Representation of convex operators and their static and dynamic sandwich extensions

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Abstract

Monotone convex operators and time-consistent systems of operators appear naturally in stochastic optimization and mathematical finance in the context of pricing and risk measurement. We study the dual representation of a monotone convex operator when its domain is defined on a subspace of L_p, with $p \in [1, \infty]$, and we prove a sandwich preserving extension theorem. These results are then applied to study systems of such operators defined only on subspaces. We propose various sandwich preserving extension results depending on the nature of time: finite discrete, countable discrete, and continuous. Of particular notice is the fact that the extensions obtained are time-consistent.

1   Introduction

The literature on extension theorems for functionals features some fundamental results. For all we just mention two: first is the Hahn-Banach theorem and its various versions, that provides e.g. a majorant preserving extension and then the König theorem that provides a sandwich preserving one (see e.g. [18]). Both cases give results for *linear* functionals with values in $\mathbb{R}$. This paper presents sandwich preserving extension theorems for *convex* monotone operators defined in a subspace L in $L_p(\mathcal{B})$ with values in $L_p(\mathcal{A})$ ($\mathcal{A} \subseteq \mathcal{B}$), for $p \in [1, \infty]$.

Other results of this type are studied in the case of linear operators, see [2] for $p \in [1, \infty)$ and [9] for $p = \infty$. Indeed the need for working in an operator setting taking values in $L_p(\mathcal{A})$ is motivated by applications to mathematical finance: pricing and risk measurement. To explain for any two fixed points

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in time, say $s \leq t$, a financial asset with payoff $X \in L_t \subseteq L_p(\mathcal{F}_t)$ has a price $x_{s,t}(X)$ evaluated at s. This value is $x_{s,t}(X) \in L_p(\mathcal{F}_s)$, where $\mathcal{F}_s \subseteq \mathcal{F}_t$.

These price operators are linear if the market model benefits of assumptions of smoothness, such as no transaction costs, no liquidity risk, perfect clearing of the market, no constrains in trading, etc. However, they are convex (when considered from the seller's perspective, the so-called ask-prices) when such market model assumptions are not fulfilled. Convex operators of such form also appear as value processes in the case of dynamic stochastic optimization and often this is in fact a way to obtain such price processes.

It is reasonable to have domain of these operators defined on a subspace of the corresponding L_p space. In fact, in general, not all positions are actually available for purchase in the market. It is only in the idealistic assumption of a complete market that we find that all positions are always feasible, i.e. the subspace is actually the whole L_p space. Strictly speaking, though less discussed in the literature, also the risk measurement is usually performed more reasonably on a subspace of L_p. In this case the subspace represents those risks for which there is grounded measurement in terms of e.g. statistical knowledge, time series analysis, and general good information. For risks outside this set, a too conservative evaluation of risk can be made with the result of higher (even too high to be competitive) hedging prices asked. This is relevant from an insurance perspective.

When dealing with a dynamic approach to pricing, we consider an information flow represented by a filtration $(\mathcal{F}_t)_{t \in [0,T]}$ $(T < \infty)$ and then a system of price operators is naturally appearing: $(x_{s,t})_{s,t \in \mathcal{T}}$, where $\mathcal{T} \subseteq [0,T]$. For the fixed times $s, t : s \leq t$ the price operator is $x_{s,t} : L_t \longrightarrow L_p(\mathcal{F}_s)$ where the domain is the subspace $L_t \subseteq L_p(\mathcal{F}_t)$. An important necessary property of these system of operators is *time-consistency*, which models the consistency of prices or measures of risk over time. Namely, for $s \leq t \leq u$ and $X \in L_u$, the evaluation $x_{s,u}(X)$ at time s is required to coincide with the two steps evaluation $x_{s,t}(x_{t,u}(X))$.

The question we address is how to extend the whole family of operators, so that the domains reach the whole $L_p(\mathcal{F}_t)$ in such manner that time-consistency is preserved together with some sandwich property. The *sandwich property* itself is a control from above and below reasonably given on such operators, as it happens, in their own context, for the Hahn-Banach and König theorems. In applications this may assume various meanings. In [3], [2], [15] there are different studies on some aspects of the fundamental theorem of asset pricing with controls on tail events, first in a multiperiod market and then in a continuous time market. In [9] the majorant and minorant operators are linked to no-good-deal dynamic bounds and the corresponding pricing measures. From the application perspective, the feasibility of these pricing rules is directly linked to the existence of the corresponding *time-consistent sandwich preserving extension* of the system of price operators.

So far this connection has been explored only for linear pricing. The present paper provides fundamental results to address some questions related to convex pricing.

We stress that to obtain a time-consistent extension it is not enough to collect all the extensions of the single operators in one family. It is only via some careful procedure of extension that we can obtain such result.

Also it is important to mention that crucial for the development of this theory is the *representation result* we obtain for convex operators defined on a subspace L of $L_p(\mathcal{B})$ taking values in $L_p(\mathcal{A})$ ($\mathcal{A} \subseteq \mathcal{B}$). Representation theorems for convex operators have been studied in the context of risk-measures in the recent years. The first results were obtained for the static case, corresponding to operators with real values ($\mathcal{A}$ trivial). Here we have to mention [17] (for $p \in [1, \infty]$) and [16] (for $p = \infty$), where the domain of the operators is the whole $L_p(\mathcal{B})$ space, and the paper [4], where a very general framework is proposed, which also includes the case of a subspace L of $L_p(\mathcal{B})$ (for $p \in [1, \infty]$) with lattice property on L. In both cases the mappings take real values. We also mention [5] and [14] for a representation in the case of operators defined on the whole $L_\infty(\mathcal{B})$ with values in $L_\infty(\mathcal{A})$ studied in the context of conditional risk measures. Our contribution in this area provides a representation theorem for convex operators defined on $L \subseteq L_p(\mathcal{B})$ (for $p \in [1, \infty]$) without requiring the lattice property.

The paper is organized as follows. In Section 2 we give a precise presentation of the operators, the spaces, and the topology we consider. Then the representation theorem is proved. Section 3 is dedicated to the sandwich extension of such convex operators. The sequel of the paper deals with time-consistent systems of operators. In Section 4 the sandwich extension is studied in the case of discrete time. In Section 5 we reach out to obtain the sandwich extension for continuous time systems of operators.

2   Convex operators in L_p and their representation

Let $(\Omega, \mathcal{B}, P)$ be a complete probability space. Here we consider $\mathcal{B}$ to be the P-completed σ-algebra generated by a countable family of sets in Ω. Also let $\mathcal{A} \subseteq \mathcal{B}$ be a P-augmented countably generated σ-algebra[1]. For example, any Borel σ-algebra of a metrizable separable space completed by the P-null events satisfies this assumption.

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[1]This assumption will be implicitly used in the sandwich extension theorems. It is not necessary for the upcoming representation theorem.

For any $p \in [1, \infty]$ we consider the $L_p(\mathcal{B}) := L_p(\Omega, \mathcal{B}, P)$ of real valued random variables with the finite norms:

$$\|X\|_p := \begin{cases} (E[|X|^p])^{1/p}, & p \in [1, \infty), \\ \operatorname{esssup}|X|, & p = \infty. \end{cases}$$

We equip these spaces with a topology. In the cases $p \in [1, \infty)$, we consider the usual topology derived from the norm. In the case $p = \infty$, we fix the weak* topology. We will denote by the superscript "+", e.g. $L_p(\mathcal{B})^+$, the cones of the non-negative random variables with the corresponding induced topology.

In the sequel we deal with a linear sub-space $L \subseteq L_p(\mathcal{B})$. We always consider L equipped with the topology induced by the corresponding $L_p(\mathcal{B})$ space. Motivated by the applications we assume that:

  i) $1 \in L$,

 ii) for the σ-algebra $\mathcal{A} \subseteq \mathcal{B}$ we have the property that $1_A X \in L$ for every $A \in \mathcal{A}$ and every $X \in L$.

Fix $p \in [1, \infty]$ and the sub-space $L \subseteq L_p(\mathcal{B})$ as above. We consider an operator

$$x : L \longrightarrow L_p(\mathcal{A}) \tag{2.1}$$

that is:

- *monotone*, i.e. for any X', $X'' \in L$,

$$x(X') \geq x(X''), \quad X' \geq X'',$$

- *convex*, i.e. for any X', $X'' \in L$ and $\lambda \in [0, 1]$,

$$x\big(\lambda X' + (1 - \lambda)X''\big) \leq \lambda x(X') + (1 - \lambda)x(X'')$$

- *lower semi-continuous*, i.e. for any $X \in L$ and any sequence $(X_n)_n$ in L with limit X,

$$\liminf_{n \to \infty} x(X_n) \geq x(X)$$

- *weak $\mathcal{A}$-homogeneous*, i.e. for all $X \in L$

$$x(1_A X) = 1_A x(X), \quad A \in \mathcal{A},$$

- *projection property*

$$x(f) = f, \quad f \in L_p(\mathcal{A}) \cap L.$$

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In particular we have $x(0) = 0$ and $x(1) = 1$.

Note that, if $p \in [1, \infty)$ and the operator x is monotone and linear (as in [2]) the assumption of weak $\mathcal{A}$-homogeneity is equivalent to $\mathcal{A}$-homogeneity, i.e. for all $X \in L$, we have

$$x(\xi X) = \xi x(X)$$

for all $\xi \in L_p(\mathcal{A})$ such that $\xi X \in L$. If $p = \infty$ and the operator is linear and semi-continuous, then the same result holds (see [9]).

2.1 Representation of a convex operator

Our first result is a representation theorem for L_p-valued convex operators of the type above. This can be regarded as a non-trivial extension of [20, Theorem 5]. The result by Rockafellar is written for functionals and can be retrieved setting $\mathcal{A}$ to be the trivial σ-algebra up to P-null events.

Theorem 2.1. *Let x be an operator of the type* (2.1). *Then the following representation holds:*

$$x(X) = \text{esssup}_{V \in \mathcal{V}}\Big\{V(X) - x^*(V)\Big\}, \quad X \in L, \tag{2.2}$$

where

$$x^*(V) := \text{esssup}_{X \in L}\Big\{V(X) - x(X)\Big\}, \quad V \in \mathcal{V},$$

and $\mathcal{V}$ is the space of the linear, non-negative, continuous, $\mathcal{A}$-homogeneous operators $V : L_p(\mathcal{B}) \longrightarrow L_p(\mathcal{A})$ such that $E[x^*(V)] < \infty$.
Moreover, the operator x also admits representation in the form:

$$x(X) = \text{esssup}_{V \in \mathbb{V}}\Big\{V(X) - x^*(V)\Big\}, \quad X \in L, \tag{2.3}$$

where

$$x^*(V) := \text{esssup}_{X \in L}\Big\{V(X) - x(X)\Big\}, \quad V \in \mathbb{V}, \tag{2.4}$$

and $\mathbb{V}$ is the space of the linear, non-negative, continuous, $\mathcal{A}$-homogeneous operators $V : L_p(\mathcal{B}) \longrightarrow L_p(\mathcal{A})$.

For future reference we borrow the terminology proper of the literature on risk measures and we call the operator x^* *minimal penalty.*

Before the proof of the theorem we present a couple of technical lemmas.

Lemma 2.2. *If* $V = 1_A V_1 + 1_{A^c} V_2$, *for* $V_1, V_2 \in \mathcal{V}$, $A \in \mathcal{A}$, *and* $A^c := \Omega \setminus A$, *then*

$$x^*(V) = 1_A x^*(V_1) + 1_{A^c} x^*(V_2). \qquad (2.5)$$

Moreover the set $\{V(X) - x^*(V) : V \in \mathcal{V}\}$ *is a lattice upward directed.*

Proof. For any $X, Y \in L$ we have

$$1_A\big(V_1(X) - x(X)\big) + 1_{A^c}\big(V_2(Y) - x(Y)\big) = V(1_A X + 1_{A^c} Y) - x(1_A X + 1_{A^c} Y)$$
$$\leq \operatorname{esssup}_{Z \in L}\{V(Z) - x(Z)\}.$$

Hence,

$$1_A \operatorname{esssup}_{X \in L}\{V_1(X) - x(X)\} + 1_{A^c} \operatorname{esssup}_{Y \in L}\{V_2(Y) - x(Y)\}$$
$$\leq \operatorname{esssup}_{Z \in L}\{V(Z) - x(Z)\}.$$

Namely, we have $1_A x^*(V_1) + 1_{A^c} x^*(V_2) \leq x^*(V)$. On the other hand, for any $Z \in L$, we have

$$V(Z) - x(Z) = 1_A\big(V_1(Z) - x(Z)\big) + 1_{A^c}\big(V_2(Z) - x(Z)\big)$$
$$\leq 1_A \operatorname{esssup}_{Z \in L}\{V_1(Z) - x(Z)\} + 1_{A^c} \operatorname{esssup}_{Z \in L}\{V_2(Z) - x(Z)\}.$$

Therefore, $x^*(V) \leq 1_A x^*(V_1) + 1_{A^c} x^*(V_2)$. So (2.5) holds.
To prove the lattice property, it is enough to consider for any $V_1, V_2 \in \mathcal{V}$, the set $A := \{V_1(X) - x^*(V_1) \geq V_2(X) - x^*(V_2)\} \in \mathcal{A}$ and $V = 1_A V_1 + 1_{A^c} V_2$. From (2.5) we have that:

$$V(X) - x^*(V) = 1_A\big(V_1(X) - x^*(V_1)\big) + 1_{A^c}\big(V_2(X) - x^*(V_2)\big)$$
$$= \sup\big\{V_1(X) - x^*(V_1), V_2(X) - x^*(V_2)\big\}.$$

By this we end the proof. $\qquad \square$

Lemma 2.3. *For any* $V \in \mathbb{V}$, *the set* $\{V(X) - x(X) : X \in L\}$ *is a lattice upward directed.*

Proof. We consider $X_1, X_2 \in L$ and we set $A := \{V(X_1) - x(X_1) \geq V(X_2) - x(X_2)\} \in \mathcal{A}$. Consider $X = 1_A X_1 + 1_{A^c} X_2$. Then

$$V(X) - x(X) = 1_A\big(V(X_1) - x(X_1)\big) + 1_{A^c}\big(V(X_2) - x(X_2)\big)$$
$$= \sup\big\{V(X_1) - x(X_1), V(X_2) - x(X_2)\big\}.$$

By this we end the proof. $\qquad \square$

We are now ready to prove Theorem 2.1.

Proof. Define
$$h(X) := E\Big[x(X)\Big], \quad X \in L.$$

Note that h is a non-negative, monotone, convex, lower semi-continuous, and proper (i.e., $h(X) > -\infty$ and finite for some X, see [20, p. 1]) functional. Thus, by application of [20, Theorem 4 and Theorem 5], we have the representation
$$h(X) = \sup_{v \in L^*} \Big\{v(X) - h^*(v)\Big\} \tag{2.6}$$

where
$$h^*(v) := \sup_{X \in L} \Big\{v(X) - h(X)\Big\}$$

is the *Fenchel transform of h* and L^* is the set of continuous linear forms on L. Note that we can restrict to $v \in L^*$ such that $h^*(v) < \infty$, and in this case v is a non-negative linear form. Recall that we always consider the usual L_p-norm topology in the case $p \in [1, \infty)$ and the weak* topology in the case $p = \infty$. Then we distinguish the two cases.

If $p \in [1, \infty)$, the classical Hahn-Banach theorem guarantees that $v(X)$, $X \in L$, can be extended to a non negative continuous linear form $v(X)$, $X \in L_p(\mathcal{B})$, and the extension admits the representation
$$v(X) = E\Big[fX\Big], \quad X \in L_p(\mathcal{B}),$$

for some $f \in L_q(\mathcal{B})$ with $q = p(p-1)^{-1}$ and $f \geq 0$.

If $p = \infty$, then we refer to a version of the Hahn-Banach theorem for locally convex topological spaces as in [10, Chapter II] and we proceed as follows. Recall that the weak* topology on $L_\infty(\mathcal{B})$, defined by the family of semi-norms
$$p_g(\cdot) := E\Big[g \cdot\Big], \quad \text{for every } g \in L_1(\mathcal{B}) : g \geq 0,$$

is locally convex. For every non-negative linear form v on L, continuous for the weak* topology, there is a semi-norm p_g such that
$$|v(X)| \leq p_g(X).$$

Hence, applying the above mentioned corollary, we can extend v to a non-negative weak* continuous linear form on $L_\infty(\mathcal{B})$. The extension admits the representation
$$v(X) = E\Big[fX\Big], \quad X \in L_\infty(\mathcal{B}),$$

for some $f \in L_1(\mathcal{B})$ such that $f \geq 0$.

Therefore for $p \in [1, \infty]$, the convex functional $h(X)$, $X \in L$, in (2.6) can be rewritten as:
$$\begin{aligned} h(X) &= \sup_{f \in L_q(\mathcal{B}): f \geq 0} \Big\{E[fX] - h^*(E[f \cdot])\Big\} \\ &= \sup_{f \in \mathcal{W}} \Big\{E[fX] - h^*(E[f \cdot])\Big\} \end{aligned} \tag{2.7}$$

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where
$$\mathcal{W} := \big\{ f \in L_q(\mathcal{B}) : \ f \geq 0 \, , h^*(E[f\cdot]) < \infty \big\}. \qquad (2.8)$$

Note that $\mathcal{W} \neq \emptyset$, because h is real valued.

We remark immediately that $E[f|\mathcal{A}] = 1$, for every $f \in \mathcal{W}$. Indeed, consider any $A \in \mathcal{A}$ and $X = 1_A$. For any $\alpha \in \mathbb{R}$ we have

$$E[f \, \alpha 1_A] - h^*(E[f\cdot]) \leq h(\alpha 1_A) = E[x(\alpha 1_A)] = \alpha E[1_A].$$

Hence,
$$\alpha\big(E[f1_A] - P(A) \big) \leq h^*(E[f\cdot]) < \infty.$$

Thus taking $\alpha \to \pm\infty$, we see that $E[f1_A] = P(A)$, $A \in \mathcal{A}$. Namely $E[f|\mathcal{A}] = 1$.

For every $f \in \mathcal{W}$, denote

$$V(X) := E[fX|\mathcal{A}], \quad X \in L_p(\mathcal{B}). \qquad (2.9)$$

Hereafter we show that $V \in \mathcal{V}$. First of all note that the operator V is naturally non-negative, linear, and $\mathcal{A}$-homogeneous. It is also continuous. Indeed for the case $p \in [1, \infty)$ it is immediate from the conditional Hölder inequality.

For the case $p = \infty$, we consider a neighborhood of $E[fX|\mathcal{A}]$ for the weak* topology:

$$\mathcal{U} := \big\{ Y \in L_\infty(\mathcal{A}) : \forall g_i \in L_1(\mathcal{A}), \ i = 1, ..., h, \quad |E\big[g_i E[fX|\mathcal{A}] - E\big[g_i Y\big]| < \epsilon \big\}$$

Since $f \geq 0$ and $E[f|\mathcal{A}] = 1$, then $g_i f \in L_1(\mathcal{B})$, $i = 1, ..., h$, and the set

$$\tilde{\mathcal{U}} := \big\{ Z \in L_\infty(\mathcal{B}) : \ \forall g_i \in L_1(\mathcal{A}), \ i = 1, ..., h, \quad |E\big[g_i fX\big] - E\big[g_i fZ\big]| < \epsilon \big\}$$

is a neighborhood of X in $L_\infty(\mathcal{B})$ in the weak* topology and for all $Z \in \tilde{\mathcal{U}}$, $E[fZ|\mathcal{A}] \in \mathcal{U}$. This proves the continuity of V for the weak* topology. Thus V belongs to $\mathbb{V}$.

Define $x^*(V) := \mathrm{esssup}_{X \in L}\{V(X) - x(X)\}$, for V in (2.9). We show that $E[x^*(V)] < \infty$. From the lattice property of Lemma 2.3, from [19, Prop VI 1.1.], and the monotone convergence theorem, we have:

$$\begin{aligned}
E[x^*(V)] &= \sup_{X \in L} \big\{ E[V(X)] - E[x(X)] \big\} \\
&= \sup_{X \in L} \big\{ E[fX] - h(X) \big\} \\
&= \sup_{X \in L} \big\{ v(X) - h(X) \big\} = h^*(v) < \infty.
\end{aligned}$$

Then we can conclude that V as in (2.9) is an element of $\mathcal{V}$.

We are now ready to prove the representation (2.2). For every $V \in \mathcal{V}$, define

$$x^*(V) := \mathrm{esssup}_{X \in L}\big\{ V(X) - x(X) \big\}.$$

Note that from $x(0) = 0$, we have that $x^*(V) \geq 0$. For every $V \in \mathcal{V}$ and $X \in L$, we have

$$x^*(V) \geq V(X) - x(X)$$

or, equivalently,

$$x(X) \geq V(X) - x^*(V).$$

Thus

$$x(X) \geq \operatorname{esssup}_{V \in \mathcal{V}} \left\{ V(X) - x^*(V) \right\}. \tag{2.10}$$

To conclude we need to show the reverse inequality. To this aim it is enough to show that

$$E\left[x(X) \right] \leq E\left[\operatorname{esssup}_{V \in \mathcal{V}} \left\{ V(X) - x^*(V) \right\} \right], \quad X \in L. \tag{2.11}$$

Indeed we have:

$$
\begin{aligned}
E[x(X)] = h(X) &= \sup_{f \in \mathcal{W}} \left\{ E[fX] - h^*(E[f\cdot]) \right\} \\
&= \sup_{f \in \mathcal{W}} \left\{ E[E[fX|\mathcal{A}]] - h^*(E[f\cdot]) \right\} \\
&\leq \sup_{V \in \mathcal{V}} \left\{ E[V(X)] - E[x^*(V)] \right\} \\
&= E[\operatorname{esssup}_{V \in \mathcal{V}} \left\{ V(X) - x^*(V) \right\}]
\end{aligned}
$$

where the last equality is due to the lattice property of Lemma 2.2 and [19, Proposition VI.1.1]. We have then proved the representation (2.2).

To prove the representation (2.3), we note that $\mathcal{V} \subseteq \mathbb{V}$. From equation (2.2) we have

$$
\begin{aligned}
x(X) &= \operatorname{esssup}_{V \in \mathcal{V}} \left\{ V(X) - x^*(V) \right\} \\
&\leq \operatorname{esssup}_{V \in \mathbb{V}} \left\{ V(X) - x^*(V) \right\}.
\end{aligned}
$$

From the definition of x^* in (2.4) we have that $x^*(V) \geq V(X) - x(X)$, that is $x(X) \geq V(X) - x^*(V)$, for every $V \in \mathbb{V}$ and $X \in L$. So, we conclude that

$$x(X) \geq \operatorname{esssup}_{V \in \mathbb{V}} \left\{ V(X) - x^*(V) \right\}.$$

By this we end the proof. $\qquad\square$

Corollary 2.4. *Let x be of type* (2.1). *Then the following representation holds:*

$$x(X) = \operatorname{esssup}_{f \in \mathbb{D}} \left\{ E\left[fX | \mathcal{A} \right] - x^*(E\left[f \cdot | \mathcal{A} \right]) \right\}, \quad X \in L, \tag{2.12}$$

where

$$\mathbb{D} := \left\{ f \in L_q(\mathcal{B}) : \ f \geq 0, \ E[f|\mathcal{A}] = 1 \right\} \tag{2.13}$$

with $q = p(p-1)^{-1}$ *and*

$$x^*\big(E\big[f \cdot |\mathcal{A}\big]\big) := \mathrm{esssup}_{X \in L}\Big\{E\big[fX|\mathcal{A}\big] - x(X)\Big\}, \quad f \in \mathbb{D}.$$

Proof. From [2, Theorem 1.1], for the case $p \in [1, \infty)$, and [9, Theorem 3.3 and Proposition 3.14], for $p = \infty$, we know that there is a one-to-one relationship between $\mathbb{V}$ and $\mathbb{D}$. Then the results follow directly from the representation (2.3). □

The representations of convex functionals were studied in the recent literature of risk measures in the case when the σ-algebra $\mathcal{A}$ is trivial. In [17] the representation is studied for convex risk measures (i.e. convex, monotone, lower semicontinuous, and translation invariant functionals) defined on the *whole* $L_p(\mathcal{B})$ with $p \in [1, \infty]$ with values in $(-\infty, \infty)$. In [16] the study is carried on for $p = \infty$. In both cases it is crucial that the functionals are defined on the whole space. In [4], the representation is studied for convex, monotone, order continuous functionals defined on Fréchet lattices and taking values in $(-\infty, \infty]$. This allows for a very general setup, however the assumption on the lattice property is crucial. If $\mathcal{A}$ is non-trivial, then we can refer to [5] and [14] for studies on the representation of convex operators in the context of conditional risk measures (i.e. convex, monotone, lower semicontinuous, translation invariant operators) defined on the *whole* $L_\infty(\mathcal{B})$. Our contribution to this area provides a representation of convex operators defined on a subspace $L \subseteq L_p(\mathcal{B})$ with values in $L_p(\mathcal{A})$, $p \in [1, \infty]$, without the assumption of the lattice property.

3   Sandwich extension of a convex operator

In the sequel we consider a criterion for the existence of an extension $\bar{x}$ of the convex operator x to the whole $L_p(\mathcal{B})$. The given x lies within two operators m and M. This extension is sandwich preserving. There is no uniqueness of such sandwich preserving extension, but our approach allows for an explicit representation of at least one of them, denoted $\hat{x}$, which turns out to be the maximal.

First of all we introduce the *minorant* as a superlinear operator: $m : L_p(\mathcal{B})^+ \longrightarrow L_p(\mathcal{A})^+$, i.e.,

$$m(X + Y) \geq m(X) + m(Y), \qquad X, Y \in L_p(\mathcal{B})^+,$$

$$m(\lambda X) = \lambda m(X), \qquad X \in L_p(\mathcal{B})^+, \, \lambda \geq 0,$$

and the *majorant* as a sublinear operator: $M : L_p(\mathcal{B})^+ \longrightarrow L_p(\mathcal{A})^+$, i.e.,

$$M(X + Y) \leq M(X) + M(Y), \qquad X, Y \in L_p(\mathcal{B})^+,$$

$$M(\lambda X) = \lambda M(X), \qquad X \in L_p(\mathcal{B})^+, \ \lambda \geq 0.$$

We remark that sublinearity implies $M(0) = 0$.

Moreover, in the case $p = \infty$, we say that the map $M : L_\infty(\mathcal{B})^+ \longrightarrow L_\infty(\mathcal{A})^+$ is *regular* if for every decreasing sequence $(X_n)_n$ in $L_\infty(\mathcal{B})$ with $X_n \downarrow 0$, $n \to \infty$ P-a.s, we have

$$M(X_n) \to 0, \quad n \to \infty \quad Pa.s. \tag{3.1}$$

Theorem 3.1. *Fix $p \in [1, \infty]$. Let $x : L \longrightarrow L_p(\mathcal{A})$ be of type (2.1). Consider the weak $\mathcal{A}$-homogeneous operators $m, M : L_p(\mathcal{B})^+ \to L_p(\mathcal{A})^+$ such that m is superlinear and M is sublinear and, if $p = \infty$, M is also regular. Assume the sandwich condition:*

$$m(Z) + x(X) \leq M(Y) \tag{3.2}$$

$$\forall X \in L \quad \forall \, Y, Z \in L_p(\mathcal{B})^+ : \ Z + X \leq Y.$$

Then x admits an extension $\hat{x}$ (to the whole $L_p(\mathcal{B})$), which is convex, monotone, lower-semicontinuous, weak $\mathcal{A}$-homogeneous and satisfying the projection property such that (3.2) is preserved, i.e.

$$m(Z) + \hat{x}(X) \leq M(Y) \tag{3.3}$$

$$\forall X \in L_p(\mathcal{B}) \quad \forall \, Y, Z \in L_p(\mathcal{B})^+ : \ Z + X \leq Y.$$

In particular the operator

$$\hat{x}(X) := \mathrm{esssup}_{V \in \mathcal{V}^S}\big\{V(X) - x^*(V)\big\}, \quad X \in L_p(\mathcal{B}), \tag{3.4}$$

with

$$x^*(V) := \mathrm{esssup}_{X \in L}\big\{V(X) - x(X)\big\}, \quad V \in \mathcal{V}^S, \tag{3.5}$$

is a sandwich preserving extension of x. Here above $\mathcal{V}^S$ is the set of linear, continuous, non-negative, $\mathcal{A}$-homogeneous operators on $L_p(\mathcal{B})$ such that $E[x^*(V)] < \infty$, and satisfying the sandwich condition:

$$m(X) \leq V(X) \leq M(X), \qquad X \in L_p(\mathcal{B})^+.$$

Moreover, for any other such extension $\bar{x}$, we have that

$$\hat{x}(X) \geq \bar{x}(X), \qquad X \in L_p(\mathcal{B}).$$

We call $\hat{x}$ the maximal extension.

Proof. From Theorem 2.1, for all $X \in L$,

$$x(X) = \mathrm{esssup}_{V \in \mathcal{V}}\big\{V(X) - x^*(V)\big\}.$$

Thus $\forall V \in \mathcal{V}$, the restriction of V to L satisfies:

$$m(Z) + V(X) - x^*(V) \le M(Y) \ \ \forall X \in L \ \forall Y, Z \in L_p(\mathcal{B})^+ : \ \ Z + X \le Y.$$

Then, for all $\alpha > 0$,

$$\alpha m(Z) + \alpha V(X) - x^*(V) \le \alpha M(Y)$$

Let $A = \{m(Z) + V(X) - M(Y) \ge 0\}$,

$$0 \le E(1_A(m(Z) + V(X) - M(Y))) \le \frac{1}{\alpha} E(1_A x^*(V)) < \infty.$$

Let $\alpha \to \infty$, it follows that $1_A(m(Z) + V(X) - M(Y)) = 0 \ P$ a.s.. Thus

$$m(Z) + V(X) \le M(Y) \tag{3.6}$$

for all $V \in \mathcal{V}$ and $\forall X \in L, \forall Y, Z \in L_p(\mathcal{B})^+ : Z + X \le Y$. From the sandwich extension theorem for linear operators, [9, Proposition 3.11] in case of L_p spaces $1 \le p < \infty$, and [9, Theorem 3.9] in case of L_∞ spaces (see also [2, Theorem 5.1]), every $V \in \mathcal{V}$ restricted to L, admits a sandwich preserving linear extension to the whole $L_p(\mathcal{B})$ denoted V^S which is monotone, lower semi continuous, weak $\mathcal{A}$-homogeneous, and satisfying the sandwich condition:

$$m(Z) + V^S(X) \le M(Y) \ \ \forall X \in L_p(\mathcal{B}) \ \forall Y, Z \in L_p(\mathcal{B})^+ : \ \ Z + X \le Y.$$

Define

$$\hat{x}(X) := \text{esssup}_{V \in \mathcal{V}^S}\{V(X) - x^*(V)\}, \qquad X \in L_p(\mathcal{B}),$$

where $\mathcal{V}^S$ is the set described in the statement of the theorem, and $x^*(V) = \text{esssup}_{Y \in L}(V(Y) - x(Y))$. It follows that $\hat{x}$ extends x and it is lower semi continuous, convex, monotone, weak $\mathcal{A}$-homogeneous and it also satisfies the projection property. It remains to verify that $\hat{x}$ satisfies the sandwich condition. Let $Y, Z \in L_p(\mathcal{B})^+ \ \forall X \in L_p(\mathcal{B}) : \ Z + X \le Y$,

$$
\begin{aligned}
m(Z) + \hat{x}(X) &= \text{esssup}_{V \in \mathcal{V}^S}\{m(Z) + V(X) - x^*(V)\} \\
&\le M(Y) + \text{esssup}_{V \in \mathcal{V}^S}(-x^*(V)) \\
&= M(Y) + \text{esssup}_{V \in \mathcal{V}}(-x^*(V)) = M(Y).
\end{aligned}
\tag{3.7}
$$

Now consider any other convex, monotone, lower-semicontinuous, weak $\mathcal{A}$-homogeneous extension $\bar{x}$ satisfying the sandwich condition. From Theorem 2.1 we have that

$$\bar{x}(X) = \text{esssup}_{V \in \mathcal{V}_{\bar{x}}}\{V(X) - \bar{x}^*(V)\}, \quad X \in L_p(\mathcal{B}),$$

with

$$\bar{x}^*(V) = \text{esssup}_{X \in L_p(\mathcal{B})}\{V(X) - \bar{x}(X)\}$$

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where $\mathcal{V}_{\bar{x}}$ is given by the mentioned theorem with reference to the operator $\bar{x}$. Moreover, since $\bar{x}$ satisfies the sandwich condition we can see that (3.6) holds for $\mathcal{V}_{\bar{x}}$ and that

$$\bar{x}(X) = \text{esssup}_{V \in \mathcal{V}_{\bar{x}}^S}\{V(X) - \bar{x}^*(V)\}, \quad X \in L_p(\mathcal{B}).$$

From the definition of $\bar{x}^*$ and of x^* with $\bar{x}(X) = x(X)$, $X \in L$, we can see that $\bar{x}^*(V) \geq x^*(V)$ is valid for all $V \in \mathbb{V}$. Hence $E[\bar{x}^*(V)] \geq E[x^*(V)]$, $V \in \mathbb{V}$, and in particular $\mathcal{V}_{\bar{x}}^S \subseteq \mathcal{V}^S$. Then $\bar{x}(X) \leq \text{esssup}_{V \in \mathcal{V}^S}\{V(X) - \bar{x}^*(V)\}$. On the other hand for every $V \in \mathcal{V}^S$ and $X \in L_p(\mathcal{B})$ we have $V(X) - \bar{x}(X) \leq \bar{x}^*(V)$, hence $V(X) - \bar{x}^*(X) \leq \bar{x}(X)$. Thus we conclude $\text{esssup}_{V \in \mathcal{V}^S}\{V(X) - \bar{x}^*(V)\} \leq \bar{x}(X)$ and we have proved that:

$$\bar{x}(X) = \text{esssup}_{V \in \mathcal{V}^S}\{V(X) - \bar{x}^*(V)\}.$$

Since $\bar{x}^*(V) \geq x^*(V)$ for all $V \in \mathcal{V}_{\bar{x}}^S$, then $\bar{x}(X) \leq \hat{x}(X)$ for all $X \in L_p(\mathcal{B})$. $\qquad\square$

Remark 3.1. *The above extension $\hat{x}$ (3.4) satisfies the following inequality:*

$$\forall X \in L_p(\mathcal{B})^+ \quad m(X) \leq -\hat{x}(-X) \leq \hat{x}(X) \leq M(X).$$

This inequality is in fact equivalent to (3.3) for every convex, monotone, lower semi continuous, weak $\mathcal{A}$-homogeneous operator defined on the whole $L_p(\mathcal{B})$. The first assertion follows from equation (3.3) applied one time with $(Z, X, Y) = (X, -X, 0)$ and the other time with $(Z, X, Y) = (0, X, X)$. The second assertion follows from the convexity of $\hat{x}$.

Corollary 3.2. *For every $V \in \mathcal{V}^S$, the penalty (3.5) in the representation (3.4) of the extension $\hat{x}$ of the operator x, satisfies $x^*(V) = \tilde{x}(V)$, where*

$$\tilde{x}(V) := \text{esssup}_{X \in L_p(\mathcal{B})}\{V(X) - \hat{x}(X)\}. \tag{3.8}$$

Moreover, define $\mathbb{V}^S$ as the set of elements in $\mathbb{V}$ satisfying the sandwich condition (3.6). Then the extension (3.4) can be rewritten as:

$$\hat{x}(X) = \text{esssup}_{V \in \mathbb{V}^S}\{V(X) - \tilde{x}(V)\}, \qquad X \in L_p(\mathcal{B}). \tag{3.9}$$

Furthermore, we can also give the representation:

$$\hat{x}(X) = \text{esssup}_{f \in \mathbb{D}^S}\{E[fX|\mathcal{A}] - \tilde{x}(E[f \cdot |\mathcal{A}])\}, \qquad X \in L_p(\mathcal{B}), \tag{3.10}$$

with

$$\mathbb{D}^S := \left\{ f \in \mathbb{D} : m(X) \leq E[fX|\mathcal{A}] \leq M(X), \forall X \in L_p(B) \right\}. \tag{3.11}$$

The penalty $\tilde{x}$ is called minimal penalty following the terminology of risk measures.

Proof. Fix $V \in \mathcal{V}^S$. From the definition we have $x^*(V) \leq \tilde{x}(V)$. On the other hand, from (3.4), we have $\hat{x}(X) \geq V(X) - x^*(V)$, for all $X \in L_p(\mathcal{B})$. Hence, $x^*(V) \geq V(X) - \hat{x}(X)$, for all $X \in L_p(\mathcal{B})$. Thus $x^*(V) \geq \tilde{x}(V)$. This proves (3.8). For the proof of (3.9) we apply the same arguments as for the proof of (2.3) in Theorem 2.1. And for the proof of (3.10) we apply the same arguments as for the proof of (2.12) in Corollary 2.4. $\qquad\square$

Definition 3.1. *The operator m is* non degenerate *if it satisfies $E(m(1_B)) > 0$ for all $B \in \mathcal{B}$ such that $P(B) > 0$.*

Lemma 3.3. *Assume that m is non degenerate. Every $f \in \mathbb{D}^S$ such that $E(\tilde{x}(E(f \cdot |\mathcal{A})) < \infty$ belongs to*

$$\mathbb{D}^e := \{ f \in \mathbb{D} \,|\, f > 0 \; P\text{a.s.} \}.$$

Proof. Let $B \in \mathcal{B}$ such that $P(B) > 0$. It follows from the Remark 3.1 that for all real $\lambda > 0$, $\hat{x}(-\lambda 1_B) \leq -m(\lambda 1_B)$. From the representation (3.10) of $\hat{x}(-\lambda 1_B)$, we get $\hat{x}(-\lambda 1_B) \geq E(-\lambda 1_B f) - \tilde{x}(E(f \cdot |\mathcal{A}))$. It follows that for all $\lambda > 0$, $E(1_B f) \geq E(m(1_B)) - \frac{E(\tilde{x}(E(f.|\mathcal{A}))}{\lambda}$. Letting $\lambda \to \infty$, the result follows from $E(m(1_B)) > 0$, being m non degenerate. $\qquad\square$

We deduce the following result from Corollary 3.2 and Lemma 3.3.

Corollary 3.4. *Assume that m is non degenerate, then $\hat{x}$ admits the following representation*

$$\hat{x}(X) = \operatorname{esssup}_{f \in \mathbb{D}^{S,e}} \{ E[fX|\mathcal{A}] - \tilde{x}(E[f \cdot |\mathcal{A}]) \}, \qquad X \in L_p(\mathcal{B}), \quad (3.12)$$

with

$$\mathbb{D}^{S,e} := \mathbb{D}^S \cap \mathbb{D}^e. \qquad (3.13)$$

The following result can be regarded as an extension of [2, Theorem 5.2] to the case of convex operators.

Corollary 3.5. *If the minorant m and the majorant M in Theorem 3.1 are linear operators:*

$$m(X) = E[m_0 X|\mathcal{A}], \quad X \in L_p(\mathcal{B})^+,$$
$$M(X) = E[M_0 X|\mathcal{A}], \quad X \in L_p(\mathcal{B})^+$$

for some random variables $m_0, M_0 \in L_q(\mathcal{B})^+$: $q = p(p-1)^{-1}$ such that $0 \leq m_0 \leq M_0$. The extension (3.4) $\hat{x}$ can be written as:

$$\hat{x}(X) = \operatorname{esssup}_{f \in \mathcal{D}} \{ E[fX|\mathcal{A}] - x^*(E[f \cdot |\mathcal{A}]) \}, \qquad X \in L_p(\mathcal{B}), \quad (3.14)$$

where

$$\mathcal{D} := \left\{ f \in L_q(\mathcal{B}) : 0 \leq m_0 \leq f \leq M_0, E[f|\mathcal{A}] = 1 \right\}.$$

Proof. This is a direct application of Corollary 3.2. □

We now prove that under the sandwich condition the esssup in (3.4) is attained. This will be a consequence of the following compactness result.

Lemma 3.6. *Let M be sublinear, monotone, weak $\mathcal{A}$-homogeneous, and, if $p = \infty$, regular. Let $\mathcal{K} = \{f \in \mathbb{D} : \ 0 \leq E(f \cdot |\mathcal{A}) \leq M\}$. Identifying $f \in \mathcal{K}$ with the linear form $E(f\cdot)$ on $L_p(\mathcal{B})$, $\mathcal{K}$ is a compact subset of the ball of radius $E(M(1)^q)^{\frac{1}{q}}$ of $L'_p(\mathcal{B})$, $1 \leq p \leq \infty$ equiped with the weak* topology $\sigma(L'_p, L_p)$. In case $p = \infty$, $\mathcal{K}$ is furthermore contained in $L_1(\mathcal{B})$.(Notice that if $p < \infty$, $L'_p = L_q$ with $q = p(p-1)^{-1}$.)*

Moreover, with the notations of Theorem 3.1, the set $\mathbb{D}^S$ is compact for the weak* topology .

Proof. $\mathcal{K}$ is a subset of the ball of radius $E(M(1)^q)^{\frac{1}{q}}$ of $L'_p(\mathcal{B})$. As this bounded ball is compact for the weak* topology (Banach Alaoglu theorem), it is enough to prove that $\mathcal{K}$ is closed for the weak* topology. Denote $\overline{\mathcal{K}}$ the weak* closure of $\mathcal{K}$. Let $\Psi \in \overline{\mathcal{K}}$. Ψ is a positive continuous linear form on $L_p(\mathcal{B})$.

In case $p \in [1, \infty)$, Ψ is represented by an element of $L_q(\mathcal{B})$ for $q = p(p-1)^{-1}$ (Riesz representation theorem).

We detail the case $p = \infty$. We first prove that Ψ defines a measure on $(\Omega, \mathcal{B})$. Let $(X_n)_n$ be any sequence of elements of $L_\infty(\mathcal{B})$ decreasing to 0 P a.s. From the regularity of M, $\forall \epsilon > 0$, there is n_0 such that $\forall n \geq n_0$, $E(M(X_n)) \leq \epsilon$. Denote U the neighborhood of Ψ defined as $U = \{\phi \in L'_\infty(\mathcal{B}), \ |\Psi(X_{n_0}) - \phi(X_{n_0})| \leq \epsilon\}$. Since $\Psi \in \overline{\mathcal{K}}$, there is $\phi \in U \cap \mathcal{K}$. For such ϕ, $0 \leq \phi(X_n) \leq E(M(X_n)) \leq \epsilon$. It follows that $|\Psi(X_{n_0})| \leq 2\epsilon$. As Ψ is a non negative linear functional and the sequence $(X_n)_n$ is decreasing to 0, it follows that $0 \leq \Psi(X_n) \leq 2\epsilon$ for every $n \geq n_0$. From Daniell Stone Theorem, see e.g. [16, Theorem A48], it follows that Ψ defines a probability measure on $(\Omega, \mathcal{B})$. This probability measure is absolutely continuous with respect to P and this gives the existence of some $g \in L_1(\mathcal{B})$ such that $\Psi = E(g\cdot)$ (Radon Nikodym Theorem). For all $X \in L_\infty(\mathcal{A})$, the equality $\Psi(X) = E(X)$ is obtained similarly making use of the neighborhood of Ψ: $U_X = \{\phi \in L'_\infty(\mathcal{B}), \ |\Psi(X) - \phi(X)| \leq \epsilon\}$. It follows that $E(g|\mathcal{A}) = 1$. The inequality $E(fX1_A) \leq E(M(X)1_A)$ for $X \in L_p(\mathcal{B})$ and $A \in \mathcal{A}$ is obtained similarly and hence $\Psi = E(g.)$ where g belongs to $\mathcal{K}$. This proves the compactness of $\mathcal{K}$ for the weak* topology.

$\mathbb{D}^S$ is equal to $\{f \in \mathcal{K} : E(m(X)1_A) \leq E(fX1_A), \forall X \in L_p(\mathcal{B}), \forall A \in \mathcal{A}\}$. Thus $\mathbb{D}^S$ is a closed subset of $\mathcal{K}$ for the weak* topology. □

Proposition 3.7. *Assume the hypothesis of Theorem 3.1. For every $X \in L_p(\mathcal{B})$, there is some f_X in $\mathbb{D}^S$ (depending on X) such that*

$$\hat{x}(X) = E(f_X X|\mathcal{A}) - \tilde{x}(E(f_X \cdot |\mathcal{A})). \tag{3.15}$$

Proof. We start from the representation (3.10) given in Corollary 3.2:

$$\hat{x}(X) = \text{esssup}_{f \in \mathbb{D}^S} \big\{ E[fX|\mathcal{A}] - \tilde{x}(E[f \cdot |\mathcal{A}]) \big\}, \qquad X \in L_p(\mathcal{B}). \quad (3.16)$$

From the lattice property proved in Lemma 2.2, it follows that $E(\hat{x}(X)) = \sup_{f \in \mathbb{D}^S}[E(fX) - E(\tilde{x}(E[f \cdot |\mathcal{A}]))]$. From the definition of $\tilde{x}$ and the lattice property proved in Lemma 2.3, it follows that $E(\tilde{x}(E[f \cdot |\mathcal{A}]))$ is a lower semi continuous function of $f \in \mathbb{D}^S$ for the weak* topology and thus we deduce from the compactness of $\mathbb{D}^S$ (see Lemma 3.6) that the upper semi continuous function $E(fX) - E(\tilde{x}(E[f \cdot |\mathcal{A}]))$, $f \in \mathbb{D}^S$, has a maximum attained for some f_X (which may not be unique). From equation (3.16) it then follows that f_X satifies (3.15). □

4   Sandwich extensions of discrete time systems

We equip the probability space $(\Omega, \mathcal{B}, P)$ with the right-continuous P- augmented filtration $(\mathcal{F}_t)_{t \in [0,T]}$. We assume that, for all t, $\mathcal{F}_t$ is generated by a countable family of events, by which we mean that $\mathcal{F}_t$ is the smallest σ-algebra containing the countable family and all P-null events.

Let $p \in [1, \infty]$. For any time $t \in [0, T]$ $(T > 0)$, consider the linear sub-space:

$$L_t \subseteq L_p(\mathcal{F}_t), \quad L_t \subseteq L_T. \quad (4.1)$$

Let $\mathcal{T} \subseteq [0, T]$ such that $0, T \in \mathcal{T}$. In the sequel we denote $(x_{s,t})_{s,t \in \mathcal{T}}$ on $(L_t)_{t \in \mathcal{T}}$ the system of operators $x_{s,t} : L_t \longrightarrow L_s$ of the type (2.1), for $s, t \in \mathcal{T}$: $s \leq t$.

In financial applications these operators represent a time-consistent system for ask prices in a market with friction. The time s is the price evaluation time of an asset which has payoff at t and the prices are defined on the domain L_t of purchasable assets. Note that, in general, $L_t \subset L_p(\mathcal{F}_t)$ for some $t \in [0, T]$ and $L_t = L_p(\mathcal{F}_t)$ for all $t \in [0, T]$ in a complete market.

Definition 4.1. *The system* $(x_{s,t})_{s,t \in \mathcal{T}}$, *is* time-consistent *(or* $\mathcal{T}$ *time-consistent) if for all* $s, t, u \in \mathcal{T}$: $s \leq t \leq u$

$$x_{s,u}(X) = x_{s,t}\big(x_{t,u}(X)\big), \quad (4.2)$$

for all $X \in L_u$.

Time-consistency is a natural assumption for such system of operators representing, e.g., price processes. This concept models the reasonable equivalence of the price evaluation for an asset with payoff at time u, say, when the evaluation is performed either in one step, i.e. the straight evaluation of the asset at time s, or in two steps, i.e. first an evaluation at time $t : t \leq u$ and then at $s : s \leq t \leq u$. This concept is also proper of a consistent risk measurements and it is studied for dynamic risk measures (where it is called *strong* time-consistency in [1]), see e.g. [11], [6].

Remark 4.1. *For any $s \leq t \leq T$, x_{st} is the restriction to L_t of x_{sT}.*

Indeed let $X \in L_t$, then $x_{tT}(X) = X$, by the projection property. Thus by time-consistency we have $x_{sT}(X) = x_{st}(x_{tT}(X)) = x_{st}(X)$, for all $X \in L_t$.

In the sequel we discuss extension of dynamic systems of operators which will be sandwich preserving. We deal with systems of superlinear and sublinear operators: each one representing the minorant and majorant of one of the operators to be extended. Motivated by applications, a modification of the concept of time-consistency is also necessary. Examples of studies of such minorants and majorants are found in [2], [15], and [9]. It is in this last paper that the general concept of *weak time-consistency* is introduced for the first time in connection with no-good deal bounds. We are now considering again this general definition in this context of convex operators also in view of upcoming applications to the study of ask prices in the context of risk-indifference pricing.

Definition 4.2.
- *The family $(m_{s,t})_{s,t \in \mathcal{T}}$ of weak $\mathcal{F}_s$-homogeneous, superlinear operators $m_{s,t} : L_p(\mathcal{F}_t)^+ \to L_p(\mathcal{F}_s)^+$ is* weak time-consistent *if, for every $X \in L_p(\mathcal{F}_t)^+$,*

$$m_{r,s}(m_{s,t}(X)) \geq m_{r,t}(X), \quad \forall r \leq s \leq t. \tag{4.3}$$

- *The family $(M_{s,t})_{s,t \in \mathcal{T}}$ of weak $\mathcal{F}_s$-homogeneous, sublinear operators $M_{s,t} : L_p(\mathcal{F}_t)^+ \to L_p(\mathcal{F}_s)^+$ is* weak time-consistent *if, for every $X \in L_p(\mathcal{F}_t)^+$,*

$$M_{r,s}(M_{s,t}(X)) \leq M_{r,t}(X), \quad \forall r \leq s \leq t. \tag{4.4}$$

Note that the operators $m_{s,t}$, $M_{s,t}$ are not required to satisfy the projection property.

Definition 4.3. *We say that the family $(m_{s,t}, M_{s,t})_{s,t \in \mathcal{T}}$ satisfies the* mM1-condition *if they are weak time-consistent families of superlinear, respectively sublinear, weak $\mathcal{F}_s$-homogeneous operators such that $m_{s,t}$, $M_{s,t} : L_p(\mathcal{F}_t)^+ \longrightarrow L_p(\mathcal{F}_s)^+$, $m_{0,T}$ is non degenerate, and $M_{s,t}$ is also regular if $p = \infty$.*

Definition 4.4. *We say that the system of operators $(x_{s,t})_{s,t \in \mathcal{T}}$ satisfies the* sandwich condition *when*

$$m_{s,t}(Z) + x_{s,t}(X) \leq M_{s,t}(Y) \tag{4.5}$$

$$\forall X \in L_t \quad \forall Y, Z \in L_p(\mathcal{F}_t)^+ : \quad Z + X \leq Y,$$

for some families of operators $(m_{s,t})_{s,t \in \mathcal{T}}$ and $(M_{s,t})_{s,t \in \mathcal{T}}$ with $m_{s,t}$, $M_{s,t} : L_p(\mathcal{F}_t)^+ \longrightarrow L_p(\mathcal{F}_s)^+$.

4.1 Finite discrete time systems

First of all we consider a finite set $\mathcal{T} := \{s_1, ..., s_K : 0 = s_0 \leq ... \leq s_K\}$.

For $s \leq t$, denoted $\mathbb{D}_{s,t}^S$ the set (3.11) corresponding to $\mathcal{A} = \mathcal{F}_s$, $\mathcal{B} = \mathcal{F}_t$, and to the minorant $m_{s,t}$ and majorant $M_{s,t}$. Analogously for $\mathbb{D}_{s,t}^{S,e} := \mathbb{D}_{s,t}^S \cap \mathbb{D}^e$, cf. (3.13).

Proposition 4.1. *Let us consider the time-consistent system $\big(x_{s,t}\big)_{s,t \in \mathcal{T}}$ on $(L_t)_{t \in \mathcal{T}}$ satisfying the sandwich condition (4.5) with $(m_{s,t}, M_{s,t})_{s,t \in \mathcal{T}}$ fulfilling mM1. For any $i < j$, consider the operators:*

$$\hat{x}_{s_i,s_j}(X) := \operatorname{esssup}_{f \in \mathcal{Q}_{i,j}} \big\{ E[fX|\mathcal{F}_{s_i}] - \alpha_{s_i,s_j}(f) \big\}, \quad X \in L_p(\mathcal{F}_{s_j}), \quad (4.6)$$

with the penalty

$$\alpha_{s_i,s_j}(f) := \sum_{l=i}^{l=j-1} E[\alpha_{s_l,s_{l+1}}(g_{l+1})|\mathcal{F}_i] \tag{4.7}$$

where

$$\alpha_{s_l,s_{l+1}}(g_{l+1}) := \operatorname{esssup}_{X \in L_{s_{l+1}}} \Big\{ E[g_{l+1}X|\mathcal{F}_{s_l}] - x_{s_l,s_{l+1}}(X) \Big\}$$

and

$$\mathcal{Q}_{i,j} := \{ f \in L_q(\mathcal{F}_{s_j})^+ : f = \Pi_{l=i}^{j-1} g_{l+1}, \ g_{l+1} \in \mathbb{D}_{s_l,s_{l+1}}^{S,e} \}$$

with $q = p(p-1)^{-1}$. For all $s \leq t$ in $\mathcal{T}$, the operator $\hat{x}_{s,t}$ extends $x_{s,t}$ on $L_p(\mathcal{F}_t)$. This family of operators $\big(\hat{x}_{s,t}\big)_{s,t \in \mathcal{T}}$ is a time-consistent sandwich preserving extension. Moreover $\big(\hat{x}_{s,t}\big)_{s,t \in \mathcal{T}}$ is maximal, in the sense that, if $\big(\bar{x}_{s,t}\big)_{s,t \in \mathcal{T}}$ is another such family we have that: for all $i < j$,

$$\hat{x}_{s_i,s_j}(X) \geq \bar{x}_{s_i,s_j}(X), \quad X \in L_p(\mathcal{F}_{s_j}).$$

Note that from Corollary 3.2, $\alpha_{s_l,s_{l+1}}(g_{l+1}) = \tilde{x}_{s_l,s_{l+1}}\big(E[g_{l+1} \cdot |\mathcal{F}_{s_l}]\big)$, where $\tilde{x}_{s_s,s_{l+1}}$ is the minimal penalty, see (3.8).

Proof. From Theorem 3.1, for every $i \leq K - 1$, we consider the maximal extension $\hat{x}_{s_i,s_{i+1}}$ of $x_{s_i,s_{i+1}}$. The operator $m_{0,T}$ is non degenerate. It follows from the weak time-consistency of $(m_{s,t})_{s,t \in \mathcal{T}}$ that for all $0 \leq s \leq t \leq T$, the operator $m_{s,t}$ is also non degenerate. From Corollary 3.4, $\hat{x}_{s_i,s_{i+1}}$ admits a representation

$$\hat{x}_{s_i,s_{i+1}}(X) = \operatorname{esssup}_{g \in \mathbb{D}_{s_i,s_{i+1}}^{S,e}} \big\{ E[gX|\mathcal{F}_{s_i}] - \alpha_{s_i,s_{i+1}}(g) \big\} \tag{4.8}$$

$$\mathbb{D}_{s_i,s_{i+1}}^{S,e} = \{ g \in L_q(\mathcal{F}_{s_{i+1}})^+ : E[g|\mathcal{F}_{s_i}] = 1, \ g > 0 \ P \ a.s.$$
$$m_{s_i,s_{i+1}}(X) \leq E[gX|\mathcal{F}_{s_i}] \leq M_{s_i,s_{i+1}}(X), \forall X \in L_p(\mathcal{F}_{s_{i+1}})^+ \}$$

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and

$$\alpha_{s_i,s_{i+1}}(g) := \tilde{x}_{s_i,s_{i+1}}(E[g \cdot |\mathcal{F}_{s_i}])$$
$$= \text{esssup}_{Y \in L_{s_{i+1}}} \left\{ E[gY|\mathcal{F}_{s_i}] - x_{s_i s_{i+1}}(Y) \right\}. \quad (4.9)$$

For any $i < j$ define

$$\hat{x}_{s_i,s_j}(X) := \text{esssup}_{f \in \mathcal{Q}_{i,j}} \left\{ E[fX|\mathcal{F}_{s_i}] - \alpha_{s_i,s_j}(f) \right\}$$

with the penalty

$$\alpha_{s_i,s_j}(f) := \sum_{l=i}^{l=j-1} E[\alpha_{s_l,s_{l+1}}(g_{l+1})|\mathcal{F}_{s_i}],$$

for $f = g_{i+1}g_{i+2}\cdots g_j$ and $\mathcal{Q}_{i,j}$ as in the statement. Note that for any $f \in \mathcal{Q}_{i,j}$ and any set $A \in \mathcal{F}_{s_i}$ we have $Q(A) := E[f1_A] = P(A)$. We remark that the penalties $(\alpha_{i,j})_{i<j}$ satisfy the cocycle condition for the time instants in $\mathcal{T}$.

The operator $x_{s_i,s_{i+1}}$ is weak $\mathcal{F}_{s_i}$-homogeneous, then $\alpha_{s_i,s_{i+1}}$ is local[2]. Observe that, for $A \in \mathcal{F}_{s_i}$, $1_A E[f_1 X|\mathcal{F}_{s_i}] = 1_A E[f_2 X|\mathcal{F}_{s_i}] \ \forall X \in L_p(\Omega, \mathcal{F}_{s_{i+1}}, P)$ is equal to $1_A f_1 = 1_A f_2$. Now we consider an argument by induction and we assume that α_{s_i,s_j} is local. First of all recall that any element $\tilde{f} \in \mathcal{Q}_{s_i,s_{j+1}}$ can be of the form fg where $f \in \mathcal{Q}_{s_i,s_j}$ and $g \in \mathcal{Q}_{s_j,s_{j+1}}$. We consider $1_A f_1 g_1 = 1_A f_2 g_2$. Then $E[1_A f_1 g_1|\mathcal{F}_{s_j}] = E[1_A f_2 g_2|\mathcal{F}_{s_j}]$, which implies that $1_A f_1 = 1_A f_2$. This in turns implies

$$1_A \alpha_{s_i,s_j}(f_1) = 1_A \alpha_{s_i,s_j}(f_2). \quad (4.10)$$

Notice that $f_1 > 0$ P a.s.. It follows that $1_A g_1 = 1_A g_2$. Hence

$$1_A \alpha_{s_j,s_{j+1}}(g_1) = 1_A \alpha_{s_j,s_{j+1}}(g_2). \quad (4.11)$$

From (4.10) and (4.11) we conclude that:

$$1_A \alpha_{s_i,s_{j+1}}(f_1 g_1) = 1_A \alpha_{s_i,s_{j+1}}(f_2 g_2)$$

by the definition of $\alpha_{s_i,s_{j+1}}$ (4.7). Hence $\alpha_{s_i,s_{j+1}}$ is local as well. The cocycle condition and the local property together imply the time-consistency of the system of operators $(\hat{x}_{s,t})_{s,t \in \mathcal{T}}$, see [6, Theorem 4.4].

To conclude we show that the family $(\hat{x}_{s,t})_{s,t \in \mathcal{T}}$ constitute a maximal extension. Indeed we have that, for all i,

$$\hat{x}_{s_i,s_{i+1}}(X) \geq \bar{x}_{s_i,s_{i+1}}(X), \quad X \in L_p(\mathcal{F}_{s_{i+1}}).$$

---

[2]i.e. for $f, g \in \mathcal{Q}_{s_i,s_{i+1}}$ and for $A \in \mathcal{F}_{s_i}$, the assertion $1_A E[fX|\mathcal{F}_{s_i}] = 1_A E[gX|\mathcal{F}_{s_i}] \ \forall X \in L_p(\Omega, \mathcal{F}_{s_{i+1}}, P)$ implies $1_A \alpha_{s_i,s_{i+1}}(f) = 1_A \alpha_{s_i,s_{i+1}}(g)$. See [6, Definition 4.1].

We proceed then by induction on h such that $j = i + h$. Let $i < l < j$

$$\hat{x}_{s_i,s_j}(X) = \hat{x}_{s_i,s_l}(\hat{x}_{s_l,s_j}(X)) \geq \bar{x}_{s_i,s_l}(\hat{x}_{s_l,s_j}(X))$$
$$\geq \bar{x}_{s_i,s_l}(\bar{x}_{s_l,s_j}(X)) = \bar{x}_{s_i,s_j}(X), \quad X \in L_p(\mathcal{B}).$$

By this we end the proof. □

Corollary 4.2. *For each $X \in L_p(\mathcal{F}_{s_j})$, there exists f_X in $\mathcal{Q}_{i,j}$ such that*

$$\hat{x}_{s_i,s_j}(X) = E\big[f_X X | \mathcal{F}_{s_i}\big] - \alpha_{s_i,s_j}(f_X).$$

Proof. For $i = j - 1$ apply Proposition 3.7:

$$\hat{x}_{s_{j-1},s_j}(X) = E\big[f_{X,j} X | \mathcal{F}_{s_{j-1}}\big] - \alpha_{s_{j-1},s_j}(f_{X,j}).$$

From Lemma 3.3, $f_{X,j}$ belongs to $\mathbb{D}_{s_i,s_{i+1}}^{S,e}$. From the time-consistency of $(\hat{x}_{s,t})_{s,t\in\mathcal{T}}$ and the definition of α_{s_i,s_j} in (4.7) we have

$$f_X = \prod_{l=i}^{j-1} f_{X,l+1}.$$

By this we end the proof. □

4.2 Countable discrete time systems

Let us now consider a countable set $\mathcal{T} \subset [0,T]$, with $0,T \in \mathcal{T}$, and a sequence of finite sets $(\mathcal{T}_n)_{n=1}^\infty$: $\mathcal{T}_n \subseteq \mathcal{T}_{n+1}$, such that $\mathcal{T} = \cup_{n=1}^\infty \mathcal{T}_n$. Let us consider the time-consistent system $(x_{s,t})_{s,t\in\mathcal{T}}$ on $(L_t)_{t\in\mathcal{T}}$ satisfying the sandwich condition (4.5) with $(m_{s,t}, M_{s,t})_{s,t\in\mathcal{T}}$ fulfilling mM1.

Lemma 4.3. *For any n, let $(x_{s,t}^n)_{s,t\in\mathcal{T}_n}$ be the maximal time-consistent sandwich preserving extensions of $(x_{s,t})_{s,t\in\mathcal{T}_n}$. Now consider $s,t,\in\mathcal{T}$. Let $n_0 \in \mathbb{N}$ such that $s,t \in \mathcal{T}_{n_0}$. Then, for any $n > n_0$ and $X \in L_p(\mathcal{F}_t)$, the sequence $(x_{s,t}^n(X))_{n>n_0}$ is non increasing P a.s. Hence it admits a limit*

$$\hat{x}_{s,t}(X) := \lim_{n\to\infty} x_{s,t}^n(X). \tag{4.12}$$

Moreover, for $n > n_0$, let $\alpha_{s,t}^n$ be the minimal penalty associated to $x_{s,t}^n$. This penalty has representation

$$\alpha_{s,t}^n(Q) := \operatorname{esssup}_{X\in L_p(\mathcal{F}_s)}(E_Q(X|\mathcal{F}_s) - x_{s,t}^n(X)), \tag{4.13}$$

for all probabilility measure $Q \sim P$, where $\alpha_{s,t}^n(Q) = \alpha_{s,t}^n(f)$ with $f = \frac{dQ}{dP}$. Then, for all $Q \sim P$, the sequence $(\alpha_{s,t}^n(Q))_{n>n_0}$ is non negative and non decreasing P a.s.. Hence it admits a limit

$$\hat{\alpha}_{s,t}(Q) := \lim_{n\to\infty} \alpha_{s,t}^n(Q). \tag{4.14}$$

Proof. The extensions $(x_{s,t}^n)_{s,t\in\mathcal{T}_n}$ are maximal over all other sandwich preserving extensions time-consistent on $\mathcal{T}_n$. Then, for $s,t,\in\mathcal{T}$ and $n > n_0$, we can regard the extension $x_{s,t}^{n+1}$ as another sandwich preserving extension of $x_{s,t}$, $(x_{s,t}^{n+1})_{s,t\in\mathcal{T}_n}$ is time-consistent on $\mathcal{T}_n$. Thus $x_{s,t}^n(X) \geq x_{s,t}^{n+1}(X)$.
From Corollary 4.2, $x_{s,t}^n$ admits a representation with equivalent probability measures. The result for $\alpha_{s,t}^n(Q)$, $Q \sim P$, is then an immediate consequence of equation (4.13). $\qquad\square$

Theorem 4.4. *Let us consider the discrete time-consistent system $\big(x_{s,t}\big)_{s,t\in\mathcal{T}}$ on $(L_t)_{t\in\mathcal{T}}$ satisfying the sandwich condition (4.5) with mM1. Then each operator in this family admits an extension to the whole $L_p(\mathcal{F}_t)$ with values in $L_p(\mathcal{F}_s)$ satisfying the sandwich condition and such that the family of extensions is time-consistent. In particular, the family of operators $(\hat{x}_{s,t})_{s,t\in\mathcal{T}}$ given in Lemma 4.3 is a time-consistent and sandwich preserving extension of $\big(x_{s,t}\big)_{s,t\in\mathcal{T}}$. Moreover, for any $s \leq t$, the operators $\hat{x}_{s,t}$ (4.12) and $\hat{\alpha}_{s,t}$ (4.14) satisfy the relationship:*

$$\hat{x}_{s,t}(X) = \mathrm{esssup}_{Q\sim P}(E_Q[X|\mathcal{F}_s] - \hat{\alpha}_{s,t}(Q))$$
$$= \mathrm{esssup}_{f\in\mathbb{D}_{s,t}^{S,e}}(E[fX|\mathcal{F}_s] - \hat{\alpha}_{s,t}(f)), \quad X \in L_p(\mathcal{F}_t). \qquad (4.15)$$

Moreover, for all X there is $f_X \in \mathbb{D}_{s,t}^{S,e}$ such that

$$\hat{x}_{s,t}(X) = E(f_X X|\mathcal{F}_s) - \hat{\alpha}_{s,t}(f_X). \qquad (4.16)$$

This extension is maximal, in the sense that, for any other such extension $\big(\bar{x}_{s,t}\big)_{s,t\in\mathcal{T}}$ we have that: for all $s < t \in \mathcal{T}$,

$$\hat{x}_{s,t}(X) \geq \bar{x}_{s,t}(X), \quad X \in L_p(\mathcal{F}_t).$$

Also for all $s,t \in \mathcal{T}$, $\hat{\alpha}_{s,t}$ is the minimal penalty associated to $\hat{x}_{s,t}$.

Proof. In Lemma 4.3 we have defined, for all $s,t \in \mathcal{T}$,

$$\hat{x}_{s,t}(X) := \lim_{k\to\infty} x_{s,t}^k(X), \qquad X \in L_p(\mathcal{F}_t),$$

where $x_{s,t}^k$ is the maximal extension of $x_{s,t}$ on $\mathcal{T}_k$ and for $f \in \mathbb{D}_{s,t}^{S,e}$ with $s,t \in \mathcal{T}_k$, we have set $\hat{\alpha}_{s,t}(f) := \hat{\alpha}_{s,t}(Q)$, $\alpha_{s,t}^k(f) = \alpha_{s,t}^k(Q)$, where $f = dQ/dP$, with

$$\hat{\alpha}_{s,t}(f) := \lim_{k\to\infty} \alpha_{s,t}^k(f), \qquad f \in \mathbb{D}_{s,t}^{S,e}$$

Step 1: Proof of the representations (4.15), (4.16) *and the sandwich property.*
Let $s,t \in \mathcal{T}$. Fix $X \in L_p(\mathcal{F}_t)$. For every k, from Corollary 4.2, there is $f_{X,k} \in \mathbb{D}_{s,t}^{S,e}$ such that

$$x_{s,t}^k(X) = E(f_{X,k}X|\mathcal{F}_s) - \alpha_{s,t}^k(f_{X,k}). \qquad (4.17)$$

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From Lemma 3.6 the set $\mathbb{D}_{s,t}^{S}$ is compact for the weak* topology, thus there is a subsequence of $(f_{X,k})_k$ converging to $f_X \in \mathbb{D}_{s,t}^{S}$. Without loss of generality we can assume that the sequence $(f_{X,k})_k$ itself has the limit f_X (for the weak* topology). Fix $n > n_0$. From equation (4.13), $\alpha_{s,t}^{n}$ is lower semi continuous for the weak* topology thus

$$\alpha_{s,t}^{n}(f_X) \leq \liminf_{k\to\infty} \alpha_{s,t}^{n}(f_{X,k}).$$

From Lemma 3.3 it follows that $f_X \in \mathbb{D}_{s,t}^{S,e}$. From Lemma 4.3, for given k, the sequence $(\alpha_{s,t}^{n}(f_{X,k}))_n$ is non decreasing. Therefore for every $k \geq n$, $\alpha_{s,t}^{n}(f_{X,k}) \leq \alpha_{s,t}^{k}(f_{X,k})$. Thus by (4.17),

$$\alpha_{s,t}^{n}(f_X) \leq \liminf_{k\to\infty} \Big(E(f_{X,k}X|\mathcal{F}_s)) - x_{s,t}^{k}(X) \Big).$$

Passing to the limit as $k \to \infty$, we get the inequality

$$\alpha_{s,t}^{n}(f_X) \leq E(f_X X|\mathcal{F}_s)) - \hat{x}_{s,t}(X).$$

Letting $n \to \infty$

$$\hat{\alpha}_{s,t}(f_X) \leq E(f_X X|\mathcal{F}_s)) - \hat{x}_{s,t}(X). \tag{4.18}$$

On the other hand, for every $Q \sim P$, and $Y \in L_p(\mathcal{F}_t)$, for every n, $\alpha_{s,t}^{n}(Q) \geq E_Q(Y|\mathcal{F}_s)) - x_{s,t}^{n}(Y)$. Passing to the limit this gives:

$$\hat{\alpha}_{s,t}(Q) \geq E_Q(Y|\mathcal{F}_s)) - \hat{x}_{s,t}(Y). \tag{4.19}$$

It follows that

$$\hat{x}_{s,t}(X) = E(f_X X|\mathcal{F}_s)) - \hat{\alpha}_{s,t}(f_X).$$

Then from the above equation and (4.19) we have proved the representations (4.15) and (4.16).

Notice that the sandwich condition follows from the sandwich condition for $x_{s,t}^{n}$ passing to the limit for $n \to \infty$, see (4.12).

Step 2: Time-consistency.
From (4.15), $\hat{x}_{s,t}$ is lower semi continuous. From the definition of $\hat{x}_{s,t}$ as the limit of $x_{s,t}^{n}$ it follows that $\hat{x}_{s,t}$ extends $x_{s,t}$ for every $s, t \in \mathcal{T}$. On the other hand for every $r \leq s \leq t$ in $\mathcal{T}$ and every n large enough such that r, s, t belong to $\mathcal{T}_n$, we already know that $(x_{s,t}^{n})_{s,t\in\mathcal{T}}$ is $\mathcal{T}_n$ time-consistent. We recall that the minimal penalty of a time-consistent family of operators satisfies the local property [6, Lemma 2.3] and the cocycle condition [6, Theorem 2.5]. The family of penalties $(\alpha_{s,t}^{n})_{s,t\in\mathcal{T}}$ satisfies the cocycle condition:

$$\alpha_{r,t}^{n}(Q) = \alpha_{r,s}^{n}(Q) + E_Q(\alpha_{s,t}^{n}(Q)|\mathcal{F}_r), \quad Q \sim P.$$

Hence, passing to the limit for the non decreasing sequence $(\alpha_{s,t}^{n})_{s,t\in\mathcal{T}}$ we get the cocycle condition:

$$\hat{\alpha}_{r,t}(Q) = \hat{\alpha}_{r,s}(Q) + E_Q(\hat{\alpha}_{s,t}(Q)|\mathcal{F}_r), \quad Q \sim P.$$

The local property is obtained in the same way. And thus from [6, Theorem 4.4], $(\hat{x}_{s,t})_{s,t \in \mathcal{T}}$ is time-consistent.

Step 3: Maximality of $\hat{x}_{s,t}$.

Notice that if another family $\overline{x}_{s,t}$ satisfies all the above properties. Necessarily for all $s, t \in \mathcal{T}$ and n large enough such that $s, t \in \mathcal{T}_n$, from the maximal property of $x_{s,t}^n$ it follows that $\overline{x}_{s,t}(X) \leq x_{s,t}^n(X)$. Thus passing to the limit we get the maximality for $\hat{x}_{s,t}$.

Step 4: Minimality of $\hat{\alpha}_{s,t}$.

To see that $\hat{\alpha}_{s,t}$ is the minimal penalty associated to $\hat{x}_{s,t}$, we proceed as follows. From Lemma 4.3, the sequence $(x_{s,t}^n(X))_{n > n_0}$ is non increasing P a.s.. We already know that $\alpha_{s,t}^n$ is the minimal penalty associated to $(x_{s,t}^n(X))$.Thus

$$\alpha_{st}^n(Q) = \text{esssup}_{X \in L_p(\mathcal{F}_t)} \big(E_Q(X|\mathcal{F}_s) - x_{s,t}^n(X) \big)$$
$$\leq \text{esssup}_{X \in L_p(\mathcal{F}_t)} \big(E_Q(X|\mathcal{F}_s) - \hat{x}_{s,t}(X) \big).$$

Passing to the limit, we have

$$\hat{\alpha}_{st}(Q) \leq \text{esssup}_{X \in L_p(\mathcal{F}_t)} \big(E_Q(X|\mathcal{F}_s) - \hat{x}_{s,t}(X) \big).$$

On the other hand, for all X we have

$$\alpha_{st}^n(Q) \geq E_Q(X|\mathcal{F}_s) - x_{s,t}^n(X).$$

Passing to the limit, we have

$$\hat{\alpha}_{st}(Q) \geq E_Q(X|\mathcal{F}_s) - \hat{x}_{s,t}(X), \ \ \forall X \in L_p(\mathcal{F}_t).$$

Hence

$$\hat{\alpha}_{st}(Q) = \text{esssup}_{X \in L_p(\mathcal{F}_t)} \big(E_Q(X|\mathcal{F}_s) - \hat{x}_{s,t}(X) \big).$$

$\square$

5 Sandwich extensions of continuous time systems of operators

In this section we study sandwich preserving extensions for a system of operators $(x_{s,t})_{s,t \in [0,T]}$. These extensions are time-consistent. We stress that to obtain a time-consistent extension it is not enough to collect all the extensions of single operators in one family. Time-consistency is achieved with some careful procedure of extension involving the representation of the operators and an appropriate passage from discrete to continuous time. For this we first define the system of majorant and minorant operators serving as bounds in the sandwiches.

Definition 5.1. *We say that the family $(m_{s,t}, M_{s,t})_{s,t \in [0,T]}$ satisfies the mM2-condition if*

1. *mM1 is satisfied (Definition 4.3);*

2. $\mathrm{esssup}_{s \leq T}(M_{s,T}(X))$ *belongs to* $L_p(\mathcal{F}_T)^+$ *for all* $X \in L_p(\mathcal{F}_T)^+$;

3. *for every* $X \in L_p(\mathcal{F}_t)^+$,

$$m_{s,t}(X) = lim_{t' > t, t' \downarrow t} m_{st'}(X); \tag{5.1}$$

4. *for every* $X \in L_p(\mathcal{F}_t)^+$,

$$M_{s,t}(X) = lim_{t' > t, t' \downarrow t} M_{st'}(X); \tag{5.2}$$

5. *for every* $X \in L_p(\mathcal{F}_t)^+$,

$$m_{s,t}(X) \leq \limsup_{s' > s, s' \downarrow s} m_{s't}(X); \quad M_{s,t}(X) \geq \liminf_{s' > s, s' \downarrow s} M_{s't}(X); \tag{5.3}$$

Let $\mathcal{T}$ be a countable dense subset of $[0, T]$ containing 0 and T.

Definition 5.2. *A system* $(x_{s,t})_{s,t \in [0,T]}$ *on* $(L_t)_{t \in [0,T]}$ *is right-continuous if for all* t, *all* $X \in L_t$, *and all sequences* $(s_n)_n$, $s < s_n \leq t$, $s_n \downarrow s$, $x_{s,t}(X) = \lim_{n \to \infty} x_{s_n,t}(X)$, *where the convergence is* P *a.s.*

Lemma 5.1. *Assume mM2 condition. Let* $(\hat{x}_{s,t})_{s,t \in \mathcal{T}}$ *and* $(\hat{\alpha}_{s,t})_{s,t \in \mathcal{T}}$ *be as in Lemma 4.3. There is a probability measure* Q_0 *equivalent to* P *such that for all* $s, t \in \mathcal{T}$, $0 \leq s \leq t \leq T$, $\hat{\alpha}_{s,t}(Q_0) = 0$.

Proof. From Theorem 4.4, there is a probability measure Q_0 such that $0 = \hat{x}_{0,T}(0) = -\hat{\alpha}_{0,T}(Q_0)$. From Lemma 3.3, Q_0 is equivalent to P. It follows from the $\mathcal{T}$-cocycle condition and the non negativity of the penalty that $\hat{\alpha}_{s,t}(Q_0) = 0$ for all $s \leq t$ in $\mathcal{T}$. $\qquad\square$

Proposition 5.2. *The notations are those of Lemma 5.1.*

1. *For all* $X \in L_p(\mathcal{F}_T)$, $(\hat{x}_{s,T}(X))_{s \in \mathcal{T}}$ *is a* Q_0-*supermartingale.*

2. *For every sequence* $(s_n)_n$ *in* $\mathcal{T}$ *decreasing to* $s \in \mathcal{T}$, $E_{Q_0}(\hat{x}_{s_n,T}(X))$ *has the limit* $E_{Q_0}(\hat{x}_{s,T}(X))$, *for all* $X \in L_p(\mathcal{F}_T)$.

Proof. 1. It follows easily from Lemma 5.1, the $\mathcal{T}$ time-consistency, and the representation of $\hat{x}_{s,T}(X)$ (4.15).

2. The proof is inspired by the one of Lemma 4 in [8]. The main differences are the facts that here $\hat{x}_{s,t}$ is only defined for s, t in $\mathcal{T}$ and that the operator is defined on $L \subset L_p(\mathcal{F}_t)$ $1 \leq p \leq \infty$, while in [8] the dynamic risk measure $\rho_{s,t}$ was defined on $L_\infty(\mathcal{F}_t)$ and time-consistency was considered for all real indexes.

Let $X \in L_p(\mathcal{F}_T)$ and $s \in \mathcal{T}$. From Theorem 4.4, there is an $f_X \in \mathbb{D}_{s,T}^{S,e}$ such that $\hat{x}_{s,T}(X) = E(f_X X | \mathcal{F}_s) - \hat{\alpha}_{s,T}(f_X)$. Let R_X be the probability measure such that $\frac{dR_X}{dP} = f_X$. It follows from the cocycle condition that

$$\hat{x}_{s,T}(X) = E_{R_X}(X|\mathcal{F}_s) - \hat{\alpha}_{s,T}(R_X)$$
$$= E_{R_X}[E_{R_X}(X|\mathcal{F}_{s_n}) - \hat{\alpha}_{s_n,T}(R_X)]|\mathcal{F}_s) - \hat{\alpha}_{ss_n}(R_X). \qquad (5.4)$$

Furthermore $E_{R_X}(X|\mathcal{F}_{s_n}) - \hat{\alpha}_{s_n,T}(R_X) \leq \hat{x}_{s_n,T}(X)$ and the penalties are non negative. Hence, we have that

$$\hat{x}_{s,T}(X) \leq E_{R_X}(\hat{x}_{s_n,T}(X)|\mathcal{F}_s) \qquad (5.5)$$

and

$$\hat{x}_{s,T}(X) \leq E_{R_X}(\hat{x}_{s_n,T}(X)|\mathcal{F}_s) = E(f_X \hat{x}_{s_n,T}(X)|\mathcal{F}_s)$$
$$= E(E(f_X|\mathcal{F}_{s_n})\hat{x}_{s_n,T}(X)|\mathcal{F}_s).$$

Let $g \in L_q(\mathcal{F}_s)$ be the Radon Nykodym derivative of the restriction of Q_0 to $\mathcal{F}_s$. Taking the Q_0 expectation, g being $\mathcal{F}_s$-measurable, we obtain

$$E_{Q_0}(\hat{x}_{s,T}(X)) \leq E[gE(f_X|\mathcal{F}_{s_n})\hat{x}_{s_n,T}(X)] \qquad (5.6)$$
$$= E(g(\hat{x}_{s_n,T}(X)) + E[\hat{x}_{s_n,T}(X)(g(E(f_X|\mathcal{F}_{s_n}) - 1))]$$

The density f_X belongs to $\mathbb{D}_{s,T}^S$ and g belongs to $\mathbb{D}_{0,s}^S$, so gf_X belongs to $L_q(\mathcal{F}_T)$ and $g(E(f_X|\mathcal{F}_{s_n}) - 1)$ has limit 0 in $L_q(\mathcal{F}_T)$.

For all X, $|\hat{x}_{s_nT}(X)| \leq \hat{x}_{s_n,T}(|X|) \leq M_{s_n,T}(|X|)$. From property 2. in Definition 5.1, $\sup_n E((|\hat{x}_{s_n,T}(X)|^p) < \infty$. It follows from Hölder inequality that $\epsilon_n(X) := E[\hat{x}_{s_n,T}(X)(g(E(f_X|\mathcal{F}_{s_n}) - 1))]$ has limit 0. Similarly,

$$\delta_n(X) := -E_{Q_0}(\hat{x}_{s_n,T}(X)) + E(g\hat{x}_{s_n,T}(X))$$
$$= -E\big([E(\frac{dQ_0}{dP}|\mathcal{F}_{s_n}) + E(\frac{dQ_0}{dP}|\mathcal{F}_s)]\hat{x}_{s_n,T}(X)\big)$$

has limit 0. Observe that, from the Q_0-supermartingale property (point 1.), if follows that

$$E_{Q_0}(\hat{x}_{s_n,T}(X)) \leq E_{Q_0}(\hat{x}_{s,T}(X)).$$

Then from (5.6) we obtain that

$$E_{Q_0}(\hat{x}_{s,T}(X)) - \epsilon_n(X) - \delta_n(X) \leq E_{Q_0}(\hat{x}_{s_n,T}(X)) \leq E_{Q_0}(\hat{x}_{s,T}(X)).$$

This proves the result.

$\square$

Theorem 5.3. *Let us consider a right-continuous time-consistent system of operators $\left(x_{s,t}\right)_{s,t\in[0,T]}$ of type (2.1) defined on $(L_t)_{t\in[0,T]}$ satisfying the sandwich condition with mM2.*
Then there is a right-continuous, time-consistent, sandwich preserving extension $\left(\hat{x}_{s,t}\right)_{s,t\in[0,T]}$ defined on the whole $(L_p(\mathcal{F}_t))_{t\in[0,T]}$. One such extension can be represented as

$$\hat{x}_{s,t}(X) = \operatorname{esssup}_{R\in\mathcal{R}}[E_R(X|\mathcal{F}_s) - \hat{\alpha}_{s,t}(R)], \quad X \in L_p(\mathcal{F}_t), \tag{5.7}$$

with

$$\mathcal{R} := \{R \sim P : \ \hat{\alpha}_{0,T}(R) < \infty\} \tag{5.8}$$

and $\hat{\alpha}_{0,T}$ is the minimal penalty associated to $\hat{x}_{0,T}$ as in Theorem 4.4. Also for any $X \in L_p(\mathcal{F}_t)$, there exists $R_X \in \mathcal{R}$ such that

$$\hat{x}_{s,t}(X) = E_{R_X}(X|\mathcal{F}_s) - \hat{\alpha}_{s,t}(R_X) \quad \forall s \leq t.$$

Furthermore for all $t > 0$, and all $X \in L_p(\mathcal{F}_t)$, $\hat{x}_{s,t}(X)_{0\leq s\leq t}$ admits a càdlàg version.

Proof. Step 1: Definition of the extension $\hat{x}_{s,t}$ for indices in the whole $[0,T]$.
Let $\mathcal{T}$ be a countable dense subset of $[0,T]$ containing 0 and T. Let $\left(\hat{x}_{s,t}\right)_{s,t\in\mathcal{T}}$ be the time-consistent extension of $\left(x_{s,t}\right)_{s,t\in\mathcal{T}}$ constructed in Section 4.2. Let $X \in L_p(\mathcal{F}_t)$. From Proposition 5.2 and from the Modification Theorem (Chapter VI Section 1 in [13]) applied to the Q_0-supermartingale $(\hat{x}_{s,T}(X))_{s\in\mathcal{T}}$, it follows that $(\hat{x}_{s,T}(X))_{s\in\mathcal{T}}$ admits a modification which can be extended into a càdlàg process $(\hat{x}_{s,T}(X))_{s\in[0,T]}$ defined for all $s \in [0,T]$. Notice that from Remark 4.1, $x_{s,t}$ coincides on L_t with the restriction of $x_{s,T}$. For $0 \leq s \leq t \leq T$, we define $\hat{x}_{s,t}$ to be the restriction of $\hat{x}_{s,T}$ on $L_p(\mathcal{F}_t)$. It follows that $\hat{x}_{s,t}$ is an extension of $x_{s,t}$ to the whole $L_p(\mathcal{F}_t)$. If $s,t \in \mathcal{T}$ this extension coincides also with the construction of $\hat{x}_{st}$ given in the previous section. To see this, consider the $\mathcal{T}$ time-consistency of $\hat{x}_{st}$ (of the previous section) and its projection property.
Step 2: Extension of the penalty on a set $\mathcal{R}$ of probability measures and right-continuity.
Consider the penalties $(\hat{\alpha}_{s,t})_{s,t\in\mathcal{T}}$ associated to $(\hat{x}_{s,t})_{s,t\in\mathcal{T}}$ given in Section 4.2. Define
$$\mathcal{R} := \{R \sim P : \ \hat{\alpha}_{0,T}(R) < \infty\}$$

Let X in $L_p(\mathcal{F}_T)$. From Theorem 4.4 (see (4.16)), there is a probability measure $R_X \sim P$ such that

$$\hat{x}_{0,T}(X) = E_{R_X}(X) - \hat{\alpha}_{0,T}(R_X). \tag{5.9}$$

Thus $\mathcal{R}$ is non empty.

Let $s \in [0,T]$ and consider $(s_n)_n \subset \mathcal{T}$, $s_n \downarrow s$. For every probability measure $R \in \mathcal{R}$, the sequence $E_R(\hat{\alpha}_{s_n,T}(R)|\mathcal{F}_s)$ is increasing. Thus it admits a limit. From Section 4.2 we can see that, if s belongs to $\mathcal{T}$, this limit is equal to $\hat{\alpha}_{sT}(R)$. Indeed it is enough to exploit the representation as minimal penalty. Then by right-continuity of the filtration and the càdlàg extension of Step 1 we have

$$\hat{\alpha}_{s,T}(R) \geq \lim_{n \to \infty} E_R[\hat{\alpha}_{s_n,T}(R)|\mathcal{F}_s]$$
$$\geq E_R[\text{esssup}_{X \in L_p(\mathcal{F}_T)} \lim_{n \to \infty} (E_R[X|\mathcal{F}_{s_n}] - \hat{x}_{s_n,T}(R)|\mathcal{F}_s] = \hat{\alpha}_{s,T}(R).$$

If $s \notin \mathcal{T}$ we can define $\hat{\alpha}_{sT}(R)$ as the limit of $E_R(\hat{\alpha}_{s_n,T}(R)|\mathcal{F}_s) < \infty$ R a.s.. Moreover, for $r,s \in \mathcal{T}$, due to the $\mathcal{T}$ time-consistency proved in Section 4.2, $\hat{\alpha}_{r,s}(R)$ satisfies the cocycle condition on $\mathcal{T}$. Note that $\sup_{s \leq T} E_R(\hat{\alpha}_{sT}(R)) = \hat{\alpha}_{0T}(R) < \infty$. Then we can define for all $0 \leq r \leq s \leq T$

$$\hat{\alpha}_{r,s}(R) := \hat{\alpha}_{rT}(R) - E_R(\hat{\alpha}_{s,T}(R)|\mathcal{F}_r) \tag{5.10}$$

Thus $\hat{\alpha}_{r,s}(R)$ is now defined for all indices in $[0,T]$ and all $R \in \mathcal{R}$. Moreover $\hat{\alpha}_{r,s}(R)$ satisfies the cocycle condition. It follows also from the right-continuity of $\hat{\alpha}_{s,T}(R)$ in s and from the cocycle condition that for all $R \in \mathcal{R}$, $\hat{\alpha}_{r,s}(R)$ is also right-continuous in s.

Step 3: Representation of $\hat{x}_{r,t}(X)$.

Let $0 \leq r \leq t \leq T$. Let $X \in L_p(\mathcal{F}_t)$. Let $R_X \in \mathcal{R}$ such that (5.9) is satisfied. Now we prove that, for all $r \leq t$,

$$\hat{x}_{r,t}(X) = E_{R_X}(X|\mathcal{F}_r) - \hat{\alpha}_{r,t}(R_X). \tag{5.11}$$

In fact, making use of the cocycle condition and then of the $\mathcal{T}$ time-consistency, we get that for all $r \in \mathcal{T}$,

$$\hat{x}_{0,T}(X) = E_{R_X}(E_{R_X}(X|\mathcal{F}_r) - \hat{\alpha}_{r,T}(R_X)) - \hat{\alpha}_{0,r}(R_X)$$
$$\leq E_{R_X}(\hat{x}_{r,T}(X)) - \hat{\alpha}_{0,r}(R_X) \leq \hat{x}_{0,r}(\hat{x}_{r,T}(X)) = \hat{x}_{0,T}(X).$$

Thus every inequality in the expression above is an equality. In particular

$$\hat{x}_{r,T}(X) = E_{R_X}(X|\mathcal{F}_r) - \hat{\alpha}_{r,T}(R_X), \quad R_X \text{ a.s. } \forall r \in \mathcal{T}, \tag{5.12}$$

and

$$\hat{x}_{0,r}(\hat{x}_{r,T}(X)) = E_R(\hat{x}_{r,T}(X)) - \hat{\alpha}_{0,r}(R_X), \quad R_X \text{ a.s. } \forall r \in \mathcal{T}. \tag{5.13}$$

Consider a sequence $(r_n)_n \subset \mathcal{T}$, $r_n \downarrow r$, with $r \in [0, T]$. Passing to the limit in the corresponding equation to (5.12), we can see that

$$\hat{x}_{r,T}(X) = E_{R_X}(X|\mathcal{F}_r) - \hat{\alpha}_{r,T}(R_X), \quad R_X \text{ a.s. } \forall r \in [0, T]. \quad (5.14)$$

We have assumed that $X \in L_p(\mathcal{F}_t)$ thus $\hat{x}_{t,T}(X) = X$. It follows then from (5.14) applied with $r = t$, that $\hat{\alpha}_{t,T}(R_X) = 0$.
Then, again from (5.14) and the cocycle condition for $R_X \in \mathcal{R}$, it follows that, for all $X \in L_p(\mathcal{F}_t)$,

$$\hat{x}_{r,t}(X) = \hat{x}_{r,T}(X) = E_{R_X}(X|\mathcal{F}_r) - \hat{\alpha}_{r,t}(R_X) \; R_X \text{ a.s. } \forall r, t \in [0, T]. \quad (5.15)$$

Step 4: Another representation of $\hat{x}_{r,t}$ for all $0 \le r \le t \le T$.
Let $X \in L_p(\mathcal{F}_t)$. We will prove that $\hat{x}_{r,t}(X)$ admits the following representation:

$$\begin{aligned} \hat{x}_{r,t}(X) &= \text{esssup}_{R \in \mathcal{R}}[E_R(X|\mathcal{F}_r) - \hat{\alpha}_{r,t}(R)] \\ &= E_{R_X}(X|\mathcal{F}_r) - \hat{\alpha}_{r,t}(R_X), \end{aligned} \quad (5.16)$$

where R_X satisfies equation (5.9). For all r, t in $\mathcal{T}$, being $\hat{\alpha}_{r,t}$ the minimal penalty, we have that for all $R \in \mathcal{R}$,

$$\hat{\alpha}_{r,t}(R) \ge E_R(X|\mathcal{F}_r) - \hat{x}_{r,t}(X), \; \forall X \in L_p(\mathcal{F}_t). \quad (5.17)$$

The above equation can also be written

$$\hat{\alpha}_{r,t}(R) \ge E_R(X|\mathcal{F}_r) - \hat{x}_{r,T}(X), \; \forall X \in L_p(\mathcal{F}_t). \quad (5.18)$$

Exploiting the right-continuity of the filtration and of $\hat{\alpha}_{r,t}$ and $\hat{x}_{r,t}$ as discussed in Steps 1 and 2, we can see that, passing to the limit in r and then to the limit in t, equation (5.18) and thus also (5.17) are satisfied for all $r \le t$ in $[0, T]$. Equation (5.16) follows then from (5.15) and (5.17).

Step 5 : Time consistency of $\hat{x}_{s,t}$.
Set $0 \le r \le s \le t \le T$. Let $X \in L_p(\mathcal{F}_t)$. Let $R_X \in \mathcal{R}$ such that (5.9) is satisfied. From (5.16), for all $Y \in L_p(\mathcal{F}_s)$, $\hat{x}_{r,s}(Y) \ge E_{R_X}(Y|\mathcal{F}_r) - \hat{\alpha}_{r,s}(R_X)$. With $Y = \hat{x}_{s,t}(X)$, making use of equation (5.15) and of the cocycle condition, it follows that

$$\hat{x}_{r,s}(\hat{x}_{s,t}(X)) \ge \hat{x}_{r,t}(X), \quad \forall r \le s \le t. \quad (5.19)$$

On the other hand, let $R_Y \in \mathcal{R}$ such that

$$\hat{x}_{0,T}(Y) = E_{R_Y}(Y) - \alpha_{0,T}(R_Y) \quad (5.20)$$

(cf. (5.9)). From equation (5.15), it follows that for all $r, s : r \le s$,

$$\hat{x}_{r,s}(Y) = E_{R_Y}(Y|\mathcal{F}_r) - \hat{\alpha}_{r,s}(R_Y) \; R_Y \text{ a.s.} \quad (5.21)$$

We already know that $(\hat{x}_{s,t})_{s,t\in\mathcal{T}}$ satisfies $\mathcal{T}$ time consistency, and the sandwich condition. From the right-continuity of $(\hat{x}_{s,t}(X))_{s\in[0,t]}$ for all $t \geq 0$ (see Step 1) and the hypothesis mM2, item 3, it follows that $\mathrm{esssup}_{s\leq t}|\hat{x}_{s,t}(X)|$ belongs to $L_p(\mathcal{F}_t)^+$. Let $s_n \in \mathcal{T}$ such that $s_n \downarrow s$. From the right-continuity and the dominated convergence theorem, it follows that $E_{R_Y}(\hat{x}_{s,t}(X)|\mathcal{F}_r) = \lim_{n\to\infty} E_{R_Y}(\hat{x}_{s_n,t}(X)|\mathcal{F}_r)$. The probability measure R_Y belongs to $\mathcal{R}$ thus from step 2, $\hat{\alpha}_{r,s}(R_Y) = \lim_{n\to\infty} \hat{\alpha}_{r,s_n}(R_Y)$. From equation (5.21), we then get that

$$\hat{x}_{r,s}(\hat{x}_{s,t}(X)) = \hat{x}_{r,s}(Y) = \lim_{n\to\infty}[E_{R_Y}(\hat{x}_{s_n,t}(X)|\mathcal{F}_r) - \hat{\alpha}_{r,s_n}(R_Y)]. \quad (5.22)$$

From equation (5.16), for all n,

$$E_{R_Y}(\hat{x}_{s_n,t}(X)|\mathcal{F}_r) - \hat{\alpha}_{r,s_n}(R_Y) \leq \hat{x}_{r,s_n}(\hat{x}_{s_n,t}(X)).$$

In the case r belongs to $\mathcal{T}$, applying the $\mathcal{T}$ time consistency, we have that the right-hand side of the above equation is equal to $\hat{x}_{r,t}(X)$. Thus

$$E_{R_Y}(\hat{x}_{s_n,t}(X)|\mathcal{F}_r) - \hat{\alpha}_{r,s_n}(R_Y) \leq \hat{x}_{r,t}(X). \quad (5.23)$$

For a general $r \in [0,T]$, the corresponding equation to (5.23) is obtained by right-continuity. Then from these equations together with (5.22), we deduce that $\hat{x}_{r,s}(\hat{x}_{s,t}(X)) \leq \hat{x}_{r,t}(X)$. This, together with (5.19) gives the time-consistency.

Step 6: Sandwich and projection property.
When $r, t \in \mathcal{T}$, it follows from Theorem 4.4 that $\hat{x}_{rt}$ extends x_{rt} and satisfies the sandwich condition. These properties extend to all $r \in [0,T]$ making use of the right-continuity and condition mM2 in $x_{r,t}$, $m_{r,t}$ and $M_{r,t}$. They extend then to every $t \in [0,T]$ using the right-continuity of $m_{r,t}$ and $M_{r,t}$ in t (see mM2), and the fact that $x_{r,t}$ is the restriction of $x_{r,T}$.
For all $r \leq t \leq T$, the projection property for $\hat{x}_{r,t}$ follows from equation (5.15) $\qquad\square$

Corollary 5.4. *The extension $(\hat{x}_{st})_{s,t\in[0,T]}$ is maximal, in the sense that, for any other such extension $(\bar{x}_{s,t})_{s,t\in[0,T]}$ we have that: for all $s < t$,*

$$\hat{x}_{s,t}(X) \geq \bar{x}_{s,t}(X), \quad X \in L_p(\mathcal{F}_t).$$

Furthermore for all $s, t \in [0,T]$, and all $R \in \mathcal{R}$, $\hat{\alpha}_{s,t}(R)$ is the minimal penalty associated to $\hat{x}_{s,t}$, i.e.

$$\hat{\alpha}_{s,t}(R) = \mathrm{esssup}_{X\in L_p(\mathcal{F}_t)}[E_R(X|\mathcal{F}_s) - \hat{x}_{s,t}(X)]. \quad (5.24)$$

Define for all $R \sim P$ $\hat{\alpha}_{s,t}(R)$ by the formula (5.24). Then $(\hat{x}_{s,t})_{s,t\in[0,T]}$ admits the following representation where $\hat{\alpha}_{s,t}$ is the minimal penalty:

$$\hat{x}_{s,t}(X) = \mathrm{esssup}_{Q\sim P}[E_Q(X|\mathcal{F}_s) - \hat{\alpha}_{s,t}(Q)]. \quad (5.25)$$

Proof. The maximality of $(\hat{x}_{st})_{s,t\in[0,T]}$ among all the extensions satisfying the required properties follows from Theorem 4.4 for $s,t \in \mathcal{T}$. For $s \in [0,T]$, we apply right-continuity. For $t \in [0,T]$ we apply the fact that $\hat{x}_{s,t} = \hat{x}_{s,T}$ and also $\bar{x}_{s,t} = \bar{x}_{s,T}$. (see Remark 4.1 and step 1 in the proof of the Theorem).

Note that $\hat{x}_{s,t}(X + Y) = \hat{x}_{s,t}(X) + Y$ for all $X \in L_p(\mathcal{F}_t)$ and $Y \in L_p(\mathcal{F}_s)$. This property is known as $L_p(\mathcal{F}_s)$-*translation invariance*. Thus $(\hat{x}_{st})_{s,t\in[0,T]}$ is up to a minus sign a time-consistent dynamic risk measure. Denote $\beta_{s,t}$ the minimal penalty associated to $\hat{x}_{s,t}$:

$$\beta_{s,t}(Q) = \operatorname{esssup}_{X\in L_p(\mathcal{F}_t)}[E_Q(X|\mathcal{F}_s) - \hat{x}_{s,t}(X)].$$

It follows from Delbaen et al [12] appendix, that the minimal penalty $\beta_{s,t}(Q)$ is right-continuous both in s and t, for all $Q \sim P$. Note that for all $R \in \mathcal{R}$, $\hat{\alpha}_{s,t}(R)$ is also right continuous both in s and t (Theorem 5.3). Furthermore we know from Section 4.2 that $\hat{\alpha}_{s,t}(R)$ is the minimal penalty for $\hat{x}_{s,t}$ for all $s,t \in \mathcal{T}$. This implies that for all $s,t \in [0,T]$, and all $R \in \mathcal{R}$, $\hat{\alpha}_{s,t}(R) = \beta_{s,t}(R)$. $\qquad\square$

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