any homotopy disks by honest disks, so we may assume that $M$ is irreducible. Consider now once more Ranicki’s $S^n_\epsilon$–functors. Since in this case $Wh(\pi) = 0$ [Wa2], there are exact sequences

\[
(4.1) \quad \cdots \rightarrow S_n^\epsilon(K(C, 1)) \oplus UNil_{n+1}(\Phi) \rightarrow S_n^\epsilon(K(A, 1)) \oplus S_n^\epsilon(K(B, 1)) \rightarrow
\]

\[
\rightarrow S_n^\epsilon(K(\pi_1, 1)) \rightarrow S_{n-1}^\epsilon(K(C, 1)) \oplus UNil_n(\Phi) \rightarrow \cdots
\]

or

\[
(4.2) \quad \cdots \rightarrow S_n^\epsilon(K(C, 1)) \oplus UNil_{n+1}(\Phi) \rightarrow S_n^\epsilon(K(A, 1)) \rightarrow
\]

\[
\rightarrow S_n^\epsilon(K(\pi_1, 1)) \rightarrow S_{n-1}^\epsilon(K(C, 1)) \oplus UNil_n(\Phi) \rightarrow \cdots
\]

in the respective cases. (See [R3].) Consider now the amalgamated free product case — the other case is similar.

The interesting case is when all the subgroups are nontrivial: the remaining case is covered by Kneser’s theorem and Theorem 6. Note that $M$ itself is a $K(\pi, 1)$,
hence $S^n_2(K(\pi, 1)) = 0$ for all $n$ by the remark following the proof of Theorem 10. But for the same reason it then suffices to find connected, compact, aspherical 3-manifolds with nonempty boundaries with fundamental groups isomorphic to $A, B$ and $C$. We first remark that by a theorem of Scott [Sc2], the subgroups $A, B$ and $C$ are actually finitely presented. Let $U, V$ and $W$ be covering spaces of $M$ with fundamental groups $A, B$ and $C$. These are again aspherical manifolds, and if they are compact, they will also have nonempty boundary, such that Theorem 10a) applies. In general they will be noncompact, but then we can find compact submanifolds of the same homotopy types [GHM]. A compact submanifold of a noncompact manifold must have nonempty boundary, so again we can use Theorem 10a). Hence all the $S^n_2$-terms in (4.1) vanish, so the $UNil$-terms must vanish as well. □

Remarks. 1) This result answers a question in [R2], pp. 685–686 in the case of manifolds with nonempty boundary. Namely: in analogy with the vanishing of Waldhausen’s $Nil(\Phi)$ and $Wh_*(\pi)$ for irreducible 3-manifolds which are sufficiently large (cf. [Wa2]), $UNil_*(\Phi)$ and $S_2^*(M)$ should be trivial for the same class of manifolds.

2) For closed 3-manifolds we get a similar result for the manifolds satisfying the condition of Theorem 10b), or, more generally, admitting a metric of nonpositive curvature.

3) The $UNil$-groups are known to vanish if Cappell’s “square-root closed” conditions holds [C1]. The above theorem covers many examples where all the $UNil$-groups vanish but the square-root closed condition fails! Such examples may be constructed e. g. by gluing Seifert fiber spaces with more than one boundary component and with singular fibers of even order along (some, but not all) boundary tori.

Remarks on the Borel conjecture. Possible counterexamples.

The most successful approach to the Borel conjecture comes from the work of Farrell–Hsiang and Farrell–Jones, and depends on differential geometric properties of the manifold — the most general result being for manifolds of nonpositive curvature [FJ]. The following theorem shows that the conjecture is true also for many manifolds outside this class:

**Theorem 12.** In every dimension $\geq 5$ there are infinitely many manifolds $M^n$ for which the Borel conjecture holds, but which do not admit any metrics of nonpositive curvature.

**Proof.** The examples we construct will be of the form $N^3 \times T^{n-3}$, where $T^{n-3}$ is a torus of dimension $n - 3$, and $N^3$ is a graph manifold i.e. a union of Seifert fiber spaces glued along boundary components (tori). Note that the Whitehead groups of such manifolds will be trivial, so to prove that the Borel conjecture holds, it suffices to show that $S^*(M)$ is trivial.

Let $(X, \partial X), (Y, \partial Y)$ be Seifert fibered 3-manifolds with $\partial X \approx \partial Y \approx T^2$. Assume that in the standard representation of $X$ and $Y$ as Seifert fiber spaces, say $\{g; 1, (\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m)\}$ with $0 < \beta_i < \alpha_i$, all the $\alpha_i$'s are odd. Let $h : \partial X \to \partial Y$ be a diffeomorphism, and form the manifold $N_h = X \cup_h Y$ by
gluing X and Y along the boundaries using h. As shown in [L] (see Example 4.1),
there are canonical choices of linearly independent elements \{x_1, x_2\} in \(H_1(\partial X; \mathbb{Z})\)
and \{y_1, y_2\} in \(H_1(\partial Y; \mathbb{Z})\) such that \(N_h\) admits a metric of nonpositive curvature
if and only if \(h_* : H_1(\partial X; \mathbb{Z}) \to H_1(\partial Y; \mathbb{Z})\) satisfies \(h_* \{x_1, x_2\} = \{\pm y_1, \pm y_2\}\). (See
the proof of Lemma 4.5 for a more precise statement in one special case.) Consequently there are infinitely many choices of gluings h such that \(N_h\) does not admit
any metrics of nonpositive curvature.

Let \(N = N_h\) for some such h, and let \(M^n = N \times T^{n-3}\). By [KL], Corollary 2.7,
none of the manifolds \(M^n\) can have metrics of nonpositive curvature either. But \(M^n\)
has an obvious separating torus \(T^{n-1}\), and our choice of invariants \(\alpha_i\) implies that
the square root closed condition is satisfied for the splitting of \(M^n\) into \(X \times T^{n-3}\) and
\(Y \times T^{n-3}\). Hence the UNil-terms in the sequence 4.1 vanish. Moreover, it is well
known that all \(S_*^k(T^{n-1})\) are trivial, so it only remains to show that \(S_*^k(X \times T^{n-3})\)
and \(S_*^k(Y \times T^{n-3})\) are trivial. But the manifolds \(T^{n-1}, X \times T^{n-3}\) and \(Y \times T^{n-3}\)
have metrics with nonpositive curvature (with flat boundary), so the \(S_*^k\)-functor vanishes on all of them. Therefore \(S_*^k(M)\) also has to be trivial. \(\Box\)

To the best of our knowledge, all results concerning the Borel conjecture (including
Theorem 12) go in the direction of enlarging the class of manifolds for which it
holds. In particular, there is virtually no discussion of possible approaches to
the construction of counterexamples. We now present one such approach, which we
find particularly appealing.

Our idea is based on the construction in the proof of Theorem 12, but without
the conditions on the Seifert invariants. Let \(\Phi\) be the Van Kampen presentation
\(\pi_1(N)\) as amalgamated product \(\pi_1(X) \ast_{\pi_1(\partial \Sigma)} \pi_1(Y)\), i.e. the pushout diagram

\[
\begin{array}{ccc}
Z \times Z & \longrightarrow & \pi_1(X) \\
\downarrow & & \downarrow \\
\pi_1(Y) & \longrightarrow & \pi_1(N)
\end{array}
\]

(Notation as in the proof of Theorem 12.) Then \(\Phi \times \mathbb{Z}^m\) is the presentation of \(\pi_1(N \times T^m)\).
By Waldhausen's results [Wa2] and the fundamental theorem in algebraic K-theory,
\(Wh(G)\) and \(K_0(\mathbb{Z}[G])\) are trivial for all groups \(G\) in these diagrams, hence
\(L_2^*(G) = L_2^*(G) = L_2^*(G)\) and \(UNil_k^*(\Phi \times \mathbb{Z}^k) = UNil_k^*(\Phi \times \mathbb{Z}^k) = UNil_k^*(\Phi \times \mathbb{Z}^k)\)
for all \(k\). Shaneson's splitting theorem and Cappell's Mayer–Vietoris sequences for
\(L\)-groups then combine to give a Shaneson type splitting of the UNil–groups as well:

\[UNil_k(\Phi \times \mathbb{Z}^m) \approx UNil_k(\Phi \times \mathbb{Z}^{m-1}) \oplus UNil_{k-1}(\Phi \times \mathbb{Z}^{m-1}).\]

In particular, if \(m \geq 3\), \(UNil_l(\Phi)\) is a direct summand of \(UNil_k(\Phi \times \mathbb{Z}^m)\) for all \(l\)
and \(k\). This leads to the following observation:

**Proposition 4.4.** The Borel conjecture holds for \(N^3 \times T^m\) for some \(m \geq 3\) if and
only if \(UNil_k(\Phi) = 0\) for all \(k\).

**Proof.** If the conjecture holds, then \(S(N \times T^m)\) vanishes, hence \(UNil_{4+m}(\Phi \times \mathbb{Z}^m)\),
being a direct summand in \(S(N \times T^m)\), also vanishes. But then all \(UNil_k(\Phi) = 0\),
by repeated use of 4.3.
The other direction follows from the Mayer–Vietoris sequence (4.1) by the same argument as in the proof of Theorem 12.

For \( p \) and \( q \) relatively prime, let \( X_{p,q} \) denote the complement of a \((p,q)\) torus knot, and let \( N \) be obtained by gluing two copies of \( X_{p,q} \) along the boundary tori. Even with these simple examples we get infinitely many sufficiently large manifolds without metrics of nonpositive curvature, and if either \( p \) or \( q \) is even, the square root closed condition is not satisfied for the decomposition \( \Phi \), and we don’t know if \( UNil(\Phi) \) vanishes. However, we also need to rule out the possibility that there might be other incompressible surfaces in \( N \), giving rise to decompositions such that the condition holds.

**Lemma 4.5.** For every pair \( \{p, q\} \) there are infinitely many choices of gluings such that \( N \) has no metric of nonpositive curvature and no other incompressible surface than the connecting torus (up to isotopy).

**Proof.** Any incompressible torus in \( N \) must be isotopic to the connecting torus, so we have to rule out the existence of surfaces of higher genus.

Let \( l_i, m_i, f_i \), \( i = 1, 2 \), be the longitude, meridian and Seifert fiber of the two knot complements, let \( h : T^2_i \approx T^2_j \) be the gluing homeomorphism, and let \[
\begin{bmatrix}
a & c \\
b & d
\end{bmatrix}
\]
be the matrix of \( h_* : H_1(T^2_i) \approx H_1(T^2_j) \) with respect to the bases \( \{l_i, m_i\} \). With suitable orientations we then have \( l_1 \cdot m_i = 1 \) (intersection numbers in \( T \)) and \( f_1 = pq \cdot m_i + l_i \).

Neumann [N1][N2] has given conditions for the existence of nonabelian surface subgroups of \( \pi_1 \) of graph manifolds. In our case it follows from his result that there is such a subgroup if and only if \( |\frac{I}{pq}| \leq 1 \), where \( I \) is the intersection number of the two Seifert fibers in the gluing torus.

But this intersection number is easy to compute:

\[
|I| = |h_*(f_1) \cdot f_2| = |(pq)^2 \cdot c + (a - d) \cdot pq - b|,
\]

hence

\[
|\frac{I}{pq}| = |pq \cdot c + (a - d) - \frac{b}{pq}|.
\]

In this special case, the precise condition for existence of a metric of nonpositive curvature as given in Example 4.1 in [L] is \( h_* \{\{l_1, f_1\} \} = \{\pm l_2, \pm f_2\} \). This is easily seen to imply that

\[
\begin{bmatrix}
a & c \\
b & d
\end{bmatrix} = \pm I, \quad \text{or} \quad \begin{bmatrix}
a & c \\
b & d
\end{bmatrix} = \pm \begin{bmatrix}1 & 0 \\
pq & -1
\end{bmatrix}.
\]

But there are obviously infinitely many choices of matrices \[
\begin{bmatrix}
a & c \\
b & d
\end{bmatrix}
\]
not of this form such that \( |\frac{I}{pq}| > 1 \), e.g. \[
\begin{bmatrix}1 & c \\
0 & 1
\end{bmatrix}
\]
for all \( c \neq 0, c \in \mathbb{Z} \).

We now have examples of aspherical manifolds which do not satisfy any of the known sufficient conditions for the Borel conjecture to hold.
**Question 4.6:** Are all the $UNil_k(\Phi)$–groups trivial for all of these examples?

If not, it follows from Proposition 4.4 that there exists a counterexample to the Borel conjecture of the form $N \times T^l$. (In fact, for all $l \geq 3$!)

We are not able to answer Question 4.6, but we offer the following remarks on how to detect a possible example.

Recall from Cappell’s work [C2] that $UNil$ measures the failure of excision for $L$–theory. In our case this means that for diagram 4.3 (and $X = Y = X_{p,q}$) there is a natural splitting $L_n(N) \cong \widehat{L}_n(N) \oplus UNil_n(\Phi)$, where $\widehat{L}_n(N)$ sits in a Mayer–Vietoris sequence

$$\cdots \to L_n(T^2) \to L_n(X) \oplus L_n(X) \to \widehat{L}_n(N) \to L_{n-1}(T^2) \to \cdots$$

When $X$ is a knot complement, it is known that the homology equivalence $X \to S^1$ induces isomorphisms on $L$–groups. (This can be found in [AFR], or proved using results of the present paper.) Moreover, the surgery obstruction maps are isomorphisms $[T^2 \times D^{n-2}]/\partial; G/TOP] \approx L_n(T^2)$ and $[(S^1 \times D^{n-1})/\partial; G/TOP] \approx L_n(S^1)$, and the induced maps are the obvious ones. This observation allows us to compute $\widehat{L}_n(N)$ completely: If the gluing map has the matrix $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ as in the proof of Lemma 4.5, then

$$\widehat{L}_n(N) = \mathbb{Z}, \mathbb{Z}/2b, \mathbb{Z}/2, \mathbb{Z} \quad \text{for } n = 0, 1, 2, 3 \pmod{4} \quad \text{and } b \text{ odd, and}$$

$$\widehat{L}_n(N) = \mathbb{Z} + \mathbb{Z}/2, \mathbb{Z}/b + \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z} + \mathbb{Z}/2 \quad \text{for } n = 0, 1, 2, 3 \pmod{4} \quad \text{and } b \text{ even.}$$

($N$ itself will be a homology lens space of type $(b,d)$. Note that $\widehat{L}_n(N)$ will not depend on $(p,q)$.) Hence there is not much beyond the signature and Kervaire invariant, and anything else we can construct in $L_n(N)$ has to be in $UNil_n(\Phi)$. Our feeling is that one should be able to detect such elements by mapping $\pi_1(N)$ to dihedral groups or binary dihedral groups (generalized quaternion groups) $D_{2m}$ or $\tilde{D}_{2m} = Q_{4m}$. For the situation we are interested in (no square root closed condition), we may assume that $p$ is even and $q$ is odd. Then it is easy to construct nontrivial homomorphisms $\pi_1(N) \to D_{2q}$ (or $\tilde{D}_{2q}$) for suitable choices of the gluing map. Such homomorphisms are built up from homomorphisms $\pi_1(X_{p,q}) \to D_{2q}$ (or $\tilde{D}_{2q}$), and these are easily classified. (For example, there are no nontrivial (i.e. with nonabelian image) homomorphisms $\pi_1(X_{p,q}) \to D_{2m}$ if $p$ and $q$ are both odd.) Then one checks that we can find such homomorphisms which match up if and only if in the gluing matrix $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $b$ is even in the dihedral case and divisible by 4 in the quaternion case. These conditions are e.g. satisfied by all the examples $\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$ as in Proposition 4.5. $L_n^{\epsilon}(D_{2q})$ and $L_n^{\epsilon}(\tilde{D}_{2q})$ have been explicitly calculated by Wall ([W2], see also [HT]), and the 2–ranks of their torsion subgroups grow rapidly with $m$. Therefore there is a good chance that they will catch images of nontrivial elements of $UNil_n(\Phi)$, which is only 2–torsion [C2].

**Remarks.** (1) The choice of knot complements $X_{p,q}$ in the construction was only for specificity and simplicity of computations. One could, of course, replace $X_{p,q}$ by many other Seifert fiber spaces. Also, we can glue $X_{p,q}$ to $X_{r,s}$ for certain choices of $(p, q) \neq (r, s)$. 


(2) The multiplication by tori in the construction can also be replaced by, for example, taking powers of $N$. To be more specific, if $UNil_2(\Phi) \neq 0$, then the Borel conjecture fails for $N \times N$, if $UNil_2(\Phi) \neq 0$, then it fails for $N \times N \times N$, etc.

(3) The computations of $L_n^\alpha(D_{2m})$ and $L_n^\alpha(\tilde{D}_{2m})$ are very explicit (see [HT] and the references there). This suggests the possibility of lifting nontrivial elements in these groups not in the image of $\tilde{L}_n(N)$ to $L_n(N)$, where they necessarily would have to come from $UNil_n(\Phi)$. In particular, lifting one of the copies of $\mathbb{Z}/4$ in $L_n^\alpha(D_{2m})$ would establish a counterexample to the Borel conjecture, and at the same time give an example of a $UNil$-group with elements of order 4.

REFERENCES


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