

A continuous auction model with insiders and random time of information release

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November 6, 2014

Abstract

In a unified framework we study equilibrium in the presence of an insider having information on the signal of the firm value, which is naturally connected to the fundamental price of the firm related asset. The fundamental value itself is announced at a future random (stopping) time. We consider the two cases in which this release time of information is known and not known, respectively, to the insider. Allowing for very general dynamics, we study the structure of the insider's optimal strategies in equilibrium and we discuss market efficiency. With respect to market efficiency, we show that in the case the insider knows the release time of information, the market is fully efficient. In the case the insider does not know this random time, we see that there is no full efficiency, but there is nevertheless an equilibrium where the sensitivity of prices is decreasing in time according with the probability that the announcement time is greater than the current time. In other words, the prices become more and more stable as the announcement approaches.

Key words: Market microstructure, equilibrium, insider trading, stochastic control, semimartingales, enlargement of filtrations.

JEL-Classification C61· D43· D44· D53· G11· G12· G14

MS-Classification 2010: 60G35, 62M20, 93E10, 94Axx

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1 Introduction

Models of financial markets with the presence of an insider or informational asymmetries have a large literature, see e.g. Karatzas and Pikovsky (1996), Amendiger et al. (1998), Imkeller et al. (2001), Corcuera et al. (2004), Biagini and Øksendal (2005), (2006), Kohatsu-Higa (2007), Di Nunno et al. (2006, 2008), Biagini et al. (2012) and the references therein. In most of these models prices are fixed exogenously, i.e. the insider does not affect the stock price dynamics, and the privileged information is a functional of the stock price process: the maximum, the final value, etc. As pointed by Danilova (2010), in an equilibrium situation market prices are determined by the demand of market participants, so in such a situation the privileged information cannot be a functional of the stock price process because this implies the knowledge of future demand and it is unrealistic. Then the privileged information is exogenous like the value of the fundamental price, or some signal of it, or the announcement time of the release of the fundamental price, which evolves independently of the demand. The questions considered in this paper deal with the existence of an equilibrium and the properties of the insider's optimal strategies. Moreover another question studied is the efficiency of the market, namely the conditions in which market prices converge to the fundamental one. These problems have been addressed in different works, with different degrees of generality, and with very different types of insider's privileged information and demands of the uninformed traders.

The original model is due to Kyle (1985), he considers three kind of actors in the market: market makers, uninformed traders and one insider who knows the fundamental or liquidation value of an asset at certain fixed released time, there is also, in the model, a price function establishing the relation between market prices and the total demand. He works in the discrete time setting, and with Gaussian random walks as noises. Back (1992) extends the work to the continuous time case. These are the two seminal papers. From then there has been several generalizations of the model: Back and Pedersen (1998) who consider a *dynamic* fundamental price and Gaussian noises with time varying volatility; Cho (2003) who considers pricing functions depending on the path of the demand process and studies what happens when the informed trader is risk-averse; Lasserre (2004) who considers a multivariate setting; Aase et al. (2012a), (2012b) who put emphasis in filtering techniques to solve the equilibrium problem; Campi and Çetin (2007), who consider a defaultable bond, in the place of the stock in the Kyle-Back model, and the default time, as privileged information; Danilova (2010) where the author considers non-regular pricing rules; Caldentey and Stacchetti (2010) who consider a random release time, and Campi et al. (2013) where again the authors consider a defaultable bond and the privileged information is not the default time anymore, but some dynamic signal related with it, see example 23 below for more details. The list could be completed with the references in the mentioned papers.

The present paper extends the previous contributions in different ways. We consider general noises for the demand processes, general pricing rules, random release times, and general dynamic information, all in the same model. Then, we study in detail which are the necessary conditions needed to have an equilibrium. These conditions are new in the literature. Specifically we consider the very general case in which an insider has access to some signal related to the firm value, which is in fact released at some stopping time. We first consider the case where the insider knows the random time of release of information and then the case where this is also unknown to her. We study these two situations in the same framework with the purpose of analyzing equilibrium and efficiency of the market.

Except for the multivariate setting of Lasserre (2004) and the risk-aversion considered in previous works, this is a general setting for the previous extensions of the Kyle-Back model, as we show through different examples.

Our study shows explicitly how *equilibrium* is a specific state of the market induced by the interplay of agents with different roles and asymmetric information. Indeed, the market makers set rational prices which are assumed to be a function H of the aggregate demand and time. For such H given, the insider optimizes her position to maximize her expected wealth. The necessary conditions for the existence of an equilibrium show how this optimization is possible only for some given pricing rules and under some available information flows.

In this study we show that the presence of the insider can be beneficial to the market from an efficiency point of view. In fact, if the insider knows the random release time, then the market is efficient. However, if this time of release is unknown also to the insider, then the market is *not* fully efficient, nevertheless equilibrium can be reached if the sensitivity of prices decreases in time according to the survival probability of the announcement. In other words, the prices become more stable as the announcement time approaches.

As far as we know this generality of the insider's information together with the presence of a random time of release has never been studied before. Moreover, our contribution includes also very general dynamics for the demand process. In fact the insider's demand is allowed to be a general semimartingale. The present paper includes also various examples in which we give explicit insider's optimal strategies for a given pricing rule and define the concept of admissibility for pricing rules and insider strategies. Here we show how our results, coupled with the mathematical tools of enlargement of filtrations or filtering techniques, allow to finding the insider's optimal strategy in various cases presented in the literature, but here treated in a unified framework.

The paper is structured as follows. In the next section we describe the model that gives rise the stock prices and we discuss the insider's optimal strategies for a given pricing rule and define the concept of admissibility

for pricing rules and insider strategies. In section 3 and 4 we discuss what happens when the release time is known to the insider or not, respectively. Finally, in section 5, we give some examples.

2 The model and equilibrium

We consider a market with two assets, a stock of a firm and a bank account with interest rate r equal to zero for the sake of simplicity. With abuse of terminology we will just write prices even though they are sometimes “discounted” prices. The trading is continuous in time over the period $[0, \infty)$ and it is order driven. There is a (possibly random) release time $\tau < \infty$ a.s., when the *fundamental value* of the stock is revealed. The fundamental value process, that we shall define in a precise way later, is denoted by V . We shall denote the market price of the stock at time t by P_t . Just after the revelation time the price of the stock coincides with the fundamental value. Then we consider P_t defined only on $t \leq \tau$. In principle, it is possible that $P_t \neq V_t$ if $t \leq \tau$, stopping our studies at this (random) time of release.

We assume that all the random variables and processes mentioned are defined in the same, *complete*, probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and that the filtrations are *complete and right-continuous* by taking, if it is necessary, the usual augmentation of them, as we shall specify below.

There are three kinds of traders. A *large* number of liquidity traders, who trade for liquidity or hedging reasons, an informed trader or insider, who has privileged information about the firm and can deduce its fundamental value, and the market makers, who set the market price and clear the market.

2.1 The agents and equilibrium

Let X be the demand process of the informed trader. At time t , her information is given by \mathcal{H}_t and her flow of information is given by the filtration $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$. It is natural to assume that X is an \mathbb{H} -predictable process. The informed trader, like any other trader, observes the market prices P and, in addition, she has access to the firm value, having access to some *signal process* η directly related to the firm value. Moreover, she will have *some* knowledge about the random time τ . In the sequel we will consider two cases:

- $\mathcal{H}_t = \bar{\sigma}(P_s, \eta_s, \tau, 0 \leq s \leq t)$, i.e. the informed trader has knowledge of the time of release of information
- $\mathcal{H}_t = \bar{\sigma}(P_s, \eta_s, \tau \wedge s, 0 \leq s \leq t)$, i.e. the informed trader has no knowledge of this release time, but she will instantly know when it happens.

Here $\bar{\sigma}$ denotes the usual augmentation of a natural filtration σ (see [32], Ch. I, Def. 4.13). That is, e.g.,

$$\bar{\sigma}(P_s, \eta_s, \tau, 0 \leq s \leq t) := \bigcap_{r>t} (\sigma(P_s, \eta_s, \tau, 0 \leq s \leq r) \cup \mathcal{N}),$$

where \mathcal{N} is the family of \mathbb{P} -null sets in \mathcal{F} , and $\sigma(P_s, \eta_s, \tau, 0 \leq s \leq r)$ is the natural filtration generated by P, η , and τ .

In both the cases above, the insider has access to the fundamental value V which, in terms of the insider's information flow, is assumed to be a càdlàg \mathbb{H} -martingale such that $\sigma_V^2(t) := \frac{d[V, V]_t^c}{dt}$ is well defined (where $[V, V]^c$ indicates the continuous part of the quadratic variation of V) and V is taken in the form:

$$V_t = \mathbb{E}(f(\eta_\tau) | \mathcal{H}_t), \quad t \geq 0, \quad (1)$$

where f is a non-negative deterministic function and η is some signal process related to the firm value. The explicit presence of f gives a structure to the relationship between the type of signal and the fundamental value, see Example 22 and Remark 10.

Let Z be the *aggregate* demand process of the liquidity traders. We recall that these are a large number of traders motivated by liquidity or hedging reasons. They are perceived as constituting noise in the market, thus also called *noise* traders. From the insider's perspective we assume that Z is a continuous \mathbb{H} -martingale, independent of η and V , such that that $\sigma_Z^2(t) := \frac{d[Z, Z]_t}{dt}$ is well defined.

Market makers clear the market giving the market prices. They rely on the information given by the total aggregate demand $Y := X + Z$ which they observe and, just like the noise traders, they instantly know about the time of release of information when that occurs. Hence their information flow is: $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, where $\mathcal{F}_t = \bar{\sigma}(Y_s, \tau \wedge s, 0 \leq s \leq t)$.

Due to the competition among market makers, the market prices are *rational*, or *competitive*, in the sense that

$$P_t = \mathbb{E}(V_t | \mathcal{F}_t), \quad 0 \leq t \leq \tau.$$

Finally we suppose that market makers give market prices through a pricing rule, which consists of a formula, here assumed of the form:

$$P_t = H(t, \xi_t), t \geq 0 \quad (2)$$

involving

$$\xi_t := \int_0^t \lambda(s) dY_s,$$

where $\lambda \in C^1$ is a strictly positive deterministic function, $H \in C^{1,2}$, $H(t, \cdot)$ is strictly increasing for all $t \geq 0$. Note that $\mathcal{F}_t = \bar{\sigma}(P_s, \tau \wedge s, 0 \leq s \leq t)$, for all t . We have the following definition.

Definition 1 Denote the class of such pairs (H, λ) above by \mathfrak{H} . An element of \mathfrak{H} is called a pricing rule.

The informed trader is assumed risk-neutral and she aims at maximizing her expected final wealth. Let W be the wealth process corresponding to insider's portfolio X . We introduce the following definitions.

Definition 2 A strategy X is called optimal with respect to a price process P if it maximizes $\mathbb{E}(W_\tau)$.

Definition 3 Let $(H, \lambda) \in \mathfrak{H}$ and consider a strategy X . The triple (H, λ, X) is an (a local) equilibrium, if the price process $P := H(\cdot, \xi)$ is rational, given X , that is

$$P_t = \mathbb{E}(V_t | \mathcal{F}_t), \quad 0 \leq t \leq \tau, \quad (3)$$

and the strategy X is (locally) optimal, given (H, λ) .

Remark 4 It is important to remark that the effect of the total demand in prices is due not only to the function λ , but also to the function H . In fact, as we shall see later, in some equilibrium cases, see Proposition 13,

$$dP_t = \partial_2 H(t, \xi_t) \lambda(t) dY_t,$$

and some authors give the name market depth to the quantity

$$\frac{1}{\partial_2 H(t, \xi_t) \lambda(t)}.$$

Here and in the sequel $\partial_i H$ denotes the derivative with respect to the i^{th} variable. So, to say that the market depth is constant is not equivalent to say that $\lambda(t)$ is constant. Only if the equilibrium pricing rule is linear, the two statements are equivalent. See Back and Pedersen (1998).

2.2 Wealth and admissible strategies

To illustrate the relationship among the processes V, P, X , and W we first consider a multi-period model where trades are made at times $i = 1, 2, \dots, N$, and where $\tau = N$ is random. If at time $i - 1$, there is an order to buy $X_i - X_{i-1}$ shares, its cost will be $P_i(X_i - X_{i-1})$, so, there is a change in the bank account given by

$$-P_i(X_i - X_{i-1}).$$

Then the total (cumulated) change at $\tau = N$ is

$$-\sum_{i=1}^N P_i(X_i - X_{i-1}),$$

and due to the convergence of the market and the fundamental prices at time $\tau = N$, there is the extra income: $X_N V_N$. So, the total wealth W_τ at τ is

$$\begin{aligned} W_\tau &= -\sum_{i=1}^N P_i(X_i - X_{i-1}) + X_N V_N \\ &= -\sum_{i=1}^N P_{i-1}(X_i - X_{i-1}) - \sum_{i=1}^N (P_i - P_{i-1})(X_i - X_{i-1}) + X_N V_N. \end{aligned}$$

Consider now the continuous time setting where we have the processes X, P , and V , and we take N trading periods, where N is random and the trading times are: $0 \leq t_1 \leq t_2 \leq \dots \leq t_N = \tau$, then we have

$$W_\tau = -\sum_{i=1}^N P_{t_{i-1}}(X_{t_i} - X_{t_{i-1}}) - \sum_{i=1}^N (P_{t_i} - P_{t_{i-1}})(X_{t_i} - X_{t_{i-1}}) + X_{t_N} V_{t_N},$$

so if the time between trades goes to zero we will have

$$\begin{aligned} W_\tau &= X_\tau V_\tau - \int_0^\tau P_{t-} dX_t - [P, X]_\tau \\ &= \int_0^\tau X_{t-} dV_t + \int_0^\tau V_{t-} dX_t + [V, X]_\tau - \int_0^\tau P_{t-} dX_t - [P, X]_\tau \\ &= \int_0^\tau (V_{t-} - P_{t-}) dX_t + \int_0^\tau X_{t-} dV_t + [V, X]_\tau - [P, X]_\tau, \end{aligned} \tag{4}$$

where (and throughout the whole article) $P_{t-} = \lim_{s \uparrow t} P_s$ a.s. We shall assume that X is an \mathbb{H} -predictable càdlàg semimartingale, so that the stochastic integrals above can be seen as Itô's integrals. Moreover, by applying Itô's formula to $P_t = H(t, \xi_t)$, $t \geq 0$, where ξ is now a càdlàg \mathbb{H} -semimartingale, we can see that P is also an \mathbb{H} -semimartingale.

In the next subsection we discuss the characterization of an insider's optimal strategy in equilibrium in terms of fundamental value and insider information. Namely, we consider a process X that is optimal in the sense that it maximizes

$$J(X) := \mathbb{E}(W_\tau) = \mathbb{E}\left(\int_0^\tau (V_{t-} - H(t, \xi_{t-})) dX_t + \int_0^\tau X_{t-} dV_t + [V, X]_\tau - [P, X]_\tau\right),$$

for a pricing rule $(H, \lambda) \in \mathfrak{H}$. However for technical and modelling reasons, we require additional properties

to the triplet (H, λ, X) . In this way, we characterize the *admissible* triplets (H, λ, X) as those processes X (that include, by hypothesis, the process $X \equiv 0$) and price functions $(H, \lambda) \in \mathfrak{H}$ satisfying:

(A1) $X_t = M_t + A_t + \int_0^t \theta_s ds$, for all $t \geq 0$, where M is a continuous \mathbb{H} -martingale, A a finite variation

\mathbb{H} -predictable process with $A_t = \sum_{0 < s \leq t} (X_s - X_{s-})$, and θ a càdlàg \mathbb{H} -adapted process

(A2) $\mathbb{E} \left(\int_0^\tau (\partial_2 H(s, \xi_s))^2 (\sigma_Z^2(s) ds + \sigma_M^2(s) ds) \right) < \infty$, where $\sigma_M^2(s) := \frac{d[M, M]_s}{ds}$

(A3) $\mathbb{E} \left(\int_0^\tau \partial_2 H(s, \xi_s) |\theta_s| ds \right) < \infty$

(A4) $\mathbb{E} \left(\sum_0^\tau \partial_2 H(s, \xi_{s-}) |\Delta X_s| \right) < \infty$, $\Delta X_s := X_s - X_{s-}$

(A5) $\mathbb{E} \left(\int_0^\tau |X_s|^2 d[V, V]_s \right) < \infty$

(A6) $\mathbb{E} \left(\int_0^\tau \lambda(s) |\partial_{22} H(s, \xi_s)| (\sigma_M^2(s) + |\sigma_{M, Z}(s)|) ds \right) < \infty$, where $\sigma_{M, Z}(s) := \frac{d[M, Z]_s}{ds}$.

Recall that ∂_i indicates the derivative w.r.t. the i^{th} argument.

Remark 5 *Note that, since $(X_t)_{t \geq 0}$ is a càdlàg \mathbb{H} -predictable process, its martingale part cannot have jumps, see Corollary 2.31 in Jacod and Shiryaev (1987). Similarly, we have chosen Z to be continuous before.*

We can recall the essential assumptions of the model as follows. Our stochastic basis is a complete filtered space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{H}, \mathbb{P})$, where $\mathbb{F} \subseteq \mathbb{H}$ are the filtrations defined in the subsection 2.1. Roughly speaking, \mathbb{F} contains information about market prices or total demand and \mathbb{H} includes also information about the fundamental value V . We have market prices $P_t = H(t, \xi_t)$, where H is a $\mathbb{C}^{1,2}$ function, $H(t, \cdot)$ strictly increasing, $\xi_t = \int_0^t \lambda(s) dY_s$, and λ is a \mathbb{C}^1 strictly positive function. From the rationality assumption (3) we have that P is an \mathbb{F} -martingale. The total demand process is given by $Y = X + Z$, with Z an \mathbb{H} -continuous martingale such that $\sigma_Z^2(s) := \frac{d[Z, Z]_s}{ds}$. The fundamental value V is a càdlàg \mathbb{H} -martingale such that $\sigma_V^2(t) := \frac{d[V, V]_t^c}{dt}$ and V has the structure (1). Finally the release time τ is a stopping time with respect to \mathbb{F} and \mathbb{H} .

2.3 The optimality condition

In the sequel we will consider two kinds of stopping times: τ bounded, or τ finite but independent of (V, P, Z) . In both cases, by the assumptions that V is an \mathbb{H} -martingale and X an \mathbb{H} -predictable càdlàg

\mathbb{H} -semimartingale satisfying (A5) we have that $\mathbb{E}(\int_0^\tau X_t dV_t) = 0$. In fact, if τ is bounded we can apply Doob's Optional Sampling Theorem and if τ is finite but independent of (V, P, Z) , we have that

$$\mathbb{E} \left(\int_0^\tau X_t dV_t \right) = \mathbb{E} \left(\mathbb{E} \left(\int_0^\tau X_s dV_s \middle| \tau = t \right) \middle|_{t=\tau} \right) = \mathbb{E} \left(\mathbb{E} \left(\int_0^t X_s dV_s \right) \middle|_{t=\tau} \right) = 0.$$

Hence,

$$J(X) := \mathbb{E}(W_\tau) = \mathbb{E} \left(\int_0^\tau (V_{t-} - H(t, \xi_{t-})) dX_t + [V, X]_\tau - [P, X]_\tau \right). \quad (5)$$

First, note that

$$\begin{aligned} \int_0^\tau (V_{t-} - H(t, \xi_{t-})) dX_t + [V, X]_\tau - [P, X]_\tau &= \int_0^{\tau-} (V_{t-} - H(t, \xi_{t-})) dX_t + [V, X]_{\tau-} - [P, X]_{\tau-} \\ &\quad + (V_\tau - H(\tau, \xi_\tau)) \Delta X_\tau. \end{aligned}$$

Then suppose that X is (locally) optimal and we modify only the the last jump of this strategy, by taking $(1 + \varepsilon\gamma)\Delta X_\tau$ with γ an $\mathcal{H}_{\tau-}$ -measurable and bounded random variable and $\varepsilon > 0$ small enough. We recall that $\mathcal{H}_{\tau-} := \mathcal{H}_0 \vee \sigma(A \cap (\tau > t) : A \in \mathcal{H}_t, t \geq 0)$ (see, e.g., Revuz and Yor (1999), page 46). Denote $X^{(\varepsilon)}$ this new strategy.

Then we obtain

$$\frac{d}{d\varepsilon} J(X^{(\varepsilon)}) \Big|_{\varepsilon=0} = \mathbb{E} \left(\gamma \left((V_\tau - H(\tau, \xi_\tau)) \Delta X_\tau - \lambda(\tau) \partial_2 H(\tau, \xi_\tau) (\Delta X_\tau)^2 \right) \right),$$

so

$$\mathbb{E} \left((V_\tau - H(\tau, \xi_\tau)) \Delta X_\tau - \lambda(\tau) \partial_2 H(\tau, \xi_\tau) (\Delta X_\tau)^2 \middle| \mathcal{H}_{\tau-} \right) = 0. \quad (6)$$

Now we modify the strategy X by taking an \mathbb{H} -adapted càdlàg process β such that $X + \varepsilon \int_0^\cdot \beta_s ds$ is admissible, with $\varepsilon > 0$ small enough.

We have

$$\begin{aligned}
0 &= \frac{d}{d\varepsilon} J(X, \varepsilon \int_0^\cdot \beta_s ds) \Big|_{\varepsilon=0} \\
&= \frac{d}{d\varepsilon} \mathbb{E} \left(\int_0^\tau (V_{t-} - H(t, \int_0^{t-} \lambda(s)(dX_s + \varepsilon\beta_s ds + dZ_s)))(dX_t + \varepsilon\beta_t dt) \right) \Big|_{\varepsilon=0} \\
&\quad - \frac{d}{d\varepsilon} \mathbb{E} \left([V, X, \varepsilon \int_0^\cdot \beta_s ds]_\tau - [H(\cdot, \int_0^\cdot \lambda(s)(dX_s + \varepsilon\beta_s ds + dZ_s), X, \varepsilon \int_0^\cdot \beta_s ds)]_\tau \right) \Big|_{\varepsilon=0} \\
&= \mathbb{E} \left(\int_0^\tau (V_{t-} - H(t, \xi_t)) \beta_t dt \right) - \mathbb{E} \left(\int_0^\tau \partial_2 H(t, \xi_{t-}) \left(\int_0^t \lambda(s) \beta(s) ds \right) dX_t \right) \\
&\quad - \mathbb{E} \left(\left[\partial_2 H(\cdot, \xi, \cdot) \left(\int_0^\cdot \lambda(s) \beta(s) ds \right), X \right]_\tau \right) \\
&= \mathbb{E} \left(\int_0^\tau \left((V_t - H(t, \xi_t)) - \lambda(t) \int_{t \wedge \tau}^\tau \partial_2 H(s, \xi_{s-}) dX_s \right) \beta_t dt \right) \\
&\quad - \mathbb{E} \left(\int_0^\tau \left(\int_0^t \lambda(s) \beta(s) ds \right) d[\partial_2 H(\cdot, \xi, \cdot), X]_t \right) \\
&= \mathbb{E} \left(\int_0^\tau \left((V_t - H(t, \xi_t)) - \lambda(t) \left(\int_{t \wedge \tau}^\tau \partial_2 H(s, \xi_{s-}) dX_s + [\partial_2 H(\cdot, \xi, \cdot), X]_t^\tau \right) \right) \beta_t dt \right),
\end{aligned}$$

where $[\cdot, \cdot]_t^\tau = [\cdot, \cdot]_\tau - [\cdot, \cdot]_t$. Since we can take $\beta_t = \alpha_u \mathbf{1}_{(u, u+h]}(t) \alpha_u$, with α_u \mathcal{H}_u -measurable and bounded, we have

$$\mathbb{E} \left(\int_u^{u+h} \left(\mathbb{E}(\mathbf{1}_{[0, \tau]}(t) (V_t - H(t, \xi_t)) | \mathcal{H}_t) - \lambda(t) \mathbb{E} \left(\int_{t \wedge \tau}^\tau \partial_2 H(s, \xi_{s-}) dX_s + [\partial_2 H(\cdot, \xi, \cdot), X]_{t \wedge \tau}^\tau \Big| \mathcal{H}_t \right) \right) dt \Big| \mathcal{H}_u \right) = 0 \tag{7}$$

and this means that the process $\Xi_t, t \geq 0$:

$$\Xi_t := \int_0^t \left(\mathbb{E}(\mathbf{1}_{[0, \tau]} V_u | \mathcal{H}_u) - \mathbb{E}(\mathbf{1}_{[0, \tau]} H(u, \xi_u) | \mathcal{H}_u) - \lambda(u) \mathbb{E} \left(\int_{u \wedge \tau}^\tau \partial_2 H(s, \xi_{s-}) dX_s + [\partial_2 H(\cdot, \xi, \cdot), X]_{u \wedge \tau}^\tau \Big| \mathcal{H}_u \right) \right) du$$

is an \mathbb{H} -martingale with bounded variation. In particular this implies that, for a.a. $t \geq 0$,

$$\mathbb{E}(\mathbf{1}_{[0, \tau]}(t) V_t | \mathcal{H}_t) - \mathbb{E}(\mathbf{1}_{[0, \tau]}(t) H(t, \xi_t) | \mathcal{H}_t) - \lambda(t) \mathbb{E} \left(\int_{t \wedge \tau}^\tau \partial_2 H(s, \xi_{s-}) dX_s + [\partial_2 H(\cdot, \xi, \cdot), X]_{t \wedge \tau}^\tau \Big| \mathcal{H}_t \right) = 0, a.s.$$

Since τ is an \mathbb{H} -stopping time, then for a.a. t and for a.a. $\omega \in \{\tau \geq t\}$, or equivalently a.s. on the stochastic interval $[0, \tau]$, we can write

$$V_t - H(t, \xi_t) - \lambda(t) \mathbb{E} \left(\int_t^\tau \partial_2 H(s, \xi_s) d^- X_s \Big| \mathcal{H}_t \right) = 0. \tag{8}$$

Where we have used a shorthand notation by means of $d^- X_s$ as the *backward* integral in the sense of Revuz and Yor (1999) (see page 144), here extended to semimartingales with jumps. As a summary we have the

following necessary condition to help identifying good candidates as insider's optimal strategies.

Theorem 6 *An admissible triple (H, λ, X) such that X is locally optimal for the insider satisfies equations (6) and (8) a.s. in $[0, \tau]$.*

Remark 7 *Note that (6) and (8) are also true in the case that $\lambda(t)$ is a piecewise strictly positive constant function including the situation treated in Danilova (2010).*

In the sequel we study two different cases of knowledge of τ from the insider's perspective. First the case in which the insider knows τ , the exact time of release of information about the firm value, then we study the case when the insider does not know τ .

3 Case when τ is known to the insider

Let $\sigma(\tau)$ be the σ -algebra generated by τ . Then we consider the case in which $\sigma(\tau) \subseteq \mathcal{H}_0$. At any time t , the insider relies on the information given by:

$$\mathcal{H}_t = \bar{\sigma}(P_s, \eta_s, \tau, 0 \leq s \leq t).$$

Moreover, we assume that τ is bounded, so the analysis here below is consistent with the one of the previous section.

3.1 Necessary conditions for the equilibrium

Our first observation is that optimal strategies lead the market price to the fundamental one, making the market be efficient. In fact we have the following proposition.

Proposition 8 *If τ is known to the insider and (H, λ, X) is admissible with X locally optimal, then the optimal strategy X has no jump at τ and the market is efficient, i.e.*

$$V_{\tau-} = H(\tau, \xi_{\tau-}) = H(\tau, \xi_{\tau}) = P_{\tau} \quad a.s.$$

Proof. By the assumptions (A1) and (A2), equation (8) can be rewritten:

$$\begin{aligned}
& V_t - H(t, \xi_t) - \lambda(t) \mathbb{E} \left(\int_t^\tau \partial_2 H(s, \xi_s) d^- X_s \mid \mathcal{H}_t \right) \\
&= V_t - H(t, \xi_t) - \lambda(t) \int_t^\tau \mathbb{E}(\partial_2 H(s, \xi_s) \theta_s \mid \mathcal{H}_t) ds \\
&\quad - \lambda(t) \sum_t^\tau \mathbb{E}(\partial_2 H(s, \xi_s) \Delta X_s \mid \mathcal{H}_t) \\
&\quad - \lambda(t) \mathbb{E} \left(\int_t^\tau \lambda(s) \partial_{22} H(s, \xi_s) (\sigma_M^2(s) + \sigma_{Z,M}(s)) ds \mid \mathcal{H}_t \right) \\
&= 0, \text{ a.s. on } [0, \tau].
\end{aligned}$$

Now by assumption (A3) and Corollary (2.4) in Revuz and Yor (1999), we have that

$$\lim_{t \uparrow \tau} \mathbb{E} \left(\int_t^\tau \partial_2 H(s, \xi_s) |\theta_s| ds \mid \mathcal{H}_t \right) = 0.$$

Analogously for the term

$$\lim_{t \uparrow \tau} \lambda(t) \mathbb{E} \left(\int_t^\tau \lambda(s) \partial_{22} H(s, \xi_s) (\sigma_M^2(s) + \sigma_{Z,M}(s)) ds \mid \mathcal{H}_t \right) = 0, \text{ a.s.}$$

Whereas

$$\begin{aligned}
& \lim_{t \uparrow \tau} \lambda(t) \sum_t^\tau \mathbb{E}(\partial_2 H(s, \xi_s) \Delta X_s \mid \mathcal{H}_t) \\
&= \lambda(\tau) \partial_2 H(\tau, \xi_\tau) \Delta X_\tau,
\end{aligned}$$

and consequently

$$V_{\tau-} - H(\tau, \xi_{\tau-}) - \lambda(\tau) \partial_2 H(\tau, \xi_\tau) \Delta X_\tau = 0. \tag{9}$$

Now consider equation (6) and recall that $\mathcal{H}_0 \subseteq \mathcal{H}_{\tau-}$. Since V is an \mathbb{H} -martingale and $\tau \in \mathcal{H}_0$, then $\mathbb{E}(V_\tau \mid \mathcal{H}_{\tau-}) = V_{\tau-}$ (see Revuz and Yor (1999), Ch. 2, Prop. 2.7). Moreover, since X is \mathbb{H} -predictable, Z is continuous, and $\tau \in \mathcal{H}_0$, we have

$$\mathbb{E} \left((H(\tau, \xi_\tau) \Delta X_\tau + \lambda(\tau) \partial_2 H(\tau, \xi_\tau) (\Delta X_\tau)^2) \mid \mathcal{H}_{\tau-} \right) = H(\tau, \xi_\tau) \Delta X_\tau + \lambda(\tau) \partial_2 H(\tau, \xi_\tau) (\Delta X_\tau)^2.$$

Therefore equation (6) gives

$$(V_{\tau-} - H(\tau, \xi_\tau)) \Delta X_\tau - \lambda(\tau) \partial_2 H(\tau, \xi_\tau) (\Delta X_\tau)^2 = 0. \text{ a.s.} \tag{10}$$

and if $\Delta X_\tau \neq 0$, it turns out that

$$V_{\tau-} - H(\tau, \xi_\tau) - \lambda(\tau) \partial_2 H(\tau, \xi_\tau) \Delta X_\tau = 0.$$

Comparing the above equation with (19) we have that $H(\tau, \xi_\tau) = H(\tau, \xi_{\tau-})$, which contradicts $\Delta X_\tau \neq 0$, being H strictly increasing in the second variable. This shows that a (locally) optimal strategy X has no jump at τ and, by (19), that $V_{\tau-} = H(\tau, \xi_{\tau-}) = H(\tau, \xi_\tau)$. ■

Remark 9 *In Aase et al. (2012a) it was already observed that market efficiency, that is the convergence of market prices to the fundamental ones, is a consequence of the optimality of the insider's strategy. Here we obtain an extension of this result for a more general behavior of the fundamental value and the demand process of the noise traders.*

Remark 10 *This efficiency situation is also the case in Campi and Çetin (2007). In our notation they have the signal $\eta = \bar{\tau}$, with $\bar{\tau}$ an \mathbb{H} -stopping time, $V_t = \mathbf{1}_{\{\bar{\tau} > 1\}}$ and the release time is $\tau = \bar{\tau} \wedge 1$. So, $\tau \in \mathcal{H}_0$ and it is bounded. Then, they obtain*

$$\mathbf{1}_{\{\bar{\tau} > 1\}} - H(\bar{\tau} \wedge 1, \xi_{\bar{\tau} \wedge 1}) = 0, a.s.$$

They also assume that $\bar{\tau}$ is the first passage time of a standard Brownian motion independent of Z .

Remark 11 *If we take $V_t \equiv V$ and $\tau \equiv 1$ then we are in Back's framework (1992). There it is shown that market prices converge to V when $t \rightarrow 1$.*

Hereafter we consider necessary conditions for the existence of an equilibrium. These conditions show the synergy between the optimal insider strategy and the pricing rule in an equilibrium state. Note that one cannot use these conditions to (uniquely) identify a pricing rule. The choice of pricing rules is not unique. In the next subsection we will study necessary and sufficient conditions for the existence of an equilibrium for a wide class of pricing rules. Before that we have the following.

Proposition 12 *Consider an admissible triple (H, λ, X) . If (H, λ, X) is a local equilibrium, we have:*

- (i) $H(\tau, \xi_\tau) = V_{\tau-}$ a.s.
- (ii) $\partial_1 H(t, \xi_t) + \frac{1}{2} \partial_{22} H(t, \xi_t) \lambda^2(t) \mathbb{E}(\sigma_Z^2(t) - \sigma_M^2(t) | \mathcal{F}_t) = 0$ a.s. on $[0, \tau)$.

Proof. (i) It is just Proposition 8. (ii) By using Itô's formula on $\frac{H(t, \xi_t)}{\lambda(t)}$, with (A2) applied, we have

$$\begin{aligned}
& \mathbb{E} \left(\int_t^\tau \frac{1}{\lambda(s)} \partial_2 H(s, \xi_{s-}) d\xi_s \middle| \mathcal{H}_t \right) \\
&= \mathbb{E} \left(\frac{H(\tau, \xi_\tau)}{\lambda(\tau)} \middle| \mathcal{H}_t \right) - \frac{H(t, \xi_t)}{\lambda(t)} \\
&- \mathbb{E} \left(\int_t^\tau \left(-\frac{\lambda'(s)}{\lambda^2(s)} H(s, \xi_s) + \frac{\partial_1 H(s, \xi_s)}{\lambda(s)} + \frac{1}{2} \partial_{22} H(s, \xi_s) \lambda(s) \sigma_Y^2(s) \right) ds \middle| \mathcal{H}_t \right) \\
&- \mathbb{E} \left(\sum_{t \leq s \leq \tau} \left(\frac{\Delta H(s, \xi_s)}{\lambda(s)} - \partial_2 H(s, \xi_{s-}) \Delta X_s \right) \middle| \mathcal{H}_t \right),
\end{aligned}$$

where $\sigma_Y^2(s) := \frac{d[Y, Y]_s^c}{ds}$. Now X is locally optimal, given (H, λ) , by the equation (8) and the Proposition 8 we can write:

$$\begin{aligned}
0 &= V_t - \lambda(t) \mathbb{E} \left(\frac{V_\tau}{\lambda(\tau)} \middle| \mathcal{H}_t \right) \\
&+ \lambda(t) \int_t^\tau \mathbb{E} \left(-\frac{\lambda'(s)}{\lambda^2(s)} H(s, \xi_s) + \frac{\partial_1 H(s, \xi_s)}{\lambda(s)} + \frac{1}{2} \partial_{22} H(s, \xi_s) \lambda(s) \sigma_Y^2(s) \middle| \mathcal{H}_t \right) ds \\
&+ \lambda(t) \sum_{t \leq s \leq \tau} \mathbb{E} \left(\frac{\Delta H(s, \xi_s)}{\lambda(s)} - \partial_2 H(s, \xi_s) \Delta X_s \middle| \mathcal{H}_t \right) \\
&- \lambda(t) \int_t^\tau \mathbb{E} (\lambda(s) \partial_{22} H(s, \xi_s) (\sigma_M^2(s) + \sigma_{Z, M}(s)) ds \middle| \mathcal{H}_t).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
0 &= \frac{V_t}{\lambda(t)} - \frac{V_t}{\lambda(\tau)} \\
&+ \int_t^\tau \mathbb{E} \left(-\frac{\lambda'(s)}{\lambda^2(s)} H(s, \xi_s) + \frac{\partial_1 H(s, \xi_s)}{\lambda(s)} + \frac{1}{2} \partial_{22} H(s, \xi_s) \lambda(s) (\sigma_Y^2(s) - 2\sigma_{M, Y}(s)) \middle| \mathcal{H}_t \right) ds \\
&+ \sum_{t \leq s \leq \tau} \mathbb{E} \left(\frac{\Delta H(s, \xi_s)}{\lambda(s)} - \partial_2 H(s, \xi_s) \Delta X_s \middle| \mathcal{H}_t \right). \tag{11}
\end{aligned}$$

where $\sigma_{M, Y}(t) := \frac{d[M, Y]_t}{dt} = \sigma_M^2(t) + \sigma_{M, Z}(t)$. By taking increments of the different terms of the previous expression when we have an infinitesimal increment of time, we can identify the predictive and martingale parts. In fact

$$d \left(\frac{V_t}{\lambda(t)} - \frac{V_t}{\lambda(\tau)} \right) = -\frac{\lambda'(t)}{\lambda^2(t)} V_t dt + \left(\frac{1}{\lambda(t)} - \frac{1}{\lambda(\tau)} \right) dV_t,$$

$$\begin{aligned}
& d \int_t^\tau \mathbb{E} \left(-\frac{\lambda'(s)}{\lambda^2(s)} H(s, \xi_s) + \frac{\partial_1 H(s, \xi_s)}{\lambda(s)} + \frac{1}{2} \partial_{22} H(s, \xi_s) \lambda(s) (\sigma_Y^2(s) - 2\sigma_{M, Y}(s)) \middle| \mathcal{H}_t \right) ds \\
&= \left(\frac{\lambda'(t)}{\lambda^2(t)} H(t, \xi_t) - \frac{\partial_1 H(t, \xi_t)}{\lambda(t)} - \frac{1}{2} \partial_{22} H(t, \xi_t) \lambda(t) (\sigma_Y^2(t) - 2\sigma_{M, Y}(t)) \right) dt + dM_t^{(t)},
\end{aligned}$$

where, for fixed u , $M^{(u)}$ is the martingale

$$M_t^{(u)} := \mathbb{E} \left(\int_u^\tau \left(-\frac{\lambda'(s)}{\lambda^2(s)} H(s, \xi_s) + \frac{\partial_1 H(s, \xi_s)}{\lambda(s)} + \frac{1}{2} \partial_{22} H(s, \xi_s) \lambda(s) (\sigma_Y^2(s) - 2\sigma_{M,Y}(s)) \right) ds \middle| \mathcal{H}_t \right), t \geq 0$$

and

$$\begin{aligned} & d \sum_{t \leq s \leq \tau} \mathbb{E} \left(\frac{\Delta H(s, \xi_s)}{\lambda(s)} - \partial_2 H(s, \xi_s) \Delta X_s \middle| \mathcal{H}_t \right) \\ &= -\frac{\Delta H(t, \xi_t) - \partial_2 H(t, \xi_t) \Delta \xi_t}{\lambda(t)} + dL_t^{(t)}, \end{aligned}$$

with

$$L_t^{(u)} := \mathbb{E} \left(\sum_{u \leq s \leq \tau} \frac{\Delta H(s, \xi_s)}{\lambda(s)} - \partial_2 H(s, \xi_s) \Delta X_s \middle| \mathcal{H}_t \right).$$

Therefore we have that

$$\begin{aligned} 0 &= \frac{\lambda'(t)}{\lambda^2(t)} V_t - \frac{\lambda'(t)}{\lambda^2(t)} H(t, \xi_t) + \frac{\partial_1 H(t, \xi_t)}{\lambda(t)} + \frac{1}{2} \partial_{22} H(t, \xi_t) \lambda(t) (\sigma_Y^2(t) - 2\sigma_{M,Y}(t)) \\ &+ \frac{\Delta H(t, \xi_t) - \partial_2 H(t, \xi_t) \Delta \xi_t}{\lambda(t)}, \text{ a.s. on } [0, \tau]. \end{aligned} \quad (12)$$

Then a.s on $[0, \tau]$, the continuous and jump parts of the r.h.s of the previous equation will be equal to zero.

So

$$\frac{\Delta H(t, \xi_t) - \partial_2 H(t, \xi_t) \Delta \xi_t}{\lambda(t)} = 0, \text{ a.s. on } [0, \tau] \quad (13)$$

and

$$0 = \frac{\lambda'(t)}{\lambda^2(t)} V_t - \frac{\lambda'(t)}{\lambda^2(t)} H(t, \xi_t) + \frac{\partial_1 H(t, \xi_t)}{\lambda(t)} + \frac{1}{2} \partial_{22} H(t, \xi_t) \lambda(t) (\sigma_Y^2(t) - 2\sigma_{M,Y}(t)). \quad (14)$$

Now, since we are in a local equilibrium, prices are rational, given X , so by taking conditional expectations w.r.t \mathcal{F}_t we have

$$\begin{aligned} 0 &= \frac{\lambda'(t)}{\lambda^2(t)} (\mathbb{E}(V_t | \mathcal{F}_t) - \mathbb{E}(H(t, \xi_t) | \mathcal{F}_t)) + \frac{\partial_1 H(t, \xi_t)}{\lambda(t)} + \frac{1}{2} \partial_{22} H(t, \xi_t) \lambda(t) \mathbb{E} (\sigma_Y^2(t) - 2\sigma_{M,Y}(t) | \mathcal{F}_t) \\ &= \frac{\partial_1 H(t, \xi_t)}{\lambda(t)} + \frac{1}{2} \partial_{22} H(t, \xi_t) \lambda(t) (\sigma_Y^2(t) - 2\mathbb{E}(\sigma_{M,Y}(t) | \mathcal{F}_t)). \end{aligned} \quad (15)$$

■

Proposition 13 *If the pricing rule $H(t, \cdot)$ is linear, for any t , or the optimal strategy X is absolutely*

continuous, then we have: (i) Y is a local martingale (ii) If $V_t \neq P_t$ a.s. on $[0, \tau)$, then $\lambda(t) = \lambda_0$,

Proof. (i) In those cases, from (13) and (15) we have

$$dP_t = dH(t, \xi_t) = \lambda(t)\partial_2 H(t, \xi_{t-})dY_t,$$

and, since P is a martingale and $\lambda(t)\partial_2 H(t, y) > 0$, we have that Y is a local martingale. (ii) From (13) and (15) we have that

$$\frac{\lambda'(t)}{\lambda^2(t)}V_t - \frac{\lambda'(t)}{\lambda^2(t)}H(t, \xi_t) = 0,$$

then $V_t \neq H(t, \xi_t)$ on $[0, \tau)$ implies that $\lambda'(t) = 0$. ■

3.2 Characterization of the equilibrium

In this subsection we shall give necessary and sufficient conditions to guarantee that (H, λ, X) is an equilibrium in the context of pricing rules satisfying

$$0 = \partial_1 H(t, y) + \frac{1}{2}\partial_{22} H(t, y)\lambda(t)^2\sigma_Z^2(t), \text{ a.a. } t \geq 0, y \in \mathbb{R}. \quad (16)$$

Note that this condition is close to condition (ii) in Proposition 12, that is a necessary condition for the equilibrium. We shall also assume that $\sigma_Z^2(t)$ is deterministic and that V is continuous. Then, when the release time τ is known and independent of Z , we have the following necessary and sufficient conditions for the equilibrium:

Theorem 14 Consider an admissible triple (H, λ, X) with (H, λ) satisfying (16). If

- (i) $\lambda(t) = \lambda_0$,
- (ii) $H(\tau, \xi_\tau) = V_\tau$
- (iii) $[X^c, X^c] \equiv 0$,
- (iv) $Y = X + Z$ is an \mathbb{F} -local martingale without jumps ,

then (H, λ, X) is an equilibrium. If $V_t \neq P_t$ for all $t \leq \tau$, the conditions (i)-(iv) above are also necessary.

Proof. Assume (i)-(iv). The proof follows the same steps as in Corcuera et al. (2014). Set, for $T \in [0, \infty)$,

$$i(y, v, T)(\omega) := \int_y^{H^{-1}(T, \cdot)(v)} \frac{v - H(T, x)}{\lambda_0} dx,$$

and

$$I(t \wedge \tau, y, v) := E(i(y + \lambda_0(Z_\tau - Z_{t \wedge \tau}), v, \tau) | \tau), \quad t \geq 0.$$

First note that

$$\mathbb{E}(H(\tau, y + \lambda_0(Z_\tau - Z_{t \wedge \tau})) | \tau) = H(t \wedge \tau, y).$$

In fact, by (16) and (A2) (also for $X \equiv 0$), $(H(t \wedge \tau, \lambda_0 Z_{t \wedge \tau}))_{t \geq 0}$ is an \mathbb{H} -martingale, so, since Z and τ are independent, Z has independent increments, and τ is bounded, we have that,

$$H(t \wedge \tau, y) = \mathbb{E}(H(\tau, \lambda_0 Z_\tau) | \lambda_0 Z_{t \wedge \tau} = y, \tau) = \mathbb{E}(H(\tau, y + \lambda_0(Z_\tau - Z_{t \wedge \tau})) | \tau), \text{ for all } t \geq 0.$$

$(I(t \wedge \tau, Z_{t \wedge \tau}, v))_{t \geq 0}$ is also an \mathbb{H} -martingale. In fact, since Z and τ are independent and Z has independent increments:

$$\begin{aligned} I(t \wedge \tau, y, v) &= \mathbb{E}(i(y + \lambda_0(Z_\tau - Z_{t \wedge \tau}), v, \tau) | \tau) \\ &= \mathbb{E}(i(\lambda_0 Z_\tau, v, \tau) | \lambda_0 Z_{t \wedge \tau} = y, \tau), \end{aligned}$$

and we have that

$$\begin{aligned} \partial_2 I(t \wedge \tau, y, v) &= \mathbb{E}(\partial_1 i(y + \lambda_0(Z_\tau - Z_{t \wedge \tau}), v, \tau) | \tau) \\ &= \mathbb{E}\left(-\frac{v - H(\tau, y + \lambda_0(Z_\tau - Z_{t \wedge \tau}))}{\lambda_0} \Big| \tau\right) = -\frac{v - H(t \wedge \tau, y)}{\lambda_0}. \end{aligned} \quad (17)$$

We can take the derivative under the integral sign because $H(\tau(\omega), \cdot)$ is monotone and $\mathbb{E}(H(\tau, \lambda_0 Z_\tau)) < \infty$ and, from (16) we obtain

$$\partial_{12} I + \frac{1}{2} \partial_{222} I \lambda_0^2 \sigma_Z^2(t) = 0$$

so

$$\partial_1 I + \frac{1}{2} \partial_{22} I \lambda_0^2 \sigma_Z^2(t) = C(t, v),$$

where $C(t, v)$ is a function depending only on t and v . Now since $(I(t \wedge \tau, Z_{t \wedge \tau}, v))_{t \geq 0}$ is a martingale, it turns out that $C(t, v) = 0$ a.a. $t \geq 0$. Then we obtain that

$$\partial_1 I + \frac{1}{2} \partial_{22} I \lambda_0^2 \sigma_Z^2(t) = 0. \quad (18)$$

Now, consider any admissible strategy X , by using Itô's formula, we have

$$\begin{aligned}
I(\tau, \xi_\tau, V_\tau) &= I(0, 0, V_0) + \int_0^\tau \partial_3 I(t, \xi_t, V_t) dV_t + \int_0^\tau \partial_1 I(t, \xi_t, V_t) dt \\
&\quad + \int_0^\tau \partial_2 I(t, \xi_{t-}, V_t) d\xi_t + \frac{1}{2} \int_0^\tau \partial_{22} I(t, \xi_t, V_t) d[\xi^c, \xi^c]_t \\
&\quad + \int_0^\tau \partial_{23} I(t, \xi_t, V_t) d[\xi^c, V]_t + \frac{1}{2} \int_0^\tau \partial_{33} I(t, \xi_t, V_t) \sigma_V^2 dt \\
&\quad + \sum_{0 \leq t \leq \tau} (\Delta I(t, \xi_t, V_t) - \partial_2 I(t, \xi_{t-}, V_t) \Delta \xi_t).
\end{aligned}$$

By construction, $\xi_0 = 0$, by (i) $d\xi_t = \lambda_0 dY_t$. Now we have that

$$d[\xi^c, \xi^c]_t = \lambda_0^2 d[X^c, X^c]_t + 2\lambda_0^2 d[X^c, Z]_t + \lambda_0^2 \sigma_Z^2(t) dt.$$

Also by (17) and the fact that V and Z are independent,

$$\partial_{23} I(t, \xi_t, V_t) d[\xi^c, V]_t = -\frac{1}{\lambda_0} d[\xi^c, V]_t = -d[X, V]_t,$$

then using (17) and (18), and the fact that Z has not jumps, we get

$$\begin{aligned}
I(\tau, \xi_\tau, V_\tau) &= I(0, 0, V_0) + \int_0^\tau \partial_3 I(t, \xi_t, V_t) dV_t + \int_0^\tau (P_{t-} - V_t) (dX_t + dZ_t) \\
&\quad + \frac{1}{2} \int_0^\tau \partial_{22} I(t, \xi_t, V_t) \lambda_0^2 d[X^c, X^c]_t - [X, V]_\tau + \frac{1}{2} \int_0^\tau \partial_{33} I(t, \xi_t, V_t) \sigma_V^2 dt \\
&\quad + \int_0^\tau \partial_{22} I(t, \xi_t, V_t) \lambda_0^2 d[X^c, Z^c]_t + \sum_{0 \leq t \leq \tau} (\Delta I(t, \xi_t, V_t) - \partial_2 I(t, \xi_{t-}, V_t) \lambda_0 \Delta X_t)
\end{aligned}$$

Subtracting $[P, X]_\tau$ from both sides and rearranging the terms, we obtain

$$\begin{aligned}
&\int_0^\tau (V_t - P_{t-}) dX_t - [P, X]_\tau + [X, V]_\tau - \left(I(0, 0, V_0) + \frac{1}{2} \int_0^\tau \partial_{33} I(t, \xi_t, V_t) \sigma_V^2 dt \right) \\
&= -I(\tau, \xi_\tau, V_\tau) + \int_0^\tau \partial_3 I(t, \xi_{t-}, V_t) dV_t + \int_0^\tau (P_t - V_t) dZ_t \\
&\quad + \frac{1}{2} \int_0^\tau \partial_{22} I(t, \xi_t, V_t) \lambda_0^2 d[X^c, X^c]_t + \int_0^\tau \partial_{22} I(t, \xi_t, V_t) \lambda_0^2 d[X^c, Z^c]_t \\
&\quad + \sum_{0 \leq t \leq \tau} (\Delta I(t, \xi_t, V_t) - \partial_2 I(t, \xi_{t-}, V_t) \lambda_0 \Delta X_t) - [P, X]_\tau.
\end{aligned} \tag{19}$$

We have that

$$[P, X]_\tau = [P^c, X^c]_\tau + \sum_{0 \leq t \leq \tau} \Delta P_t \Delta X_t.$$

Then Itô's formula for H shows that the continuous local martingale part of P is $\int \frac{\partial H}{\partial y}(t, \xi_t) d\xi_t^c$, so by using (17), we obtain

$$\begin{aligned} [P^c, X^c]_\tau &= \left[\int_0^\tau \partial_1 H(t, \xi_t) d\xi_t^c, X^c \right]_\tau = \int_0^\tau \partial_1 H(t, \xi_t) d[\xi^c, X^c]_t \\ &= \int_0^\tau \partial_{22} I(t, \xi_t, V_t) \lambda_0^2 d[X^c, X^c]_t + \int_0^\tau \partial_{22} I(t, \xi_t, V_t) \lambda_0^2 d[X^c, Z]_t, \end{aligned}$$

and

$$\begin{aligned} \lambda_0 \partial_2 I(t, \xi_{t-}, V_t) \Delta X_t + \Delta P_t \Delta X_t &= (P_{t-} - V_t) \Delta X_t + \Delta P_t \Delta X_t \\ &= (P_t - V_t) \Delta X_t = \lambda_0 \partial_2 I(t, \xi_t, V_t) \Delta X_t. \end{aligned}$$

Substituting the above relationships in the right-hand side of the equation (19), it becomes

$$\begin{aligned} &-I(\tau, \xi_\tau, V_\tau) + \int_0^\tau \partial_3 I(t, \xi_t, V_t) dV_t + \int_0^\tau (P_t - V_t) dZ_t - \frac{1}{2} \int_0^\tau \partial_{22} I(t, \xi_t, V_t) \lambda_0^2 d[X^c, X^c]_t \\ &+ \sum_{0 \leq t \leq \tau} (I(t, \xi_t, V_t) - I(t, \xi_{t-}, V_t) - \lambda_0 \partial_2 I(t, \xi_t, V_t) \Delta X_t). \end{aligned}$$

Now it is important to note that $\partial_{33} I(t, y, v)$ does not depend on y and so $\partial_{33} I(t, \xi_t, V_t)$ does not depend on ξ . Then $I(0, 0, V_0) + \frac{1}{2} \int_0^\tau \partial_{33} I(t, \xi_t, V_t) \sigma_V^2 dt$ is actually fixed ω , a lower bound for *any* strategy. Then we will show that, taken the expectation, the right-hand side of (19) is non-positive. The result follows from the following points.

1. We know that $\lambda_0 \partial_{22} I(\tau, \xi_\tau, V_\tau) = \partial_2 H(\tau, \xi_\tau) > 0$ and that $\lambda_0 \partial_2 I(\tau, \xi_\tau, V_\tau) = -V_\tau + H(\tau, \xi_\tau)$ so by hypothesis (ii) we have a maximum value of $-I(\tau, \xi_\tau, V_\tau)$ for our strategy.
2. The processes $\int_0^\cdot \partial_3 I(t, \xi_t, V_t) dV_t$ and $\int_0^\cdot (P_t - V_t) dZ_t$ are \mathbb{H} -martingales, hence they have null expectation.
3. By (17) and H being increasing monotone, we have that $\partial_{22} I > 0$, and the measure $d[X^c, X^c] \geq 0$, so

$$-\frac{1}{2} \int_0^\tau \partial_{22} I(t, \xi_t, V_t) \lambda_0^2 d[X^c, X^c]_t \leq 0,$$

and by hypothesis (iv) we obtain the maximum value for our strategy.

4. $\partial_{22} I > 0$ (convexity) implies that

$$I(t, x+h, v) - I(t, x, v) - \partial_2 I(t, x+h, v)h \leq 0.$$

So,

$$\sum_{0 \leq t \leq \tau} (I(t, \xi_{t-} + \lambda_0 \Delta X_t, V_t) - I(t, \xi_{t-}, V_t) - \partial_2 I(t, \xi_t, V_t) \lambda_0 \Delta X_t) \leq 0,$$

and has its maximum if and only if $\Delta X_t = 0$, which is assumed at *(iv)*.

5. Assumption *(iv)* and *(ii)* together with condition (A2) guarantee the rationality of prices. In fact from condition (A2) and (16) we have that $H(\cdot \wedge \tau, \xi_{\cdot \wedge \tau})$ is an \mathbb{F} -martingale, then from *(ii)*, and on the set $\{t \leq \tau\}$ we have

$$\mathbb{E}(H(\tau, \xi_\tau) | \mathcal{F}_t) = \mathbb{E}(V_\tau | \mathcal{F}_t) = \mathbb{E}(\mathbb{E}(V_\tau | \mathcal{H}_t) | \mathcal{F}_t) = \mathbb{E}(V_t | \mathcal{F}_t).$$

Conversely, if (H, λ, X) is an equilibrium, by point 3. $\sigma_M^2 = 0$ and now by (14)

$$0 = \frac{\lambda'(t)}{\lambda^2(t)} V_t - \frac{\lambda'(t)}{\lambda^2(t)} H(t, \xi_t),$$

so if $V_t \neq P_t$, we have that $\lambda(t) = \lambda_0$. Condition *(ii)* is *(i)* in Proposition 12. Also from (16), (13) and (15)

$$dP_t = dH(t, \xi_t) = \lambda_0 \partial_2 H(t, \xi_{t-}) dY_t,$$

and, since P is an \mathbb{F} -martingale and $\lambda_0 \partial_2 H(t, y) > 0$, we have that Y is an \mathbb{F} -local martingale. Finally from point 4, $\Delta X_t = 0$ and $Y = X + Z$ is a local martingale without jumps. ■

4 Case when τ is unknown to the insider

In this section we consider the case when the insider does not know the precise time τ of release of information. Namely, the insider's information flow is given by:

$$\mathcal{H}_t = \sigma(P_s, \eta_s, \tau \wedge s, 0 \leq s \leq t).$$

Moreover we assume that τ finite is independent of (V, P, Z) , that $\mathbb{P}(\tau > t) > 0$ for all $t \geq 0$ and that τ has a density. Going back to Proposition 6, we can see that, on $[0, \tau]$, equation (8) can be written as:

$$V_t - H(t, \xi_t) - \lambda(t) \mathbb{E} \left(\int_t^\infty \mathbf{1}_{[0, \tau]}(s) (\partial_2 H(s, \xi_s) d^- X_s) | \mathcal{H}_t \right) = 0.$$

Here we recall that the optimal total demand X for the insider satisfies (A1) - (A6). Then we have

$$\begin{aligned}
0 &= V_t - H(t, \xi_t) - \lambda(t) \mathbb{E} \left(\int_t^\infty \mathbb{P}(\tau > s | \mathcal{H}_t) (\partial_2 H(s, \xi_s) d^- X_s) \middle| \mathcal{H}_t \right) \\
&= V_t - H(t, \xi_t) - \frac{\lambda(t)}{\mathbb{P}(\tau > t)} \mathbb{E} \left(\int_t^\infty \mathbb{P}(\tau > s) (\partial_2 H(s, \xi_s) d^- X_s) \middle| \mathcal{H}_t \right) \\
&= V_t - H(t, \xi_t) - \frac{\lambda(t)}{\mathbb{P}(\tau > t)} \mathbb{E} \left(\int_t^\infty \mathbb{P}(\tau > s) \partial_2 H(s, \xi_s) \theta_s ds \middle| \mathcal{H}_t \right) \\
&\quad - \frac{\lambda(t)}{\mathbb{P}(\tau > t)} \mathbb{E} \left(\sum_t^\infty \mathbb{P}(\tau > s) \partial_2 H(s, \xi_s) \Delta X_s \middle| \mathcal{H}_t \right) \\
&\quad - \frac{\lambda(t)}{\mathbb{P}(\tau > t)} \mathbb{E} \left(\int_t^\infty \lambda(s) \mathbb{P}(\tau > s) \partial_{22} H(s, \xi_s) (\sigma_M^2(s) + \sigma_{Z,M}(s)) ds \middle| \mathcal{H}_t \right) \text{ on } [0, \tau]. \tag{20}
\end{aligned}$$

First of all we note that, by assumption (A3), and Corollary (2.4) in Revuz and Yor (1999) we have that

$$\lim_{t \rightarrow \infty} \mathbb{E} \left(\int_t^\infty \mathbb{P}(\tau > s) \partial_2 H(s, \xi_s) |\theta_s| ds \middle| \mathcal{H}_t \right) = 0, \quad \text{a.s.}$$

Analogously for $\mathbb{E} \left(\sum_t^\infty \mathbb{P}(\tau > s) \partial_2 H(s, \xi_{s-}) |\Delta X_s| \middle| \mathcal{H}_t \right)$ and $\mathbb{E} \left(\int_t^\infty \lambda(s) \mathbb{P}(\tau > s) \partial_{22} H(s, \xi_s) (\sigma_M^2(s) + \sigma_{Z,M}(s)) ds \middle| \mathcal{H}_t \right)$.

Then, from (20), we have that

$$\lim_{t \rightarrow \infty} \frac{(V_t - H(t, \xi_t)) \mathbb{P}(\tau > t)}{\lambda(t)} = 0, \quad \text{a.s.} \tag{21}$$

Applying the Itô's formula to $\frac{H(t, \xi_t) \mathbb{P}(\tau > t)}{\lambda(t)}$, $t \leq T$, and studying the limit for $T \rightarrow \infty$, we have

$$\begin{aligned}
&\mathbb{E} \left(\int_t^\infty \mathbb{P}(\tau > s) \partial_2 H(s, \xi_{s-}) dX_s \middle| \mathcal{H}_t \right) \\
&= \lim_{T \rightarrow \infty} \mathbb{E} \left(\frac{H(T, \xi_T) \mathbb{P}(\tau > T)}{\lambda(T)} \middle| \mathcal{H}_t \right) - \frac{H(t, \xi_t) \mathbb{P}(\tau > t)}{\lambda(t)} \\
&\quad - \mathbb{E} \left(\int_t^\infty \left(\partial_s \left(\frac{\mathbb{P}(\tau > s)}{\lambda(s)} \right) H(s, \xi_s) + \frac{\mathbb{P}(\tau > s)}{\lambda(s)} \partial_1 H(s, \xi_s) \right. \right. \\
&\quad \left. \left. + \frac{1}{2} \partial_{22} H(s, \xi_s) \mathbb{P}(\tau > s) \lambda(s) \sigma_Y^2(s) \right) ds \middle| \mathcal{H}_t \right) \\
&\quad - \mathbb{E} \left(\sum_t^\infty \frac{\mathbb{P}(\tau > s) \Delta H(s, \xi_s)}{\lambda(s)} - \mathbb{P}(\tau > s) \partial_2 H(s, \xi_{s-}) \Delta X_s \middle| \mathcal{H}_t \right). \tag{22}
\end{aligned}$$

Moreover, by (21), we have

$$\begin{aligned}
\lim_{T \rightarrow \infty} \mathbb{E} \left(\frac{H(T, \xi_T) \mathbb{P}(\tau > T)}{\lambda(T)} \middle| \mathcal{H}_t \right) &= \lim_{T \rightarrow \infty} \mathbb{E} \left(\frac{V_T \mathbb{P}(\tau > T)}{\lambda(T)} \middle| \mathcal{H}_t \right) \\
&= V_t \lim_{T \rightarrow \infty} \frac{\mathbb{P}(\tau > T)}{\lambda(T)} := cV_t, \tag{23}
\end{aligned}$$

where we assume that $\lim_{T \rightarrow \infty} \frac{\mathbb{P}(\tau > T)}{\lambda(T)} = c < \infty$. By substituting (22) and (23) into (20), we obtain the equation

$$\begin{aligned}
0 &= V_t \left(c - \frac{\mathbb{P}(\tau > t)}{\lambda(t)} \right) - \mathbb{E} \left(\int_t^\infty \left(\partial_s \left(\frac{\mathbb{P}(\tau > s)}{\lambda(s)} \right) H(s, \xi_s) \right. \right. \\
&\quad \left. \left. + \frac{\mathbb{P}(\tau > s)}{\lambda(s)} \partial_1 H(s, \xi_s) + \frac{1}{2} \partial_{22} H(s, \xi_s) \mathbb{P}(\tau > s) \lambda(s) (\sigma_Y^2(s) - 2\sigma_{M,Y}(s)) \right) ds \middle| \mathcal{H}_t \right) \\
&\quad - \mathbb{E} \left(\sum_t^\infty \frac{\mathbb{P}(\tau > s) \Delta H(s, \xi_s)}{\lambda(s)} - \mathbb{P}(\tau > s) \partial_2 H(s, \xi_s) \Delta X_s \middle| \mathcal{H}_t \right). \tag{24}
\end{aligned}$$

In the same way we did for the stochastic process appearing in the r.h.s. of the equation (11) we can identify the predictive and martingale parts and we will obtain that

$$\begin{aligned}
0 &= \partial_t \left(\frac{\mathbb{P}(\tau > t)}{\lambda(t)} \right) (V_t - H(t, \xi_t)) + \\
&\quad + \frac{\mathbb{P}(\tau > t)}{\lambda(t)} \partial_1 H(t, \xi_t) + \frac{1}{2} \partial_{22} H(t, \xi_t) \mathbb{P}(\tau > t) \lambda(t) (\sigma_Y^2(t) - 2\sigma_{M,Y}(t)) \\
&\quad + \left(\frac{\mathbb{P}(\tau > t) \Delta H}{\lambda(t)} - \frac{\mathbb{P}(\tau > t)}{\lambda(t)} \partial_2 H(t, \xi_t) \Delta \xi_t \right). \tag{25}
\end{aligned}$$

Now since we are in a local equilibrium prices are rational and by taking conditional expectations w.r.t \mathcal{F}_t , we obtain

$$\begin{aligned}
0 &= \frac{\mathbb{P}(\tau > t)}{\lambda(t)} \partial_1 H(t, \xi_t) + \frac{1}{2} \partial_{22} H(t, \xi_t) \mathbb{P}(\tau > t) \lambda(t) (\sigma_Y^2(t) - 2\mathbb{E}(\sigma_{M,Y}(t) | \mathcal{F}_t)) \\
&\quad + \left(\frac{\mathbb{P}(\tau > t) \Delta H}{\lambda(t)} - \frac{\mathbb{P}(\tau > t)}{\lambda(t)} \partial_2 H(t, \xi_t) \Delta \xi_t \right). \tag{26}
\end{aligned}$$

So we have proved the following results:

Proposition 15 *Consider an admissible triple (H, λ, X) . Assume that $\lim_{t \rightarrow \infty} \frac{\mathbb{P}(\tau > t)}{\lambda(t)} = c < \infty$. If (H, λ, X) is a local equilibrium, we have:*

$$\partial_1 H(t, \xi_t) + \frac{1}{2} \partial_{22} H(t, \xi_t) \lambda^2(t) \mathbb{E}(\sigma_Z^2(t) - \sigma_M^2(t) | \mathcal{F}_t) = 0 \quad \text{a.s. on } [0, \tau].$$

Proposition 16 *Consider an admissible triple (H, λ, X) . If (H, λ, X) is a local equilibrium, $\sigma_Z^2(t)$ is deterministic and satisfies (16) and $\lim_{t \rightarrow \infty} \frac{\mathbb{P}(\tau > t)}{\lambda(t)}$, we have:*

$$(i) \ Y \text{ is a local martingale} \tag{27}$$

$$(ii) \ \text{If } V_t \neq P_t \text{ a.s. on } [0, \tau), \text{ then } \lambda(t) = c\mathbb{P}(\tau > t), \quad \text{a.a.t } t \geq 0 \quad (c > 0). \tag{28}$$

Remark 17 Analogously to the Proposition 13, we have that the same result is true when $\sigma_Z^2(t)$ is not deterministic but $H(t, \cdot)$ is linear or the strategies are absolutely continuous, in both cases (27) and (28) are also true.

Remark 18 Here we can draw conclusions similar to the one in Cho (2003) where he considers a risk-averse insider (and a deterministic release time). Cho concludes that, in equilibrium, a risk-averse insider would do most of her trading early to avoid the risk that the prices get closer to the asset value, unless the trading conditions become more favorable over time. Similarly in our case, when the (risk-neutral) insider does not know the release time of information, she would trade early in order to use her piece of information before the announcement time comes. This behavior would continue unless the price pressure decreases over time providing more favorable trading also at a later time. A similar conclusion is obtained by Baruch (2002), who studies exactly the same problem about the effect of risk-aversion for the insider, by assuming that the noise trading is a Brownian motion with time varying instantaneous variance.

Example 19 We can consider the context of Caldentey and Stacchetti (2010) where the authors assume that V and Z are arithmetic Brownian motion with variances σ_V and σ_Z respectively, and τ follows an exponential distribution with scale parameter μ , independent of (V, P, Z) . Then, by Proposition 16, we have that, for a.a. t and a.a. $\omega \in \{t < \tau\}$,

$$V_t - H(t, \xi_t) - \lambda(t) \mathbb{E} \left(\int_t^\infty e^{-\mu(s-t)} \partial_2 H(s, \xi_s) dX_s \middle| \mathcal{H}_t \right) = 0.$$

And to have a local equilibrium, provided that $V_t - H(t, \xi_t) \neq 0$, we need $\lambda(t) = \lambda_0 e^{-\mu t}$.

5 Explicit insider's optimal strategies

In this section we shall apply our results to explicitly find the insider's optimal strategy in equilibrium. We will show how our general framework serves different models known in the literature presented as extensions of the Kyle-Back model. In order to perform the explicit computations we will use techniques of enlargements of filtrations.

To explain how enlargement of filtration enters the topic we consider a total demand $Y = Z + X$ in equilibrium given by:

$$Y_t = Z_t + \int_0^t \theta(\eta_t; Y_u, 0 \leq u \leq s) ds, \quad 0 \leq t \leq T. \quad (29)$$

Here X is absolutely continuous process with respect to the Lebesgue measure. We recall that Z is perceived by the insider as an \mathbb{H} -martingale independent of $V. = E(f(\eta_\tau)|\mathcal{H}.)$ and η . So since $\mathbb{F}^{Y,\eta} \subseteq \mathbb{H}$ and Z is adapted to $\mathbb{F}^{Y,\eta}$, it is also an $\mathbb{F}^{Y,\eta}$ -martingale. Here $\mathbb{F}^{Y,\eta} = (\mathcal{F}_t^{Y,\eta})_{t \geq 0}$ is the filtration $\mathcal{F}_t^{Y,\eta} := \bar{\sigma}(Y_s, \eta_s, 0 \leq s \leq t)$. On the other hand, Y is a local martingale when in equilibrium, as for the cases of Theorem 14, Proposition 12 and Proposition 16. Consequently (29) becomes the Doob-Meyer decomposition of Y when we enlarge the filtration \mathbb{F}^Y with the process η . We are then into a problem of *enlargement of filtrations*. However, in our problem Z is fixed in advance and we want to obtain Y as a function of Z , fixed η , so we look in fact for *strong* solutions of (29), whereas the results on enlargement of filtrations provide weak solutions. In this sense the celebrated Yamada-Watanabe's theorem is the result, when Z is Gaussian, that can be used to obtain strong solutions from weak solutions. See, for instance, Theorem 1.5.4.4. in Jeanblanc *et al.* (2009). These various examples correspond to different models that are extensions of the Kyle-Back model and where the results about enlargement of filtrations can be applied, but we do not enter, however, into details to show that the solutions of the corresponding stochastic differential equations appearing in the equilibrium equation, are in fact strong solutions.

Example 20 (*Back (1992)*) Assume that Z is a Brownian motion with variance σ^2 , $V. \equiv V_1$ and the release time $\tau = 1$. In equilibrium, if the strategy of the insider is optimal $V_1 = H(1, Y_1)$. Since $H(1, \cdot)$ can be chosen freely because it is the boundary condition of equation (16) and if V_1 has a continuous cumulative distribution function, we can assume w.l.o.g that $Y_1 \equiv N(0, \sigma^2)$. It is assumed that V_1 (and consequently Y_1) is independent of Z . Then, see Example 1, page 306, in Jeulin and Yor (1985), we have that

$$Y_t = Z_t + \int_0^t \frac{Y_1 - Y_s}{1 - s} ds,$$

is a Brownian motion with variance σ^2 . Hence, prices are rational and we recognize the equilibrium strategy to be

$$X_t = \int_0^t \frac{Y_1 - Y_s}{1 - s} ds, \quad 0 \leq t < 1.$$

Example 21 (*Aase et al. (2012a)*) Assume that $\tau = 1$ and suppose that Z is given by

$$Z_t = \int_0^t \sigma_s dW_s$$

where σ is deterministic and $V. \equiv Y_1$ is a $N(0, \int_0^1 \sigma_s^2 ds)$ independent of Z . Then by Jeulin (1980), page 51,

$$Y_t = Z_t + \int_0^t \frac{Y_s - Y_1}{\int_t^1 \sigma_u^2 du} \sigma_s^2 ds,$$

has the same law as Z . Then

$$X_t = \int_0^t \frac{Y_s - Y_1}{\int_t^1 \sigma_u^2 du} \sigma_s^2 ds$$

is the optimal strategy.

Example 22 (Campi and Çetin (2007)) If we want the aggregate process Y to be a Brownian motion that reaches the value -1 for the first time at time $\bar{\tau}$, and Z is also a Brownian motion then, by Example 3 in Jeulin and Yor (1985), page 306,

$$Y_t = Z_t + \int_0^t \left(\frac{1}{1 + Y_s} - \frac{1 + Y_s}{\bar{\tau} - s} \right) \mathbf{1}_{[0, \bar{\tau}]}(s) ds,$$

so, in this case $\eta_t \equiv \bar{\tau}$, $V_t \equiv \mathbf{1}_{\{\bar{\tau} > 1\}}$ and the release time is $\bar{\tau} \wedge 1$.

Example 23 (Back and Pedersen (1998), Wu (1999), Danilova (2010)) The insider receives a continuous signal

$$\eta_t = \eta_0 + \int_0^t \sigma_s dW_s,$$

where σ_s is deterministic, η_0 is a zero mean normal random variable, W is a Brownian motion, both independent of the Brownian motion Z , $\tau = 1$. It is assumed that $\text{var}(\eta_1) = \text{var}(\eta_0) + \int_0^1 \sigma_s^2 ds = 1$, then

$$Y_t = Z_t + \int_0^t \frac{\eta_s - Y_s}{\text{var}(\eta_s) - s} ds, \quad 0 \leq t \leq 1.$$

is a Brownian motion. This result can be obtained by the following proposition.

Set

$$V_t = V_0 + \int_0^t \sigma_s dW_s^1, \quad 0 \leq t \leq 1,$$

where σ_s is a deterministic function, V_0 is a zero mean normal random variable, and (W^1, W^2) is a 2-dimensional Brownian motion independent of V_0 .

Proposition 24 Assume that $\text{Var}(V_1) = 1$ and that

$$\int_0^t \frac{ds}{\text{Var}(V_s) - s} < \infty \text{ for all } 0 \leq t < 1,$$

then

$$B_t = W_t^2 + \int_0^t \frac{V_s - B_s}{\text{Var}(V_s) - s} ds, \quad 0 \leq t \leq 1$$

is a Brownian motion with $B_1 = V_1$.

Proof. Denote $v_r := \text{Var}(V_r)$

$$B_t = \int_0^t \exp\left(-\int_u^t \frac{1}{v_r - r} dr\right) dW_u^2 + \int_0^t \exp\left(-\int_u^t \frac{1}{v_r - r} dr\right) \frac{V_u}{v_u - u} du,$$

so B is a centered Gaussian process, and for $s \leq t < 1$,

$$\begin{aligned} E(B_t B_s) &= \exp\left(-\int_s^t \frac{1}{v_r - r} dr\right) \\ &+ E\left(\int_0^t \int_0^s \exp\left(-\int_u^t \frac{1}{v_r - r} dr\right) \exp\left(-\int_v^s \frac{1}{v_r - r} dr\right) \frac{V_u V_v}{(v_u - u)(v_v - v)} dudv\right) \\ &= \exp\left(-\int_s^t \frac{1}{v_r - r} dr\right) \int_0^s \exp\left(-2\int_u^s \frac{1}{v_r - r} dr\right) du \\ &+ \int_s^t \int_0^s \exp\left(-\int_u^t \frac{1}{v_r - r} dr\right) \exp\left(-\int_v^s \frac{1}{v_r - r} dr\right) \frac{v_v}{(v_u - u)(v_v - v)} dudv \\ &+ 2 \int_0^s \int_0^u \exp\left(-\int_u^t \frac{1}{v_r - r} dr\right) \exp\left(-\int_v^s \frac{1}{v_r - r} dr\right) \frac{v_v}{(v_u - u)(v_v - v)} du. \end{aligned}$$

Then, since

$$\int_0^s \exp\left(-\int_v^s \frac{1}{v_r - r} dr\right) \frac{v_v}{v_v - v} dv = s,$$

and

$$2 \int_0^s \exp\left(-2\int_v^s \frac{1}{v_r - r} dr\right) \frac{v_v}{v_v - v} dv = 2s + \int_0^s \exp\left(-2\int_u^s \frac{1}{v_r - r} dr\right) du$$

we obtain that $E(B_t B_s) = s$. So for $0 \leq t < 1$ we have that (B_t) is a standard Brownian motion. On the other hand

$$\begin{aligned} E(B_t V_t) &= E\left(\int_0^t \exp\left(-\int_u^t \frac{1}{v_r - r} dr\right) \frac{V_u V_t}{v_u - u} du\right) \\ &= \int_0^t \exp\left(-\int_u^t \frac{1}{v_r - r} dr\right) \frac{v_u}{v_u - u} du \\ &= t, \end{aligned}$$

therefore

$$\begin{aligned} E((B_t - V_t)^2) &= E(B_t^2) + E(V_t^2) - 2E(B_t V_t) \\ &= t + v_t - 2t = v_t - t, \end{aligned}$$

and, since by hypothesis $v_1 = 1$, this means that

$$\lim_{t \rightarrow 1} B_t \stackrel{L^2}{=} V_1,$$

then for all $0 \leq t < 1$

$$E \left(\int_0^t \frac{|V_s - B_s|}{v_s - s} ds \right) < \int_0^t \frac{E \left((V_s - B_s)^2 \right)^{\frac{1}{2}}}{v_s - s} ds = \int_0^t \sqrt{v_s - s} ds < \sqrt{2},$$

and this implies, by the monotone convergence theorem, that

$$\lim_{t \rightarrow 1} \int_0^t \frac{|V_s - B_s|}{v_s - s} ds = \int_0^1 \frac{|V_s - B_s|}{v_s - s} ds < \infty$$

and that $B_1 = \lim_{t \rightarrow 1} B_t$ is well defined. Now, we have, by the uniqueness of the limit in probability, that $V_1 = B_1$ a.s. ■

Another view of the problem of finding the equilibrium strategy is the following. Market makers observe Y with dynamics

$$dY_t = dZ_t + \theta(V_t, Y_s, 0 \leq s \leq t) dt,$$

V is not observed. Then, the dynamics of $m_t := \mathbb{E}(V_t | \mathcal{F}_t^Y)$ can be obtained in certain cases, basically when Z and V are Gaussian diffusions, from the filtering theory, see for instance Theorem 12.1 in Liptser and Shiryaev (1978). Now we can try to deduce $\theta(V_t, Y_s, 0 \leq s \leq t)$ from the equilibrium condition: $P_t = m_t$. Even if V is not a Gaussian diffusion, but can be written in the form $V_t = h(D_t)$ where h is a strictly increasing function and D is a Gaussian diffusion, we can still apply the filtering results for the couple (Y, D) .

In the following example we use the filtering approach to find the equilibrium strategy.

Example 25 (*Caldentey and Stacchetti (2010)*) *The release time τ is unknown (so we cannot apply Proposition 24),*

$$dV_t = \sigma_v(t) dB_t^v, V_0 \sim N(P_0, \Sigma_0), \quad dZ_t = \sigma_z(t) dB_t^z, Z_0 = 0.$$

B^v and B^z being independent Brownian motions, $\sigma_v(t)$ and $\sigma_z(t)$ deterministic functions. Then, if we look for pricing rules such that

$$dP_t = \lambda_t dY_t$$

and strategies

$$dX_t = \beta_t(V_t - P_t)dt$$

with β_t deterministic, we have

$$dP_t = \lambda_t \beta_t (V_t - P_t)dt + \lambda_t \sigma_z(t) dB_t^z.$$

Let denote $m_t = E(V_t | \mathcal{F}_t^Y)$, by standard filtering results (see for instance Lipster and Shiriyayev (2001)) we have

$$dm_t = \frac{\Sigma_t \beta_t}{\lambda_t \sigma_z^2(t)} (dP_t - \lambda_t \beta_t (m_t - P_t)dt), \quad \frac{d}{dt} \Sigma_t = \sigma_v^2(t) - \frac{(\Sigma_t \beta_t)^2}{\sigma_z^2(t)},$$

where Σ_t is the filtering error. Now, we can recover the identity $P_t = m_t$, if and only if we impose $\Sigma_t \beta_t = \lambda_t \sigma_z^2(t)$ (remember that by construction $P_0 = m_0 = E(V_0)$). Then

$$\Sigma_t = \Sigma_0 + \int_0^t \sigma_v^2(s)ds - \int_0^t \sigma_z^2(s) \lambda_s^2 ds, \quad \beta_t = \frac{\lambda_t \sigma_z^2(t)}{\Sigma_t}.$$

Note that in particular we obtain that

$$Y_t = Z_t + \int_0^t \frac{\lambda_s \sigma_z^2(s) (V_s - \int_0^s \lambda_u dY_u)}{\Sigma_s} ds,$$

is the Doob-Meyer decomposition of the martingale Y in the filtration generated by (Z, V) . Now if we assume $\sigma_z^2(t) = \sigma_z^2$, independent of t , and we take into account that in the equilibrium $\lambda_t = \lambda_0 e^{-\mu t}$, we have that

$$\Sigma_t = \Sigma_0 + \int_0^t \sigma_v^2(s)ds - \sigma_z^2 \frac{\lambda_0^2}{2\mu} (1 - e^{-2\mu t}), \quad \beta_t = \frac{\sigma_z^2 \lambda_0 e^{-\mu t}}{\Sigma_t}.$$

However λ_0 is not determined. We need an additional condition to fix λ_0 . One possibility is to impose that

$$\lim_{t \rightarrow \infty} \Sigma_t = 0.$$

In such a case

$$0 = \Sigma_0 + \int_0^\infty \sigma_v^2(s)ds - \sigma_z^2 \frac{\lambda_0^2}{2\mu},$$

and

$$\lambda_0 = \sqrt{\frac{2\mu(\Sigma_0 + \int_0^\infty \sigma_v^2(s)ds)}{\sigma_z^2}}.$$

Note that if $\sigma_v^2(t) = \sigma_v^2$ there is no solution! Another possibility, according with Proposition 16, is to take T

such that

$$\Sigma_t = 0, \text{ for all } t \geq T$$

and then $P_t = V_t$ for $t \geq T$. But this implies, for $\sigma_v^2(t) = \sigma_v^2$,

$$\begin{aligned} 0 &= \Sigma_0 + \sigma_v^2 T - \sigma_z^2 \frac{\lambda_0^2}{2\mu} (1 - e^{-2\mu T}) \\ &= \Sigma_0 + \sigma_v^2 T - \sigma_z^2 \frac{\lambda_T^2}{2\mu} (e^{2\mu T} - 1). \end{aligned}$$

Now if we assume a smooth transition from the absolutely continuous strategy then $\sigma_v^2 - \sigma_z^2 \lambda_t^2 = 0$ for all $t \geq T$ and $\lambda_t = \lambda_T = \frac{\sigma_v}{\sigma_z}$, for all $t \geq T$. Finally

$$dP_t = \lambda_t dY_t = \lambda_t dX_t + \lambda_t dZ_t = dV_t, \quad t \geq T$$

so

$$dX_t = \frac{\sigma_z}{\sigma_v} dV_t - dZ_t,$$

and T is the solution of

$$\Sigma_0 + \sigma_v^2 T = \frac{\sigma_v^2}{2\mu} (e^{2\mu T} - 1).$$

This is exactly what Caldentey and Stacchetti (2010) obtain. It is important to remark that the authors obtain a limit of optimal strategies when passing from the discrete version of the model to the continuous one. This limit strategy is such that there is an endogenously determined time T such that, if $t \leq T$, then the limit strategy is absolutely continuous with respect to the Lebesgue measure and, if $t > T$, the strategy is not of bounded variation. In this case an insider's optimal strategy, between times T and τ , would yield to giving out the full information to the market by making the market prices match the fundamental value. They claim that this limit strategy is not optimal for the continuous time model and that we need to consider the discrete time model to realize about its existence. However this limit strategy can be obtained has a limit of strategies for the continuous model when we restrict the class of strategies to the set of absolutely continuous strategies and we try to maximize the wealth. In fact if we have a sequence of strategies $(X^{(n)})_{n \geq 1}$, their corresponding wealth is given by

$$W_\tau^{(n)} = X_\tau^{(n)} V_\tau^{(n)} - \int_0^\tau P_t^{(n)} dX_t^{(n)} - [P^{(n)}, X^{(n)}]_\tau.$$

Then, if we assume that $(X^{(n)}, P^{(n)}, V^{(n)}) \xrightarrow[n \rightarrow \infty]{u.c.p} (X, P, V)$ we obtain that

$$X_\tau^{(n)} V_\tau^{(n)} - \int_0^\tau P_{t-}^{(n)} dX_t^{(n)} \xrightarrow[n \rightarrow \infty]{u.c.p} X_\tau V_\tau - \int_0^\tau P_{t-} dX_t$$

but in general

$$[P^{(n)}, X^{(n)}]_\tau \not\rightarrow [P, X]_\tau,$$

For instance if $X^{(n)}$ is a bounded variation process X is not necessarily a bounded variation one. Then the gain limit for this limit of strategies after T , on the set $\{\tau > T\}$, is given by

$$\begin{aligned} V_\tau X_\tau - V_T X_T - \int_T^\tau P_{t-} dX_t &= \int_T^\tau X_{t-} dV_t + \int_T^\tau V_{t-} dX_t + \int_T^\tau d[V, X]_t - \int_T^\tau P_{t-} dX_t \\ &= \int_T^\tau (V_{t-} - P_{t-}) dX_t + \int_T^\tau d[V, X]_t + \int_T^\tau X_{t-} dV_t. \end{aligned}$$

Now if we take the (conditional) expectation, last term of the right-hand side cancels and we obtain that

$$\mathbb{E} \left(V_\tau X_\tau - V_T X_T - \int_T^\tau P_{t-} dX_t \middle| \mathcal{H}_T \right) = \mathbb{E} \left(\int_T^\tau (V_{t-} - P_{t-}) dX_t + \int_T^\tau d[V, X]_t \middle| \mathcal{H}_T \right).$$

Finally, since for the limit strategy $V_{t-} = P_{t-}$, $t > T$, in the conditions of Example 19, we obtain that there is a profit after T given by

$$\mathbb{E} \left(\int_T^\infty e^{-\mu(t-T)} d[V, X]_t \middle| \mathcal{H}_T \right) = \sigma_z \sigma_v \int_T^\infty e^{-\mu(t-T)} dt = \frac{\sigma_z \sigma_v}{\mu} > 0.$$

Now we can justify the condition $\dot{\Sigma}_T = 0$. The expected wealth for the insider with this kind of strategies is given by

$$\begin{aligned} J(X) &= \mathbb{E} \left(\int_0^{T \wedge \tau} (V_t - P_t) \theta_t dt \right) + \mathbb{E} \left(\int_{T \wedge \tau}^\tau d[V, X]_t \right) = \mathbb{E} \left(\int_0^{T \wedge \tau} \beta_t (V_t - P_t)^2 dt \right) + \mathbb{E} \left(\int_{T \wedge \tau}^\tau d[V, X]_t \right) \\ &= \mathbb{E} \left(\int_0^T \mathbf{1}_{[0, \tau]}(t) \beta_t (V_t - P_t)^2 dt \right) + \mathbb{E} \left(\int_T^\infty \mathbf{1}_{[0, \tau]}(t) d[V, X]_t \right) = \int_0^T \mathbb{P}(\tau > t) \beta_t \Sigma_t dt + \int_T^\infty \mathbb{P}(\tau > t) \frac{\sigma_v^2}{\lambda_t} dt \\ &= \int_0^T e^{-\mu t} \beta_t \Sigma_t dt + \sigma_v^2 \int_T^\infty \frac{e^{-\mu t}}{\lambda_t} dt = \sigma_z^2 \int_0^T e^{-\mu t} \lambda_t dt + \sigma_v^2 \int_T^\infty \frac{e^{-\mu t}}{\lambda_t} dt. \end{aligned}$$

Then if we impose that T is optimal, we have the condition

$$\sigma_z^2 e^{-\mu T} \lambda_T - \sigma_v^2 \frac{e^{-\mu T}}{\lambda_T} = 0,$$

that is

$$\lambda_T = \frac{\sigma_v}{\sigma_z},$$

and this is equivalent to $\dot{\Sigma}_T = 0$. Note that other equilibria are possible by taking $\lambda_t \neq \lambda_T$ when $t > T$.

Remark 26 It can be proved that the linearity of the strategies assumed in the previous example implies that the equilibrium pricing rules have to be linear as well. This interesting result can be seen also in Aase et al. (2012a).

Example 27 Another interesting example is that of Campi et al. (2013). There, authors consider a defaultable stock. The default time is modeled as the first time that a Brownian motion, say B , hits the barrier -1 , as in the above Example 22. However in this case the default time, $\delta = \inf\{t \geq 0, B_t = -1\}$, is not known by the insider, but it is a stopping time for every trader. Instead, she observes the process $(B_{r(t)})$ where $r(t)$ is a deterministic, increasing function with $r(t) > t$ for $t \in (0, 1)$, $r(0) = 0$, and $r(1) = 1$. This circumstance allows the insider to know in advance the default time. The horizon of the market is $t = 1$. They also consider a payoff of the kind $f(B_1)$ in case of no default. Note that $\delta = r(\tau)$, where $\tau = \inf\{t \geq 0, B_{r(t)} = -1\}$. Then, in this example the release time $r(\tau)$, the signal is $\eta_t = B_{r(t)}$ and the fundamental value is

$$V_t = \mathbf{1}_{\{\tau > t\}} \mathbb{E}(f(B_1) | B_{r(t)}).$$

Moreover the aggregate demand of noise traders follows a Brownian motion, say W , so $Z = W$. Even though τ , and consequently, δ is not known for the insider, they are predictable stopping times, and, by an extension of the case considered in section 3, we will have that, the price pressure is constant and that the optimal strategy moves prices to the fundamental one:

$$\lim_{\delta_n \uparrow \delta} P_{\delta_n} = V_\delta,$$

where (δ_n) is any increasing sequence of stopping times that grows to δ . To find the explicit form of an equilibrium strategy is not straightforward. However, if $\tau \leq s \leq V(\tau)$ an equilibrium strategy is obtained from a strong solution of

$$Y_s = W_s + \int_0^s \left(\frac{1}{1 + Y_u} - \frac{1 + Y_u}{V(\tau) - u} \right) (u) du,$$

as we deduce from Example 22 above, the difficult part is to see what happens until time τ . It requires a quite involved use of enlargement of filtrations and filtering techniques. See Campi et al. (2013b) for the details.

Acknowledgement. We would like to thank José Fajardo for helpful discussions and advice and also

anonymous reviewers of an earlier version of this work for their comments. This research was partially carried out at CAS - Centre for Advanced Study at the Norwegian Academy of Science and Letters, Research group SEFE.

References

- [1] Aase, K.K., Bjuland, T., Øksendal, B. (2012a) Strategic insider trading equilibrium: A filter theory approach. *Afrika Matematika* 23 (2), 145-162.
- [2] Aase, K.K., Bjuland, T., Øksendal, B. (2012b) Partially informed noise traders. *Mathematics and Financial Economics* 6, 93-104
- [3] Amendinger, J., Imkeller, P., Schweizer, M. (1998) Additional logarithmic utility of an insider. *Stochast. Proc. Appl.* 75, 263-286.
- [4] Back, K., Pedersen, H. (1998) Long-lived information and intraday patterns. *Journal of Financial Markets* 1, 385-402.
- [5] Back, K. (1992) Insider trading in continuous time. *The Review of Financial Studies*, 5 (3), 387-409.
- [6] Back, K. (1993) Asymmetric information and options. *The Review of Financial Studies*, Vol. 6 No. 3:435-472.
- [7] Baruch, S. (2002) Insider trading and risk aversion. *Journal of Financial Markets*, 5, 451-464.
- [8] Biagini, F., Øksendal, B. (2005) A general stochastic calculus approach to insider trading. *App. Math. & Optim.* 52, 167-181
- [9] Biagini, F., Øksendal, B. (2006) Minimal variance hedging for insider trading. *International Journal of Theoretical & Applied Finance* 9, 1351-1375.
- [10] Biagini, F., Hu, Y., Meyer-Brandis, T. Øksendal, B. (2012) Insider trading equilibrium in a market with memory. *Mathematics and Financial Economics* 6, 229-247.
- [11] Caldentey R., Stacchetti, E. (2010) Insider trading with a random deadline. *Econometrica*, Vol. 78, No. 1, 245-283.
- [12] Campi, L., Çetin, U. (2007) Insider trading in an equilibrium model with default: a passage from reduced-form to structural modelling. *Finance and Stochastics*, Vol. 4:591-602.

- [13] Campi, L., Çetin, U., Danilova, A. (2011) Dynamic Markov bridges motivated by models of insider trading. *Stochastic Processes and Their Applications*, 121(3), 534-567.
- [14] Campi, L., Çetin, U., Danilova, A. (2013) Equilibrium model with default and dynamic insider information. *Finance and Stochastics*. 17 (347), pp. 565-585.
- [15] Campi, L., Çetin, U., Danilova, A. (2013b) Explicit construction of a dynamic Bessel bridge of dimension 3. *Electron. J. Probab.* **18**, 30, 1-25.
- [16] Corcuera, J.M., Imkeller, P., Kohatsu-Higa, A., Nualart, D. (2004) Additional utility of insiders with imperfect dynamical information. *Finance and Stochastics*, 8, 437-450.
- [17] Corcuera, J.M., Di Nunno, G., Farkas, G., Øksendal, B. (2014) Kyle-Back's model with Lévy noise. Preprint.
- [18] Cho, K. (2003). Continuous auctions and insider trading: uniqueness and risk aversion. *Finance and Stochastics*, Vol. 7:47–71.
- [19] Danilova, A. (2010) Stock market insider trading in continuous time with imperfect dynamic information. *Stochastics: an international journal of probability and stochastic processes*, 82 (1), 111-131.
- [20] Di Nunno, G., Kohatsu-Higa, A., Meyer-Brandis, T., Øksendal, B., Proske, F., and Sulem, A. (2008) Anticipative stochastic control for Lévy processes with application to insider trading. *Mathematical Modelling and Numerical Methods in Finance*. Handbook of Numerical Analysis, Bensoussan and Zhang (eds.). North Holland.
- [21] Di Nunno, G., Meyer-Brandis, T., Øksendal, B., and Proske, F. (2006) Optimal portfolio for an insider in a market driven by Lévy processes. *Quantitative Finance* 6 (1), 83–94.
- [22] Imkeller, P., Pontier, M., Weisz, F. (2001) Free lunch and arbitrage possibilities in a financial market with an insider. *Stochast. Proc. Appl.* **92**, 103-130.
- [23] Jeanblanc, M., Yor, M., Chesney, M. (2009) *Mathematical Methods for Financial Markets*. Springer-Verlag. London.
- [24] Jacod, J. and Shiryaev, A.N. (1987) *Limit Theorems for Stochastic Processes*. Springer-Verlag. Berlin.
- [25] Jeulin, Th. (1980) *Semi-martingales et grossissement d'une filtration*, Lect. Notes in Maths. 833. 1980 Springer-Verlag. Berlin.

- [26] Jeulin, Th. and Yor, M., editors. (1985) *Grossissement de filtrations: exemples et applications*. Lect. Notes in Maths. 1118. Springer-Verlag. Berlin.
- [27] Karatzas, I. Pikovski, I. (1996) Anticipative portfolio optimization, *Adv. Appl. Prob.* **28**, 1095-1122.
- [28] Kohatsu-Higa, A. (2007) Models for insider trading with finite utility. *Paris-Princeton Lectures on Mathematical Finance Series: Lect. Notes in Maths*, 1919, 103-172. Springer-Verlag. Berlin.
- [29] Kyle, A. S. (1985) Continuous auctions and insider trading. *Econometrica*, Vol. 53 No. 6:1315-1335.
- [30] Lasserre, G. (2004) Asymmetric information and imperfect competition in a continuous time multivariate security model. *Finance and Stochastics*, Vol. 8, No. 2:285-309.
- [31] Liptser, Robert S.; Shiryaev, Albert N. (2001) *Statistics of Random Processes II. Applications*. Translated from the 1974 Russian original by A. B. Aries. Second, revised and expanded edition. Applications of Mathematics (New York), 6. Stochastic Modelling and Applied Probability. Springer-Verlag, Berlin.
- [32] Revuz, D. and Yor, M. (1999) Continuous Martingales and Brownian motion. 3rd edition. Springer-Verlag, New York.