THE FORWARD DYNAMICS IN ENERGY MARKETS – INFINITE DIMENSIONAL MODELING AND SIMULATION

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ABSTRACT. In this paper an infinite dimensional approach to model energy forward markets is introduced. Similar to the Heath-Jarrow-Morton framework in interest rate modeling, a first order hyperbolic stochastic partial differential equation models the dynamics of the forward price curves. These equations are analyzed, and in particular regularity and no-arbitrage conditions in the general situation of stochastic partial differential equations driven by an infinite dimensional martingale process are studied. Both arithmetic and geometric forward price dynamics are studied, as well as accounting for the delivery period of electricity forward contracts. A stable and convergent numerical approximation in form of a finite element method for hyperbolic stochastic partial differential equations is introduced and applied to some examples with relevance to energy markets.

1. INTRODUCTION

The Heath-Jarrow-Morton (HJM) framework was originally introduced to model products in fixed income markets. Rather than to base the analysis of price processes on the spot rate(s), Heath, Jarrow and Morton [34] proposed to model the forward rates directly. This approach, however, is not limited to interest-rate derivatives: forward markets represent a considerable portion of the trading on energy and commodities. Therefore, it is natural to apply the HJM approach to such settings (see Benth *et al.* [16], Koekebakker and Ollmar [37], etc). To introduce models for the forward price dynamics directly rather than via spot prices circumvents the difficulty of linking spot and forward prices. In a market like electricity, the spot-forward relationship is far from straightforward.

The HJM modeling paradigm is naturally embedded into an infinite dimensional framework. In fixed-income theory, there is already an extensively developed theory in this direction (see Filipović [29], Carmona and Tehranchi [20], and the references therein). The purpose of this paper is to analyze a class of HJM models for forwards in energy markets within an infinite dimensional framework. Further, we develop convergent approximations

of the resulting infinite dimensional equations with the purpose of simulating sample paths or functionals of the solution, such as moments, via Monte Carlo methods.

Energy markets have several distinct features that qualify them for a separate study within the HJM approach. First, in markets such as gas and electricity, contracts do not deliver at fixed maturities. By the very nature of these commodities, delivery must take place over an agreed period of time. The holder of such a commodity receives a flow of gas/electricity for some prescribed period. This introduces a new parameter: the *length-of-delivery*. The presence of such a crucial element further complexifies the existing theory for forward rates in money markets, which only present a fixed maturity as parameter. We shall dub *flow forwards* those contracts that deliver over the span of some time period, and present an infinite dimensional framework for both such forwards and those delivering at a fixed maturity time. As it turns out, the "classical" forward contracts with fixed time of delivery of the underlying is an important modeling tool for the price dynamics of flow forwards.

A precise model of electricity markets requires an infinite number of stochastic processes driving the forward curve. In interest rate theory, as suggested in Carmona and Tehranchi [20], three driving noise sources (Wiener processes) and their linear combinations cover more than 99% of the variation in the data. In these markets there seem to be a strong connection among various maturities, which may explain why low-dimensional noise processes are sufficient to model the yield curve. In contrast, the picture in electricity markets is very different. In Koekebakker and Ollmar [37] the authors show, using principal component analysis, that more than 12 factors are necessary to cover only 95% of the variation in the data. This suggests that there is a sizable number of noise sources in the forward market, where each contract has a large degree of idiosyncratic risk associated to it. Hence, an infinite dimensional approach to modeling the noise in the forward curves of energy markets is in order.

There are other arguments in favor of using infinite dimensional processes as the driving noise of the forward curves in energy markets, in particular markets where the underlying commodity is not storable (such as electricity). As it is explained in Benth and Meyer-Brandis [15], in such markets forward-looking information may be relevant to the forward markets, but not to the spot market. In electricity markets, say, there is a vast stream of information that impacts forward prices, including weather prediction, power plant shutdowns, production plans, reservoir situations, etc. This translates into a large number of noise sources that affect the forward curve at different maturities, and therefore create a high-dimensional noise pattern. Using a completely different approach to forward price modeling, namely the certainty equivalence principle, Benth, Cartea and Kiesel [11] show that no-arbitrage models can only be obtained by introducing idiosyncratic risk to each traded forward. If one accepts such a pricing principle, then one is forced to deal with as many sources of noise as maturity times, which would as well support the choice of infinitely many sources of noise in our model.

In Benth *et al.* [16] and Frestad *et al.* [32], it is shown that the (log-)returns of flow forward prices in electricity markets are non-normally distributed, independent of the delivery period. The normal inverse Gaussian (NIG) distribution has been used successfully to fit

the observed return data. This points towards using Lévy processes to model the noise driving the forward curve, which, together with the above observation of high dimensional noise in these markets, suggests the use of an infinite dimensional Lévy processes as the driving noise source in the forward curve dynamics.

In this paper, we separate into two classes of forward models, additive and multiplicative. Inspired by the existing models in the literature and the "statistical" modeling view, we take simple, infinite dimensional dynamics for the forward curve evolution specified under the Musiela parametrization. In fact, our infinite dimensional dynamics is specified in terms of a drift and volatility term structure, as well as a noise modeling the dependency structure over delivery times (spatial dependency, that is). This resembles the "statistical" modeling view of asset prices, first advocated by Eberlein and Keller [26] for stock prices using an exponential Lévy process and recently applied to electricity forward pricing by Andresen *et al.* [1]. As forward prices are positive, we also treat exponential models in this paper, which we refer to as *multiplicative*. The infinite dimensional process driving the forward curve will be explicitly representable, and not derived as a solution to a stochastic differential equation (in infinite dimensions). For a general introduction to Hilbert space valued stochastic differential equations with Brownian noise we refer the reader to Da Prato and Zabczyk [22], Chow [21], Prévôt and Röckner [41] and to Peszat and Zabczyk [40] for the case of Lévy processes.

Additive forward price models have been used with great success in energy markets, see for example Benth *et al.* [12]. Similar approaches in interest-rate theory are presented in Ekeland and Taflin [27], Peszat and Zabczyk [40], Filipović [29] and Carmona and Tehranchi [20]. Here we extend these approaches to energy markets. We cover as well several other models, for example the multiplicative ones by Audet *et al.* [2] and Schwartz [47]. Particularly, the model presented in Audet *et al.* [2] is further investigated as it is the basis for some numerical examples. We also note that the ambit fields suggested as a model for energy forwards in Barndorff-Nielsen *et al.* [3] is nicely corresponding to our framework. Audet *et al.* [2] applied their model to price European options on electricity forwards. Recently, this model was generalized to Lévy random fields by Hepperger [35], who develops efficient numerical schemes for solving the integro-partial differential equations describing the price of options on forwards.

Flow forwards are linked to forwards with fixed delivery time through a simple integral. Both classes of contracts must satisfy a martingale condition in order to avoid an arbitrage dynamics for the prices. We derive this explicitly for both the additive and multiplicative models in the case of fixed delivery forwards, and for the additive case for flow forwards. Furthermore, flow forwards must satisfy an additional consistency condition with respect to the delivery period in order to prevent arbitrage opportunities between contracts with overlapping delivery. We characterize the volatility term structures such that the flow forward dynamics satisfies the arbitrage consistency condition. A particular example is discussed in detail, leading back to a term structure for the volatility incorporating the Samuelson effect for forwards with a delivery period.

One of the main contributions of the paper is to provide a numerical method for simulating the forward curves. We face the problem of numerically solving a first-order hyperbolic

partial differential equation, which we approximate via a Finite Element method. This method provides great flexibility in the choice of the nodal points of the approximation, which will be the various times to maturity of the forward contracts. For first-order hyperbolic problems with non-smooth data, a standard Galerkin approximation will lead to instabilities, thus we chose a Petrov–Galerkin method (SUPG method). Convergence results for such schemes for deterministic equations and a thorough introduction to Finite Element methods for deterministic first-order hyperbolic equations can be found, e.g., in [43, 36, 48, 24]. In the stochastic case, L^2-convergence was proved in Barth [5]. To calculate expectations of functionals of the forward prices, such as moments or payoffs, a fast multilevel Monte Carlo algorithm was introduced in Barth and Lang [9] for hyperbolic stochastic partial differential equations. There the authors showed convergence of the multilevel Monte Carlo algorithm and its advantages over a singlelevel Monte Carlo method. The method discussed and applied here can also be used in fixed-income theory for simulating forward rates. We have included several numerical examples with relevance to energy forward pricing. In particular, we look at a forward price model which resembles the one suggested by Audet *et al.* [2], but where we also consider Lévy-noise driven fields. We choose Lévy fields which are of normal inverse Gaussian type, relating our cases back to the statistical analysis and modeling in Frestad *et al.* [32] and Andresen et al. [1]. In particular, the implied spot price dynamics is simulated from the fields and we discuss the effect of spatial correlation on the paths.

The results are presented as follows: We describe our infinite dimensional framework for forward contracts delivering at a fixed time of maturity in Section 2, and present regularity results on the smoothness of the model. We analyze the features of additive and multiplicative models. Further, we briefly discuss an extension to these popular models. In Section 3 we develop an infinite dimensional model directly for flow forwards. This approach satisfies the no-arbitrage condition presented in Benth *et al.* [16]. Finally, in Section 4 we develop a numerical scheme and present some simulation results.

2. Infinite dimensional modeling of forward prices

Here we present a general infinite dimensional model for the forward price $f(t, T)$ at time $t \geq 0$ of a contract delivering the underlying commodity at time $T \geq t$, where $t, T \in [0, \tau]$ and $\tau < +\infty$. Much of the theory developed for such price dynamics can be transferred to flow forwards, which we shall study in more detail in the next section. We provide conditions for an arbitrage-free price dynamics for both additive and multiplicative models.

As it is standard in interest rate theory, we consider the so-called *Musiela parametrization* (see [20]), where we set $x = T - t$, the time to maturity, and consider the price $g(t, x)$ of a forward contract at time $t \geq 0$ with $x \geq 0$ time left to maturity. As we see, $g(t, x)$ is a function of "time and space" $(t, x) \in [0, \tau] \times [0, \tau]$, and it is therefore natural to look for a stochastic dynamics in time $t \mapsto g(t, \cdot)$, where $g(t, \cdot)$ takes values in some appropriate function space. We place ourselves within the theory of Hilbert space valued stochastic processes when modeling g, which is done under the risk-neutral measure. Thus, we must

choose dynamics for g such that the process

$$t \mapsto f(t,T) = g(t, T-t) \quad (t \in [0,T])$$

is a martingale for every maturity T. This leads to *martingale conditions* on the parameters in the specification of g (see [20], for example, for the similar situation in interest-rate theory and [23] on the fundamental theorem of asset pricing).

It is not obvious whether the model for g should be additive or multiplicative, that is, if we should model the dynamics of g directly rather than $\ln(g)$. In Benth *et al.* [12] additive spot-price models were introduced and justified for energy markets. This leads to additive forward (and, specifically flow-forward) price dynamics. Furthermore, in Bernhardt, Klüppelberg and Meyer-Brandis [17], a statistical analysis of Singapore electricity spot data showed evidence in favor of additive models, in particular continuous-time autoregressive moving-average processes. In view of these results, one may be tempted to model g directly. On the other hand, forward prices are naturally positive, thus it may be advantageous to model $\ln(g)$ instead. This would also be in line with some classical examples of forward price models like the one by Audet *et al.* [2]. In this paper we discuss both alternatives.

In the mathematical specifications of our model we follow the general set-up presented by Peszat and Zabczyk [40] (in particular Chapters 8 and 9). Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a probability space equipped with a filtration $\mathcal{F} = (\mathcal{F}_t, t \geq 0)$, satisfying "the usual condition". The process M is a (possibly discontinuous) square-integrable martingale taking values in a separable Hilbert space $(U, (\cdot, \cdot)_U)$. This class of processes is denoted by $\mathcal{M}^2(U)$. Observe that we denote the probability measure on $(\Omega, \mathcal{F})$ by $\mathbb{Q}$, indicating that we model the forward curve dynamics directly in the risk-neutral world.

The process M is used as an integrator in the random evolution of the forward curve, and for this purpose we restrict ourselves to the following subclass of $\mathcal{M}^2(U)$ of martingales with bounded stochastic covariance:

$$\mathcal{M}_b^2(U) := \{M \in \mathcal{M}^2(U) : \exists Q \in L_1^+(U) \text{ s. t. } \forall t \geq s \geq 0, \ \langle\langle M, M \rangle\rangle_t - \langle\langle M, M \rangle\rangle_s \leq (t-s)Q\},$$

where $L_+^1(U)$ is the space of nuclear, symmetric, and positive definite linear operators on U. For every $Q \in L_1^+(U)$, there exists an orthonormal basis $(e_n, n \in \mathbb{N})$ of U consisting of the eigenvectors of Q. This leads to the spectral representation $Qe_n = \gamma_n e_n$, where γ_n is the eigenvalue of Q corresponding to e_n. We use the notation $\mathcal{H} = Q^{1/2}(U)$, where the (Hilbert-Schmidt) operator $Q^{1/2}$ is the pseudo square root of Q. Note that $\operatorname{Tr} Q < +\infty$ since Q is nuclear.

We define the following stochastic partial differential equation with additive noise, i.e. a Hilbert-space-valued stochastic differential equation for $t \in [0, \tau]$

$$(2.1) \qquad dX(t) = (AX(t) + a(t))\, dt + b(t)\, dM(t)$$

with initial condition $X(0) = X_0 \in L^2(\Omega, H)$, where H denotes a separable Hilbert space, not necessarily equal to U. Later, we shall introduce conditions on the Hilbert spaces which are natural in our context. The stochastic differential equation is defined on a finite time interval $0 \leq t \leq \tau$, where $\tau < +\infty$. We assume further, that the linear (differential) operator A generates a C_0-semigroup S of contractions on H. When we link X to g, we

show subsequently that A is a first order differential operator with S being the right-shift semigroup. We suppose that a is a mapping from $[0, \tau]$ into H which has $\mathbb{Q}$-a.s. Bochner-integrable trajectories in the finite interval $[0, \tau]$. The process $b : [0, \tau] \to L(\mathcal{H}, H)$ is for every $t \in [0, \tau]$ a linear operator from $\mathcal{H}$ to H.

Further, we denote by $\| \cdot \|_{L_{HS}(\mathcal{H}, H)}$ the Hilbert–Schmidt norm on the space of Hilbert–Schmidt operators from $\mathcal{H}$ to H. Equation (2.1) is only well defined if X_0 is $\mathcal{F}_0$-measurable. It is well known (see Peszat and Zabczyk [40], Filipovic, Tappe and Teichmann [30] and Marinelli [39]) that (under appropriate assumptions on the coefficients) a unique mild solution X in H to Equation (2.1) exists, and that the predictable process $X : [0, \tau] \times \Omega \to H$ is given by

$$(2.2) \qquad X(t) = S(t)X_0 + \int_0^t S(t-s)a(s)\,ds + \int_0^t S(t-s)b(s)\,dM(s)\,.$$

Next, we introduce the forward price dynamics based on the process X.

As we discussed earlier, both additive and multiplicative models are of interest in energy forward markets. We introduce both classes here, based on the general infinite dimensional stochastic process $X(t)$ presented in Equation (2.1) (or equivalently, Equation (2.2)).

We remark that the model of X in Equation (2.1) is not the most general one from the mathematical point of view, with no dependency on the states in the drift a and volatility b. In fact, we have taken the "statistical" modeling view here where one specifies directly the drift and volatility, as well as the noise structure. The crucial points from a statistical viewpoint is to capture the dependency between contracts as well as the volatility term structure. Typically, the existing models in energy can be traced back to the form Equation (2.2) (see, e.g., Barndorff-Nielsen *et al.* [3]). For example, this is the case with the model of Audet *et al.* [2] and Andresen *et al.* [1]. The former can be represented as the exponential of a Wiener field modulated by a volatility structure, while the latter corresponds to the exponential of a Lévy field of normal inverse Gaussian type. We shall come back to this with more details below.

So far, we have formulated X as a stochastic process with values in some abstract Hilbert space. Since our interest is to model the dynamics of curves $t \mapsto g(t, x)$, where $g(t, x)$ is the forward price at time t of a contract with time-to-maturity $x \geq 0$, we must introduce a function space H. A minimum requirement (see, e.g., Carmona and Tehranchi [20] and Filipović [29]) is the following assumption:

Assumption 2.1. *H is a space of real-valued continuous functions in $x \in [0, \tau] \subset \mathbb{R}_+$, on which the evaluation functional δ_x is continuous, that is, belongs to the dual H^*.*

Further, the Musiela parametrization tells us that $f(t, T) = g(t, T - t)$, where $f(t, T)$ is the forward price at time t for a contract maturing at time T. Thus, looking at the differential of f introduces naturally a derivative of g with respect to x in the drift. Thus, we only consider an operator A being the first-order differential operator $A = \partial / \partial x$. Obviously, under Assumption 2.1 with H being a space of continuous functions on a subset D of $\mathbb{R}_+$, the domain of A, $\mathcal{D}(A)$, is in general not equal to H. However, $\mathcal{D}(A)$ is a Hilbert space with

norm $\|f\|_{D(A)} = (\|f\|_H^2 + \|Af\|_H^2)^{\frac{1}{2}}$. Generated by A is the shift operator $S(t)$ which acts on elements $f \in H$ as $S(t)f(x) = f(x + t)$, for $x \in D$. We have, the following assumption:

Assumption 2.2. *The operator A is the generator of the C_0-semigroup of contractions $S(t), t \geq 0$.*

Remark 2.3. *Assumption 2.2 is quite restrictive, but in the previously described setting sufficient. Equation* (2.2) *is also defined for the more general classes of semigroups.*

In the rest of this Section we suppose that the separable Hilbert space H and the operator A satisfy Assumptions 2.1 and 2.2.

Next, we give examples of Hilbert spaces H which satisfy our conditions. The Hilbert space H is a function space over $D \subset \mathbb{R}_+$. If we assume that $H = L^2(D)$, then for all Sobolev spaces $H^s(D)$ with $s > \frac{1}{2}$ the evaluation functional δ is an element of the dual space $(H^s(D))^*$, also called $H^{-s}(D)$. For integers s the norm in $H^s(D)$ is given by:

$$\|f\|_{H^s(D)}^2 = \int_D \left(f^2 + \sum_{i=1}^s \left(\frac{d^i f}{dx^i} \right)^2 \right) dx \, .$$

The right shift semigroup S is well defined on $H^s(D)$, since $H^s(\mathbb{R})$ and H^s_- are invariant under the shift translation. Here, H^s_- denotes the subset of $H^s(\mathbb{R})$ containing all functions f with support $(-\infty, 0]$, such that $f(t) = 0$ for all $t > 0$.

Note that for $s > \frac{1}{2}$ all functions in $H^s(D)$ converge to zero at $\tau \to \infty$. In an additive model this would mean that in the far end of the market, forward prices are essentially zero, which seems like an unreasonable property. Most spot models considered for energy markets predict forward prices being constant in the far end of the forward curve, however, not being equal to zero (see e.g. Ornstein-Uhlenbeck models discussed in Benth *et al.* [16]). To deal with this issue, Ekeland and Taflin introduced in [27] a special class of Sobolev spaces allowing for non-zero asymptotic behaviour. We define

$$U^s(D) := \{ h \mid h = \hat{h} + \hat{\alpha}, \, \hat{h} \in H^s(D) \text{ and } \hat{\alpha} \in \mathbb{R} \} \, ,$$

equipped with the norm

$$\|h\|_{U^s(D)}^2 = \|\hat{h}\|_{H^s(D)}^2 + \hat{\alpha}^2 \, .$$

The decomposition $h = \hat{h} + \hat{\alpha}$ is unique, so $U^s(D)$ has the Hilbert space structure $H^s(D) \oplus \mathbb{R}$, and its dual is given by $(U^s(D))^* = (H^s(D))^* \oplus \mathbb{R}$. The shift semigroup S can be extended to $U^s(D)$ by defining for $\hat{\alpha} \in \mathbb{R}$, $S(t)\hat{\alpha} = \hat{\alpha}$.

We have the following regularity result for the mild solution X, given in Equation (2.2):

Lemma 2.4. *If $\mathbb{E}\|X_0\|_{H^1(D)}^2 < \infty$ and for a constant $C < \infty$, it holds*

$$\sup_{t \in [0, \tau]} \left(\|a(t)\|_{H^1(D)}^2 + \|b(t)\|_{L_{HS}(\mathcal{H}; H^1(D))}^2 \right) \leq C \, ,$$

then, the unique solution X to Equation (2.2) *satisfies* $\sup_{t \in [0, \tau]} \mathbb{E}\|X(t)\|_{H^1(D)}^2 < \infty$.

Proof. By the definition of the mild solution we may write

$$\mathbb{E}\|X(t)\|_{H^1(D)}^2 = \mathbb{E}\|S(t)X_0 + \int_0^t S(t-s)a(s)\,ds + \int_0^t S(t-s)b(s)\,dM(s)\|_{H^1(D)}^2\,.$$

With the Itô isometry (see [40]) for square integrable (noncontinuous) martingales and since S is a C_0-semigroup it follows that

$$\mathbb{E}\|X(t)\|_{H^1(D)}^2$$

$$\leq 3(\mathbb{E}\|X_0\|_{H^1(D)}^2 + \mathbb{E}\|\int_0^t S(t-s)a(s)\,ds\|_{H^1(D)}^2 + \mathbb{E}\|\int_0^t S(t-s)b(s)\,dM(s)\|_{H^1(D)}^2)$$

$$\leq C(\tau)(\mathbb{E}\|X_0\|_{H^1(D)}^2 + \int_0^t \|a(s)\|_{H^1(D)}^2\,ds + \int_0^t \|b(s)\|_{L_{HS}(\mathcal{H},H^1(D))}^2\,ds)$$

$$< \infty\,,$$

where the constant C only depends on τ. $\qquad\square$

We remark that the regularity result is valid for more general Hilbert spaces and, in particular, also for $U^1(D)$ resp. $U^2(D)$.

Based on the H-valued stochastic process X we define the additive forward price dynamics as

$$(2.3)\qquad\qquad g_a(t,x) = \delta_x(X(t)) = X(t,x)\,.$$

The multiplicative forward dynamics is assumed to follow

$$(2.4)\qquad\qquad g_m(t,x) = \exp(\delta_x(X(t))) = \exp(X(t,x))\,.$$

We recall that we model the dynamics under the risk-neutral probability directly, which is the common approach also in fixed-income markets. Notice that we obtain the spot price dynamics $E(t)$ in the market by considering $x = 0$, that is, $E(t) := E_a(t) = \delta_0(X(t))$ for the additive model, and $E(t) := E_g(t) = \exp(\delta_0(X(t)))$ in the case of a multiplicative forward model. The spot dynamics is of interest, and we shall investigate it numerically in Section 4.

We must impose conditions on the parameters and the operators in the dynamics of $X(t)$ to guarantee that $f(t,T)$ is a martingale. For the additive model we find the following:

Proposition 2.5. *Let $f(t,T) = g_a(t, T-t)$ for $t \leq T$. Then, under the assumptions of Lemma 2.4, $t \mapsto f(t,T)$ is a martingale if and only if $a = 0$ almost everywhere.*

Proof. Since δ_x is a linear functional on H, we have

$$g_a(t,x) = \delta_x(X(t))$$

$$= \delta_x S(t)X(0) + \delta_x \int_0^t S(t-s)a(s)\,ds + \delta_x \int_0^t S(t-s)b(s)\,dM(s)$$

$$= \delta_x S(t)X(0) + \int_0^t \delta_x S(t-s)a(s)\,ds + \int_0^t \delta_x S(t-s)b(s)\,dM(s)\,.$$

From elementary considerations, we have $\delta_x = \delta_0 S(x)$. Thus, for every $f \in H$ we find

$$\delta_x S(t-s)f = \delta_0(S(x)S(t-s))f = \delta_0 S(x+t-s)f \,.$$

Therefore, by letting $x = T - t$

$$f(t,T) = \delta_0 S(T)X(0) + \int_0^t \delta_0 S(T-s)a(s)\,ds + \int_0^t \delta_0 S(T-s)b(s)\,dM(s) \,.$$

Since $\delta_0 S(T-s)b(s)$ is a linear functional on U, and M is a U-valued martingale, it holds that the process

$$t \mapsto \int_0^t \delta_0 S(T-s)b(s)\,dM(s) \,,$$

is a real-valued martingale (see Thm. 8.7(iii) in Peszat and Zabczyk [40]). Hence, $t \mapsto f(t,T)$ is a martingale if and only if the drift is zero. The proposition holds. $\square$

We now turn our attention to the more involved geometric case. First, by definition, the forward price dynamics is given as

$$f(t,T) = g_m(t,T-t)$$

$$= \exp\left(\delta_{T-t}S(t)X(0) + \int_0^t \delta_{T-t}S(t-s)a(s)\,ds + \int_0^t \delta_{T-t}S(t-s)b(s)\,dM(s)\right)$$

$$= \exp\left(\delta_0 S(T)X(0) + \int_0^t \delta_0 S(T-s)a(s)\,ds + \int_0^t \delta_0 S(T-s)b(s)\,dM(s)\right),$$

since $\delta_x S(y) = \delta_0 S(x+y)$. Letting $\ln f(0,T) = \delta_0 S(T)X(0)$ and observing that $a(t)(x) = \delta_0 S(x)a(t)$, we find

$$(2.5) \qquad f(t,T) = f(0,T)\exp\left(\int_0^t a(s)(T-s)\,ds + \int_0^t \delta_0 S(T-s)b(s)\,dM(s)\right) \,.$$

we introduce the following short-hand notation: define the real-valued, continuous, finite variation process ϕ as

$$(2.6) \qquad \phi(t) = \int_0^t a(s)(T-s)\,ds \,,$$

and the real-valued stochastic process ψ as

$$(2.7) \qquad \psi(t) = \int_0^t \delta_0 S(T-s)b(s)\,dM(s) \,.$$

Note that by Thm. 8.7 (iii) in Peszat and Zabczyk [40], ψ is a martingale. We have the following martingale condition in the geometric case:

Proposition 2.6. *The process $t \mapsto f(t,T)$, for $t \leq T$, is a (local) martingale on $\mathbb{R}$ if and only if*

$$d\phi(t) = -\frac{1}{2}d[\psi,\psi]^c(t) - \left\{\mathrm{e}^{\Delta\psi(t)} - 1 - \Delta\psi(t)\right\} \,,$$

where $[\psi,\psi]^c$ is the continuous part of the bracket process of the martingale ψ (see Protter [42]) and $\Delta\psi(t) = \psi(t) - \psi(t-)$ is the jump of ψ at time t.

Proof. We apply Itô's Formula as formulated in Appendix D.1 in Peszat and Zabczyk [40] to Equation (2.5), which gives

$$f(t,T) = f(0,T) + \int_0^t f(s-,T) \left(d\phi(s) + d\psi(s)\right)$$
$$+ \frac{1}{2} \int_0^t f(s-,T) \, d[\psi,\psi]^c(s)$$
$$+ \sum_{s \leq t} \left\{\Delta f(s,T) - f(s-,T)\Delta(\phi(s) + \psi(s))\right\}.$$

Since $\phi(s)$ is a continuous process, we have $\Delta(\phi(s) + \psi(s)) = \Delta\psi(s)$. Furthermore,

$$\Delta f(s,T) = f(s,T) - f(s-,T) = f(s-,T)\left(e^{\Delta\psi(s)} - 1\right).$$

Hence,

$$f(t,T) = f(0,T) + \int_0^t f(s-,T) \, d\psi(s) + \int_0^t f(s-,T) \, d\phi(s)$$
$$+ \frac{1}{2} \int_0^t f(s-,T) \, d[\psi,\psi]^c(s)$$
$$+ \sum_{s \leq t} f(s-,T) \left\{e^{\Delta\psi(s)} - 1 - \Delta\psi(s)\right\}.$$

The integral with respect to ψ above gives rise to a (local) martingale, which completes the proof. □

As an example, consider first the simple case of a real-valued noise process $M = W$, W being a Brownian motion. Hence, we study the equation

$$X(t) = S(t)X(0) + \int_0^t S(t-s)a(s)ds + \int_0^t S(t-s)b(s)dW(s).$$

Then, $\phi(t) = \int_0^t S(t-s)a(s)ds$ is a continuous (deterministic) function, if a is continuous. Further, $b(s)$, for $s \in [0,\tau]$, is the multiplication operator on a Hilbert space H. So we have

$$\psi(t) = \int_0^t b(s)(T-s) \, dW(s).$$

It holds that $d[\psi,\psi]^c(t) = b^2(t)(T-t) \, dt$, and we acquire the condition

$$a(t)(x) = -\frac{1}{2}b^2(t)(x),$$

as expected. Extending this to an infinite-dimensional Q-Wiener process $M = W$, we may write

$$\psi(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \int_0^t \delta_0 S(T-s)b(s)e_n \, dW_n(s)$$

by appealing to the series representation of W, given by

$$(2.8) \qquad W(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} W_n(t) e_n \,.$$

Here, W_n are independent, real-valued Brownian motions, and λ_n are the eigenvalues of Q with eigenvectors e_n constituting a basis in U. By independence of W_n, we find

$$\begin{aligned}
[\psi, \psi]^c(t) &= \sum_{n=1}^{\infty} \lambda_n \int_0^t (\delta_0 S(T-s)b(s)e_n)^2 \, ds \\
&= \int_0^t \sum_{n=1}^{\infty} |\delta_0 S(T-s)b(s)Q^{1/2}e_n|^2 \, ds \\
&= \int_0^t \|\delta_0 S(T-s)b(s)Q^{1/2}\|_{L_{HS}(U,\mathbb{R})}^2 \, ds \,,
\end{aligned}$$

where $\|\cdot\|_{L_{HS}(U,\mathbb{R})}$ denotes the Hilbert-Schmidt norm for operators from U to $\mathbb{R}$. Thus, we get the martingale condition

$$a(t)(x) = -\frac{1}{2}\|\delta_0 S(x)b(t)Q^{1/2}\|_{L_{HS}(U,\mathbb{R})}^2 \,.$$

We give an example relating our model to the geometric Brownian motion dynamics for forward contracts proposed by Audet *et al.* [2].

Example 2.7. *Given the forward dynamics*

$$(2.9) \qquad df(t,T) = f(t,T)e^{-\alpha(T-t)}\sigma(T) \, dB_T(t) \qquad \forall t \in [0,T] \text{ and } T \in [0,\tau] \,,$$

where σ is a deterministic volatility function depending on maturity time T only, and α is a positive constant such that $\exp(-\alpha(T-t))\sigma(T)$ models the Samuelson effect in the forward market. This term may also be related to the spot price dynamics, with α as the speed of mean-reversion (see Benth et al. [16]). The noise $B_T(t)$ is a Gaussian random field, being a Wiener process for each fixed maturity T, with $t \le T$. For each fixed time $t \le T$, it has a covariance kernel given by the function $q(T,T') = \exp(-\kappa|T-T'|)$, here $\kappa > 0$ is a range parameter. Indeed, such a covariance structure was partly rejected by Frestad [31] and Andersen et al. [1], who detected additional seasonality effects in the correlation between swap contracts of different maturity in the Nordic electricity market NordPool.

With $x = T-t$, we define $W(t,x) = B_{t+x}(t)$. W may be interpreted as a Gaussian process with values in the Hilbert space $U = L^2(\mathbb{R})$ and having a covariance kernel given by $q(x,y) = \exp(-\kappa|x-y|)$. Thus, we can express $f(t,t+x)$ by

$$\begin{aligned}
\ln f(t, x+t) = {} & \ln f(0, x+t) - \frac{1}{2}\int_0^t \sigma^2(x+(t-s)+s)e^{-2\alpha(x+t-s)} \, ds \\
& + \int_0^t \sigma(x+(t-s)+s)e^{-\alpha(x+t-s)} \, dW(s,x) \,.
\end{aligned}$$

We therefore identify the volatility operator $b(t)$ as the multiplication operator on U into a Hilbert space H of functions on $\mathbb{R}$ as

$$\delta_x(b(t)(u)) = \sigma(x + t)\mathrm{e}^{-\alpha x}u(x)\,.$$

We must impose regularity conditions on σ in order to ensure that the operator $b(t)$ maps into the chosen Hilbert space. For instance, we may choose $H = L^2(\mathbb{R})$, and require that $y \mapsto \sigma(y)$ is a bounded measurable function on $\mathbb{R}$. This identifies the model of Audet et al. [2] in our infinite-dimensional framework. We note that the martingale condition also holds, giving a no-arbitrage dynamics for the forward process.

Equation (2.1) describes the risk-neutral dynamics of the log-forward price. In energy markets, forward contracts are applied for hedging purposes, and thus one is interested in the market dynamics of the forward prices. Also, if we want to estimate the parameters in the forward price model, the observed data is under the market probability and not the risk-neutral one. Obviously, if one has option prices on forwards available, one could use these to extract the "implied" parameters. However, sometimes, like in power, the option market may be rather illiquid and it is preferrable to use the forward prices observed in the market for estimation instead. To obtain the dynamics under the real-world measure $\mathbb{P}$, we must change the measure from $\mathbb{Q}$ to $\mathbb{P}$. This is what we discuss in some more detail next:

In the language of energy markets, the *market price of risk*, may be modeled as a process with values in the same space as the noise process. If U is a function space over $x \in D$, time-to-maturity, then we can have a market price of risk being a function of x. This is in contrast to the case with finite-dimensional noise, where the market price of risk must have the same dimension as the noise. In situations where one applies for example utility-based methods for pricing, one is forced to enlarge the dimensionality of the driving noise to match the number of forward contracts in the market to avoid arbitrage possibilities, this is the same as having one market price of risk per contract (see e.g. the discussion in Benth, Cartea and Kiesel [11]). By lifting the driving noise to infinite dimensions we may achieve a richer structure on the market price of risk. On the other hand, if the market price of risk has to be a function of time-to-maturity, then the market must admit an infinite dimensional noise in its dynamics.

In the case where a Lévy field $M = L$ drives the dynamics of X, we introduce an infinite-dimensional version of the Esscher transform (see Benth *et al.* [16] and Shiryaev [45] for an account on the finite-dimensional Esscher transform in finance). We suppose that L has the characteristic triplet (μ, Q, ℓ) (see Peszat and Zabczyk [40] for a definition), and denote by $\varphi(x)$ the cumulant function of L for $x \in U$, that is, from Thm. 4.27 in Peszat and Zabczyk [40],

$$\varphi(x) = \ln \mathbb{E}\left[\mathrm{e}^{\mathrm{i}(x,L(1))}\right] = \mathrm{i}(\mu,x)_U - \frac{1}{2}(Qx,x)_U + \int_U \left(\mathrm{e}^{\mathrm{i}(x,y)_U} - 1 - \mathrm{i}(x,y)_U 1_{(|y|_U \leq 1)}\right)\ell(dy)\,.$$

For a $\theta \in U$, suppose that $\varphi(-\mathrm{i}\theta)$ is well-defined, and define the real-valued stochastic process

$$Z(t) = \exp\left((\theta, L(t))_U - \varphi(-\mathrm{i}\theta)\right)\,,$$

for $t \in [0, \tau]$. Note that $Z(0) = 1$ and Z is a martingale since L has independent increments. Introduce the density process

$$\frac{d\mathbb{P}}{d\mathbb{Q}}\Big|_{\mathcal{F}_t} = Z(t), t \in [0, \tau],$$

of the probability measure $\mathbb{P}$. We show that L preserves the Lévy process property under $\mathbb{P}$. By Bayes' Formula and the independent increment property of L under $\mathbb{Q}$, we have for $t \geq s$,

$$\mathbb{E}_{\mathbb{P}}\left[e^{i(x,L(t)-L(s))_U} \mid \mathcal{F}_s\right] = \mathbb{E}\left[e^{i(x,L(t)-L(s))_U}\frac{Z(t)}{Z(s)} \mid \mathcal{F}_s\right]$$
$$= \exp\left(\varphi(x - i\theta) - \varphi(-i\theta)\right).$$

Thus, by definition of φ and the symmetry of the covariance operator Q, we find that the characteristic triplet of L under $\mathbb{P}$ is $(\mu_\theta, Q, \ell_\theta)$, with

$$(\mu_\theta, x)_U = (a, x)_U + \int_U (x, y)_U 1_{(|y|_U \leq 1)}\, \ell_\theta(dy),$$

and

$$\ell_\theta(dy) = e^{(\theta,y)_U}\, \ell(dy).$$

This defines an infinite-dimensional version of the Esscher transform, and so we introduced a market price of risk $\theta \in U$ also for the case of Lévy noise $M(t) = L(t)$. Observe that we do not change the covariance operator Q similar to the Girsanov transform. It is simple to extend the Esscher transform to θ being time-dependent, deterministic functions with values in U, integrable with respect to $L(t)$. This comes, however, at the expense of losing the Lévy property of $L(t)$ under $\mathbb{P}$. On the other hand, we preserve the property of independent increments.

We end this section with a brief discussion on the generalization of our forward price dynamics to include a multiplicative structure in the noise and the drift. To this end, we introduce the stochastic partial differential equation

$$(2.10) \qquad dX(t) = (AX(t) + a(t, X(t)))\, dt + b(t, X(t))\, dM(t)$$

with initial condition $X(0) = X_0 \in H$. In this case we suppose that a is a mapping from $[0, \tau] \times H$ into H which has $\mathbb{Q}$-a.s. Bochner-integrable trajectories in the finite interval $[0, \tau]$. The process $b : [0, \tau] \times H \to L(\mathcal{H}, H)$ is for every $t \in [0, \tau]$ and element in H a linear operator from $\mathcal{H}$ to H. We assume that the operators a and b satisfy the following linear growth and Lipschitz conditions for $u, v \in H$ and $t \in [0, \tau]$:

$$\|a(t, u)\|_H^2 \leq C_1^2 (1 + \|u\|_H^2)$$
$$\|a(t, u) - a(t, v)\|_H \leq C_1 \|u - v\|_H,$$

and

$$\|b(t, u)\|_{L_{HS}(\mathcal{H};H)}^2 \leq C_2^2 (1 + \|u\|_H^2)$$
$$\|b(t, u) - b(t, v)\|_{L_{HS}(\mathcal{H};H)} \leq C_2 \|u - v\|_H,$$

for constants $C_1, C_2 < +\infty$. Then. a unique mild solution X in H to Equation (2.10) exists (see [30] and [39]), and the predictable process $X : [0, \tau] \times \Omega \to H$ is given by

$$(2.11) \qquad X(t) = S(t)X_0 + \int_0^t S(t - s)a(s, X(s)) \, ds + \int_0^t S(t - s)b(s, X(s)) \, dM(s) \, .$$

A specification of the process X as the solution of the stochastic partial differential equation (2.10) is clearly a generalization of the dynamics defined in Equation (2.1).

We have the following regularity result, similar to Lemma 2.4

Lemma 2.8. *If $\mathbb{E}\|X_0\|_{H^1(D)}^2 < \infty$, then the unique solution X to Equation* (2.11) *satisfies* $\sup_{t \in [0, \tau]} \mathbb{E}\|X(t)\|_{H^1(D)}^2 < \infty$.

Again, we could assume that the forward curve (under the Musiela parameterization) is either defined by $g_a(t, x) = \delta_x(X(t))$ or $g_m(t, x) = \exp(\delta_x(X(t)))$ for the additive and multiplication dynamics respectively. We leave out a further analysis of these more general models here, but remark that they can be approximated and simulated with the methods discussed in Section 4 below.

3. Modeling of electricity flow forwards

In this Section we consider flow forward prices and model them using infinite dimensional dynamics. Since flow forwards deliver over a time period, we enlarge the function space to dimension 2, since now we have both time-to-maturity and length-of-delivery in the Musiela parametrization. In this case one could argue in favour of adding noise in both variables. However, a no-arbitrage condition linked to different delivery periods restricts the range of possible noise specifications in the "length-of-delivery" dimension. We provide a discussion of functions satisfying this condition.

Suppose $F(t, T_1, T_2)$ is the flow forward price at time t for a contract delivering the commodity (electricity or gas, typically) over the time period $[T_1, T_2]$, where $t, T_1, T_2 \in [0, \tau]$ and $t \leq T_1 \leq T_2$. We know that this price dynamics must be a martingale in the risk-neutral framework, i.e.

$$(3.1) \qquad\qquad\qquad\qquad t \mapsto F(t, T_1, T_2)$$

is a $\mathbb{Q}$-martingale. One can show (see [16]) that the dynamics must also satisfy an additional no-arbitrage condition, namely

$$(3.2) \qquad\qquad\qquad F(t, T_1, T_2) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} F(t, u, u) \, du \, .$$

In many electricity markets, flow forward contracts may be traded over various delivery periods, some of which may overlap. For example, in the NordPool market, one may invest in flow forwards delivering electricity in the four quarters of the year, or one may trade in a flow forward delivering over the whole year. Similarly there are contracts delivering over the different months constituting a quarter. Hence, there must be specific no-arbitrage relations between these contracts with overlapping delivery periods. In Benth *et al.* [16]

such conditions are presented, and it is shown that they imply Equation (3.2) when considering a market where *all* possible delivery periods are included. Thus, models for the flow-forward price $F(t, T_1, T_2)$, which must hold for all possible choices of T_1 and T_2, must satisfy Equation (3.2). In an infinite dimensional framework this is exactly the case.

Let us introduce the Musiela parametrization $x = T_1 - t$ and $y = T_2 - T_1$, and define

$$G(t, x, y) \triangleq F(t, t + x, t + x + y).$$

Our dynamics for the flow forwards F needs to satisfy the no-arbitrage condition (3.2). Translated into a Musiela parametrization, it takes the form,

$$(3.3) \qquad yG(t, x, y) = \int_0^y G(t, x + z, 0)\, dz.$$

Thus, any specification of $G(t, x, y)$ has to satisfy this condition in order to prevent arbitrage possibilities between flow forward contracts of different length and time to maturity. We observe that relation (3.3) immediately suggests a way to model the flow forward dynamics via the use of fixed-delivery contracts studied in the previous Section. Let $G(t, x, 0) \triangleq g(t, x)$, with g equal to g_a or g_m given in Equation (2.3), resp. Equation (2.4). Define $G(t, x, y)$ by

$$yG(t, x, y) = \int_0^y g(t, x + z)\, dz.$$

Since $g(t, \cdot)$ takes values in a function space of regular functions, the integral makes sense for each $y > 0$, if we assume, for example, $H = L^2(\mathbb{R}_+)$ and the regularity result in Lemma 2.4.

In the following we investigate functions which fulfill the no-arbitrage condition (3.2). Notice that $F(t, \cdot, \cdot)$ is symmetric in T_1, T_2. We remark that the no-arbitrage condition only affects the start of delivery and the end of delivery, and not the present time t. If we just look at those variables for the time being we find that the values of the function on the diagonal, i.e $F(t, u, u)$, characterize the function on every point $(T_1, T_2) \in [0, \tau] \times [0, \tau] \subset \mathbb{R}_+^2$. If we are given an arbitrary integrable function $\psi(u)$ with an antiderivative given by $\Psi(u)$, i.e $\Psi'(u) = \psi(u)$ an set $\psi(u) = F(t, u, u)$ we get from Equation (3.2)

$$F(t, T_1, T_2) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \psi(u)\, du = \frac{\Psi(T_2) - \Psi(T_1)}{T_2 - T_1}.$$

If we set, as an example, $\psi(u) = (n + 1)u^n$, for $n \in \mathbb{N}$, we get that

$$F(t, T_1, T_2) = \frac{T_2^{n+1} - T_1^{n+1}}{T_2 - T_1}$$

fulfills the no-arbitrage condition. The choice $\psi(u) = e^u$ gives

$$F(t, T_1, T_2) = \frac{e^{T_2} - e^{T_1}}{T_2 - T_1}.$$

It is possible to derive a similar characterization for functions fulfilling the parameterized version of the no arbitrage condition, Equation (3.3). We remark as above that this condition only affects the time to maturity x and the length of delivery y and not t.

Here we choose as above an arbitrary integrable function ψ with antiderivative Ψ and set $\psi(u) = G(t, u, 0)$, then we may write

$$G(t, x, y) = \frac{1}{y} \int_0^y \psi(x + z) dz = \frac{1}{y} \left(\Psi(x + y) - \Psi(x) \right).$$

As an example: if $\psi(u) = \mathrm{e}^u$, it follows that $G(t, x, y) = \frac{1}{y}(\mathrm{e}^{x+y} - e^x)$ is fulfilling the no-arbitrage condition. It should be mentioned that we could also choose the antiderivative Ψ as an absolutely continuous function and derive the same results.

We proceed with the Hilbert-space-valued stochastic differential equation to model directly the flow forwards. Recall the H-valued stochastic process X given by the solution of the stochastic differential equation (2.1), and with mild solution given in Equation (2.2). We suppose H is a Hilbert space of functions in (x, y), and assume that M is U-valued, with U also being a Hilbert space (again possibly a space of functions in (x, y)). We consider the evaluation functional $\delta_{(x,y)}$, and extend the assumptions on the forward models from the previous Section. The discussion of some possible choices of Hilbert spaces also extends naturally. Note, however, that in order for the evaluation functional to be continuous, we must increase the order of the Sobolev spaces to $s > 3/2$, since our functions now are defined on $\mathbb{R}_+^2$ rather than the positive real line (recall the Sobolev Embedding Theorem (Morrey's inequality)).

As it is shown in Benth and Koekebakker [13], it is not possible to model $F(t, T_1, T_2)$ using a geometric Brownian motion whose volatility depends on T_1 and T_2. Only volatility specifications depending on time t are valid. This essentially rules out all relevant models since naturally the volatility should have a dependency on the delivery period due to the well-known Samuelson effect (see [13] for more a detailed discussion). It is an established empirical fact that the volatility of forwards increases as time to maturity approaches zero (see, e.g., Frestad *et al.* [32]), and to have reasonable models one is forced to take this into account. We conclude with this that multiplicative models for flow forwards seem difficult to specify in the context of power markets. Admittedly, the result in Benth and Koekebakker [13] was proven for real-valued noise processes M, but the arguments may be generalized to infinite dimensional noises as well.

In the remaining of this Section we focus solely on additive models, since they provide the richest structures for flexible flow forward dynamics. To this end, define

(3.4) $$G(t, x, y) := \delta_{(x,y)}(X(t)).$$

Since we have that

$$F(t, T_1, T_2) = G(t, T_1 - t, T_2 - T_1),$$

the operator A is still given by $A = \partial/\partial x$. The martingale condition on $X(t)$ is derived as in Prop. 2.5, and becomes $a(t) = 0$. From now on we suppose this to hold.

We move on by studying the possible range of models G based on the specification in Equation (3.4) that will satisfy the no-arbitrage condition (3.3). First, we remark that the condition can be interpreted as a side-constraint to the stochastic differential equation, (2.1), of X, since it implicitly imposes certain structural conditions on the evolution of X. Next, since $G(t, x, y)$ is fully specified by $G(t, z, 0)$ via the no-arbitrage condition, the

variations in *length-of-delivery y* is simply an integral of those in *time-to-maturity.* Hence, there cannot be any separate, independent noisy behaviour. As we have discussed in the Introduction, there is ample statistical evidence for noisy behaviour of forward prices along time-to-maturity x. We are not aware of any studies of noise in y, the length-of-delivery, and how this relates to x.

In order to study the range of volatility specifications b in more detail, we restrict ourselves to driving noise processes M which fulfill the Itô isometry:

$$\mathbb{E}\| \int_0^t S(t-s)b(s)dM(s)\|_H^2 = \mathbb{E} \int_0^t \|S(t-s)b(s)\|_{L_{HS}(\mathcal{H},H)}^2 ds \,.$$

We remark that for the class of square integrable martingales $\mathcal{M}_b^2(U)$, introduced in Section 2, this is in general not true. However, for many processes, such as the important class of Lévy processes the isometry holds.

Proposition 3.1. *Let S_2 be the shift operator in the second variable on H, and suppose the initial forward curve $G(0,x,y)$ satisfies the no-arbitrage condition (3.3). Then $G(t,x,y)$ fulfills the no-arbitrage condition (3.3) if and only if for every $0 \le s \le t \le \tau$ and $x,y \ge 0$, the volatility process b has the property*

$$(3.5) \qquad \left\{ S_2(y) - \frac{1}{y} \int_0^y S(z)\, dz \right\} b(s)u \in ker\left(\delta_{0,0}S(t-s+x)\right),$$

for all $u \in U$, where the integration of S is understood as a Bochner integral.

Proof. By the continuity of $\delta_{x,y}$, we have that

$$G(t,x,y) = \delta_{x,y}S(t)X(0) + \int_0^t \delta_{x,y}S(t-s)b(s)\, dM(s)\,.$$

It follows from the no-arbitrage condition (3.2) that

$$G(t,x,y) = \frac{1}{y} \int_0^y G(t,x+z,0)dz$$
$$= \frac{1}{y} \int_0^y \delta_{x+z,0}S(t)X(0)\, dz + \frac{1}{y} \int_0^y \int_0^t \delta_{x+z,0}S(t-s)b(s)\, dM(s)\, dz\,.$$

If we compare the first terms, we may write

$$\delta_{x,y}S(t)X(0) = \frac{1}{y} \int_0^y \delta_{x+z,0}S(t)X(0)\, dz\,,$$

or, by definition,

$$G(0,x,y) = \frac{1}{y} \int_0^y G(0,x+z,0)\, dz\,.$$

By assumption, this is supposed to hold.

Next, comparing the terms with the stochastic integral, we find by linearity of the operators

$$\int_0^t \delta_{x,y}S(t-s)b(s)\, dM(s) = \frac{1}{y} \int_0^y \int_0^t \delta_{x+z,0}S(t-s)b(s)\, dM(s)\, dz$$

$$= \int_0^t \frac{1}{y} \int_0^y \delta_{x+z,0} S(t-s) b(s) \, dz \, dM(s) \,.$$

Hence, if the integral process with respect to M satisfies the Itô isometry, we find the operator equality

$$\delta_{x,y} S(t-s) b(s) = \frac{1}{y} \int_0^y \delta_{x+z,0} S(t-s) b(s) \, dz \,.$$

It is easily verified that, for $v \in H$

$$\delta_{x,y} S(t-s) v = \delta_{0,y} S(t-s+x) v = \delta_{0,0} S(t-s+x) S_2(y) v \,.$$

Applying this and the semigroup property of S, we get,

$$\delta_{0,0} S(t-s+x) S_2(y) b(s) = \frac{1}{y} \int_0^y \delta_{x+z,o} S(t-s) b(s) \, dz$$

$$= \frac{1}{y} \int_0^y \delta_{0,0} S(x+z) S(t-s) b(s) \, dz$$

$$= \frac{1}{y} \int_0^y \delta_{0,0} S(t-s+x+z) b(s) \, dz$$

$$= \delta_{0,0} \left\{ \frac{1}{y} \int_0^y S(t-s+x+z) b(s) \, dz \right\}$$

$$= \delta_{0,0} S(t-s+x) \left\{ \frac{1}{y} \int_0^y S(z) \, dz \right\} b(s) \,.$$

This is equivalent to

$$\delta_{0,0} S(t-s+x) \left\{ S_2(y) b(s) - \frac{1}{y} \int_0^y S(z) \, dz b(s) \right\} = 0 \,.$$

This proves the Proposition. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We note that a sufficient condition for the volatility b to ensure no-arbitrage is that

$$S_2(y) b(t) = \frac{1}{y} \int_0^y S(z) \, dz b(t) \,.$$

Recall b is an operator mapping U to H, while the shift operators S and S_2 maps H into itself. Thus, for $h \in U$ we have that, after evaluating the function $b(t)h$ in (u,v),

$$(b(t)h)(u, v+y) = \frac{1}{y} \int_0^y (b(t)h)(u+z, v) \, dz \,.$$

We now discuss this condition for the special case when the driving noise M of $X(t)$ is one-dimensional, that is, $M(t)$ is a real-valued square-integrable martingale. Further we need the integral with respect to M to fulfill the Itô isometry.

By definition of G, we have (assuming a drift-free equation)

$$G(t, x, y) = X(0)(x+t, y) + \int_0^t b(s)(x+t-s, y) \, dM(s) \,.$$

Appealing to the stochastic Fubini Theorem (see Peszat and Zabczyk [40]), implicitly assuming that b and M are sufficiently regular for this theorem to hold, we find from Equation (3.2) that the volatility b must satisfy

$$(3.6) \qquad \int_0^t b(s)(t-s+x,y)dM(s) = \frac{1}{y}\int_0^y \int_0^t b(s)(t-s+x+z,0)\,dM(s)\,dz\,.$$

Let us specialize to the case of volatilities which are additive and with a multiplicative structure between x and y, that is, we suppose that

$$b(t)(x,y) = \alpha(t,x)\beta(t,y)\,,$$

for two positive functions α and β. To avoid technicalities, we assume them to be smooth. The no-arbitrage condition imposes

$$(3.7) \qquad \int_0^t \frac{\beta(s,y)}{\beta(s,0)}dM(s) = \int_0^t \frac{1}{y}\int_0^y \frac{\alpha(s,t-s+x+z)}{\alpha(s,t-s+x)}\,dz\,dM(s)\,.$$

The left-hand side of Equation (3.7) is independent of x, which means that the right-hand side must be a function independent of x as well. This holds in particular if there exists a positive smooth function $k(s,t,z)$ such that

$$\frac{\alpha(s,t-s+x+z)}{\alpha(s,t-s+x)} = k(s,t,z)\,.$$

We note that $k(s,t,0) = 1$. In light of the Samuelson effect, it is reasonable to have a decaying volatility in time-to-maturity, meaning that $x \mapsto b(t)(x,y)$ is decreasing. This implies that $z \mapsto k(s,t,z)$ decreases as well, if we impose the Samuelson effect into the model. We find,

$$\frac{\alpha(s,t-s+x+z)-\alpha(s,t-s+x)}{z} = \frac{k(s,t,z)-k(s,t,0)}{z}\alpha(s,t-s+x)\,,$$

and passing to the limit $z \downarrow 0$ we write (for $s = t$),

$$\partial_x\alpha(t,x) = \partial_z k(t,t,0)\alpha(t,x)\,.$$

Here we introduce the notation $\partial_x = \partial/\partial x$. Putting $c(t) := \partial_z k(t,t,0)$, we conclude that

$$(3.8) \qquad\qquad\qquad \alpha(t,x) = \alpha(t,0)e^{c(t)x}\,.$$

If we take the Samuelson effect in the volatility structure into account, $z \mapsto k(s,t,z)$ is decreasing. This implies that $c(t)$ is non-positive. With α as in Equation (3.8), we find

$$(3.9) \qquad \int_0^t \beta(s,y)dM(s) = \int_0^t \frac{\beta(s,0)}{c(s)y}\left(e^{c(s)y}-1\right)dM(s)\,.$$

In conclusion, a sufficient class of volatility structures $b(t,x,y)$ satisfying the no-arbitrage condition (3.6) is given by

$$b(t,x,y) = \frac{h(t)}{c(t)y}\left(e^{c(t)(x+y)} - e^{c(t)x}\right)\,,$$

for smooth functions h and c. If the Samuelson effect holds, then c and h are non-positive. Using that $x = T_1 - t$ and $y = T_2 - T_1$, we find the volatility structure of the forward price dynamics to be

$$\sigma(t, T_1, T_2) := b(t, T_1, T_2 - T_1) = \frac{h(t)}{c(t)(T_2 - T_1)} \left(\mathrm{e}^{c(t)(T_2 - t)} - \mathrm{e}^{c(t)(T_1 - t)} \right) .$$

We observe that for $c(t) = -\kappa$, this volatility structure is resulting from a one-factor spot price model being an Ornstein-Uhlenbeck process with speed of mean-reversion κ and "spot volatility" $h(t)$ (see Schwartz [47] and Benth *et al.* [16] for details).

We remark that this result is a concrete example for the structure of the forward dynamics imposed by the no-arbitrage condition, which we derived in the beginning of this Section. Due to the additive structure of the model here we can drive the structure only for the volatility. For the model to be conform with the no-arbitrage condition the initial condition has to have a similar structure.

As we have mentioned before, reasonable arbitrage-free geometric models seem hard to construct in the framework of flow forwards. One approach to avoid the complicating no-arbitrage condition in Equation (3.3) is to model *time to maturity* and *length of delivery* simultaneously by letting x be the time to some point *within* the delivery period. Thus, one introduces a dynamics $g(t, x)$ with $x = \tau - t$, for a given $\tau \in [T_1, T_2]$. This approach was implicitly suggested and used in Bjerksund *et al.* [18] and applied for example in Audet *et al.* [2]. As we see, this approach brings us back to the forward price dynamics for contracts with fixed time of delivery. It is important to notice that we only obtain a flow forward price dynamics up to time τ. In some markets, like the NordPool power market, one cannot trade the forwards within their delivery period, so in fact $t \leq T_1 \leq \tau$ is the natural modelling horizon. In other markets, like the German power exchange EEX, the flow forwards can be traded within the delivery period, however the liquidity is very low. As we recall from Section 2, we have conditions ensuring an arbitrage-free dynamics of the geometric model which allow for a rich class of possible processes. Moreover, by choosing $\tau > T_1$ we also incorporate the empirical fact that flow forward prices do not converge to the spot price as time to delivery goes to zero.

4. Simulations and numerics

In this Section we present a numerical method for the simulation of forward curves based on numerical solution of the stochastic partial differential equation (2.1). We restrict our attention to forward contracts with fixed maturity time. Both the additive $g_a(t, x)$ in Equation (2.3) and the multiplicative $g_m(t, x)$ in Equation (2.4) are covered by our numerical approach, since we solve for $X(t)$. However, in the examples that we present we shall be concerned with $g_m(t, x)$. The method we propose does not have any restriction on the volatility structure as in Bjørk and Gombani [19] to allow for finite dimensional representations. We equilibrate the error for the finite dimensional realization of the infinite dimensional noise with the errors introduced by the space and time discretization of the stochastic partial differential equation. Moreover, we remark that it is not restricted to

energy markets, but provides an alternative approach to simulating forward rates in fixed-income markets for example, where the HJM approach has been applied.

First, we introduce a space and time approximation for the simulation of a sample path. To calculate functions of the solution, such as moments, we refer to the multilevel Monte Carlo method as introduced in Barth and Lang [9]. The multilevel Monte Carlo method exhibits significant improvement over the Monte Carlo method in terms of computational complexity. This is especially important if we want to solve stochastic partial differential equations, since their computation is expensive. We have to solve for each sample a partial differential equation, which entails in solving a (possibly very large) system of equations in every time-step.

To simulate the dynamics of the first-order hyperbolic, stochastic partial differential equation (2.1) we need an approximation in space, leading to the semidiscretized dynamics, as well as an approximation in time (full discrete form). For the finite dimensional approximation in space we introduce a Petrov–Galerkin method (SUPG method), which is well known to yield reasonable results for deterministic, first-order hyperbolic differential equations. A Galerkin approximation would provide convergence as well, but oscillations might occur when the solution is not smooth enough. Since the solution to a stochastic partial differential equation is not necessarily smooth we have to introduce an artificial diffusion via a Petrov–Galerkin approximation. There are several publications on convergence of these methods for deterministic equations (c.f. [38] or [24]). In general, a finite element method is preferable since it allows for more freedom when choosing the nodal points of the approximation than, for instance, a method based on finite differences.

We are facing the initial boundary value problem

$$u_t = Au + \xi \,,$$

with initial condition $u(0, \cdot) = v$ in some convex domain $D \subset \mathbb{R}^d$ with Lipschitz boundary ∂D. Here u_t denotes the partial derivative with respect to t. For a given vector field $\mathbf{a}$, the first order differential operator A is defined as

$$A\phi(x) := \sum_{i=1}^{d} a_i(x)\phi_{x_i}(x) \,.$$

The inflow boundary is the set

$$\partial D^- := \{x \in \partial D : \mathbf{a}(x) \cdot \mathbf{n}(x) > 0\} \,,$$

where $\mathbf{n}(x)$ denotes the exterior normal to ∂D at x. To have a well posed problem we impose a boundary condition on the inflow boundary and the initial condition $u(0, \cdot) = v$ matching the boundary condition in ∂D^-. The source term ξ is in our case the sum of the drift and the random field. As in the deterministic case, we introduce a finite-dimensional finite element space $\mathcal{S}_h$ as an approximation for the space variable $x \in D$. In the case studied we have $\mathcal{D} = [0, \tau] \subset \mathbb{R}_+$. $\mathcal{S}_h$ is a family of spaces of piecewise linear finite element functions with respect to a partition of D, without any boundary conditions on the functions in $\mathcal{S}_h$, thus $\mathcal{S}_h \subset H^1(D)$. The expectation of the solution $X(t)$, for $t \in [0, \tau]$ is, according to Lemma 2.4, an element of $H^1(D)$ if we have certain regularity on the initial

condition. For first-order hyperbolic equations we just have essential boundary conditions at the inflow boundary ∂D^-, which leads to the following definition for the finite element space:

$$\mathcal{S}_h^- = \{\chi \in \mathcal{S}_h : \chi = c \text{ at } \partial D^-\}$$

for some real constant c. The parameter h is a measurement of the fineness of the approximation. It denotes the longest side of any element of the triangulation in the space approximation. Translated into the problem at hand, the inflow boundary is the long end of the market. The constant c is, in this context, the value at the long end of the market, i.e. the value contracts reach when time-to-delivery tends to infinity.

The deterministic, semidiscrete, first order hyperbolic differential equation converges with rate $h^{3/2}$ for the SUPG method, see, e.g., Larsson and Thomée [38]. As indicated in Barth [5] and Barth and Lang [6], convergence of the stochastic partial differential equation only depends on the convergence of the corresponding partial differential equation and the compatibility of the scheme with the Itô isometry or a martingale inequality. A convergence result for this kind of equations was derived in Barth [5].

The main difference between SUPG and Galerkin methods is the test functions used. In a Galerkin approximation, one uses the same functions as test functions and trial functions. In contrast, to implement our Petrov–Galerkin method we choose as test functions $\chi + \gamma \hat{\chi}$, where χ is, for instance, the hat function basis and $\hat{\chi}$ denotes the (weak) derivative of the basis function. The trial functions remain χ. The parameter γ has to be chosen according to the problem, leading to a numerical scheme which is problem dependent. This could be circumvented with a Discontinuous Galerkin method (DG method). However, the parameter does not depend on the noise, but rather on the continuity of the initial and/or the boundary condition.

For the fully discrete problem we have to approximate the derivative in t as well. There are several options for this approximation. We use a Crank–Nicolson approach, which is known to be unconditionally stable. As a discretization of $[0, \tau]$ we introduce the equidistant time stepping $0 = t_0 < t_1 < \ldots < t_N = \tau$, where $t_i = \frac{\tau}{N}i$, for $i = 1, \ldots, N$, $N \in \mathbb{N}$. We couple the number of time steps with the degrees of freedom h in the space discretization, to have an equilibrated error in space and time, as suggested in Barth and Lang [9].

For the simulations we assume the driving noise process to be a Gaussian random field W or a random field of correlated Lévy processes L. For the latter, according to Peszat and Zabczyk [40] we have

$$L(t)(x) = \sum_{k=1}^{\infty} (L(t), e_k)_H e_k(x)$$

where $L(t)$ is a square integrable Lévy process in H and $(e_k, k \in \mathbb{N})$ is an orthonormal basis of H. The series on the right-hand side converges in L^2. For the finite-dimensional approximation of the infinite dimensional noise we truncate the sum after K terms. The rate of convergence of this truncated sum should be higher than the convergence of the full approximation scheme. In the case of a Gaussian random field we truncate the expansion (2.8). Results on how to truncate these series can be found in [7]. There the authors show
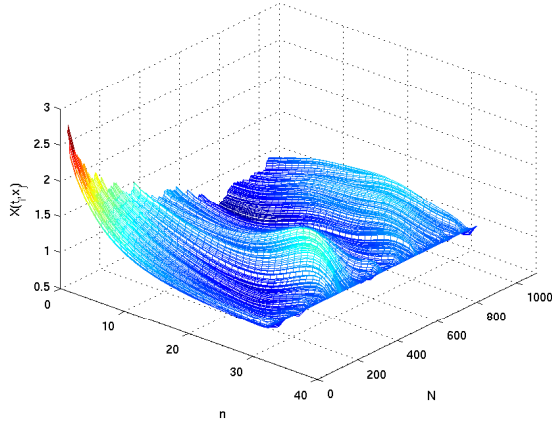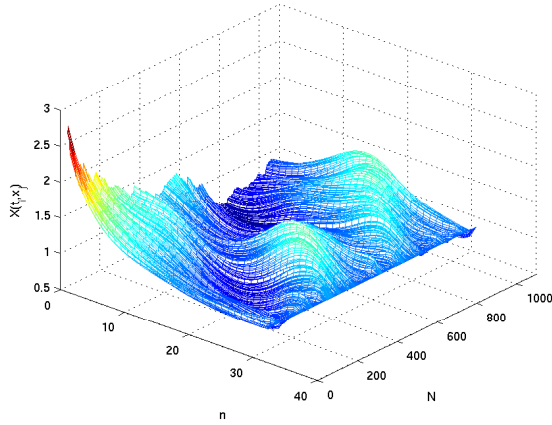
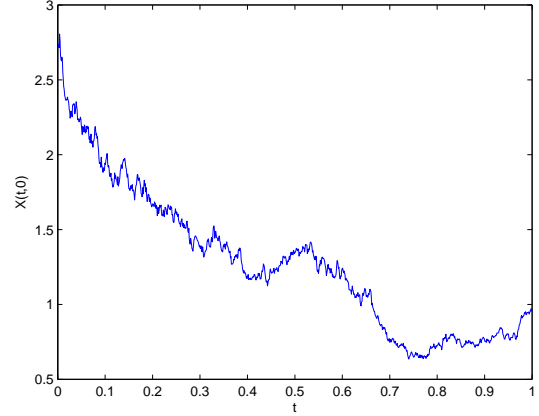(a) Forward curve, $(X(t_i, x_j), i = 0, \ldots, N, j = 0, \ldots, n)$
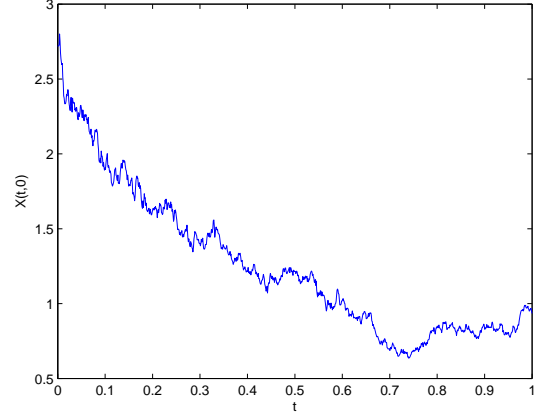
(b) Spot curve, $X(t, 0)$

FIGURE 1. Forward surface and corresponding spot curve for $\alpha = 4$ and $\kappa = 2$ driven by a Gaussian random field

(a) Forward curve, $(X(t_i, x_j), i = 0, \ldots, N, j = 1, \ldots, n)$
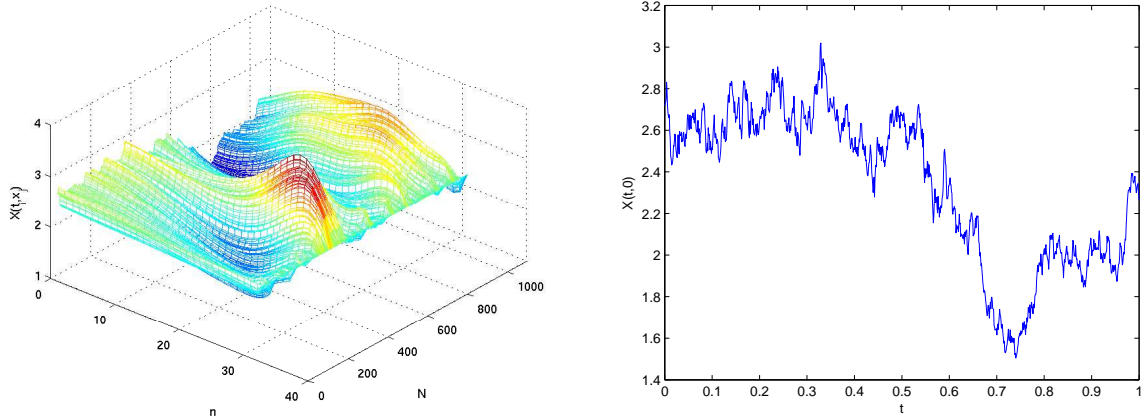
(b) Spot curve, $X(t, 0)$

FIGURE 2. Forward surface and corresponding spot curve for $\alpha = 4$ and $\kappa = 10$ driven by a Gaussian random field

how the truncation depends on the decay of the eigenvalues of the covariance operator Q in order to equilibrate the error of the truncation and the error of the space and time discretization. Thus, we have to simulate K one-dimensional Gaussian processes or Lévy processes, which are correlated via an operator Q.

As a first example we simulate a multiplicative model $g_m(t, x)$ inspired by Audet *et al.* [2], that is, $g_m(t, x) = \exp(\delta_x(X(t)))$ with

$$dX(t) = (AX(t) + a(t))dt + b(t)dM(t),$$

and $a(t)(x) = \mathrm{e}^{-2\alpha x}\sigma^2(x + t), (b(t) u)(x) = \delta_x(b(t)u)$ with $b(t)(x) = \mathrm{e}^{-\alpha x}\sigma(x + t)$ as introduced in Example 2.7. If the noise M is one-dimensional, this forward price dynamics would be implied by a spot dynamics given as an exponential Ornstein-Uhlenbeck process with speed of mean-reversion α and volatility $\sigma(t)$ (see e. g. Benth *et al.* [16]). In the numerical examples, we choose $\sigma(t)$ to be time independent. With an infinite-dimensional noise driving the forward curve, we open for idiosyncratic risk in each maturity. Following Audet *et al.* [2], we first specify M to be a Gaussian random field, later a random field of correlated NIG processes on $L^2(\mathbb{R})$. Both with covariance kernel function given by $\rho(x, y) = \exp(-\kappa|x-y|^2)$. For the Gaussian covariance kernel ρ the number of independent processes is chosen to be the number of nodal points of the approximation. In Barth and Lang [7], and in a more general form in Barth and Lang [8], this is shown to be sufficient to meet the error of the space and time discretization. The exponential covariance kernel given by $\rho(x, y) = \exp(-\kappa|x-y|^2)$ has eigenvalues which decay much slower. Consequently a truncation of Equation (2.8) has to involve much more terms, so that the overall error is not dominated by the truncation error of the noise process.
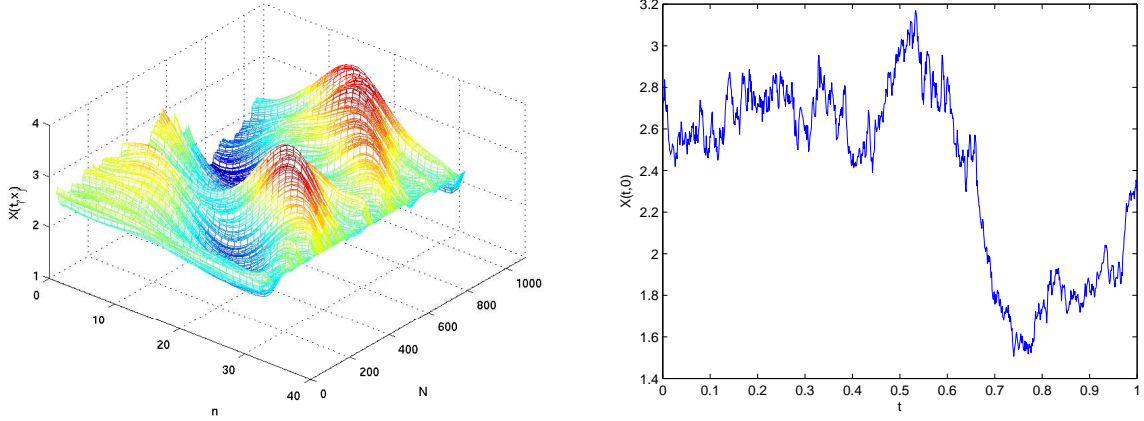

(a) Forward curve, $(X(t_i, x_j), i = 0, \ldots, N, j = 1, \ldots, n)$

(b) Spot curve, $X(t, 0)$

FIGURE 3. Forward surface and corresponding spot curve for $\alpha = 0.2$ and $\kappa = 2$ driven by a Gaussian random field

Let $x, y \in \{0 = x_1 < x_2 < \ldots < x_n = \tau\}$, be the set of the nodal points of the approximation. Simulations are presented for two different range parameters of the covariance kernel: $\kappa = 2$ and $\kappa = 10$. This is done to have first a stronger correlation, such that all points in D are correlated and the latter to have correlation in practice only over fractions of D. In the latter case one would expect a much rougher random field than in the former.

(a) Forward curve, $(X(t_i, x_j), i = 0, \ldots, N, j = 1, \ldots, n)$

(b) Spot curve, $X(t, 0)$

FIGURE 4. Forward surface and corresponding spot curve for $\alpha = 0.2$ and $\kappa = 10$ driven by a Gaussian random field

We chose, also for simplicity, the nodal points of the approximation $\{x_i\}$, $i = 1, \ldots, n$, to be equidistant, with $n = 2^5$. We assume $D = [0,1]$, i.e. $\tau = 1$. The step size for the approximation in time is $\Delta t = \tau/N$, where $N = 2^{10}$. To have a reasonable initial condition $g_m(0, x)$ for the case of a Gaussian random field, we consider the one-dimensional spot price model by Schwartz, where the spot price is $S(t) = \exp(Y(t))$ for

$$dY(t) = -\alpha Y(t)\, dt + \sigma\, dB(t)\,,$$

for a Brownian motion B. In this case, the initial forward curve would be

$$f(0, T) = S(0)^{e^{-\alpha T}} e^{\frac{\sigma^2}{4\alpha}(1 - e^{-2\alpha T})}\,.$$

Thus, in view of the close connection between this simple case and our infinite-dimensional model, we assume that

$$X(0, x) = \ln g_m(0, x) = e^{-\alpha x} + \frac{\sigma^2}{4\alpha}(1 - e^{-2\alpha x})\,.$$

Note that, in the analogy of the one-dimensional spot model, we have implicitly assumed that $\ln S(0) = 1$. The inflow condition is set to

$$X(0, 1) = X(t, 1) := e^{-\alpha} + \frac{\sigma^2}{4\alpha}(1 - e^{-2\alpha})$$

matching the initial condition. The parameter α is chosen to be $\alpha = 4$, and $\alpha = 0.2$, in line with the model example in Audet et $al.$ [2]. For simplicity we fix σ to 0.5. For the choice $\alpha = 0.2$ we have nearly a linearly decaying initial condition, whereas for $\alpha = 4$ we see a clear exponential decay. Both cases give a forward market in so-called $backwardation$, which is the normal situation in a market where the producers are hedging using forward

contracts and therfore accepting a discount in price compared to the current level of the spot. We simulate n independent Gaussian processes and correlate the field with the truncated expansion (2.8).

The outflow boundary $\mathrm{e}^{X(t,0)}$ is the spot price curve. Notice that we did not make any assumption on the spot. We display the resulting spot curve with the corresponding surface.

In Figures 1 and 2 a path of the forward surface driven by a Gaussian random field is pictured with $\alpha = 4$. The smoothing effect of the stronger correlation in Figure 1 is obvious. In the weakly correlated case ($\kappa = 10$) we have overall a stronger effect of the noise. Clearly our smooth choice of a covariance kernel leads to smooth forward surfaces. The corresponding spot curves reflect the Brownian motion. Noteably, the shape of the forward curves change with time, and can take more complex structures than simple backwardation. We can see for example humps, which is empirically observed in several forward markets (see for example Geman [33] for a discussion of humps in the oil market). These humps will in our case be a reflection of the high-dimensionality of the noise (see Benth [10] for a discussion of multi-factor spot models and humps). The simulated forward curves are very smooth, and may not be reasonable for the variations observed in power markets where there is a high degree of idiosyncratic risk between delivery times. The logarithmic spot dynamics decays rapidly towards a level 1, reflecting the strongly backwardated forward curve.

We compare these figures with Figures 3 and 4 and see the effect of a relatively flat initial condition. For both simulations exactly the same random numbers were used. We see a clear difference in the resulting spot curves. The flat initial condition together with the constant inflow boundary condition lead to a flat solution of the (deterministic) transport equation. However, the noise introduces fluctuations into the dynamics, which are transported by the transport equation, but with no effect from the boundary conditions (in space and time), i.e. the term structure and the long end of the market. In all four figures we can nicely see the transport of the information from one timestep to the next. This means that a certain forward price profile is merely the same shifted at a later point in time. We also note that the logarithmic spot dynamics shows a mean-reverting behaviour around the level indicated by the long-term value of the forward curve, in line with the structure of the forward curve model.

As a second example we simulate a random field of correlated centered NIG processes. We use the approach suggested in Rydberg [44] and Schoutens [46] to simulate a NIG process, with parameters $\mu = 0$, $\widetilde{\alpha}$, β and δ, as a time-changed Brownian motion. We simulate one IG process $I(t)$, where $I \sim IG(at, b)$ with $a = 1$ and $b = \delta\sqrt{\widetilde{\alpha}^2 + \beta^2}$, as a subordinator for the whole field. In fact, we correlate n independent Brownian motions, which are subordinated by this IG process, as in the case of the Gaussian random field. The generation of the random field of correlated centered NIG processes with one subordinating process I for the whole field ensures that the marginal process in each point $x \in D$ is again a one dimensional NIG process. This is an important feature, since the spot dynamics
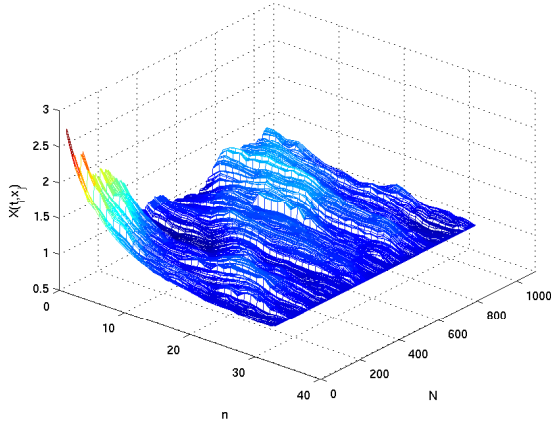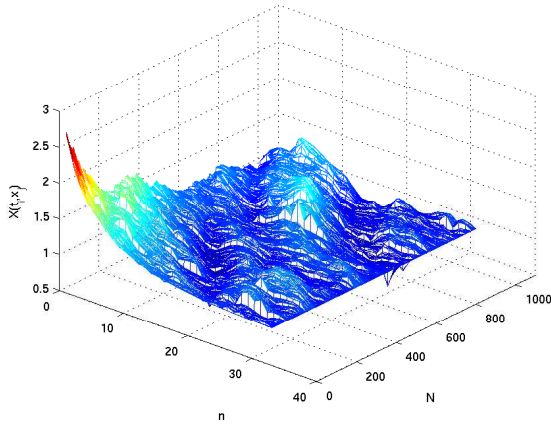
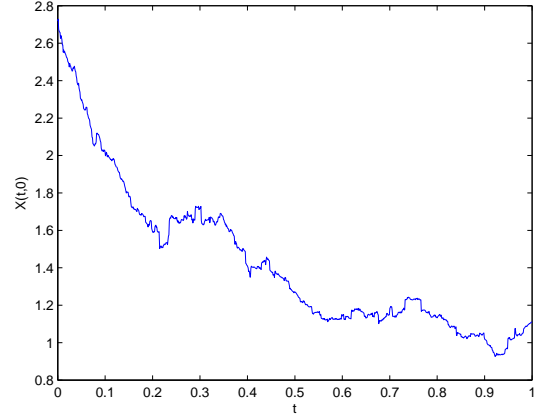(a) Forward curve, $(X(t_i, x_j), i = 0, \ldots, N, j = 1, \ldots, n)$
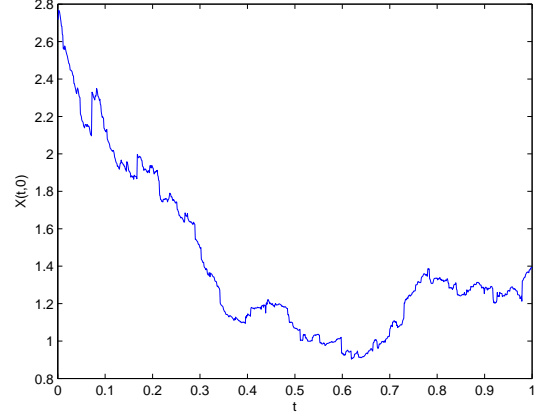
(b) Spot curve, $X(t, 0)$

FIGURE 5. Forward surface and corresponding spot curve for $\alpha = 4$ and $\kappa = 2$ driven by a random field of correlated NIG processes

(a) Forward curve, $(X(t_i, x_j), i = 0, \ldots, N, j = 1, \ldots, n)$

(b) Spot curve, $X(t, 0)$

FIGURE 6. Forward surface and corresponding spot curve for $\alpha = 4$ and $\kappa = 10$ driven by a random field of correlated NIG processes

should be NIG distributed. The NIG field is given with the described method by

$$L^{NIG}(t) = \beta\delta^2 I(t) + \delta W(I(t)),$$

where W is a Gaussian random field. This random field has the expansion given by

$$L^{NIG}(t) = \beta\delta^2 I(t) + \delta \sum_{i=1}^{\infty} \sqrt{\lambda_i} e_i W_i(I(t)),$$

where, as before, $((\lambda_i, e_i), i \in \mathbb{N})$ is the basis formed by the spectral decomposition of the covariance operator Q and $(W_i, i \in \mathbb{N})$ is a sequence of one-dimensional Brownian motions. So in an arbitrary point $x \in D$ we obtain again an NIG process.

For the example we set $\widetilde{\alpha} = 10$,[1] $\beta = 0$ and $\delta = 1$. The jump height can be further influenced by a change of the parameter $\widetilde{\alpha}$.

To calculate initial and inflow boundary conditions for the Lévy driven stochastic partial differential equation we consider a one-dimensional spot price model, where the spot price is $S(t) = \exp(\hat{Y}(t))$ for

$$d\hat{Y}(t) = -\alpha\hat{Y}(t)\,dt + dL(t)\,,$$

for a NIG process L, whose increments are $NIG(0, \widetilde{\alpha}, \beta, \delta)$-distributed. The corresponding jump intensity measure of the NIG process is given by:

$$\nu(dz) = \frac{\delta\widetilde{\alpha}}{\pi} \frac{e^{\beta z} K_1(\widetilde{\alpha}|z|)}{|z|} dz,$$

where $K_1(z)$ denotes the modified Bessel function of the third kind of order 1.

We have two ways to derive the forward curve leading to the same result: Once we apply Prop. 4.6 in Benth *et al.* [16] and find

$$f(0, T) = S(0)^{e^{-\alpha T}} \exp\left(\int_0^T c(e^{-\alpha s})\,ds \right)\,,$$

where c is the cumulant function of $L(1)$, that is, the logarithm of the moment generating function. For a NIG-variable, it is (see p. 53 in [16])

$$c(u) = \mu u + \delta\left(\sqrt{\widetilde{\alpha}^2 - \beta^2} - \sqrt{\widetilde{\alpha}^2 - (\beta + u)^2} \right)\,.$$

We remark that we have derived the price of the forward without any risk adjustment. This could easily be incorporated, but we refrain from doing so here.

On the other hand we perform a similar calculation as for the Gaussian case: Then the dynamics of the forward curve reads

$$f(0, T) = S(0)^{e^{-\alpha T}} e^{\frac{\delta\widetilde{\alpha}}{\alpha\pi}(1 - e^{-\alpha T})\int_1^\infty e^{\beta z} K_1(\widetilde{\alpha}|z|)dz}\,.$$

With $\beta = 0$ and $\delta = 1$ we may write

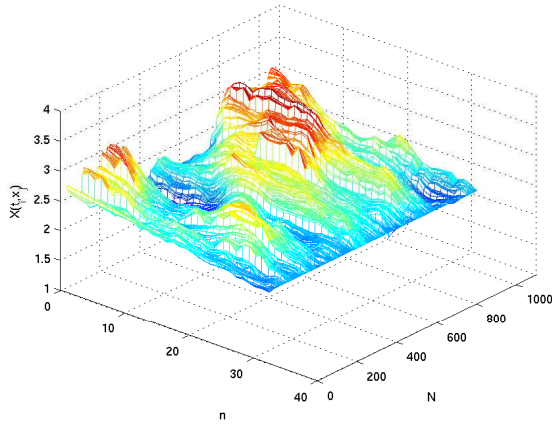$$f(0, T) = S(0)^{e^{-\alpha T}} e^{\frac{K_0(\widetilde{\alpha})}{\alpha\pi}(1 - e^{-\alpha T})}\,.$$

As before we get, assuming $\ln S(0) = 1$, the initial condition for the infinite dimensional model as

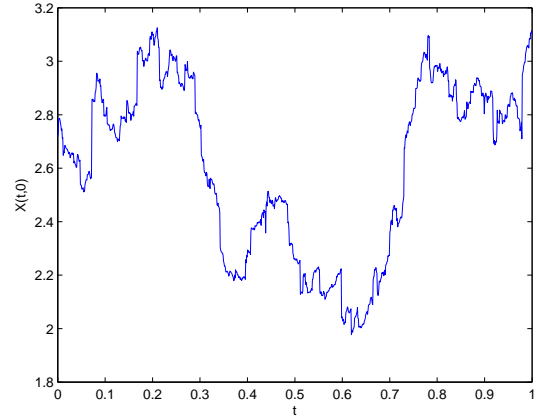$$X(0, x) = e^{-\alpha x} + \frac{K_0(\widetilde{\alpha})}{\alpha\pi}(1 - e^{-\alpha x})$$

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[1]This parameter is usually denoted α in the NIG distribution, but that notation was not available to us.

and thus as the inflow boundary condition

$$X(t,1) = \mathrm{e}^{-\alpha} + \frac{K_0(\widetilde{\alpha})}{\alpha\pi}(1 - \mathrm{e}^{-\alpha}).$$
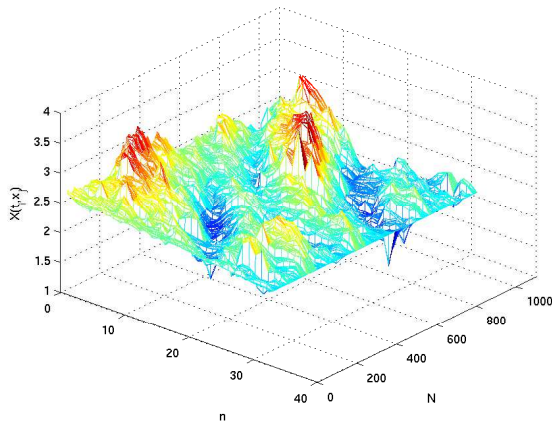

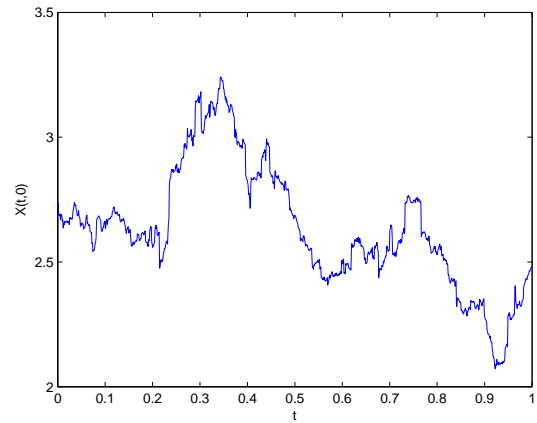(a) Forward curve, $(X(t_i, x_j), i = 0, \ldots, N, j = 1, \ldots, n)$

(b) Spot curve, $X(t,0)$

FIGURE 7. Forward surface and corresponding spot curve for $\alpha = 0.2$ and $\kappa = 2$ driven by a random field of correlated NIG processes

(a) Forward curve, $(X(t_i, x_j), i = 0, \ldots, N, j = 1, \ldots, n)$

(b) Spot curve, $X(t,0)$

FIGURE 8. Forward surface and corresponding spot curve for $\alpha = 0.2$ and $\kappa = 10$ driven by a random field of correlated NIG processes

In Figures 5 and 6 we plotted paths of the forward surface driven by a random field of correlated NIG processes. Again in Figure 5 the smoothing effect of the stronger correlation is visible. For both figures we chose $\alpha = 4$. The modeling of the random field as a time changed Brownian Motion with one IG process as a subordinator for the whole field causes the jump of the whole forward curve. The corresponding spot curves reflect the structure of a NIG process. The smaller the parameter κ is, the higher the correlation is and therefore the smoother the forward surface becomes. In Figure 6 (as well as in Figure 8) we see jumps of more or less single points, this is impossible with a smaller parameter κ. In Figures 7 and 8 we plotted a field of correlated NIG processes with $\alpha = 0.2$. As in the case of the Gaussian random field we see the influence of the initial and the boundary condition.

By using a NIG-Lévy field as a model for the noise, we obtain much rougher looking sample paths in the simulation. Such forward curves might seem reasonable in view of the idiosyncratic risk in power markets. Typically in general commodity markets, one expects random fluctuations along time that can have jumps, however, along time-to-delivery there is much more smoothness expected. In power this is different, and we device here a class of models with namely such a feature of roughness. In Andresen, Koekebakker and Westgaard [1] one finds clear empirical evidence for a NIG-driven random field noise for forward prices observed in the NordPool market, confirming and extending the analysis in Frestad, Benth and Koekebakker [32]. These two papers consider flow forwards, as these are the ones traded in the market. Noteworthy is, that one finds more variation the shorter the length of delivery is. This is particularly evident in Benth, Koekebakker and Ollmar [14], detecting big variations in the short end of the NordPool curve where short-term contracts are traded (down to one day of delivery). We further note that the implied logarithmic spot price curves are similar to those coming from the Gaussian cases, except that the noise seems to be different, reflecting the Lévy field chosen.

One might be interested in the approximation of the expectation of a function of the solution and not in the approximation of a path of the solution process, i.e. we want to estimate the quantity $\mathbb{E}(f(X(t)))$, where f is a Lipschitz function. In this case a multilevel Monte Carlo method could be applied, where we approximate the expectation by a combination of Monte Carlo estimators with sample sizes depended on the space (and time) discretization. In Barth and Lang [9] convergence of the approximation is proved and an optimal number of samples is calculated to reduce the overall computational effort when compared to a normal Monte Carlo method.

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