STACKELBERG EQUILIBRIA IN A CONTINUOUS TIME VERTICAL CONTRACTING MODEL WITH UNCERTAIN DEMAND AND DELAYED INFORMATION

BERNT ØKSENDAL,∗ University of Oslo

LEIF SANDAL,** Norwegian School of Economics

JAN UBoE,*** Norwegian School of Economics

Abstract

We consider explicit formulas for equilibrium prices in a continuous time vertical contracting model. A manufacturer sells goods to a retailer, and the objective of both parties is to maximize expected profits. Demand is an Itô-Lévy process, and to increase realism information is delayed. We provide complete existence and uniqueness proofs for a series of special cases, including geometric Brownian motion and the Ornstein-Uhlenbeck process, both with time-variable coefficients. Moreover, these results are operational because we are able to offer explicit solution formulas. An interesting finding is that information that is more precise may be a considerable disadvantage for the retailer.

Keywords: Vertical contracting; stochastic differential games; delayed information; Itô-Lévy processes
2010 Mathematics Subject Classification: Primary 60H30
Secondary 91A15

1. Introduction

In a newsvendor problem a retailer orders goods from a manufacturer. Demand is a random variable, and the retailer aims to find an order quantity that maximizes expected profit. In the single period problem only one such order is made, while the multiperiod problem is concerned with a sequence of orders. In this paper we consider the newsvendor problem in continuous time, where the discrete order quantity is replaced by an ordering rate, i.e., number of items ordered per time unit. The single period problem dates back to Edgeworth (1888). The basic problem is very simple but appears to have a never ending number of variations. There is now a very large literature on such problems, and for further reading we refer to the survey papers by Cachón (2003) and Qin et al. (2011).

In our paper, a retailer and a manufacturer write contracts for a specific delivery rate following a decision process in which the manufacturer is the leader who initially decides the wholesale price. Based on that wholesale price, the retailer decides on the
delivery rate. We assume a Stackelberg framework, and hence ignore cases where the retailer can negotiate the wholesale price. The contract is written at time \( t - \delta \), and goods are received at time \( t \). It is essential to assume that information is delayed. If there is no delay, the demand rate is known, and the retailer’s order rate is made equal to the demand rate. Stackelberg games of this type has been studied in Øksendal et al. (2013), and in our paper we will use Theorem 3.2.2 in Øksendal et al. (2013) to provide explicit formulas for commonly used stochastic processes, i.e., geometric Brownian motion (extended to a geometric Lévy process) and the Ornstein-Uhlenbeck (OU) process.

Stackelberg games for single period newsvendor problems have been studied extensively by Lariviere and Porteus (2001), providing quite general conditions under which unique equilibria can be found. Multiperiod newsvendor problems with delayed information have been discussed in several papers. Bensoussan et al. (2009) use a time-discrete approach and generalize several information delay models. Computational issues are not explored in their paper, and they only consider decision problems for inventory managers, disregarding any game theoretical issues. Calzolari et al. (2011) discuss filtering of stochastic systems with fixed delay, indicating that problems with delay lead to nontrivial numerical difficulties even when the driving process is Brownian motion. Kaplan (1970) is a classical paper discussing stochastic lead times in a multiperiod problem. Several authors have contributed to the discussion of stochastic lead times, we mention Song and Zipkin (1996).

The geometric Lévy process is fundamental in many models in physics, biology and finance, because it is a natural extension to the case with random coefficients of an exponential growth model, as follows: If the relative growth rate in an exponential model is assumed random and represented by a sum of a continuous noise (generated by Brownian motion) plus a jump noise (generated by a pure jump Lévy process), we arrive at a geometric Lévy process. Such processes represent natural generalizations to jumps of the classical geometric Brownian motion, which were introduced by Samuelson (1965) and later applied in the famous Black-Scholes market model by F. Black, M. Scholes and R. Merton. Regarding financial motivations and justifications for using extensions of the geometric Brownian motion to jump models based on Lévy processes, we refer to Barndorff-Nielsen O. (1998), Eberlein (2009) and the references therein.

The OU process is a widely used model for any stochastic phenomenon exhibiting mean reversion. It is is the unique non-trivial stochastic process that is stationary, Markovian and Gaussian, Maller et al. (2009). It is used in financial engineering as a model for the term structure of interest rates, Vasicek (1977), and under other variants or generalisations as a model of financial time series with applications to option pricing, portfolio optimization and risk theory, see, e.g., Nikato and Vernardos (2003), Barndorf-Nielsen and Shepard (2001), see Maller et al. (2009) and references therein. The OU process can be thought of as a continuous time interpolation of an autoregressive process of order one (AR(1) process), i.e., the series obtained by sampling OU processes at equally spaced times are autoregressive of the same order.

The paper is organized as follows. In Section 2, we formulate and discuss a general
continuous-time newsvendor problem. In Section 3, we consider the case where the demand rate is given by geometric Brownian motion and provide explicit solutions for the unique equilibria that occur in that case. The result in the constant coefficient case is quite startling as it leads to an equilibrium where the manufacturer offers a constant price $w$ and the retailer orders a fixed fraction of the observed demand rate. In Section 4 we discuss non-Markov jump diffusions and demonstrate that knowledge of the state of the system at time $t$ is not sufficient to infer the optimal order quantity. In Section 5 we provide explicit formulas for the unique equilibria that occur when demand is given by an OU process with time variable coefficients. We also compute numerical values to compare the dynamic approach with a static approach where both parties (wrongly) believe that the demand rate has a static distribution. An interesting finding is that information that is more precise can be a considerable disadvantage for the retailer. Finally, in Section 6 we offer some concluding remarks. To make the paper easier to read, complete proofs are in most cases placed in the Appendix.

2. Continuous time newsvendor problems

In this section, we formulate a continuous time newsvendor problem and use the results in Øksendal et al. (2013), to describe a set of explicit equations that we need to solve to find Stackelberg equilibria. We will assume that the demand rate for a good is given by an Itô- Lévy process of the form

$$dD_t = \mu(t, D_t, \omega)dt + \sigma(t, D_t, \omega)dB_t + \int_{\mathbb{R}} \gamma(t, D_t - \xi, \omega)\tilde{N}(dt, d\xi); \quad t \in [0, T]$$

(1)

$D_0 = d_0 \in \mathbb{R}$

Here $B_t$ denotes a Brownian motion and $\tilde{N}(dt, d\xi)$ is a compensated Poisson term. The coefficients $\mu$, $\sigma$, and $\gamma$ are assumed to satisfy standard conditions making sure that (1) has a unique solutions, see Øksendal and Sulem (2007).

At time $t - \delta$ a retailer and a manufacturer are negotiating a contract for items to be delivered at time $t$, where $\delta > 0$ is the delay time. The idea is that production takes time, and that the contract must be settled in advance. The manufacturer (leader) offers a wholesale price $w_t$ per unit. On the basis of this wholesale price, the retailer (follower) chooses a delivery rate $q_t$. The retail price $R$ we assume to be fixed. At the time when the contract is written, the demand at time $t$ is unknown, so the contract must be based on information available at time $t - \delta$.

To formalize the information, we let $\mathcal{F}_t$ denote the $\sigma$-algebra generated by $B_s$ and $\tilde{N}(s, dz)$, $0 \leq s \leq t$. Intuitively $\mathcal{F}_t$ contains all the information up to time $t$. When information is delayed, we consider the $\sigma$-algebras $\mathcal{E}_t := \mathcal{F}_{t-\delta}$, $t \in [\delta, T]$. Both the retailer and the manufacturer should base their actions on the delayed information. Technically that means that $q_t$ and $w_t$ should be $\mathcal{E}$-predictable processes.

We assume that items can be salvaged at a unit price $S \geq 0$, and that items cannot be stored, i.e., they must be sold instantly or salvaged. Assuming that sale will take
part in the time period $\delta \leq t \leq T$, the retailer will get an expected profit

$$J_2(w, q) = E \left[ \int_{\delta}^{T} (R_t - S) \min[D_t, q_t] - (w_t - S)q_t dt \right]$$  \hspace{1cm} (2)$$

When the manufacturer has a constant production cost per unit $M$, the manufacturer will get an expected profit

$$J_1(w, q) = E \left[ \int_{\delta}^{T} (w_t - M)q_t dt \right]$$  \hspace{1cm} (3)$$

The profit functions (2) and (3) sets up a stochastic Stackelberg game of a type that has been studied in Øksendal et al. (2013).

2.1. Finding Stackelberg equilibria in the newsvendor model

It is well known that under conditions similar to our assumptions above, the discrete multi-period newsvendor model can be solved by an optimization pointwise in $t$. In a single period newsvendor model with demand $D$, the retailer will order $q$ satisfying the equation

$$P(D \geq q) = \frac{w_t - S}{R - S}$$  \hspace{1cm} (4)$$

If the demand process is Markov, it is reasonable to conjecture that the retailer at time $t - \delta$ should order a quantity corresponding to the distribution of $D_t$ conditional on $E_t$. If the process is non-Markov, this is not clear, but an affirmative answer clarifying the situation is provided by Theorem 3.2.2 in Øksendal et al. (2013);

**Theorem 2.1.** (Øksendal et al. (2013)). Suppose the pair $(\hat{w}, \hat{q})$ is a Stackelberg equilibrium for the newsvendor problem defined by (3) and (2). Assume that $D_t$ as given by (1) has a continuous distribution. For any given $w_t$ with $S < M \leq w_t \leq R$ consider the equation

$$E \left[ (R - S)X_{[0, D_t]}(q_t) - w_t + S|E_t \right] = 0$$  \hspace{1cm} (5)$$

Let $q_t = \phi(w_t)$ denote the unique solution of (5), and assume that the function

$$w_t \mapsto E [(w_t - M)\phi(w_t)|E_t]$$  \hspace{1cm} (6)$$

has a unique maximum at $w_t = \hat{w}_t$. Then $\hat{q}_t = \phi(\hat{w}_t)$.

Here $X_{[0, D_t]}(q)$ denotes the indicator function for the interval $[0, D_t]$, i.e., a function that has the value 1 if $0 \leq q \leq D_t$, and is zero otherwise. To see why (5) always has a unique solution, note that $w_t$ is $E_t$-measurable and hence (5) is equivalent to

$$E \left[ X_{[0, D_t]}(q_t)|E_t \right] = \frac{w_t - S}{R - S}$$  \hspace{1cm} (7)$$

Existence and uniqueness of $q_t$ then follows from monotonicity of conditional expectation. (7) is in fact the correct generalization of (4) to the time continuous case. To avoid degenerate cases we need to know that $D_t$ has a continuous distribution. In the
next sections we will consider special cases, and we will often be able to write down explicit solutions to (5) and prove that (6) has a unique maximum. Notice that (5) is an equation defined in terms of conditional expectation. Conditional statements of this type are in general difficult to compute, and the challenge is then to state the result in terms of unconditional expectations.

3. Explicit formulas for geometric Brownian motion

In this section, we offer explicit formulas for the equilibria that occur when the demand rate is given by a geometric Brownian motion. We first consider the case with constant coefficients, and then extend the results to the case with time-dependent, deterministic coefficients. We also discuss a non-Markovian case where demand is given by a geometric Lévy process.

3.1. Geometric Brownian motion with constant coefficients

In this section we assume that $D_t$ is a geometric Brownian motion with constant coefficients, i.e., that
\begin{equation}
\quad dD_t = aD_t dt + \sigma D_t dB_t
\end{equation}
where $a, \sigma$ are constants. The explicit solution to (8) is $D_t = D_0 \exp \left[ (a - \frac{1}{2} \sigma^2) t + \sigma B_t \right]$, and it is easy to see that
\begin{equation}
D_t = D_{t-\delta} \exp \left[ \left( a - \frac{1}{2} \sigma^2 \right) \delta + \sigma (B_t - B_{t-\delta}) \right]
\end{equation}
The explicit form of (9) makes it possible to write down a closed form solution to (5). The function $G$ is the cumulative distribution of a standard normal distribution.

Proposition 3.1. Let $\Phi : [M, R] \to \mathbb{R}$ denote the function
\begin{equation}
\Phi[w] = \exp \left[ \left( a - \frac{1}{2} \sigma^2 \right) \delta + \sqrt{\sigma^2 \cdot \delta} \cdot G^{-1} \left[ 1 - \frac{w - S}{R - S} \right] \right]
\end{equation}
and let $\Psi : [M, R] \to \mathbb{R}$ denote the function $\Psi[w] = (w - M) \Phi[w]$. The function $\Psi$ is quasiconcave and has a unique maximum with a strictly positive function value. At time $t - \delta$ the retailer should observe $y = D_{t-\delta}$, and a unique Stackelberg equilibrium is obtained at
\begin{equation}
w_t^* = \text{Argmax}[\Psi] \quad \text{(constant)} \quad q_t^* = y \cdot \Phi[\text{Argmax}[\Psi]]
\end{equation}
Proof. See the Appendix.
The equilibria resulting from this situation are quite surprising. We see that the wholesale price is in fact constant. As a consequence of this, the manufacturer need not observe demand at time $t - \delta$ to settle the price. In fact she can write a contract with set wholesale price for the whole sales period. The retailer needs to observe demand, but his strategy is very simple; observe demand and order a fixed fraction of the observed demand.

As is clear from the proof, these properties originates from the multiplicative scaling of geometric Brownian motion, i.e., if the initial condition is scaled by a multiplicative factor, any sample path is scaled by the same factor. Critical fractiles are scaled accordingly, and as a consequence the optimal wholesale price will not change. It is the same type of effect driving the classical Merton’s portfolio problem in finance, Merton (1969): If the risky asset is a constant coefficient geometric Brownian motion, the optimal policy is to keep a fixed fraction in the risky asset.

3.2. Geometric Brownian motion with variable coefficients

In this section we assume that $D_t$ is a geometric Brownian motion with variable deterministic coefficients, i.e., that

$$dD_t = a(t)D_t dt + \sigma(t)D_t dB_t$$  \hspace{2cm} (12)

where $a(t), \sigma(t)$ are given deterministic functions.

**Proposition 3.2.** For $t \in [\delta, T]$, let $\Phi_t : [M, R] \to \mathbb{R}$ denote the function

$$\Phi_t[w] = \exp \left[ \hat{a}(t) + \hat{\sigma}(t) \cdot G^{-1} \left[ 1 - \frac{w - S}{R - S} \right] \right]$$  \hspace{2cm} (13)

where

$$\hat{a}(t) = \int_{t-\delta}^{t} a(s) - \frac{1}{2} \sigma^2(s) ds \quad \hat{\sigma}(s) = \sqrt{\int_{t-\delta}^{t} \sigma^2(s) dt}$$  \hspace{2cm} (14)

and let $\Psi_t : [M, R] \to \mathbb{R}$ denote the function $\Psi_t[w] = (w - M)\Phi_t[w]$. The function $\Psi_t$ is quasiconcave and has a unique maximum with a strictly positive function value. At time $t - \delta$ the retailer should observe $y = D_{t-\delta}$, and a unique Stackelberg equilibrium is obtained at

$$w^*_t = \text{Argmax}[\Psi_t] \quad q^*_t = y \cdot \Phi_t[\text{Argmax}[\Psi_t]]$$  \hspace{2cm} (15)

**Proof.** See the Appendix.

If we compare with the case with constant coefficients, we see that the wholesale price $w$ is no longer constant. Nevertheless we see that the equilibria are defined in terms of two deterministic functions $\text{Argmax}[\Psi_t]$ and $\Phi_t[\text{Argmax}[\Psi_t]]$. As in the constant coefficient case, the manufacturer need not observe demand, but can settle wholesale prices upfront for the whole sales period.
4. Geometric Lévy processes

In this section we will compute explicit Stackelberg equilibria in cases where the demand is given by non-Markov processes. We first consider a case where demand is given by

\[
dD_t = (\alpha_1 + \alpha_2 B_t)Dt + \sigma D_t dB_t
\]

(16)

where \(\alpha_1, \alpha_2\) and \(\sigma\) are constants. Solving (16), we get

\[
D_t = D_{t-\delta} \cdot \exp \left[ -\frac{1}{2} \sigma^2 \delta + \sigma (B_t - B_{t-\delta}) + \int_{t-\delta}^{t} \alpha_1 + \alpha_2 B_s ds \right]
\]

(17)

Here we get an additional difficulty as the last term, i.e., \(\int_{t-\delta}^{t} \alpha_2 B_s ds\), is not independent of \(\mathcal{E}_t\), reflecting the non-Markovian structure of the solution. To compute the conditional expectation, we need to rewrite the expression. Integration by parts gives

\[
D_t = D_{t-\delta} \cdot e^{a(t-\delta)} \cdot \exp \left[ -\frac{1}{2} \sigma^2 \delta + \int_{t-\delta}^{t} \alpha_2 (t-s) + \sigma dB_s \right]
\]

(18)

This separates the expression into a product where the first factor is \(\mathcal{E}_t\)-measurable, while the second factor is log-normal and independent of \(\mathcal{E}_t\). Using the same separation technique as before, it is then straightforward to find an explicit solution to (5), and existence and uniqueness of the corresponding Stackelberg problem follows as in the proof of Proposition 3.1.1. This technique is in fact applicable to quite general processes. A geometric Lévy process is a solution of a stochastic differential equation of the form

\[
dD(t) = D(t^-) \left( a(t, \omega) dt + \sigma(t, \omega) dB_t + \int_{\mathbb{R}} \gamma(t, z, \omega) \tilde{N}(dt, dz) \right)
\]

(19)

If we assume that \(D(0) = D_0 > 0\) and \(\gamma(t, z) > -1\), the solution satisfies \(D_t \geq 0\) for all \(t\). The explicit solution of (19) is

\[
D_t = D_0 \exp \left[ \int_0^t \left( a(s, \omega) - \frac{1}{2} \sigma^2(s, \omega) + \int_{\mathbb{R}} \log[1 + \gamma(s, z, \omega)] - \gamma(s, z, \omega) \nu(dz) \right) ds \right]
\]

\[
+ \int_0^t \sigma(s, \omega) dB_s + \int_0^t \int_{\mathbb{R}} \log[1 + \gamma(s, z, \omega)] \tilde{N}(ds, dz)
\]

(20)

Now we make the additional assumption that

\[
a(s, \omega) = \alpha_1(s) + \alpha_2(s) B_s(\omega) \quad \sigma(s, \omega) = \sigma(s) \quad \gamma(s, z, \omega) = \gamma(s, z)
\]

(21)

i.e., that \(\sigma\) and \(\gamma\) are given deterministic functions, while the growth rate \(a(s, \omega)\) depends on \(\omega\) as well as \(t\), \(\alpha_1, \alpha_2\) are given deterministic functions. For each fixed \(t\),
Bernt Øksendal, Leif Sandal and Jan Ubøe

consider the random variable \( X_t \) given by

\[
X_t = \exp \left[ \int_{t-\delta}^t \left( \int_s^t \alpha_2(u) du + \sigma(s) \right) dB_s + \int_{t-\delta}^t \left( -\frac{1}{2} \sigma^2(s) + \int_{R_0}^s \log[1 + \gamma(s, z)] - \gamma(s, z) \nu(dz) \right) ds \right] \tag{22}
\]

We can then state the following proposition:

**Proposition 4.1.** Assume that demand \( D_t \) is a geometric Lévy process given by (19), where the coefficients satisfy (21). Let \( F_t \) denote the cumulative distribution of \( X_t \) given by (22), and for each fixed \( t \) let \( F_t^{-1} \) denote the inverse function of \( F_t \). Consider for each \( t \in [\delta, T] \), the functions

\[
\Phi_t[w] = F_t^{-1} \left[ 1 - \frac{w - S}{R - S} \right] \quad \Psi_t[w] = (w - M) \Phi_t[w] \tag{23}
\]

At time \( t - \delta \) the retailer should observe both the demand rate \( y = D_{t-\delta} \) and \( z = B_{t-\delta} \), and a Stackelberg equilibrium is obtained at

\[
w_t^* = \text{Argmax}[\Psi_t] \quad q_t^* = y \cdot e^{\int_{t-\delta}^t \alpha_1(s) + \alpha_2(s) z ds} \cdot \Phi_t[\text{Argmax} [\Psi_t]] \tag{24}
\]

**Proof.** See the Appendix.

Note that the value of \( z \) can be derived from the growth rate \( \alpha_1(t-\delta) + \alpha_2(t-\delta) B_{t-\delta} \).

If \( \alpha_1, \alpha_2 \) are constants, we see that the factor \( e^{\int_{t-\delta}^t \alpha_1(s) + \alpha_2(s) z ds} \) is the correction we would expect if the growth rate had stayed constant at the level we observed at time \( t - \delta \).

We notice that the structure of the solution is quite similar to the case covered in Proposition 3.2.1. The manufacturer has a pricing strategy defined in terms of a deterministic function. The retailer should observe the demand rate, adjust it by the observed growth rate, and multiply the adjusted number by a deterministic fraction.

5. The Ornstein-Uhlenbeck process

In this section we discuss equilibrium prices for the Ornstein-Uhlenbeck process. We extend the results from Øksendal et al. (2012) to the case with time variable coefficients, and also report the results from a numerical experiment where we compare the performance of a static versus a dynamic pricing strategy.

5.1. Explicit formulas for the Ornstein–Uhlenbeck process

In this section, we offer explicit formulas for the equilibria that occur when the demand rate is given by an Ornstein–Uhlenbeck process. The constant coefficient case was studied in Øksendal et al. (2012), and we here extend the formulas to the case

\[
dD_t = a(t)(\mu(t) - D_t)dt + \sigma(t)dB_t \tag{25}
\]
where \( a(t), \mu(t), \) and \( \sigma(t) \) are given deterministic functions. The increased flexibility is important in applications since it allows for scenarios where the mean reversion level \( \mu \) can have a time variable trend. The basic result can be summarized as follows:

**Proposition 5.1.** For each \( t \in [\delta, T] \), \( y \in \mathbb{R} \), let \( \Phi_{t,y} : [M, R] \to \mathbb{R} \) denote the function

\[
\Phi_{t,y}[w] = ye^{-\int_{t-\delta}^{t} a(s)ds} + \bar{\mu}(t) + \bar{\sigma}(t)G^{-1} \left[ 1 - \frac{w - S}{R - S} \right]
\]

(26)

where

\[
\bar{\mu}(t) = \int_{t-\delta}^{t} a(s)\mu(s)e^{-\int_{s}^{t} a(u)du}ds,
\]

\[
\bar{\sigma}(t) = \sqrt{\int_{t-\delta}^{t} \sigma^2(s)e^{-2\int_{s}^{t} a(u)du}ds}
\]

(27)

and let \( \Psi_{t,y} : [M, R] \to \mathbb{R} \) denote the function \( \Psi_{t,y}[w] = (w - M)\Phi_{t,y}[w] \). If \( \Phi_{t,y}[M] > 0 \), the function \( \Psi_{t,y} \) is quasiconcave and has a unique maximum with a strictly positive function value. At time \( t - \delta \) the parties should observe \( y = D_{t-\delta} \), and a unique Stackelberg equilibrium is obtained at

\[
w^*_t = \begin{cases} 
\text{Argmax}[\Psi_{t,y}] & \text{if } \Phi_{t,y}[M] > 0 \\
M & \text{otherwise}
\end{cases}
\]

\[
q^*_t = \begin{cases} 
\Phi_{t,y}[\text{Argmax}[\Psi_{t,y}]] & \text{if } \Phi_{t,y}[M] > 0 \\
0 & \text{otherwise}
\end{cases}
\]

(28)

The condition \( \Phi_{t,y}[M] > 0 \) has an obvious interpretation. The manufacturer cannot offer a wholesale price \( w \) lower than the production cost \( M \). If \( \Phi_{t,y}[M] \leq 0 \), it means that the retailer is unable to make a positive expected profit even at the lowest wholesale price the manufacturer can offer. When that occurs, the retailer’s best strategy is to order \( q = 0 \) units. When the retailer orders \( q = 0 \) units, the choice of \( w \) is arbitrary. However, the choice \( w = M \) is the only strategy that is increasing and continuous in \( y \).

**Proof.** See the Appendix.

We notice that the structure of the equilibria are quite different from the case with geometric Brownian motion. Contrary to the geometric Brownian motion case the manufacturer needs to observe the market to compute wholesale prices.

### 5.2. Numerical examples for the Ornstein–Uhlenbeck process

In this section we will compare the performance of the dynamic approach with a scenario where the retailer believes that demand has a constant distribution \( D \). A constant coefficient Ornstein–Uhlenbeck process

\[
D_t = D_0 e^{-at} + \mu(1 - e^{-at}) + \sigma e^{-at} \cdot \int_0^t e^{as}dB_s
\]

(29)

is ergodic in the sense that observations along any sample path will approach the distribution \( N\mu, \sigma^2 \). Assuming that the retailer believes the demand rate has a static distribution \( \tilde{D} \) and that he has observed that demand rate for long enough prior
to ordering, he will conclude that \( D \) is \( N(\mu, \sigma^2) \). If the manufacturer knows that the retailer will order according to a static \( N(\mu, \sigma^2) \) distribution, a fixed value for \( w \) can be computed, which optimizes the expected profit.

To examine dynamic and static approaches, we sampled paths of the Ornstein–Uhlenbeck process using the parameters \( \mu = 100, \sigma = 12, a = 0.05, D_0 = 100 \). The values for the accumulated profits

\[
\int_{\delta}^{T} (R - S) \min[D_t, q_t] - (w_t - S)q_t dt \quad \int_{\delta}^{T} (w_t - M)q_t dt
\]

were computed for different values of \( \delta \) using the values \( R = 10, M = 2, S = 1, T = 100 + \delta \), and using four different strategies:

- Dynamic approach as defined by Proposition 5.1.1.
- Static approach as defined above.
- Dynamic cooperative approach using \( w_t = M \).
- Static cooperative approach using \( w_t = M \).

Adding the expected profits in (2) and (3) it is easy to see that when \( w_t = M \) (in which case the manufacturer has zero profit) then the optimal policy for the retailer also maximizes the expected profit for the supply chain. The order quantity in the dynamic cooperative case is then found from (7) with \( w_t = M \), leading to \( q_t^* = \Phi_{t,y}(M) \) where \( \Phi_{t,y} \) is given by (26). The static cooperative case is equivalent to a single period newsvendor problem, leading to a constant order rate.

We assume that sales take place in time intervals \([\delta, 100 + \delta] \). The length of the sales period hence is 100 regardless of the value of \( \delta \). This makes it easier to compare performance using different values of \( \delta \). The results were averaged over 1000 sample paths and these averages are reported in the tables below.
Table 1: Performance of dynamic versus static strategies. Delay $\delta = 1$.

<table>
<thead>
<tr>
<th>Values over 1000 sample paths</th>
<th>Manufacturer</th>
<th>Retailer</th>
<th>Supply chain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average profit static approach</td>
<td>42 830</td>
<td>12 729</td>
<td>55 559</td>
</tr>
<tr>
<td>Average profit dynamic approach</td>
<td>61 356</td>
<td>4 073</td>
<td>65 429</td>
</tr>
<tr>
<td>Average profit static cooperation</td>
<td>-</td>
<td>-</td>
<td>73 251</td>
</tr>
<tr>
<td>Average profit dynamic cooperation</td>
<td>-</td>
<td>-</td>
<td>77 766</td>
</tr>
</tbody>
</table>

Table 2: Performance of dynamic versus static strategies. Delay $\delta = 7$.

<table>
<thead>
<tr>
<th>Values over 1000 sample paths</th>
<th>Manufacturer</th>
<th>Retailer</th>
<th>Supply chain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average profit static approach</td>
<td>42 830</td>
<td>12 457</td>
<td>55 286</td>
</tr>
<tr>
<td>Average profit dynamic approach</td>
<td>48 592</td>
<td>9 438</td>
<td>58 030</td>
</tr>
<tr>
<td>Average profit static cooperation</td>
<td>-</td>
<td>-</td>
<td>73 029</td>
</tr>
<tr>
<td>Average profit dynamic cooperation</td>
<td>-</td>
<td>-</td>
<td>74 838</td>
</tr>
</tbody>
</table>

Table 3: Performance of dynamic versus static strategies. Delay $\delta = 30$.

<table>
<thead>
<tr>
<th>Values over 1000 sample paths</th>
<th>Manufacturer</th>
<th>Retailer</th>
<th>Supply chain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average profit static approach</td>
<td>42 830</td>
<td>12 074</td>
<td>54 903</td>
</tr>
<tr>
<td>Average profit dynamic approach</td>
<td>43 225</td>
<td>11 882</td>
<td>55 106</td>
</tr>
<tr>
<td>Average profit static cooperation</td>
<td>-</td>
<td>-</td>
<td>72 648</td>
</tr>
<tr>
<td>Average profit dynamic cooperation</td>
<td>-</td>
<td>-</td>
<td>72 794</td>
</tr>
</tbody>
</table>

As we can see from these tables, the dynamic approach favors the manufacturer, the more favorable the shorter the delay. At $\delta = 30$, the effect of the dynamic approach is close to being wiped out. The same results apply for the supply chain, i.e., a dynamic approach offers improved profits and the improvement is larger when the delay is shorter. It is interesting to note, however, that the retailer has a distinct disadvantage under the dynamic approach and that this disadvantage is larger the shorter the delay.

In a cooperative setting, a dynamic approach can reward both the retailer and the manufacturer. Profits can be shared, which leads to an improved position for both parties. In a noncooperative equilibrium, more precise information can be a disadvantage for the retailer. This is due to the Stackelberg structure of the game. With more precise information, the leader has more control and can take a larger share of the profits. In the limit $\delta \to 0$, the leader is in full control. The retailer will then order the observed demand rate regardless of the price. The manufacturer offers a price marginally close to $R$ taking all profit in the limit. See also Taylor and Xiao (2010) for an interesting discussion of the single period case.

6. Concluding remarks

In this paper we have provided explicit formulas for equilibrium prices in a time continuous newsvendor model. Complete existence and uniqueness results have been stated for widely used processes like geometric Brownian motion and the Ornstein-Uhlenbeck process, both with time variable coefficients. We have also outlined how to obtain explicit expressions when demand is given by a geometric Lévy process with time variable, deterministic coefficients, including cases with random coefficients. To
our knowledge path properties of this kind has not previously been discussed in the literature.

Of particular interest is the structure of the equilibria for a geometric Brownian motion with constant coefficients. In this case the manufacturer offers a fixed wholesale price, while the retailer orders a fixed fraction of the observed demand. This result is clearly a parallel to Merton’s classical result on optimal investment in a risky and a secure asset, where the optimal policy is to keep a fixed fraction in the risky asset.

From an applied point of view, we believe that the numerical results in Section 4.2 are of general interest. We demonstrate that the retailer suffers a distinct disadvantage from having more information, and that this disadvantage is bigger the more precise is the information. Such issues may have important political implications, in particular in electricity markets, and we believe that our model offer new insights into the mechanisms governing equilibria in such markets.

Appendix A.

In this appendix, we give complete proofs for all unproved statements given in Sections 3, 4, and 5. We start with a nontrivial estimate for the standard normal distribution, which will be crucial in the proofs of unique maxima. This lemma was proved in Øksendal et al. (2013), and we refer to that paper for a proof.

Lemma A.1. In this lemma $G[x]$ is the cumulative distribution function of the standard normal distribution. Let $0 \leq m \leq 1$, and for each $m$ consider the function $h_m : \mathbb{R} \to \mathbb{R}$ defined by

\[ h_m[z] = z(1 - m - G[z]) - G'[z] \]  

Then

\[ h_m[z] < 0 \quad \text{for all } z \in \mathbb{R} \]  

Details for Proposition 3.1.1

From (9), we easily see that the statement $q_t \leq D_t$ is equivalent to the inequality

\[ \ln \left[ \frac{q_t}{D_t - \delta} \right] - (a - \frac{1}{2} \sigma^2)\delta \leq \sigma (B_t - B_{t-\delta}) \]  

The left-hand side is $\mathcal{E}_t$-measurable, while the right-hand side is normally distributed and independent of $\mathcal{E}_t$. It is then straightforward to prove that

\[ \mathbb{E} \left[ X_{[0,D_t]}(\hat{q}_t) | \mathcal{E}_t \right] = 1 - G \left[ \frac{\ln \left[ \frac{q_t}{D_t - \delta} \right] - (a - \frac{1}{2} \sigma^2)\delta}{\sqrt{\sigma^2 \delta}} \right] \]  

Hence it follows from (7) that

\[ q_t = D_{t-\delta} \cdot \exp \left[ (a - \frac{1}{2} \sigma^2)\delta + \sqrt{\delta \sigma^2} \cdot G^{-1} \left[ 1 - \frac{w - S}{R - S} \right] \right] \]
With this order quantity, the expected profit for the manufacturer is
\[ E[D_{t-\delta} \cdot (w_t - M)] \exp \left[ \left( a - \frac{1}{2} \sigma^2 \right) \delta + \sqrt{\delta \sigma^2} \cdot G^{-1} \left[ 1 - \frac{w_t - S}{R - S} \right] \right] \] (36)

In general \( w_t \) can be a random variable. If \( w^* = \text{Argmax}[\Psi] \), where
\[ \Psi[w] = (w - M) \exp \left[ \left( a - \frac{1}{2} \sigma^2 \right) \delta + \sqrt{\delta \sigma^2} \cdot G^{-1} \left[ 1 - \frac{w - S}{R - S} \right] \right] \] (37)
we know, however, that
\[ E \left[ D_{t-\delta} \cdot (w_t - M) \exp \left[ \left( a - \frac{1}{2} \sigma^2 \right) \delta + \sqrt{\delta \sigma^2} \cdot G^{-1} \left[ 1 - \frac{w_t - S}{R - S} \right] \right] \right] = E[D_{t-\delta} \cdot \Psi[w^*]] \] (38)

with equality if \( w_t = w^* \). Therefore \( w^* \) is optimal. It remains to prove that Argmax[\Psi] is unique. If we put \( b = \sqrt{\delta \sigma^2} \), it follows that \( \Psi \) is proportional to a function of the form
\[ w \mapsto (w - M) \exp \left[ b G^{-1} \left[ 1 - \frac{w - S}{R - S} \right] \right] \] (40)
where \( b > 0 \). Make a monotone change of variables using \( z = G^{-1} \left[ 1 - \frac{w - S}{R - S} \right] \). With this change of variables we see that \( \Psi \) is proportional to
\[ (R - S) \left( 1 - G[z] - \frac{M - S}{R - S} \right) \exp[b z] \] (41)

Put \( m = \frac{M - S}{R - S} \), and note that \( \Psi \) is proportional to a function
\[ (1 - m - G[z]) \exp[b z] \] (42)

For each fixed \( 0 \leq m \leq 1, b > 0 \) consider the function
\[ f_m[z] = (1 - m - G[z]) \exp[b z] \quad \text{on the interval} \quad -\infty < z \leq G^{-1}[1 - m] \] (43)

We have
\[ f'_m[z] = -G'[z] \exp[b z] + (1 - m - G[z])b \exp[b z] \] (44)

Note that \( \lim_{z \to -\infty} f_m[z] = 0, f_m[G^{-1}[1 - m]] = 0, \) and \( f'_m[G^{-1}[1 - m]] < 0 \). The function therefore has at least one strictly positive maximum. To see that the maximum is unique, find \( z_0 \) s.t. \( f'_m[z_0] = 0 \). Using \( G''[z] = -z \cdot G'[z] \), we can simplify the expression to obtain
\[ f''_m[z_0] = (z_0 - b)G'[z_0] \exp[b z] \] (45)

From Lemma A.1 and \( f''_m[z_0] = 0 \) we obtain
\[ (1 - m - G[z_0])b = G'[z_0] > (1 - m - G[z_0])z_0 \] (46)

If \( f''_m[z_0] = 0 \), we must have \( z_0 < b \), which implies \( f''_m[z_0] < 0 \). The function is therefore quasiconcave and has a unique, strictly positive maximum. It follows from
Theorem 2.1 that this is the only candidate for a Stackelberg Equilibrium. To see that
this candidate is indeed a Stackelberg Equilibrium, we argue as follows: Since $\Psi_y$ is
quasiconcave, any $w_t$ other than $\text{Argmax}[\Psi_{D_{t-\delta}}]$ will lead to strictly lower expected
profit at time $t$. As demand does not depend on $w_t$, low expected profit at one point in
time cannot be compensated by higher expected profits later on. Hence if the statement
$w_t = \text{Argmax}[\Psi_{D_{t-\delta}}]$ a.s. $\lambda \times P$ ($\lambda$ denotes Lebesgue measure on $[0, T]$) is false, any
such strategy will lead to strictly lower expected profits. The same argument applies
for the retailer, and hence a unique Stackelberg equilibrium always exists in this case.

$\square$

Details for Proposition 3.2.1

In the case with variable coefficients, we have

$$D_t = D_{t-\delta} \cdot \exp \left[ \int_{t-\delta}^{t} \mu(s) - \frac{1}{2} \sigma^2(s) ds + \int_{t-\delta}^{t} \sigma(s) dB_s \right] \quad (47)$$

Put

$$\hat{\mu}(t) = \int_{t-\delta}^{t} \mu(s) - \frac{1}{2} \sigma^2(s) ds \quad \hat{\sigma}^2(s) = \int_{t-\delta}^{t} \sigma^2(s) ds \quad (48)$$

Because the exponent in (47) is normally distributed and independent of $\mathcal{E}_t$, we obtain

$$E \left[ X_{[0,D_t]}(\hat{q}_t) | \mathcal{E}_t \right] = 1 - G \left[ \frac{w}{D_{t-\delta}} - \frac{\ln \left( \frac{\hat{q}_t}{D_{t-\delta}} \right) - \hat{\mu}(t)}{\hat{\sigma}(t)} \right] \quad (49)$$

It follows from (7) that

$$q_t = D_{t-\delta} \cdot \exp \left[ \hat{\mu}(t) + \hat{\sigma}(t) \cdot G^{-1} \left[ 1 - \frac{w - S}{R - S} \right] \right] \quad (50)$$

With this order quantity, the expected profit for the manufacturer is

$$E \left[ D_{t-\delta} \cdot (w_t - M) \exp \left[ \hat{\mu}(t) + \hat{\sigma}(t) \cdot G^{-1} \left[ 1 - \frac{w - S}{R - S} \right] \right] \right] \quad (51)$$

The calculations in the proof of Proposition 3.1.1 can now be repeated line by line for
each fixed $t$ proving the general case in Proposition 3.2.1.

$\square$
Details for Proposition 4.1.1

From (20) it follows that

\[
D_t = D_{t-\delta} \exp \left[ \int_{t-\delta}^t \left( \alpha_1(s) + \alpha_2(s)B_s(\omega) - \frac{1}{2} \sigma^2(s) + \int_{\mathbb{R}_0} \log[1 + \gamma(s, z)] - \gamma(s, z)\nu(dz) \right) ds \right. \\
\left. + \int_{t-\delta}^t \sigma(s)dB_s + \int_{t-\delta}^t \int_{\mathbb{R}_0} \log[1 + \gamma(s, z)]N(ds, dz) \right] \\
= D_{t-\delta} \cdot \exp \left[ \int_{t-\delta}^t \alpha_1(s) + \alpha_2(s)B_s(\omega)ds \right] \\
\cdot \exp \left[ \int_{t-\delta}^t -\frac{1}{2} \sigma^2(s) + \int_{\mathbb{R}_0} \log[1 + \gamma(s, z)] - \gamma(s, z)\nu(dz) \right] ds \\
+ \int_{t-\delta}^t \sigma(s)dB_s + \int_{t-\delta}^t \int_{\mathbb{R}_0} \log[1 + \gamma(s, z)]N(ds, dz) \right]
\]

The problem here is the second term \( \exp \left[ \int_{t-\delta}^t \alpha_1(s) + \alpha_2(s)B_s(\omega)ds \right] \), which is usually not independent of \( E_t \). Changing the order of integration we see that

\[
\exp \left[ \int_{t-\delta}^t \alpha_1(s) + \alpha_2(s)B_s(\omega)ds \right] = \exp \left[ \int_{t-\delta}^t \alpha_1(s) + \alpha_2(s)B_{t-\delta}ds + \int_{t-\delta}^t \int_s^t \alpha_2(u)du dB_s \right]
\]

from which it follows that

\[
D_t = D_{t-\delta} \cdot \exp \left[ \int_{t-\delta}^t \alpha_1(s) + \alpha_2(s)B_{t-\delta}ds \right] \cdot X_t
\]

where \( X_t \) is given by (22). Here the first two terms are \( E_t \)-measurable, while the last term is independent of \( E_t \). It is then straightforward to see that (23) follows from (7). \( \square \)

Details for Proposition 5.1.1

The statement \( q_t \leq D_t \) is equivalent to the inequality

\[
q_t = \left( D_{t-\delta} e^{-\int_{t-\delta}^t a(u)du} \right) + \int_{t-\delta}^t a(s)\mu(s)e^{-\int_{s}^t a(u)du} ds \leq \int_{t-\delta}^t \sigma(s)e^{-\int_{s}^t a(u)du} dB_s
\]

Using the Itô isometry, we see that the right-hand side has expected value zero and variance \( \int_{t-\delta}^t \sigma^2(s)e^{-2\int_{s}^t a(u)du} ds \). It is then straightforward to see that

\[
E \left[ X[0,D_t](\hat{q}_t) \right] = 1 - G \left[ q_t - \left( D_{t-\delta} e^{-\int_{t-\delta}^t a(u)du} \right) + \int_{t-\delta}^t a(s)\mu(s)e^{-\int_{s}^t a(u)du} ds \right] \sqrt{\int_{t-\delta}^t \sigma^2(s)e^{-2\int_{s}^t a(u)du} ds}
\]
and (26) follows trivially from (7). It remains to prove that the function \( \Psi_{t,y} \) has a unique maximum if \( \Phi_{t,y}[M] > 0 \). First put

\[
\hat{y} = \frac{D_{t_\delta} e^{-\int_{t_\delta}^{t_\delta} a(u)du} + \int_{t_\delta}^{t_\delta} a(s) \mu(s) e^{-\int_{t_\delta}^{s} a(u)du} ds}{\sqrt{\int_{t_\delta}^{t_\delta} \sigma^2(s) e^{-2\int_{t_\delta}^{s} a(u)du} ds}}
\]

and note that \( \Psi_{t,y} \) is proportional to

\[
(w - M) \left( \hat{y} + G^{-1} \left[ 1 - \frac{w - S}{R - S} \right] \right)
\]

We make a monotone change of variables using \( z = G^{-1} \left[ 1 - \frac{w - S}{R - S} \right] \). With this change of variables we see that \( \Psi_{t,y} \) is proportional to

\[
(R - S) \left( 1 - G[z] - \frac{M - S}{R - S} \right) (\hat{y} + z)
\]

Put \( m = \frac{M - S}{R - S} \), and note that \( \Psi_{t,y} \) is proportional to

\[
(1 - m - G[z])(\hat{y} + z)
\]

\( \Phi_{t,y}[M] > 0 \) is equivalent to \( \hat{y} + G^{-1}[1 - m] > 0 \), and the condition \( w \geq M \) is equivalent to \( z \leq G^{-1}[1 - m] \). Note that if \( S \leq M \leq R \), then \( 0 \leq m \leq 1 \). For each fixed \( 0 \leq m \leq 1 \), \( \hat{y} \in \mathbb{R} \) consider the function

\[
f_m[z] = (1 - m - G[z])(\hat{y} + z) \quad \text{on the interval} \quad -\hat{y} \leq z \leq G^{-1}[1 - m]
\]

If \( \hat{y} + G^{-1}[1 - m] > 0 \), the interval is nondegenerate and nonempty, and

\[
f_m'[z] = -G'[z](\hat{y} + z) + (1 - m - G[z])
\]

Note that \( f_m'[-\hat{y}] > 0 \), and that \( f_m[-\hat{y}] = f_m[G^{-1}[1 - m]] = 0 \). These functions therefore have at least one strictly positive maximum. To prove that the maximum is unique, assume that \( f_m'[z_0] = 0 \), and compute \( f_m''[z_0] \). Using \( G''[z] = -z \cdot g'[z] \), it follows that

\[
f_m''[z_0] = z_0(1 - m - G[z_0]) - 2G'[z_0] < z_0(1 - m - G[z_0]) - G'[z_0] < 0
\]

by Lemma A.1. The function is thus quasiconcave and has a unique, strictly positive maximum. It follows from Theorem 2.1 that this is the only candidate for a Stackelberg Equilibrium. Since \( \Psi_{t,y} \) is quasiconcave, the argument that we used at the end of the proof of Proposition 3.1.1 shows that a unique Stackelberg equilibrium exists also in this case.

\[\Box\]

**Acknowledgements**

The authors wish to thank Steve LeRoy and an anonymous referee for several useful comments to improve the paper. The research leading to these results has received funding from the European Research Council under the European Community’s Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement no [228087]. The research leading to these results has received funding from NFR project 196433.
References


