

N-body simulations with massive neutrinos

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Thesis submitted for the degree of
Master of Science in Astronomy

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June, 2014

Abstract

Neutrinos are very weakly interacting particles with a low mass. In fact, the mass of the neutrinos has not yet been determined by experiments or observations. Still, neutrinos are a good candidate for some of the dark matter in the universe.

In this thesis we will look at the effect of massive neutrinos on cosmological structure formation. We will use a model with cold dark matter (CDM) and a cosmological constant, and we will replace some of the CDM with neutrinos in the simulations. An N-body code is used to simulate the evolution of the model, and neutrinos are added to the code via two different methods. In the first method we use transfer functions from linear theory to generate a density field for the neutrinos. For the second method we will use a smoothing function that smooths out the perturbations to the CDM density field. This smoothed out field will approximate a neutrino density field, and will be added to the CDM field. We analyse the effect of the neutrinos on the matter power spectrum.

acknowledgements

First I would like to thank my supervisor Øystein Elgarøy for presenting a subject that has been interesting to work on, and for believing in me.

I would also like to thank Claudio Llinares for providing me with a code, and helping me with some problems related to it.

Lastly, thanks to my fellow students, who have made the my life as a student an enjoyable one.

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CHAPTER 1

Introduction

From Big Bang nucleosynthesis, we know that there is a background of neutrinos in the universe that has evolved independently ever since the neutrinos decoupled from the rest of the medium. We also know from oscillation experiments that the neutrinos have a non-zero mass. So, neutrinos are weakly interacting and massive particles, therefore they contribute to the dark matter. Unfortunately they can not make up all of the dark matter, because neutrinos have a high thermal velocity, so they form structure on much larger scales than what is observed today.

The neutrino masses are unlikely to be determined by laboratory experiments in the near future. However, it is well known that neutrinos suppress the formation of cosmic structure on small scales, and this effect has been used to put quite strong upper bounds. It is hoped that future missions to map the distribution of matter might be able to determine the masses. If this hope is to be realised, we need to understand how neutrinos affect structure formation on both linear and non-linear scales.

The aim of the work presented in this thesis was to look at a simple way of including neutrinos in an N-body code presented in [1], and to study the effect on the matter power spectrum.

1.1 Mathematical functions

Below we present some mathematical functions that will be useful later in this work. The books [8] and [7] have been used in these derivations.

1.1.1 Fourier transforms

The Fourier transform is a mathematical method that will be very useful to simplify calculations. We will transform some theoretical equations to Fourier space to get rid of spatial derivatives, and in our numerical simulations we will use the discrete Fourier transform to save computational power.

The analytical Fourier transform is

$$F(k) = \int_{-\infty}^{\infty} f(x)e^{ikx} dx,$$

where $F(k)$ is the Fourier transform of the function $f(x)$. The inverse transform is then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k)e^{-ikx} dk.$$

The discrete Fourier transform is given by

$$H_n = \sum_{k=0}^{N-1} h_k e^{2\pi i kn/N}.$$

h_k is a discrete function of N sampled values, and H_n is its transformed values. The reverse transform is almost the same, but we change the sign in the exponential and we divide by a factor N .

$$h_k = \frac{1}{N} \sum_{n=0}^{N-1} H_n e^{-2\pi i kn/N}$$

1.1.2 Distribution functions

Distribution functions describe the distributions of particles based on some parameters. These parameters are usually time, space and energy. The functions give the number density of the particle species we are interested in, as a function of the parameters that goes into the distribution function.

Below are the Fermi-Dirac and Bose-Einstein distribution functions that are only valid for a homogenous distribution of particles. That is why these functions do not depend on spatial position. Here μ is the chemical potential, T is the temperature, E is the energy and k is the Boltzmann constant.

The Fermi-Dirac distribution is defined as

$$f_{\text{FD}} = \frac{1}{e^{(E-\mu)/kT} + 1},$$

while the Bose-Einstein distribution is defined as

$$f_{\text{BE}} = \frac{1}{e^{(E-\mu)/kT} - 1}.$$

CHAPTER 2

Cosmology

In this chapter most of the background theory is presented. The derivations are mainly based on the book by Dodelson [3]. In addition [17], [13], [9], [12] and [14] are used. We start by deriving equations for a homogeneous universe, and then add fluctuations to the distribution of matter.

2.1 Short history

It is widely accepted that the universe started with the Big Bang. The universe has expanded and evolved from a hot and dense initial state. We have no theory to explain this, but the result of the Big Bang was a very hot and dense plasma of elementary particles. Before one Planck time, which is about 10^{-43} seconds, nothing is known of what happens in the universe.

After this Planck time the universe had a very short period of incredible expansion, called inflation. This inflationary period lasted from about 10^{-36} seconds to about 10^{-32} seconds, and during this short period the size of the universe increased by a factor of e^{60} . This period is also the source of small inhomogeneities in the distribution of matter, which will later evolve to stars and galaxies and clusters of galaxies.

After inflation ended, the universe had grown to cosmic scales, but it was still very dense and hot. But now elementary particles like protons and neutrons could go together to form atomic nuclei. This is called the big bang nucleosynthesis, and light elements up to lithium were created in this process. Although there were no electrons bound to the nuclei, because as soon as an electron would attach to a nucleus, a high energy photon would come along and ionize the atom again. The temperature of the universe was high enough that photons could transfer more energy than the electrons binding energy in a collision, and the electron would detach from the nucleus.

By this time the universe was a hot ionised gas, and it would be like that for some time. The universe continued to expand and cool down, and finally after 380,000 years, it had cooled down enough that the photons no longer could ionise neutral hydrogen. This is called recombination and most of the hydrogen became neutral. Since the universe consisted mostly of hydrogen this had the effect that photons could travel

freely through the cosmic gas, and the universe went from opaque to transparent. This is referred to as the photon decoupling, and from now on photons will develop independently from the rest of the cosmic medium. These photons are what we now refer to as the cosmic microwave background radiation (CMB).

As the universe continues to expand, small differences in the distribution of mass will lead to large clouds of gas that are gravitationally bound. These clouds continue to collapse, and by about 150 million years after Big Bang, the first stars were born. These stars started to radiate high energy photons, and over the next few hundred million years the hydrogen once again became ionised. This is called reionization.

From there the universe continued to expand and make structure on larger and larger scales, until it ended up as we see it today. Figure (2.1) illustrates the evolution of the universe.

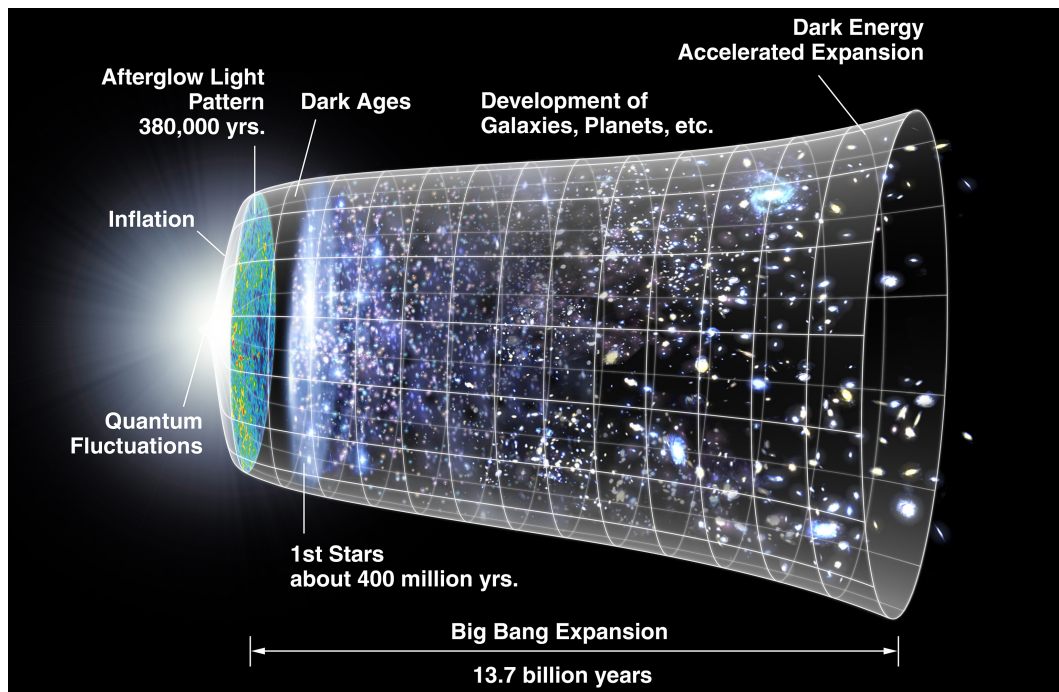


Figure 2.1: Credit: NASA/WMAP Science Team [19].

2.2 General relativity

The framework of modern cosmology is Einstein's theory of general relativity. So it is important that we understand the basic principles of this theory before we continue on to the cosmology. In general relativity, gravitation is not seen as a force, but rather a consequence of the curvature of spacetime. This curvature is described by the metric tensor $g_{\mu\nu}$. The distance between two points in any geometry is given by the invariant

quantity ds , which we can calculate using the metric tensor.

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

ds is called the space-time interval, and since it is invariant it is the same in all frames. It can also be called the proper time. Because, if we attach a clock to a particle, it would take the particle a time ds to travel between the two points. The path that gives the longest proper time is called a geodesic, and particles without any external forces always travel along geodesic lines.

To do calculations in general relativity we need the geodesic equation,

$$\frac{d^2 x^\mu}{ds^2} = -\Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds},$$

which describes the motion of a free particle, given the geometry of spacetime. The Christoffel symbol, $\Gamma_{\alpha\beta}^\mu$, which is given by the metric tensor, describes the geometry of space.

$$\Gamma_{\alpha\beta}^\mu = \frac{g^{\mu\nu}}{2} \left[\frac{\partial g_{\alpha\nu}}{\partial x^\beta} + \frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu} \right]$$

To relate energy to the space-time geometry, we need the Einstein equations. These are given as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = 8\pi G T_{\mu\nu}$$

$G_{\mu\nu}$ is called the Einstein tensor, and contains all information on the geometry of space. This is seen in the Ricci tensor, $R_{\mu\nu}$, which depends only on the Christoffel symbols.

$$R_{\mu\nu} = \Gamma_{\mu\nu,\alpha}^\alpha - \Gamma_{\mu\alpha,\nu}^\alpha + \Gamma_{\beta\alpha}^\alpha \Gamma_{\mu\nu}^\beta - \Gamma_{\beta\nu}^\alpha \Gamma_{\mu\alpha}^\beta$$

To get the Ricci scalar, \mathcal{R} , we contract the metric tensor with the Ricci tensor, $g^{\mu\nu} R_{\mu\nu}$. Finally, the energy-momentum tensor, $T_{\mu\nu}$, gives the distribution of energy. For a homogeneous and isotropic energy distribution, $T_{\mu\nu}$ is given as

$$T_{\nu}^{\mu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \quad (2.1)$$

We also need to assume a perfect fluid for this to apply. Here ρ is the energy density of all the components of the universe, and p is the pressure of the perfect fluid.

2.3 The homogeneous universe

Even though the real universe is not completely homogenous, the approximation to a homogenous universe can give us some useful information. We will not get any structure formation in these models, but they will give a picture of how the size of the universe evolves, depending on its contents. Below we will derive a couple of these models.

2.3.1 Friedmann equations

First we need to use the Einstein equations to derive the Friedmann equations, which are the equations that determine the cosmological expansion of homogeneous and isotropic space. For an expanding, flat universe the metric tensor is given by the Friedmann-Robertson-Walker metric

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & a^2(t) & 0 \\ 0 & 0 & 0 & a^2(t) \end{pmatrix}. \quad (2.2)$$

Here $a(t)$ is the scale factor, which gives the size of the universe at a time t . Today the scale factor $a_0 = 1$.

We start with the time component of the Einstein equations.

$$R_{00} - \frac{1}{2}g_{00}\mathcal{R} = 8\pi GT_{00}$$

To calculate the Ricci tensor is a bit of work, but we end up with

$$R_{00} = -3\frac{\ddot{a}}{a},$$

and

$$R_{ij} = \delta_{ij} [2\dot{a}^2 + a\ddot{a}].$$

The Ricci scalar is then

$$\mathcal{R} = -R_{00} + \frac{1}{a^2}R_{ii} = 6 \left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right].$$

Since $T_{00} = \rho$, we get

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho \quad (2.3)$$

from the time component of the Einstein equation. This is called the first Friedmann equation, and it gives the change of the scale factor $a(t)$. We get the second Friedmann equation from the space components of the Einstein equation,

$$R_{ij} - \frac{1}{2}g_{ij}\mathcal{R} = 8\pi GT_{ij}.$$

The space components of the energy-momentum tensor gives the pressure, $T_{ij} = \delta_{ij}a^2p$, so the second Friedmann equation becomes

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \quad (2.4)$$

2.3.2 Energy density

If we solve the Friedmann equations we get the expansion of the homogeneous and isotropic universe, but what about the evolution of the energy of the different components? We can use the energy-momentum tensor from the previous section, and assume that the energy of each component is conserved. In special relativity this would be $\partial T^\mu_\nu / \partial x^\mu = 0$, but in general relativity we need to use the covariant derivative.

$$T^\mu_{\nu;\mu} = \frac{\partial T^\mu_\nu}{\partial x^\mu} + \Gamma^\mu_{\alpha\mu} T^\alpha_\nu - \Gamma^\alpha_{\nu\mu} T^\mu_\alpha$$

If we look at the $\nu = 0$ component of this equation and set it to zero to get conservation of energy we get

$$\begin{aligned} \frac{\partial T^\mu_0}{\partial x^\mu} + \Gamma^\mu_{\alpha\mu} T^\alpha_0 - \Gamma^\alpha_{0\mu} T^\mu_\alpha &= 0, \\ \frac{\partial T^0_0}{\partial x^0} + \Gamma^\mu_{0\mu} T^0_0 - \Gamma^\alpha_{0\mu} T^\mu_\alpha &= 0. \end{aligned}$$

Since $\Gamma^\mu_{0\nu} = \dot{a}/a \delta_{\mu\nu}$ and $T^0_0 = -\rho$ we get

$$\frac{d\rho}{dt} + 3\frac{\dot{a}}{a}[\rho + p] = 0.$$

Now we introduce the equation of state $p = w\rho$, which relates the pressure to the density. w is a constant that determines what type of fluid we look at. We use the equation of state to eliminate pressure from our equation, and we get

$$\begin{aligned} \frac{d\rho}{dt} &= -3\frac{\dot{a}}{a}[\rho + w\rho] \\ \frac{d\rho}{dt} &= -\frac{3}{a}\frac{da}{dt}\rho[1 + w] \\ \frac{d\rho}{\rho} &= -3[1 + w]\frac{da}{a} \end{aligned}$$

We can integrate this equation from today at t_0 , to some time in the past, with initial condition given as $\rho(t_0) = \rho_0$ and $a(t_0) = a_0 = 1$. So we get

$$\int_{\rho_0}^{\rho} \frac{d\rho}{\rho} = -3[1 + w] \int_{a_0}^a \frac{da}{a},$$

which gives

$$\rho = \rho_0 \left(\frac{a}{a_0} \right)^{-3(1+w)}. \quad (2.5)$$

Now we have the evolution of the energy density in a homogeneous universe. For non-relativistic matter we have $w = 0$, so we get

$$\rho_m = \rho_{m,0} a^{-3}.$$

We see that the matter energy density goes down by a factor a^3 as the universe expands. This makes sense, because there are three spacial dimensions. For radiation $w = 1/3$ which gives

$$\rho_r = \rho_{r,0} a^{-4}.$$

In addition to the three spacial dimensions expanding, radiation also loses energy because the wavelength gets bigger as space expands.

2.3.3 Solutions of the Friedmann equations

Before we can look at solutions to the Friedmann equations, we need to introduce the cosmological constant, which we will need in some of the solutions. The first person to add a constant to the Friedmann equations was Einstein in 1916. He did this to make static solutions possible. Without the cosmological constant, the only static solutions would be a universe without any contents ($\rho = 0$), or one with zero size ($a = 0$). Both of these solutions are unrealistic, so Einstein modified the equations by adding a constant.

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3} \rho + \frac{\Lambda}{3} \quad (2.6)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p) + \frac{\Lambda}{3} \quad (2.7)$$

Here Λ represents the cosmological constant. These equations are only valid for flat space ($k = 0$), so we will not look at solutions with curved space.

It is also useful to define a couple of quantities. The first is the Hubble constant, which determines the rate of expansion of the universe. It is defined as

$$H(t) = \frac{\dot{a}}{a}.$$

If we use eq. (2.3) with all the quantities evaluated today, we can find an expression for the critical density today

$$\rho_{c0} = \frac{3H_0^2}{8\pi G},$$

where H_0 is the Hubble constant today. By this definition the critical density is the density needed for a flat universe. It is also common to write the density in terms of the critical density, and we get

$$\Omega_0 = \frac{\rho_0}{\rho_{c0}}.$$

We see that for a flat universe we have $\Omega_0 = 1$.

Einstein-de Sitter

The first solution is a solution with non-relativistic matter as the only component. We start with the Friedmann equations, where we have used eq. (2.5), and the equation of

state to remove pressure.

$$\begin{aligned}\frac{\dot{a}^2}{a^2} &= \frac{8\pi G}{3} \rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)} \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3} (1+3w) \rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)}\end{aligned}\tag{2.8}$$

We can use the expression for the critical density with the first equation to get rid of the density, and then take the square root to end up with

$$\frac{\dot{a}}{a} = H_0 \left(\frac{a}{a_0}\right)^{-3(1+w)/2}.$$

We can integrate this to get an equation, which we can solve for a . But if we set some initial condition, we can simplify the equation. We choose that $a = 0$ when $t = 0$, and we get the relation

$$\frac{2}{3(1+w)} = H_0 t_0$$

Our final solution for a is then

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}}\tag{2.9}$$

The expansion is determined by a power law. At this stage, we still have not decided on a value for w , which is 0 for matter and 1/3 for radiation. In the case of matter, which is what is called the Einstein-de Sitter model, we get

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{\frac{2}{3}}$$

This is a good approximation to the expansion in the matter dominated era.

de Sitter

The next solution we will look at was first derived by Willem de Sitter. It contains only a cosmological constant Λ . In this case the Friedmann equations become very simple.

$$\begin{aligned}\frac{\dot{a}^2}{a^2} &= \frac{\Lambda}{3} \\ \frac{\ddot{a}}{a} &= \frac{\Lambda}{3}\end{aligned}\tag{2.10}$$

We take the square root of the first equation again, and use the positive solution, since the expansion should be positive. From the definition of the Hubble parameter, we get that

$$\frac{\Lambda}{3} = H(t).$$

In this model the Hubble parameter is a constant, $H(t) = H_0$. We can solve the equation $\dot{a}/a = H_0$, and use the initial condition $a(t_0) = a_0$, to get the expansion

$$a(t) = a_0 e^{H_0(t-t_0)}. \quad (2.11)$$

Here the universe expands exponentially.

Λ CDM

Finally we have a solution with both matter and a cosmological constant. Observations have shown that this model fit well with the actual expansion of the universe. Now we will write the first Friedmann equation in a slightly different way.

$$\frac{H^2(t)}{H_0^2} = \Omega_{m0} \left(\frac{a_0}{a} \right)^3 + (1 - \Omega_{m0}) \quad (2.12)$$

Here $\Omega_\Lambda = (1 - \Omega_{m0})$ is the density parameter for the cosmological constant. As long as Ω_Λ has a positive value, the universe will always expand. At early times the universe is matter dominated, and the expansion will decelerate. But at some scale $a_{m\Lambda}$, the universe will become dominated by the cosmological constant, and then the expansion will start to accelerate. The era when the contribution to the energy density is equal for matter and the cosmological constant is given by

$$a_{m\Lambda} = a_0 \left(\frac{\Omega_{m0}}{1 - \Omega_{m0}} \right)^{1/3}$$

The Friedmann equation can be written as

$$H_0 dt = \frac{1}{\sqrt{1 - \Omega_{m0}}} \frac{\sqrt{a} da}{\sqrt{(a_{m\Lambda}/a_0)^3 + a^3}}.$$

If we integrate this, we get an equation that can be solved for a , and we end up with

$$a(t) = a_0 \left(\frac{\Omega_{m0}}{1 - \Omega_{m0}} \right)^{1/3} \left[\sinh \left(\frac{3}{2} \sqrt{1 - \Omega_{m0}} H_0 t \right) \right]^{2/3}. \quad (2.13)$$

This is a complicated function, and so it is hard to see how it behaves. But for $a < a_{m\Lambda}$, the universe expands with a deceleration. And for $a > a_{m\Lambda}$ the expansion accelerates.

2.4 Neutrinos

The neutrino was first postulated theoretically by Wolfgang Pauli in 1930. He needed a new particle to explain the conservation of energy in β -decay, which is a radioactive processes. This new particle is massless according to the standard model in particle physics, but experiments show that the particle has a mass, even though it must be small, and has not yet been determined. The neutrino is a very weakly interacting

fermion with zero charge. Therefore it is only affected by the weak force and gravity. Neutrinos belong to a group of particles called leptons. There are six leptons in the standard model, electron (e), muon (μ) and tau (τ), and their associated neutrinos ν_e , ν_μ and ν_τ respectively.

Neutrinos were detected for the first time in 1956, in an experiment that used nuclear reactors as the source for neutrinos. Over the years we have built large chambers deep underground filled with a liquid that interacts with neutrinos, that are used for detection. These detectors are based on the same principle as the 1956 experiment. Nearly all of the neutrinos detected by these chambers come from the Sun. The flux of neutrinos from the Sun is about 10^{10} particles per square centimetre per second. Despite this massive amount of particles, nearly all of them pass right through the Earth. Only very few neutrinos are detected by the detectors per day.

2.4.1 Neutrino masses

Until recently there was a problem, called the Solar neutrino problem, where the detectors only detected about 1/3 of the neutrinos that the theory of the nuclear reactions in the Sun predicted. Theories for neutrino oscillations have been around since the 1950s, but in 1998 it was confirmed via observations of neutrinos in the atmosphere, that neutrinos can oscillate between the three different flavours, via a quantum mechanical effect, when they travel through matter, such as the core of the Sun. So the electron neutrinos (ν_e) that are created in the Sun's core, can oscillate between the other two neutrino species. This solves the problem of too few detections in experiments, because the detectors could only detect one of the three neutrino flavours.

For these fluctuations to be possible, the neutrino need to have a non-zero mass, because the probability of this effect is based on the square of the difference in mass between the two neutrino species. This is interesting to cosmologists, because that means that neutrinos can be a candidate for some of the dark matter in the universe. We know that in the early universe, there were created a lot of neutrinos. And together with photons, neutrinos are the most abundant particle in the universe. So if the neutrinos have mass, they must constitute a part of the dark matter.

As the universe expands and cools down, eventually the neutrinos will decouple from the rest of the medium, and continue to evolve independently. Because the neutrinos are fermions, the distribution will follow the Fermi-Dirac distribution function. The temperature of the cosmic neutrino distribution is related to the CMB temperature by

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_{\text{CMB}}.$$

If we use this together with the Fermi-Dirac distribution function, we get the number density of relic neutrinos in the universe. For a CMB temperature $T_{\text{CMB}} = 2.726$ K, we get a number density of $n_\nu = 112\text{cm}^{-3}$. This again gives a density fraction of

$$\Omega_\nu h^2 = \frac{m_{\nu,\text{tot}}}{93.14\text{eV}},$$

where $m_{\nu,\text{tot}}$ is the total mass of all three neutrino flavours, and h is the dimensionless Hubble parameter.

The neutrinos are highly relativistic at early times, which means that they don't clump together very well. This is a problem, because observations show that structure in the universe was first formed on small scales, and then on large scales. The clumped dark matter is essential to form gravitationally bound galaxies. Cosmological models where all of the dark matter is in the form of neutrinos also creates a universe that is very different from our own. But these models are still very useful, because we can determine an upper bound for the neutrino mass, by looking at how much of the dark matter we can replace by neutrinos in the models, before we get something that is not consistent with our universe. Upper bounds on the total neutrino mass is presented in [2], and it indicates that the total neutrino mass is under 1 electron volts (eV).

2.5 Linear perturbations

We have seen how a homogeneous and isotropic universe evolves, but that is not really how the universe is. In reality there are small perturbations to the matter density. These can be described by $\rho = \rho^{(0)} + \delta\rho$, where $\rho^{(0)}$ is the average density and solution to the homogeneous equations, and $\delta\rho$ is a small correction to that average.

Since matter is no longer evenly distributed, there will also be small perturbations to the metric given in eq. (2.2). We have Ψ , which is the perturbation to the Newtonian potential, and Φ , which is the perturbation to the spacial curvature. The new metric will then look like

$$g_{\mu\nu} = \begin{pmatrix} -1 - 2\Psi & 0 & 0 & 0 \\ 0 & a^2(1 + 2\Phi) & 0 & 0 \\ 0 & 0 & a^2(1 + 2\Phi) & 0 \\ 0 & 0 & 0 & a^2(1 + 2\Phi) \end{pmatrix}. \quad (2.14)$$

This is the metric given in conformal Newtonian gauge, which is one of several gauges, or coordinate systems, in which we can write our equations. All the perturbations are functions of time t , and space x^i .

2.5.1 Boltzmann equation

When describing the distribution of matter or radiation in the universe, we use a distribution function. For the homogeneous universe, this function would be the Fermi-Dirac function for fermions and the Bose-Einstein function for bosons. These zero-order functions depend only on time t and energy E , but we will need to expand them to first order when calculating the perturbed matter distributions. And then they will also depend on space, x^i , and the magnitude and direction of the momentum, p .

The distribution function $f(x^i, E, p^i, t)$ will be needed when we derive the equations for the evolution of the perturbations. Because to find these equations, we need the

Boltzmann equation

$$\frac{df}{dt} = C[f]. \quad (2.15)$$

The left hand side is the total time derivative of the distribution function. To do anything with this we first need to change this total derivative to partial derivatives with respect to each of the variables. The right hand side is a collision term that we can ignore as long as we look at particles that does not react to other types of particles.

Cold dark matter

Cold dark matter is non-relativistic, very weakly interacting particles. Therefore we can ignore the collision term in the Boltzmann equation, which is zero.

If we write out the left hand side of eq. (2.15) using partial differentials we get

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i} \frac{dx^i}{dt} + \frac{\partial f}{\partial E} \frac{dE}{dt} + \frac{\partial f}{\partial \hat{p}^i} \frac{d\hat{p}^i}{dt}.$$

To calculate the total derivatives in this equation, we need the momentum vector defined as

$$P^\mu = \frac{dx^\mu}{d\lambda},$$

Where λ is the path of a particle. We also need the relation $P^2 = g_{\mu\nu} P^\mu P^\nu = -m^2$, where m is the mass of a dark matter particle. We also have the energy of the particle as $E = \sqrt{p^2 + m^2}$, and $p^2 = g_{ij} P^i P^j$ is a sum over the spatial components of the momentum vector.

First of all, since we are keeping the equations linear, we can neglect the last term in our Boltzmann equation. This is a second order term, because we need the first order expansion of the distribution function to differentiate with the direction of the momentum.

To get the term with x^i , we use the momentum vector and end up with

$$\frac{dx^i}{dt} = \frac{\hat{p}^i p}{aE}$$

The only remaining term is then the energy, E . We can get this from the geodesic equation, and with a little calculation we end up with

$$\frac{dE}{dt} = -\frac{da/dt}{a} \frac{p^2}{E} - \frac{p^2}{E} \frac{\partial \Phi}{\partial t} - \frac{\hat{p}^i p}{a} \frac{\partial \Psi}{\partial x^i}.$$

So far our Boltzmann equation for cold dark matter look like

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i p}{aE} \frac{\partial f}{\partial x^i} - \left[\frac{da/dt}{a} \frac{p^2}{E} + \frac{p^2}{E} \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i p}{a} \frac{\partial \Psi}{\partial x^i} \right] \frac{\partial f}{\partial E} = 0.$$

Now we can integrate this equation and look at the zero order moment and first order moment separately. After a bit of work we end up with two equations for the evolution of dark matter, namely

$$\frac{\partial \delta}{\partial t} + \frac{1}{a} \frac{\partial v^i}{\partial x^i} + 3 \frac{\partial \Phi}{\partial t} = 0$$

and

$$\frac{\partial v^j}{\partial t} + \frac{da/dt}{a} v^j + \frac{1}{a} \frac{\partial \Psi}{\partial x^j} = 0.$$

Here, δ and v^i are the perturbations to the dark matter density and velocity, respectively.

Baryons

Baryons, as dark matter, are non-relativistic massive particles, so the equations of evolution are very similar. The first equation is actually the same as for dark matter. In Fourier space it is

$$\frac{\partial \delta_b}{\partial t} + \frac{1}{a} i k v_b + 3 \frac{\partial \Phi}{\partial t} = 0.$$

The second equation also has a collision term

$$\frac{\partial v_b}{\partial t} + \frac{da/dt}{a} v_b + \frac{1}{a} i k \Psi = \frac{d\tau}{dt} \frac{4\rho_\gamma}{3\rho_b} [3i\Theta_1 + v_b].$$

Here τ is the optical depth, which tells us how far light can travel before it is scattered, and Θ_1 is the dipole moment of the photon perturbation.

Massive neutrinos

For massless neutrinos we have the same equations of evolution as for photons, except that there is no collision term in the Boltzmann equation, since neutrinos are so weakly interacting. For massive neutrinos though, we need to differ between the relativistic and non-relativistic regime. We follow [13] for the derivation of neutrino perturbation equations.

As long as the neutrinos are relativistic, we can simplify the Boltzmann equation by introducing a relative fluctuation to the distribution function f_ν .

$$\mathcal{N}(\eta, x^i, y, n^i) = \frac{f_\nu(\eta, x^i, y, n^i)}{f_{\nu 0}(\eta, y)} - 1 \quad (2.16)$$

Here $f_{\nu 0}$ is the zero-order distribution and $y = ap$, where p is the momentum. As long as the neutrinos remain relativistic, we can use the same equations as for massless neutrinos (and photons), and we get

$$\mathcal{N}(\eta, x^i, y, n^i) = -\Theta_\nu(\eta, x^i, y, n^i) \frac{d \ln f_{\nu 0}}{d \ln y}. \quad (2.17)$$

In the case of non-relativistic neutrinos we have that \dot{x}^i and \dot{y} both depend on the energy ϵ , which is defined as $\epsilon = \sqrt{y^2 + a^2 m^2}$, where m is the neutrino mass. We then get the evolution equation

$$\dot{\mathcal{N}} + \frac{y}{\epsilon} i(n_j \cdot k^i) \mathcal{N} + \frac{d \ln f_{\nu 0}}{d \ln y} \left[\dot{\phi} - \frac{\epsilon}{y} i(n_j \cdot k^i) \psi \right] = 0. \quad (2.18)$$

In the relativistic limit, $y/\epsilon = 1$, so the equation simplifies to the massless Boltzmann equation. We can take moments of this equation and we finally end up with the set below.

$$\begin{aligned}\dot{\mathcal{N}}_{\nu 0} &= -\frac{yk}{\epsilon}\mathcal{N}_{\nu 1} - \phi \frac{d \ln f_{\nu 0}}{d \ln y} \\ \dot{\mathcal{N}}_{\nu 1} &= \frac{yk}{3\epsilon}(\mathcal{N}_{\nu 0} - 2\mathcal{N}_{\nu 2}) - \frac{\epsilon k}{3y}\psi \frac{d \ln f_{\nu 0}}{d \ln y} \\ \dot{\mathcal{N}}_{\nu l} &= \frac{yk}{(2l+1)\epsilon}[\mathcal{N}_{\nu(l-1)} - (l+1)\mathcal{N}_{\nu(l+1)}]\end{aligned}\tag{2.19}$$

2.5.2 Einstein equations

Now we will use the Einstein equation to derive evolutionary equations for the Newtonian potential, Ψ , and the spatial curvature, Φ . We have already seen the Einstein equation

$$G_{\nu}^{\mu} = 8\pi G T_{\nu}^{\mu},$$

but now we will use the perturbed metric given in eq. (2.14) to get $G_{\mu\nu}$. First we can get the Christoffel symbols using the same method we used for the homogeneous universe and we get

$$\begin{aligned}\Gamma_{00}^0 &= \frac{\partial \Psi}{\partial t} \\ \Gamma_{0i}^0 &= \frac{\partial \Psi}{\partial x^i} = ik_i \Psi \\ \Gamma_{ij}^0 &= \delta_{ij} a^2 \left[H + 2H(\Phi - \Psi) + \frac{\partial \Phi}{\partial t} \right] \\ \Gamma_{00}^i &= \frac{ik^i}{a^2} \Psi \\ \Gamma_{j0}^i &= \delta_{ij} \left(H + \frac{\partial \Phi}{\partial t} \right) \\ \Gamma_{jk}^i &= i\Phi [\delta_{ij} k_k + \delta_{ik} k_j - \delta_{jk} k_i].\end{aligned}\tag{2.20}$$

We have written the equations in Fourier space to get rid of the spatial derivatives. We can now construct the Ricci tensor and scalar, which have a lot of terms. The time-time component of the Ricci tensor is

$$R_{00} = -3\frac{d^2 a/dt^2}{a} - \frac{k^2}{a^2} \Psi - 3\frac{\partial^2 \Phi}{\partial t^2} + 3H \left(\frac{\partial \Psi}{\partial t} - 2\frac{\partial \Phi}{\partial t} \right),$$

and the space-space component is

$$\begin{aligned}R_{ij} = \delta_{ij} \left[\left(2a^2 H^2 + a \frac{d^2 a}{dt^2} \right) (1 + 2\Phi - 2\Psi) + a^2 H \left(6\frac{\partial \Phi}{\partial t} - \frac{\partial \Psi}{\partial t} \right) + a^2 \frac{\partial^2 \Phi}{\partial t^2} + k^2 \Phi \right] \\ + k_i k_j (\Phi + \Psi).\end{aligned}$$

We can now find the Ricci scalar, $\mathcal{R} = g^{\mu\nu} R_{\mu\nu}$. But we will only write the first-order terms, which are a correction to the zero order Ricci scalar that we have already found.

$$\delta\mathcal{R} = -12\Psi \left(H^2 + \frac{d^2 a/dt^2}{a} \right) + \frac{2k^2}{a^2} \Psi + \frac{\partial^2 \Phi}{\partial t^2} - 6H \left(\frac{\partial \Psi}{\partial t} - 4 \frac{\partial \Phi}{\partial t} \right) + 4 \frac{k^2 \Phi}{a^2}$$

Finally we are ready to construct the Einstein tensor. The time-time component is given by

$$G_0^0 = g^{00} R_{00} - \frac{\mathcal{R}}{2}$$

and if we only look at the first-order terms we get

$$\delta G_0^0 = 6H \frac{\partial \Phi}{\partial t} + 6\Psi H^2 + 2 \frac{k^2 \Phi}{a^2}.$$

The time-time component of the energy-momentum tensor is the energy density of all the components. To first order this is $\rho(1 + \delta)$ for baryons and dark matter, and it is $\rho(1 + 4\Theta)$ for photons. If we only use the first-order terms again we get the first evolution equation for Φ and Ψ ,

$$-6H \frac{\partial \Phi}{\partial t} + 6\Psi H^2 + 2 \frac{k^2 \Phi}{a^2} = 8\pi G [\rho_{\text{cdm}} \delta + \rho_{\text{b}} \delta_{\text{b}} + 4\rho_{\gamma} \Theta_0].$$

For the second evolution equation we need to do something smart. We need to first get the space-space component of the Einstein tensor G_j^i , and then contract it with a projection operator, so we are left with only the traceless part of G_j^i . We then need to do the same for the energy-momentum tensor. The traceless part of T_j^i is called anisotropic stress, and is only affected by relativistic particles, so the second evolution equation is then

$$k^2 (\Phi + \Psi) = -32\pi G a^2 \rho_{\gamma} \Theta_0,$$

where Θ_0 is the monopole moment of the photon perturbations.

To sum up, the equations of evolution in our perturbed universe is

$$\begin{aligned} \frac{\partial \delta}{\partial t} + \frac{1}{a} i k v + 3 \frac{\partial \Phi}{\partial t} &= 0 \\ \frac{\partial v^j}{\partial t} + \frac{da/dt}{a} v^j + \frac{1}{a} i k \Psi &= 0 \\ \frac{\partial \delta_{\text{b}}}{\partial t} + \frac{1}{a} i k v_{\text{b}} + 3 \frac{\partial \Phi}{\partial t} &= 0 \\ \frac{\partial v_{\text{b}}}{\partial t} + \frac{da/dt}{a} v_{\text{b}} + \frac{1}{a} i k \Psi &= \frac{d\tau}{dt} \frac{4\rho_{\gamma}}{3\rho_{\text{b}}} [3i\Theta_1 + v_{\text{b}}] \\ -6H \frac{\partial \Phi}{\partial t} + 6\Psi H^2 + 2 \frac{k^2 \Phi}{a^2} &= 8\pi G [\rho_{\text{cdm}} \delta + \rho_{\text{b}} \delta_{\text{b}} + 4\rho_{\gamma} \Theta_0] \\ k^2 (\Phi + \Psi) &= -32\pi G a^2 \rho_{\gamma} \Theta_0 \\ \dot{\Theta} + i k \mu \Theta + \dot{\Psi} + i k \mu \Psi &= -\dot{\tau} [\Theta_0 - \Theta + \mu v_{\text{b}}] \\ \mathcal{N} + \frac{y}{\epsilon} i (n_j \cdot k^i) \mathcal{N} + \frac{d \ln f_{\nu 0}}{d \ln y} \left[\dot{\phi} - \frac{\epsilon}{y} i (n_j \cdot k^i) \psi \right] &= 0, \quad (\text{non-relativistic}) \end{aligned} \tag{2.21}$$

where all the equations is written in Fourier space. Here we have also included the perturbations to the photons, Θ .

CHAPTER 3

Simulations

Cosmological simulations have been done for as long as there were computers around to do them. Of course in those early days in the 1960s, there were neither a lot of computing power, nor any refined computational methods, so the simulations were very basic. In the type of simulation we will focus on in this thesis, namely N-body simulations, they would brute-force calculate the effect of gravity on every particle from all the other particles, for each time step in the simulation. This takes a lot of computing power, so they could only run the simulations with at most 100 particles in a grid. But still, this was enough to get some indication of how galaxies form, or how the cosmic medium evolves.

During the 70s and 80s there were a lot of advancements in both computational power and methods, that made it possible to do more advanced simulations. For example, algorithms were applied to cosmological N-body simulations that made them much more efficient. So it became possible to run simulations with as much as 10^5 particles.

Cosmological simulations are mainly used to test the validity of cosmological models. And in the 80s cosmologists realised that the simplest models were not enough to fully explain the cosmic evolution. More physics was needed, and baryonic gas dynamics is an example of new physics that was implemented. [4] and [11] are used to present the algorithms that are included in simulations of structure formation.

3.1 N-body simulations

In this thesis we will need to use simulations to evolve the dark matter from a set of initial conditions. We will see later how these initial conditions are created, but for now we need to know that they are determined from a set of cosmological parameters and come in the form of a grid of CDM particles. We will use tree-dimensional grids with 128 particles in each dimension. So we end up with 128^3 particles in our simulations. Each particle has a position and a velocity. The main purpose of the N-body code is to find the acceleration of each particle, and then use the acceleration to move the particles to the next time step. There are several different ways to do this, and we will discuss the three most popular methods.

3.1.1 PM algorithm

First we have the particle-mesh algorithm. In this algorithm we find the gravitational potential in a grid in our simulation, and then use this potential to find the acceleration of the particles. First we use the masses of the particles to get a density grid. The easiest way to do this, is to move the mass of each particle to its nearest grid point. This will also give the biggest errors, so a better way to get the density grid is to use the Cloud-in-Cell (CIC) scheme, which distributes the mass of each particle to the eight nearest grid points.

Once we have the density grid, we can use the Poisson equation to calculate the gravitational potential.

$$\nabla^2 \phi = 4\pi G \rho \quad (3.1)$$

We can Fourier transform this equation to remove the spacial derivatives, and we get

$$\phi(k, t) = -4\pi G \frac{\rho(k, t)}{k^2}, \quad (3.2)$$

where ϕ and ρ are now Fourier transformed potential and density. Once we have the potential ϕ , we can reverse Fourier transform, and use

$$g = -\nabla \phi$$

to get the gravitational acceleration g .

3.1.2 3PM algorithm

The particle-particle/particle-mesh algorithm is an evolution of the PM algorithm, where we use a direct summation of forces on a particle from other particles that are close to the current particle. This method requires more computational power, but it is still the preferred method. This is because the PM algorithm has a force resolution limited to the grid, so if there are many particles close together, or even within the same grid spacing, their forces will be inaccurately given by the PM algorithm. If we use gravitational forces from individual particles within a few grid spacings, this problem will be resolved.

3.1.3 Tree algorithm

In the tree algorithm we collect several particles into cells of a size s . If we, in the point we are interested in, have that $s/d < \theta$, where d is the distance to the cell, we can treat all the particles in the cell as one large particle that lies in the center of mass of the cell. Here θ is typically some number smaller than one. That is, if the cell is much smaller than the distance from our point of interest to the cell, then we can treat the cell as a single particle. And we can determine the accuracy by setting a value for θ . This means that particles that are close, will still contribute individually to the force calculation, as there is no cell small enough that it can encompass more than one particle.

3.1.4 Time evolution

Now that we have the gravitational acceleration of each particle, we can calculate the forces and move the particles. To do this we need integrate a set of equations of motion

$$\frac{dx}{dt} = v, \quad \frac{dv}{dt} = g.$$

Here x is the position, v is the velocity and g is the acceleration. To solve these equations numerically, and get new positions and velocities, we will need a numerical integrator. There are many different algorithms that can do the job, and we will chose the leapfrog scheme, (Hockney & Eastwood 1981)[10], which is a second order integrator.

$$x_{i+1} = x_i + v_i \Delta t + \frac{1}{2} g_i \Delta t^2 \quad (3.3)$$

$$v_{i+1} = v_i + \frac{1}{2} (g_i + g_{i+1}) \Delta t \quad (3.4)$$

The subscript i indicates the time step and Δt is the size of each time step. In the last equation we note that we need the acceleration from the next time step. To make the algorithm as effective as possible, we wait until the next time iteration to calculate the updated velocities. This is fine to do, because we only need the updated positions to get a new potential.

3.2 Initial conditions

To run simulations of structure formation we need to have a set of initial conditions to start the simulation from. These initial conditions are determined by the cosmological parameters and gaussian fluctuations set up by inflation. Initial conditions for the N-body codes consists of particle positions and velocities. To generate initial conditions we need to evolve the linearised Einstein and Boltzmann equations up to the time we want to start our N-body simulation.

3.2.1 Particle distribution

Gaussian random field initial conditions [5] are based on the assumption that perturbations to the matter density is small compared to the actual density, $\delta = \delta\rho/\rho_0 \ll 1$. We can then solve the equation for linear evolution of the CDM density perturbations

$$\ddot{\delta} + 3H\dot{\delta} - \frac{\nabla^2 \phi}{a^2} = \nabla^2 P$$

to get an expression for the perturbation as a function of a cosmological growth factor, $\delta(t) = D(t)\delta_0$. In this equation P is a pressure term. Now we can solve the Poisson equation $\nabla^2 \phi = \delta$, and get the gradient of the potential, $\nabla \phi$. Finally we can use this to solve the geodesic equation and get particle positions, and velocities by the derivative of the positions.

$$x_i = x_0 + D(t)\nabla \phi, \quad v_i = \frac{dD}{dt}\nabla \phi.$$

CHAPTER 4

Modifying the code

In this thesis we want to simulate structure formation on large scales in the universe, using a Λ CDM model. We will use an already existing N-body code developed by Claudio Llinares [15], which is based on the 3PM algorithm for calculating the gravitational potential. We will add neutrinos to this code, and study the effect the neutrinos have on formation of structure, via the power spectrum. Due to the higher thermal velocity of the neutrinos, it can be a challenge to do full N-body simulations of neutrinos together with the CDM. This is because the neutrinos will form structure on larger scales than the CDM. We will therefore look at two methods on how to implement neutrinos into the N-body code.

4.1 The power spectrum

To analyse the results of simulations of structure formation, we use the power spectrum of the density fluctuations of matter. The power spectrum $P(k)$ is defined by

$$\langle \delta(k)\delta(k') \rangle = (2\pi)^3 P(k) \delta^3(k - k'), \quad (4.1)$$

with units of $(\text{length})^3$. Here $\delta(k)$ is perturbations to the density field in Fourier space, and the brackets indicates that they are averaged over the whole distribution. And δ^3 is the Dirac delta function, which sets the restriction $k = k'$. $P(k)$ is a measure of the variance in the density distribution. Scales that have structure with densities much higher or lower than the average density, will have a high power in $P(k)$, and scales with densities close to the average, will have low power.

4.2 Creating initial conditions

The initial conditions to the N-body code is created by the publicly available COSMICS package [6]. The `lingers` code is integrating the linear theory equations up to the time the simulation is initiated, which is at $z = 50$. The Nine-Year Wilkinson Microwave Anisotropy Probe (WMAP9) data for a pure Λ CDM model [20] is used to get

Cosmological parameters	
Ω_{CDM}	0.233
Ω_{b}	0.0463
Ω_{Λ}	0.721
H_0	70.0 km/s/Mpc
σ_8	0.81
T_{CMB}	2.726 K
Y_{He}	0.24
n	0.97

Table 4.1: Input parameters for the codes that create initial conditions for the N-body code. Taken from WMAP9 [20].

the cosmological parameters. **Lingers** needs the parameters Ω_{CDM} , Ω_{b} , Ω_{Λ} and Ω_{ν} , which are density fractions of CDM, baryons, the cosmological constant and neutrinos respectively. The initial conditions are generated for a Λ CDM model, so the neutrino density fraction is set to zero. **Lingers** also needs the Hubble parameter for today H_0 , the temperature of the cosmic microwave background T_{CMB} and the fraction of baryons that are helium Y_{He} . The values are given in table (4.1). **Lingers** outputs the density and velocity divergence for all the different components.

This output is used by **grafic**, which computes the positions and velocity of the cold dark matter particles. In addition to the **lingers** output, **grafic** also takes as input the spectral index n and the normalisation factor σ_8 . Next, the size of the simulation box is set. The used value is 128 Mpc/h. How many particles are used in the simulation is determined by an input file, and is set to 128 particles in each dimension. Lastly, the **grafic** code needs a seed for a random number generator. This seed is used to create gaussian distributed random numbers, used in the random density and displacement fields in the output.

4.3 Method 1: Linear transfer functions

The first method on how to include neutrinos into the simulation was presented in [1]. It involves solving the Boltzmann equations in linear theory to get neutrino transfer functions at regular time intervals in the simulation. The transfer functions are used to make a neutrino density realisation at each time step in the simulation. The neutrino density is then added to the CDM density grid, which is created in the normal way, using particle mass and positions.

Neutrino masses				
Ω_ν	0.013	0.025	0.05	0.15
Σm_ν	0.60 eV	1.14 eV	2.28 eV	6.85 eV

Table 4.2: Mass fractions and the sum of the masses for the three species of neutrinos in electron volt (eV).

4.3.1 Implementation

Transfer functions

The first thing I did was acquire neutrino transfer functions. For this I used CAMB [16], which is a code that solves the linearised Einstein and Boltzmann equations. In the input file in CAMB I still use the WMAP9 parameters, but here I also include neutrinos. The total number of neutrino species is set to 3.046, where 2.046 species is massless and 1 is massive. The reason why the number of species is slightly over 3 is because at the time of decoupling, not all of the neutrinos decoupled at exactly the same time. The electron neutrino decoupled a little later than the other two species, which slightly raises the background neutrino temperature. I ran the code with four different neutrino masses, presented in table (4.2). Because of what I assume is a bug in the CAMB code, I could only get transfer functions for one redshift z , on each run. For each of the masses, I wrote to a file the neutrino transfer functions for the redshifts $z = 55, 50, 45, 40, 35, 30, 25, 20, 15, 10, 8, 6, 4, 3, 2, 1$ and 0.

Density grid

In a routine in the N-body code, I read the transfer functions into a two-dimensional grid over the z and k values used by CAMB. I then used linear interpolation to get values for the functions between those given in the grid.

I also read the file with random numbers created by `grafic`. They are needed to get the correct density distribution. In the N-body code I ran a loop over all the grid points, and for each point I found the density in Fourier space, using a realisation of the power spectrum. The power spectrum is given as

$$P(k) = 2\pi^2 A k T(k, z)^2, \quad (4.2)$$

where A is a scale factor with the value $A = 2.1 \cdot 10^{-9}$. With the power spectrum and the gaussian random numbers, I found the density.

$$\delta(k) = \frac{1}{\sqrt{2\pi}} P(k)^{1/2} W \quad (4.3)$$

W is a matrix with the random numbers. The factor $(2\pi)^{-1/2}$ comes from a normalisation in the Fourier transform. Next, I move the density grid around in the memory to prepare for a backwards Fourier transform. I use the fast Fourier transform code

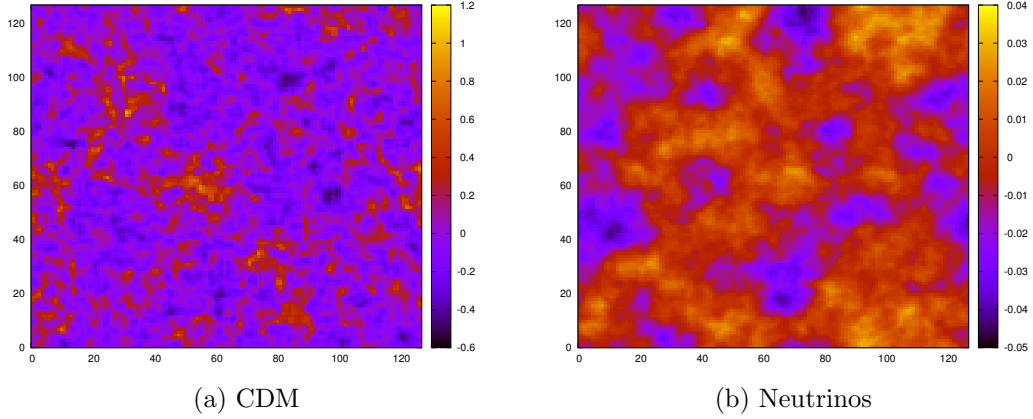


Figure 4.1: Plot of a slice of the density grids of CDM and neutrinos at the start of the simulation. The neutrino density fraction is $\Omega_\nu = 0.05$.

`fftw` [18], to do the transforms. When I have the density in real space, it is added to the density of the CDM particles.

The problem I had with this method is that the density is not distributed in the same way for neutrinos and CDM, even though the same random numbers are used in the generation of both. This is shown in figure (4.1).

4.4 Method 2: CDM diffusion

The second method consists of applying a damping routine to the already present CDM density grid in the N-body code. This damping routine is a gaussian function that will smooth out and damp the structure of the CDM grid. The damped grid will simulate the effect of massive neutrinos, because their main effect is to suppress the CDM perturbations on scales smaller than the horizon at the time the neutrinos went from being relativistic to becoming non-relativistic.

4.4.1 Implementation

The damping routine can be applied in either real space or in Fourier space. In real space the the density field is damped by

$$\delta_\nu(\mathbf{x}) = \int_{-\infty}^{\infty} \delta(\mathbf{x}') \frac{e^{-(\mathbf{x}-\mathbf{x}')^2/2\lambda^2}}{(2\pi)^{3/2}\lambda^3} d\mathbf{x}'. \quad (4.4)$$

Here δ_ν and δ are the density perturbations for neutrinos and CDM respectively. λ is the free streaming length, discussed in [13], which is given by

$$\lambda = \frac{2\pi}{0.0180\Omega_m^{1/2}} \left(\frac{1\text{eV}}{m_\nu} \right)^{1/2} \text{Mpc}/h, \quad (4.5)$$

where Ω_m is the density fraction of matter and m_ν is the total neutrino mass in electron volts. The background of neutrinos behaves like a collisionless fluid, so the neutrinos propagate through space at the speed of their thermal velocity. The free streaming length is the length which the neutrinos have propagated since decoupling. At early times, when neutrinos were relativistic particles, their velocity was very close to the speed of light. Therefore the free streaming length was equal to the Hubble radius. After the neutrinos became non-relativistic, and during matter domination in the universe, the free streaming length increases slower than the expansion of the universe, so the comoving free streaming length decrease. The comoving free streaming wavenumber k_{nr} goes to a minimum at the time of transition from relativistic to non-relativistic neutrinos. The free streaming length is defined as the wavelength that corresponds to this wavenumber, $\lambda = 2\pi/k_{\text{nr}}$. The neutrinos will not form gravitationally bound structure on scales smaller than this.

To implement equation (4.4) into the code, we need to discretise it.

$$\delta_\nu(x, y, z) = \sum_{i,j,k=0}^{\text{box}} \delta(i, j, k) \frac{e^{-(\Delta x^2 + \Delta y^2 + \Delta z^2)/2\lambda^2}}{(2\pi)^{3/2} \lambda^3} \quad (4.6)$$

Here Δx , Δy and Δz are the distance in Mpc in each spatial dimension from the grid point (x, y, z) that the neutrino density is evaluated in, to the grid points (i, j, k) that is summed over. The exponential function damps the contribution from the CDM grid points that are far from the neutrino grid point being calculated. There are two problems with this method. The first problem is that it takes too much time to do a sum over the whole simulation box for each grid point. I therefore restricted it to a sum over the three nearest neighbours in each dimension. This will cut off the exponential function before it has had time to get to a sufficiently low value, as illustrated in figure (4.2). This has the effect that the neutrino density grid will have smaller values than if the whole box was used in the sum. As the figure illustrates, if the free streaming length is smaller than the cutoff point in the sum, the majority of the exponential is included in the sum, and this simplification will not be a problem. Figure (4.3) shows the density grid for neutrinos with a free streaming length of 5 Mpc. The density is both smoothed out and damped as we wanted, but the density fluctuations are always negative, which might be because our sum over nearest neighbours is too small.

The other problem is that realistic values of the free streaming length is much larger than the box in the simulation. Using equation (4.5), we find that for a total neutrino mass of 1 eV, $\lambda = 910$ Mpc, which is five times as large as the box in the simulation. The only thing to do here is using a smaller unphysical λ in the simulation. I have used $\lambda = 5, 10, 15, 20$ and 40 in the simulation.

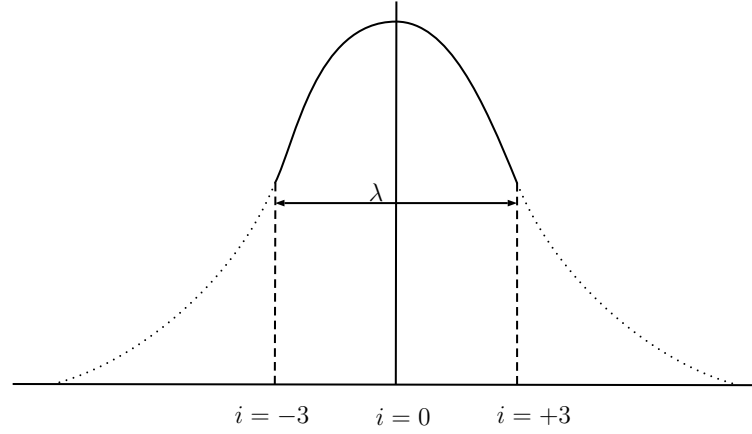


Figure 4.2: Illustration of how a sum over only a small portion of the simulation box will affect the neutrino density.

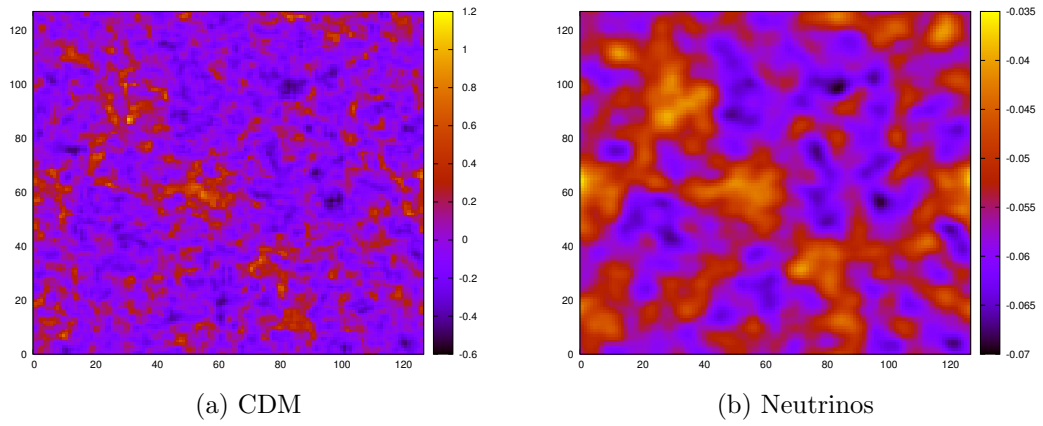


Figure 4.3: Plot of a slice of the density grids of CDM and neutrinos at the start of the simulation. In the neutrino plot, the free streaming length used is 5 Mpc.

CHAPTER 5

Results

In this chapter results are presented for simulations with different parameters.

In the first plot, we test the code by running Λ CDM simulations, without neutrinos. The mass of the CDM particles are decreased by adding to the mass fraction of the neutrinos, Ω_ν , thereby decreasing Ω_{CDM} . This is only done to decrease the CDM mass, and neutrino physics are not added to the code. The effect of the decrease in CDM mass can be seen in figure (5.1). Decreasing the mass of CDM particles will lower the power spectrum uniformly on all scales. In the case where $\Omega_\nu = 0.15$, the power spectrum falls more steeply, which indicates that the CDM does not form structure as well as expected.

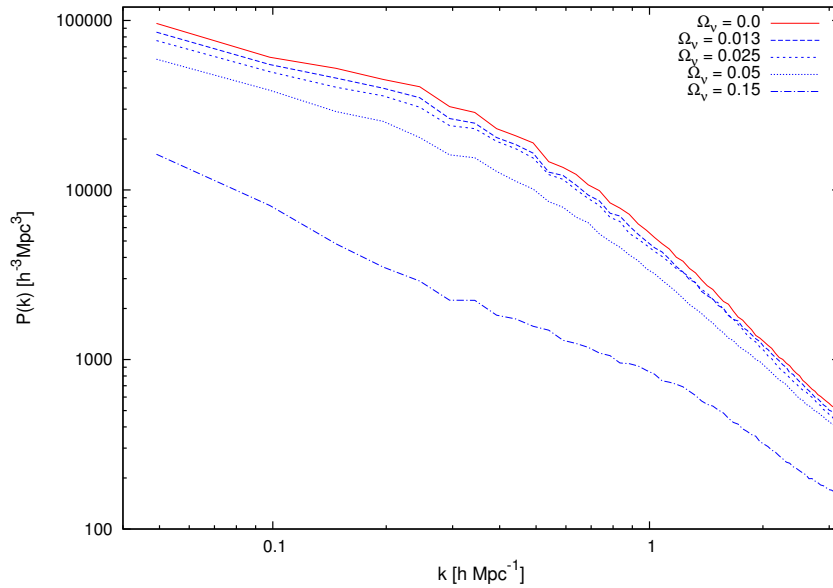
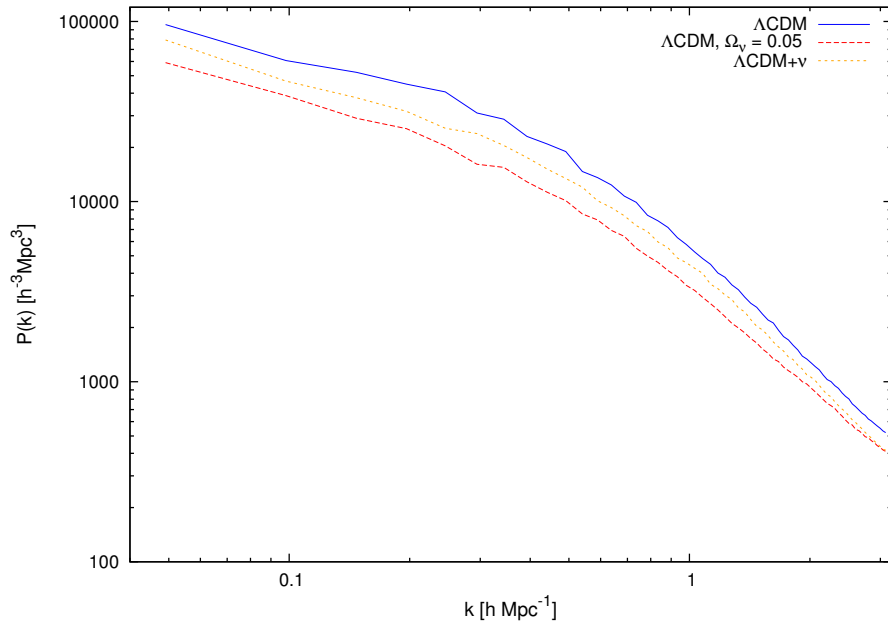
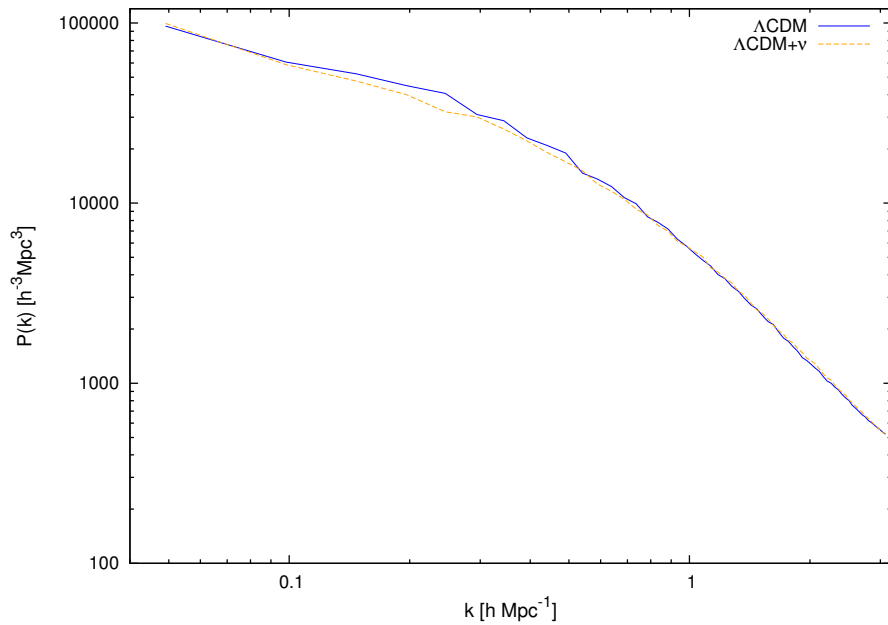


Figure 5.1: These power spectra are from simulations using only CDM, but for non-zero Ω_ν . In the simulation the CDM density fraction is given by $\Omega_{\text{CDM}} = \Omega_m - \Omega_b - \Omega_\nu$. Here Ω_ν is varied to see the effect on the power spectrum.

In the figures (5.2), (5.3), (5.4) and (5.5) we have plotted the $\Lambda\text{CDM}+\nu$ models where we use transfer functions from CAMB to get the neutrino component. Figure (5.2) shows the effect of neutrinos on the power spectrum, which is very small. I chose to use $\Omega_\nu = 0,05$, which is equal to a total neutrino mass of $m_\nu = 2.28$ eV, because this is the largest realistic neutrino mass I have used in the simulations. In figure (5.2a), I have also included the power spectrum for $\Omega_\nu = 0.05$ from figure (5.1), to compare with the simulation where neutrino physics are included. We see that for the simulation with neutrino physics included, the power spectrum is damped toward the red line on small scales, as we would expect. But figure (5.2b) shows that when the power spectrum is rescaled to start at the same scale as the spectrum for the pure ΛCDM model, the damping is not significant. Figures (5.3) and (5.4) further shows the neutrino effect for different values of total neutrino mass (see table (4.2)). I chose the different masses to get a good variation in the range of realistic total mass. I also included the mass $m_\nu = 6.85$ eV to see what would happen in a simulation with a mass that is too large to be considered realistic. None of the simulations with different masses, except for the simulation with $\Omega_\nu = 0.15$, gives any noticeable damping on the power spectrum. Figure (5.5) shows the relative difference in the power spectrum between simulations with and without neutrinos, where the curve is a least squares fit. For $\Omega_\nu = 0.013$ and $\Omega_\nu = 0.15$, it seems like the plots are dominated by noise, but for the two intermediate mass fractions, there is a tendency towards smaller relative differences on small scales, which is the opposite of what we expect.

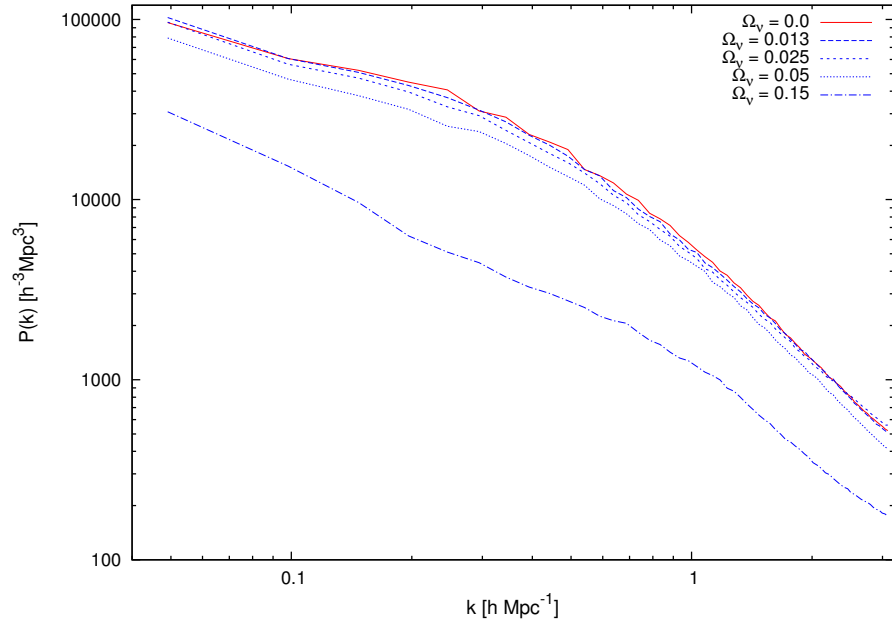


(a) Plot of a Λ CDM simulation with neutrinos (orange line), and without neutrinos (blue line). The red line is a plot with neutrinos included in the matter density, $\Omega_m = \Omega_b + \Omega_{\text{CDM}} + \Omega_\nu$, but neutrino physics are not included in the simulation. The effect is that the Λ CDM power spectrum is damped for all scales.

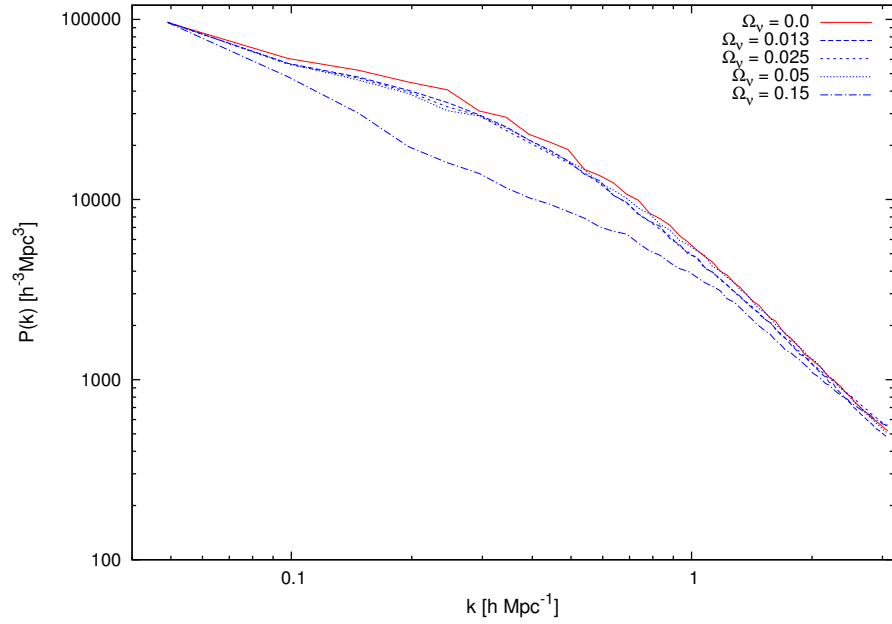


(b) This is the same plot as in figure (5.2a), but the spectra have been scaled to originate at the same value of $P(k)$.

Figure 5.2: Plots of power spectra with and without neutrinos. The total neutrino mass used is $m_\nu = 2.28$ eV.



(a) Plots of different neutrino masses.



(b) This is the same plot as in figure (5.3a), but rescaled.

Figure 5.3: Power spectra for $\Lambda\text{CDM}+\nu$ simulations.

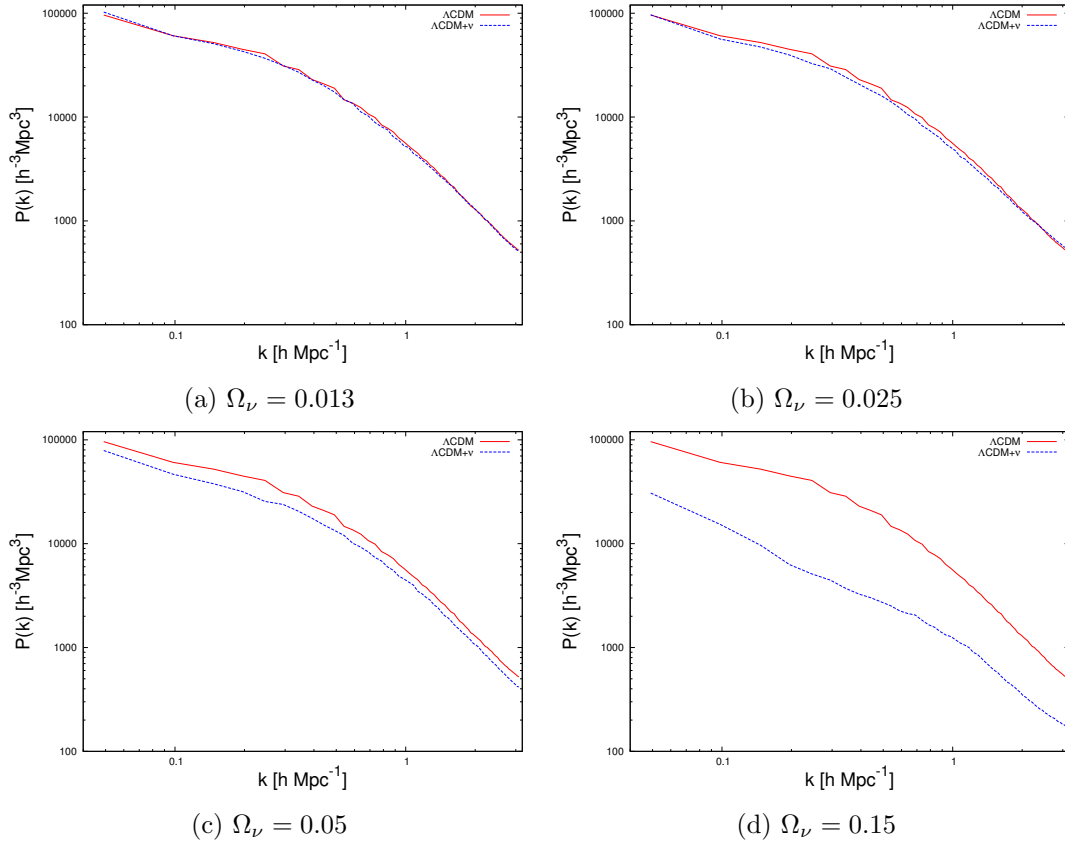


Figure 5.4: Comparison between ΛCDM and $\Lambda\text{CDM}+\nu$ for four total neutrino masses. This is a continuation of figure (5.3), where the power spectrum for each value of Ω_ν are plotted independently for clarity.

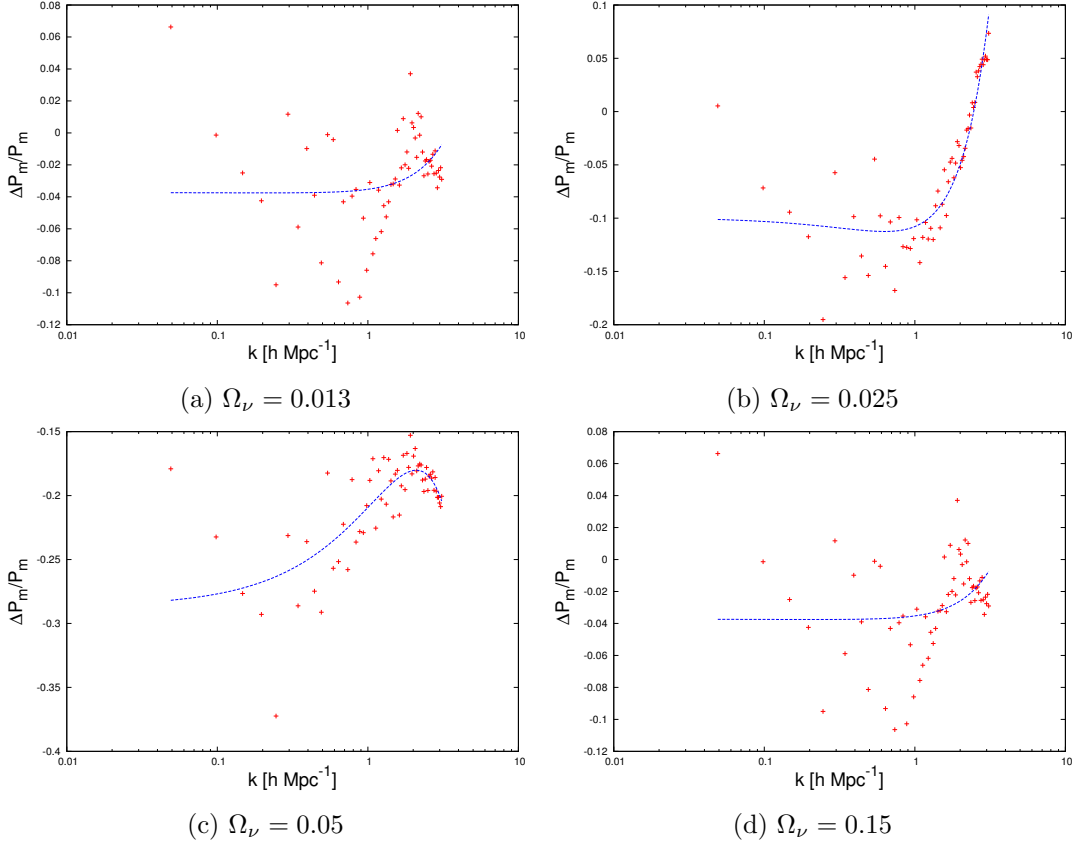


Figure 5.5: Relative difference between Λ CDM and Λ CDM+ ν . The neutrino density grid for these simulations are generated using linear transfer functions. The blue line is fitted to the points using a least squares method.

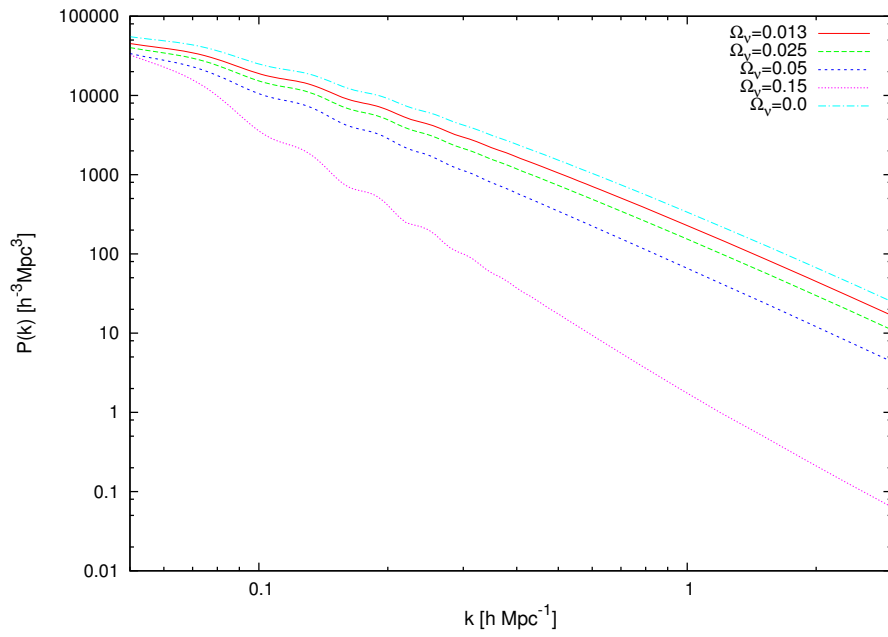
In figure (5.6) and (5.7), we have plotted the power spectra for a Λ CDM+ ν model, using linear theory. The power spectrum are calculated with the formula

$$P(k) = 2\pi^2 A k T_{\text{total}}(k)^2, \quad (5.1)$$

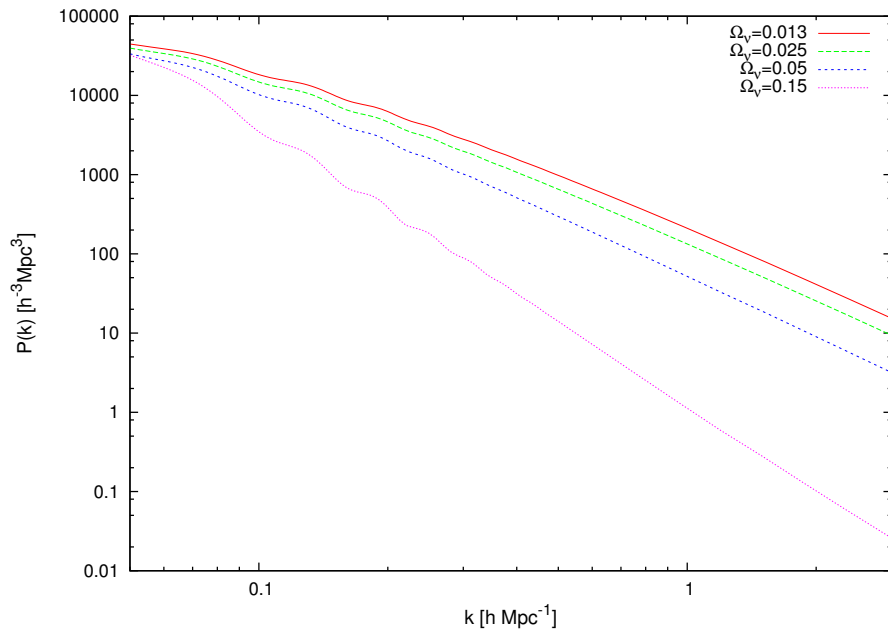
where A is a primordial scale factor that we have set to $A = 2.1 \times 10^{-9}$, and T_{total} is a linear combination of T_{CDM} and T_ν , using Ω_{CDM} and Ω_ν as weights.

$$T_{\text{total}} = \frac{\Omega_{\text{CDM}}}{\Omega_{\text{CDM}} + \Omega_\nu} T_{\text{CDM}} + \frac{\Omega_\nu}{\Omega_{\text{CDM}} + \Omega_\nu} T_\nu$$

The transfer functions for CDM and neutrinos were calculated by CAMB. The neutrino density fractions we have used are $\Omega_\nu = 0.013, 0.025, 0.05$ and 0.15 . Ω_{CDM} is given as $\Omega_{\text{CDM}} = \Omega_m - \Omega_b - \Omega_\nu$. The figures show the expected damping of $P(k)$ on small scales due to neutrinos.



(a) CDM transfer functions from CAMB used to calculate the spectra.



(b) A linear combination of CDM and neutrino transfer functions are used to calculate the spectra.

Figure 5.6: Power spectra calculated using linear theory with CAMB.

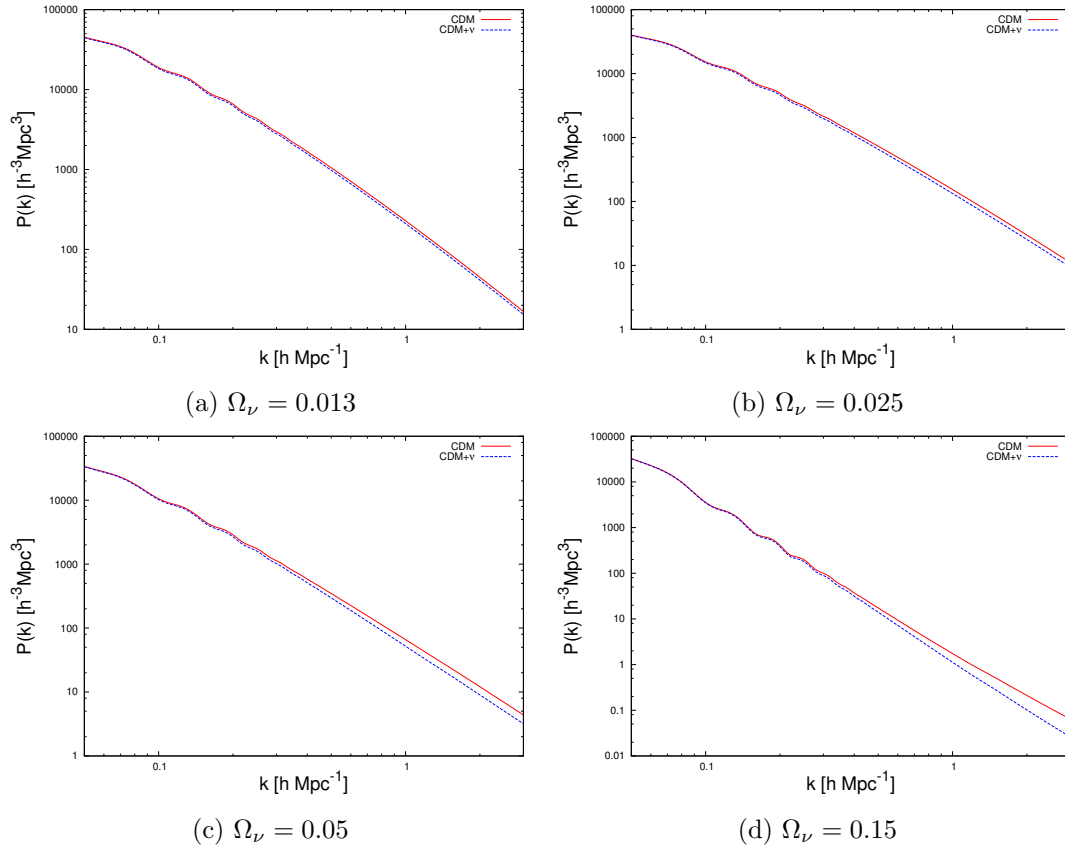


Figure 5.7: Comparison between Λ CDM and Λ CDM+ ν for four different total neutrino masses. Plotted using linear transfer functions from CAMB.

Figure (5.8) shows a comparison between the power spectra from the N-body simulations, plotted in figure (5.4) and the linear power spectra plotted directly from the CAMB data. The non-linear spectra are rescaled to fit with their linear counterparts. We see that the non-linear power spectra have more power than the linear theory. And they stay at the same scales for all the different masses, whereas the linear power spectra are damped more for larger neutrino masses.

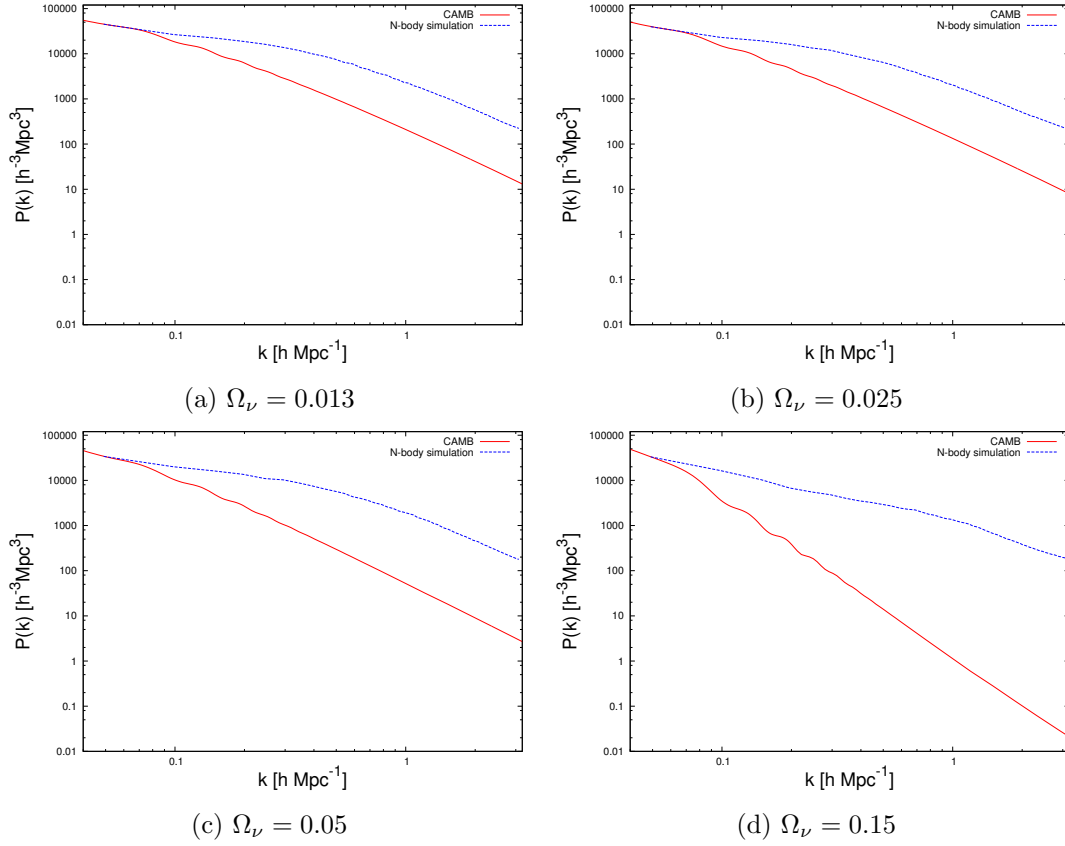
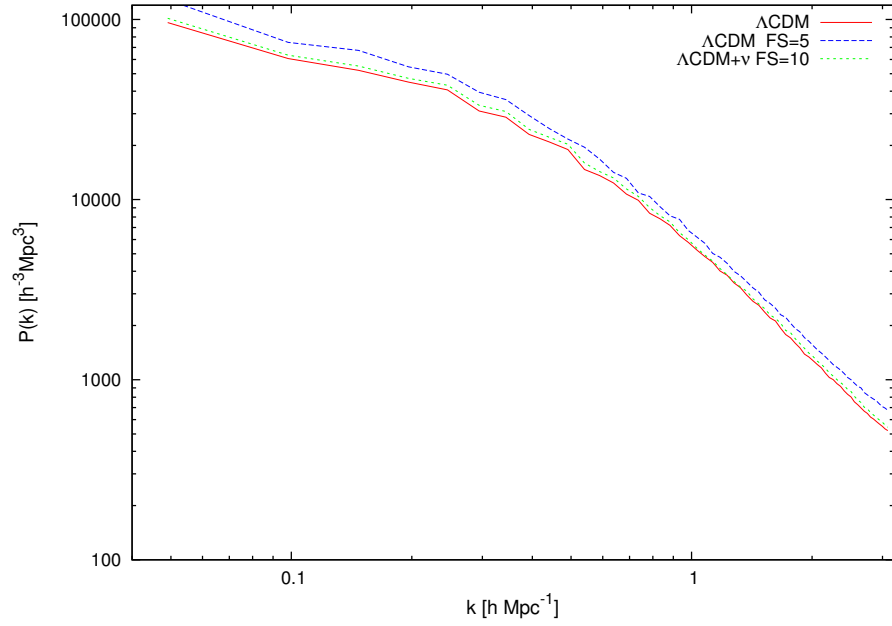


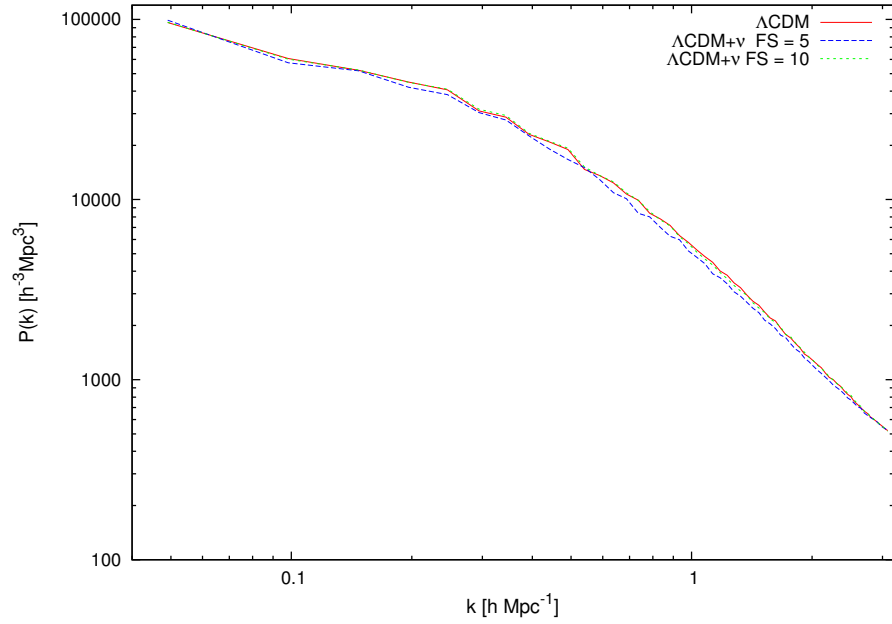
Figure 5.8: Comparison between linear and non-linear power spectra, for four total neutrino masses. The non-linear power spectra are scaled to start at the same value as the linear power spectra from CAMB.

In figures (5.9), (5.10) and (5.11) the method of diffusion of the CDM density field in the simulations have been used to produce the plots. The neutrino free streaming length (FS) is used as the variable that determines the mass of the neutrinos. But the box in the simulations is only 128 Mpc/h long in each dimension, while a free streaming length is about 900 Mpc. So because of computational limitations, we have to choose free streaming lengths that are much smaller than what is realistic. For the simulations I ran, I chose free streaming lengths of 5, 10, 15, 20 and 40 Mpc.

Figures (5.9) and (5.10) shows that for small free streaming lengths the power spectrum has a higher amplitude on all scales, and increasing the free streaming length decreases the power until it is equal to the Λ CDM power spectrum. The reason for this may be that the sum that adds up the neutrino density for each grid cell in the code, is only over the three nearest neighbours in each direction. So, the sum is over a length of 6 Mpc in the simulation box, which is too small to account for all the contributions to the neutrino density in most of the simulations. The effect is that simulations with a higher free streaming length will appear very similar.

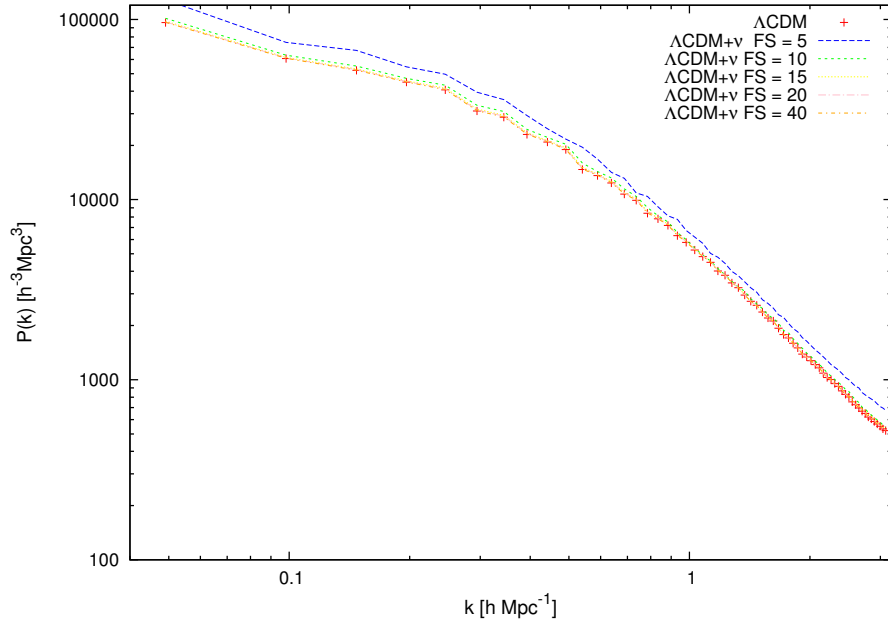


(a) A free streaming length of 5 and 10 Mpc have been used.

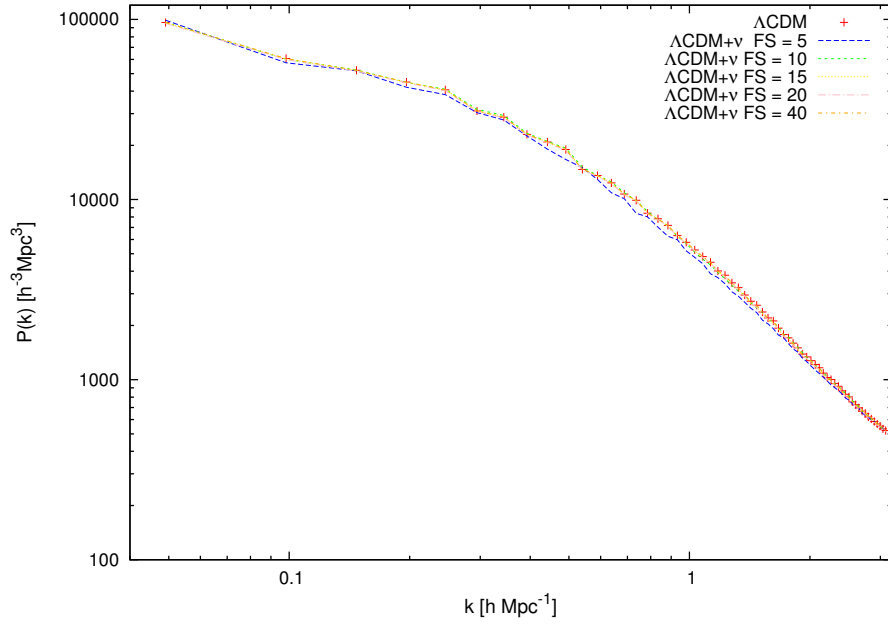


(b) This is the same plot as in figure (5.9a), but rescaled.

Figure 5.9: In these plots the method of CDM density diffusion to generate neutrinos have been used in the simulations.



(a) The free streaming lengths used are 5, 10, 15, 20 and 40 Mpc.



(b) This is the same plot as in figure (5.10a), but rescaled.

Figure 5.10: This is a continuation of figure (5.9), where more free streaming lengths are included. As the free streaming length is increased, the spectra converge fast toward the power spectrum of the Λ CDM model.

Figure (5.11) shows the relative difference between the the Λ CDM model and the power spectra from figure (5.10). What all of these plots have in common, is that the relative difference is always above zero. Which means that the neutrino power spectrum is larger than the Λ CDM power spectrum for all scales and for all free streaming lengths.

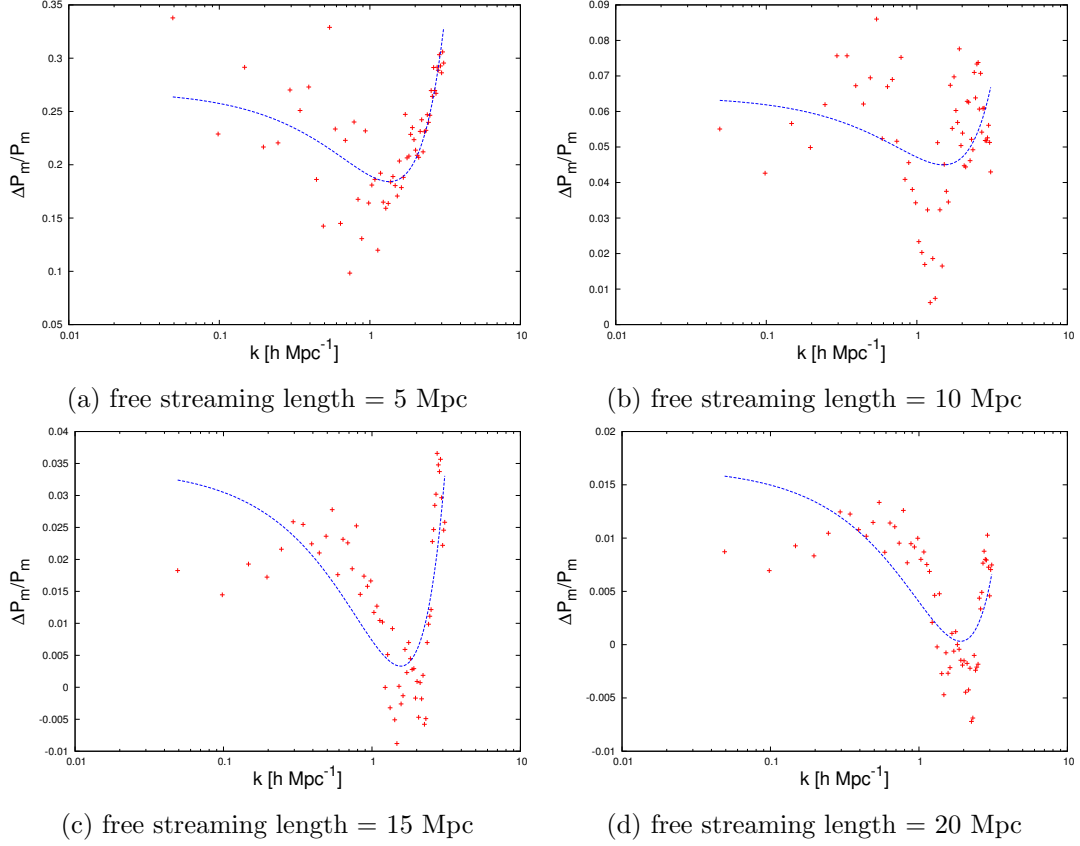


Figure 5.11: Relative difference between Λ CDM and Λ CDM+ ν . The neutrino density for these simulations are generated using diffusion of the CDM density. The blue line is fitted to the points using a least squares method.

To sum up, neither of the two methods we have implemented seem to have the effect on small scale structure formation that we would expect. For the neutrinos generated with linear transfer functions, the problem was that the neutrino grid did not align with the CDM grid in the code. It is likely that this was the cause for the power spectrum not getting damped on small scales, as expected. For the other method, it is difficult to work with free streaming lengths larger than the box size we are using.

CHAPTER 6

Conclusion

The goal of this thesis was to present a simple way to include neutrinos in N-body simulations, introduced in [1], and then include the method into an already existing code. The chosen code was written by Claudio Llinares. The neutrinos were included into the code using linear transfer functions from CAMB, by creating a density realisation from the transfer functions. Another way of including neutrinos into the code was also presented. It consisted of using a smoothing function on the CDM grid in the code to represent neutrinos. This smoothed out grid can represent the neutrinos because neutrinos free stream on scales smaller than the horizon at the time of transition from relativistic to non-relativistic particles.

Due to some problems implementing these methods exactly, the results were not as expected. As seen in figures (5.3) and (5.10), there are not much effect from neutrinos on the power spectrum. This can be compared to the results presented in [1], where we see that neutrinos are the source of a significant damping of the matter power spectrum.

The source of these unexpected results are likely the misalignment of the neutrino and CDM density grids, as seen in figure (4.1). So, going from here, we need to investigate why the neutrino grid and the CDM grid do not align. Another thing to do is increase the size of the simulation to accommodate the large free streaming length.

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