Face rings of Δ-sets with applications to degenerations of abelian surfaces

by

Jarle Stavnes

THESIS FOR THE DEGREE OF

Master in Mathematics

(Master of Science)

Det matematisk- naturvitenskapelige fakultet
Universitetet i Oslo

May 2014

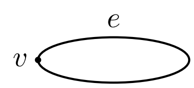
Faculty of Mathematics and Natural Sciences
University of Oslo

Introduction

The motivation behind this master's thesis is related to moduli spaces. We are ultimately interested in compactifications of the moduli space of polarized abelian surfaces. Versal spaces of deformations of degenerate abelian surfaces such as Stanley-Reisner schemes has proven useful in this regard, by the work of Jan Christophersen and Klaus Altmann (See [AC10] and [Chr10]). A generalization of Stanley-Reisner schemes, and their corresponding versal spaces of deformations, has been the object of this project.

We consider the tesselation $\{3,6\}$ of the plane. We have the translation group given by the lattice $T = \{(n, \frac{\sqrt{3}}{2}m)|(n,m) \in \mathbb{Z}^2\}$. Dividing the plane out with subgroups $\Gamma \subseteq T$ of finite index yields triangulations, or equivelar maps on the torus, both polyhedral and non-polyhedral. Any polyhedral map can be considered a simplicial complex with an associated Stanley-Reisner scheme. The appropriate generalization was to find schemes that belongs to non-polyhedral equivelar maps. It turned out this could be done in general using Δ-sets, which is a generalization of ordered simplicial complexes. Any equivelar map can be considered a Δ-set, so the main part of the thesis thus consists of a generalization of the Stanley-Reisner construction on simplicial complexes to Δ-sets.

A Δ-set is defined as a sequence of sets $M = \{M_n\}_{n=0}^{\infty}$ together with face maps $d_i : M_{n+1} \to M_n$ for $i = 0, \ldots, n+1$. M_n consists of the n-dimensional faces of M. A map $f : M \to N$ between Δ-sets is defined as a family of maps $\{f_n : M_n \to N_n\}$ for each $n \geq 0$. See Section 1.1 for a more precise definition. We will only be interested in the category of finite-dimensional Δ-sets. Any ordered simplicial complex may be considered a Δ-set. Note that maps between Δ-sets must respect dimension, while in the category of ordered simplicial complexes we are not restricted to dimension-preserving maps. For an example of a simple Δ-set which is not an ordered simplicial complex, consider *the loop* $\mathcal{L}$ composed of an edge e and a single vertex v.

I.e. $\mathcal{L}_0 = \{v\}$, and $\mathcal{L}_1 = \{e\}$: 

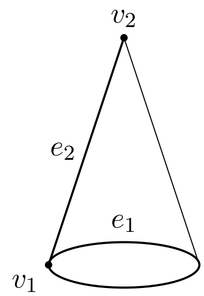
One of our first results is that the category of Δ-sets has colimits, and that any Δ-set M has an associated diagram $H_M : I_M \to \Delta$-sets of which M is the colimit, such that $H_M(X)$ is a simplex for each $X \in I_M$. The index category I_M simply consists of the faces of M. The contravariant functor R : Δ-simplices $\to$ Graded k-algebras sending an n-dimensional simplex Δ^n to the free k-algebra of $n+1$ indeterminants $k[x_0, \dots, x_n]$ (see Section 1.3 for the construction of this functor) allows us to consider the diagram $R \circ H_M : I_M \to$ Graded k-algebras. It turns out that for any ordered simplicial complex M, the limit $\varprojlim R \circ H_M$ is the Stanley-Reisner ring of M. Since we have the functor $R \circ H_M$ for any Δ-set M, this motivates the following definition of what we call the Δ-face ring of M:

Definition. *For a Δ-set M, let $\varprojlim R \circ H_M$ be the Δ-face ring of M.*

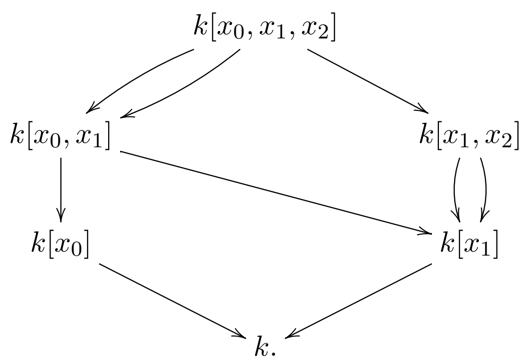
We will call this the naive definition of the Δ-face ring, since it turns out that we may construct a functor F : Δ-sets $\to$ Graded k-algebras such that $F(M) = \varprojlim R \circ H_M$ for every Δ-set M. The functorial property of this construction will be useful for proving various properties of Δ-face rings, but the precise definition of F is rather technical.

We are now able to find schemes associated to Δ-sets generalizing the notion of Stanley-Reisner schemes. Using our definition of the Δ-face ring, we get our scheme $\mathbb{P}(M) = \operatorname{Proj} F(M)$ for any Δ-set M, called the Δ-face scheme of M. In fact, we will show that it is possible to define a functor $\mathbb{P}$: Δ-sets $\to \mathrm{Sch}_k$ such that $\mathbb{P}(M) = \operatorname{Proj} F(M)$, with properties reflecting the geometric nature of Δ-sets. In short, $\mathbb{P}$ preserves dimension, irreducibility, intersections, unions and finite colimits, as well as injectivity and surjectivity of maps. Note that for Δ-sets M, $F(M)$ is not in general a standard graded k-algebra, so $\mathbb{P}(M)$ embeds naturally in weighted projective space.

The naive definition of the Δ-face rings allows us to compute them as colimits of specific diagrams. For an example of this, consider now *the cone* $\mathcal{C}$ composed of a 2-dimensional face f, two edges e_1, e_2 and two vertices v_1, v_2. I.e. $\mathcal{C}_0 = \{v_1, v_2\}$, $\mathcal{C}_1 = \{e_1, e_2\}$, and $\mathcal{C}_2 = \{f\}$:



Using that $F(\mathcal{C}) = \varprojlim R \circ H_{\mathcal{C}}$, we get that $F(\mathcal{C})$ is the limit of the diagram illustrated as follows:



We end up with the subring $F(\mathcal{C}) = k[x_0, x_1 + x_2, x_1x_2, x_1^2x_2]$ of $k[x_0, x_1, x_2]$. For the loop $\mathcal{L}$, we have $F(\mathcal{L}) = k[x_1 + x_2, x_1x_2, x_1^2x_2] \subseteq k[x_1, x_2]$. We may realize the scheme $\mathbb{P}(\mathcal{L})$ as the nodal cubic curve $V(y^3 - xyz + xz^2) \subseteq \mathbb{P}^2$, and we see in fact that $\mathbb{P}(\mathcal{C})$ is the projective cone over $\mathbb{P}(\mathcal{L})$.

We return to the question of finding versal spaces of deformations of the schemes associated to non-polyderal equivelar maps on the torus, with each such equivelar map corresponding to a Δ-set N. Our initial goal was to find the first-order deformations of the schemes $\mathbb{P}(N)$, i.e. to compute the global sections of the sheaf $\mathcal{T}^1_{\mathbb{P}(N)/k}$ of $\mathcal{O}_{\mathbb{P}(N)}$-modules. Our main tool in this regard will be to consider our Δ-set N (corresponding to non-polyhedral equivelar map defined by a subgroup $\Gamma \subseteq T$) as the quotient of a simplicial complex M (corresponding to a polyhedral equivelar map) by a finite group G acting on M. Such simplicial complexes are found by considering subgroups $\Gamma_0 \subseteq \Gamma$, such that the corresponding quotient of the tesselation $\{3, 6\}$ by Γ_0 yields a polyhedral equivelar map on the torus. G is the finite group Γ/Γ_0, with a naturally induced free action on M. This gives us a categorical group quotient $\pi : M \to N$ by the action of G on M.

Using the result that the functor $\mathbb{P}$ preserves colimits, we are able to deduce that $\mathbb{P}(\pi) : \mathbb{P}(M) \to \mathbb{P}(N)$ is a categorical group quotient in Sch_k via the induced group action of G on $\mathbb{P}(M)$. We prove that the morphism $\mathbb{P}(\pi)$ is ètale, which will allow us to compute $H^0(\mathbb{P}(N), \mathcal{T}^1_{\mathbb{P}(N)/k})$ as the G-invariants of $H^0(\mathbb{P}(M), \mathcal{T}^1_{\mathbb{P}(M)/k})$ via an induced action on the sheaf $\mathcal{T}^1_{\mathbb{P}(M)/k}$. This latter sheaf was studied in [AC10], where it was shown that a basis for $H^0(\mathbb{P}(M), \mathcal{T}^1_{\mathbb{P}(M)/k})$ is in one-to-one correspondence with the edges of M. Our approach shows that this is the case for $H^0(\mathbb{P}(N), \mathcal{T}^1_{\mathbb{P}(N)/k})$ as well.

In fact, more generally we look at the cotangent complex for rings and for schemes, and define the family of modules $\mathcal{T}^i_{X/k}$ for $i \geq 0$ for any scheme X over k. We prove that $H^n(\mathbb{P}(N), \mathcal{T}^i_{\mathbb{P}(N)/k})$ is equal to the G-invariants of $H^n(\mathbb{P}(M), \mathcal{T}^i_{\mathbb{P}(M)/k})$ for each $i, n \geq 0$.

In [Chr10], the deformation functor $\mathrm{Def}_{(\mathbb{P}(M),\mathcal{L}_M)}$: Artin rings $\to$ Set was studied, where the $\mathcal{L}_M = \mathcal{O}_{\mathbb{P}(M)}(1)$ is a very ample invertible sheaf on the Stanley-Reisner scheme $\mathbb{P}(M)$ associated to a polyhedral equivelar map on the torus. We show that also in the non-polyhedral case, $\mathcal{O}_{\mathbb{P}(N)}(1)$ is ample but not necessarily very ample, and that a basis for $H^0(\mathbb{P}(N), \mathcal{O}_{\mathbb{P}(N)}(1))$ is in one-to-one correspondence with the vertices of N. This suggests that the $\mathbb{P}(N)$'s are interesting points on the boundary of the moduli space of polarized abelian surfaces with a not very ample polarization class.

In Chapter 1, we introduce the notion of Δ-sets. We show that the category of Δ-sets has colimits, and to any Δ-set M we construct the associated diagram $H_M : I_M \to \Delta$-sets. We proceed to define the functor $F : \Delta\text{-sets} \to$ Graded k-algebras defining the Δ-face ring $F(M)$ for any Δ-set M. We also show that this is a generalization of the Stanley-Reisner ring construction.

In Chapter 2, we explore the properties of the Δ-face schemes $\mathbb{P}(M)$. We define the functor $\mathbb{P} : \Delta\text{-sets} \to \mathrm{Sch}_k$. In particular, we prove the geometry-preserving properties listed earlier.

In Chapter 3, we focus on group actions on Δ-sets, and the induced group action on Δ-face rings and Δ-face schemes. For a finite group G acting on a finite Δ-set M, we see that the categorical group quotient map $\pi : M \to M/G$ correspond to the categorical group quotient morphism $\mathbb{P}(\pi) : \mathbb{P}(M) \to \mathbb{P}(M)/G$ via the functor $\mathbb{P}$. We introduce group schemes, and prove that for finite groups, the morphism $\mathbb{P}(\pi) : \mathbb{P}(M) \to \mathbb{P}(M)/G$ is ètale when the action of G on M is free.

In Chapter 4, we look at the cotangent complex for rings and for schemes. The $\mathcal{T}^i$-modules are defined. We prove some results regarding group actions on schemes and the related group actions on the $\mathcal{T}^i$-modules as described above.

In Chapter 5, we review some of the results of Christophersen and Altmann regarding the cotangent cohomology for Stanley-Reisner schemes.

In Chapter 6, we focus on the motivating application for our theory, the tesselation $\{3,6\}$ of the plane. The versal spaces of the deformations of Stanley-Reisner schemes associated to polyhedral triangulations are known, and can be found in [Chr10]. A future goal is to extend these results to the non-polyhedral case. We apply the results of the previous chapters to compute $H^0(\mathbb{P}(N), \mathcal{T}^1_{\mathbb{P}(N)/k})$ for the non-polyhedral triangulations N of the torus.

I am grateful to my advisor Jan Christophersen for suggesting this topic and his dedicated guidance through this project.

Notes: The field k will be assumed to be algebraically closed and of characteristic 0 (characteristic 0 is assumed because we will make use of the Reynolds-operator). A diagram $H : I \to C$ between categories I and C is called connected if the index category I represented as a directed graph is connected (i.e. there are no disjoint pairs of subgraphs). The functor H may be covariant or contravariant. All calculations are done in Macaulay2.

Contents

Chapter 1

Δ-face rings

1.1 Δ-sets

A simplicial complex M is a subset of the power set of $\{0, 1, \ldots, n\}$ for some integer n such that for every $x \in M$, $y \subseteq x \Rightarrow y \in M$. A Δ-set M is a sequence of sets $\{M_n\}_{n=0}^{\infty}$ together with face maps

$$d_i \ : \ M_{n+1} \to M_n$$

for $i = 0, 1, \ldots, n+1$ and $n \geq 0$, such that

$$d_i \circ d_j = d_{j-1} \circ d_i$$

whenever $i < j$. We will sometimes write d_i^M if there is need to specify which Δ-set it belongs to. A map f between Δ-sets M and N is a collection of maps

$$\{f_n \ : \ M_n \to N_n\}_{n=0}^{\infty}$$

such that

$$f_n \circ d_i = d_i \circ f_{n+1}$$

whenever both sides of the equation are defined. This allows us to define the category of Δ-sets where the objects are Δ-sets and the morphisms are the maps between them.

The dimension of a Δ-set M is the smallest integer n for which M_{n+1} is empty. If such an n does not exist, we call M infinite-dimensional. M_0 is the set of vertices of M. Any element $f \in M_n$ for some n is called an n-face, or an n-dimensional face. If f is not in the image of any face map, f is called a maximal face. A finite Δ-set is a Δ-set with finitely many faces. As a convention we will set $M_{-1} = \{\emptyset\}$, with the trivial face map $d_0 : M_0 \to M_{-1}$.

The ordered simplicial complexes can be realized as objects in the category of Δ-sets as follows. Given an simplicial complex M, consider it a Δ-set by letting M_n consist of the n-dimensional faces. An ordering of a simplicial complex M is a partial ordering of the vertex set M_0 which restricts to a total ordering on the vertices $v_0 < v_1 < \ldots < v_n$ in any simplex $\sigma = (v_0, \ldots, v_n)$ in M_n. The face maps $d_i : M_n \to M_{n-1}$ are defined as such: $d_i(\sigma) = (v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$ for $0 \leq i \leq n$. For a standard n-simplex Δ^n with $n \geq 0$, we write the vertices as $0, 1, \ldots, n$, and order them by $0 < 1 < 2 < \ldots < n$. The empty simplex is denoted by Δ^{-1}, containing only the single element $\emptyset$. Note that a Δ-set map between ordered simplicial complexes as Δ-sets is a map of ordered simplicial complexes. We will from now on refer to ordered simplicial complexes simply as simplicial complexes. A Δ-simplex will be a Δ-set which is a simplex as a simplicial complex. By abuse of notation, when n is given, we will sometimes refer to Δ^j as an n-dimensional Δ-simplex as a means of indexing several Δ-simplices of equal dimension, where j ranges over an index set. See [Ran92] for more on Δ-sets.

Δ-set maps are more restrictive than those of simplicial complexes, in that a map may only send an n-dimensional face to an n-dimensional face. So while any simplicial complex may be viewed as a Δ-set, not every map of simplicial complexes may be viewed as a map of Δ-sets.

Example 1.1.1. Our running example will be the possibly simplest Δ-set which is not a simplicial complex. We define *the loop* $\mathcal{L}$ to be the Δ-set defined by

$$\mathcal{L}_0 = \{v\},$$
$$\mathcal{L}_1 = \{e\}.$$

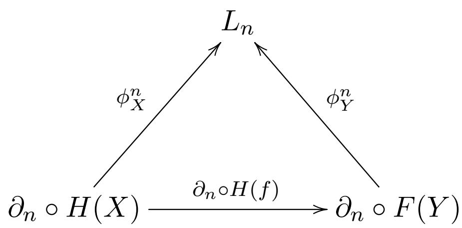
$\mathcal{L}$ consists of only a single vertex v and an edge e. The edge maps d_0 and d_1 simply sends e to v.

1.2 Properties of the category of Δ-sets

Theorem 1.2.1. *The category of Δ-sets has colimits.*

Proof. Let $H : I \to \Delta$-sets be a diagram. Define $\partial_n : \Delta\text{-sets} \to \text{Set}$ to be the functor that sends a Δ-set M to the set of n-faces, M_n, and a map $f : M \to N$ to the map $f_n : M_n \to N_n$. For each n, we have an induced diagram functor $\partial_n \circ H : I \to \text{Set}$. Let L_n be the colimit of this functor (in Set).

Then, for every $X \in I$ we have the maps $\phi_X^n : \partial_n \circ H(X) \to L_n$ such that the diagram



commutes for every morphism $f : X \to Y$. We will provide the sequence of sets $L = \{L_n\}_{n=0}^{\infty}$ with face maps $d_i^L : L_{n+1} \to L_n$. We have by the set-theoretic construction of colimits that

$$L_n = \bigsqcup_{X \in I} \partial_n \circ H(X) / \sim$$

where for every map $f : X \to Y$ and $x \in \partial_n \circ H(X)$ we have

$$\partial_n \circ H(f)(x) \sim x.$$

Define $d_i^L : L_{n+1} \to L_n$ by the following: for any $x \in \partial_{n+1} \circ H(X)$, let $d_i^L([x]) = [d_i^X(x)]$, where d_i^X are the face maps of $H(X)$ and $d_i^X(x)$ is in $H(X)_n = \partial_n \circ H(X)$. To see that this is well-defined, let $f : X \to Y$ be a morphism. We need to show that

$$d_i^X(x) \sim d_i^Y(\partial_{n+1} \circ H(f)(x)). \tag{*}$$

By properties of Δ-set maps and the definition of $\sim$, we have that

$$d_i^Y(\partial_{n+1} \circ H(f)(x)) = \partial_n \circ H(f) \circ d_i^X(x) \sim d_i^X(x)$$

(since $\partial_n \circ H(f) = H(f)_n$). Therefore $(*)$ is satisfied. The condition

$$d_i^L \circ d_j^L = d_{j-1}^L \circ d_i^L$$

is clear as it holds for all d_i^X's: For each $x \in L_{n+1}$,

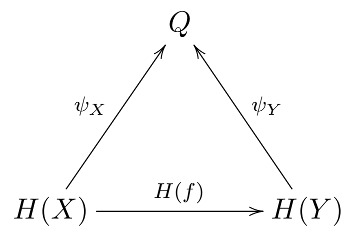
$$d_i^L \circ d_j^L([x]) = [d_i^X \circ d_j^X(x)] = [d_{j-1}^X \circ d_i^X(x)] = d_{j-1}^L \circ d_i^L([x])$$

for some $X \in I$.

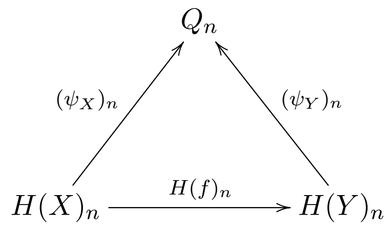
Thus L is a Δ-set. Define the maps $\phi_X : H(X) \to L$ by letting $(\phi_X)_n : H(X)_n \to L_n$ be defined as $\phi_X^n : \partial_n \circ H(X) \to L_n$. To show that these are Δ-set maps, we need to show the following:

$$d_i^L \circ (\phi_X)_{n+1}(x) = (\phi_X)_n \circ d_i^X(x)$$

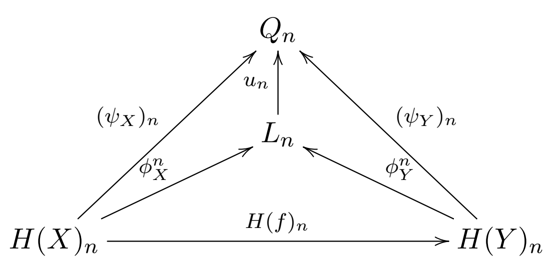
for $x \in H(X)_{n+1}$. Now, $(\phi_X)_{n+1}(x) = \phi_X^{n+1}(x) = [x]$ in L_{n+1}. So what we require is simply that $d_i^L([x]) = [d_i^X(x)]$. But this is just the definition of d_i^L. Now, suppose we have a node Q to the diagram $H : I \to \Delta$-set, with maps $\psi_X : H(X) \to Q$, such that the diagram



commutes for every morphism $f : X \to Y$. Applying the n-face functor ∂_n, we get a commutative diagram



and therefore a unique map $u_n : L_n \to Q_n$ such that the diagram



commutes for all $f : X \to Y$. I.e. $u_n \circ \phi_X^n = (\psi_X)_n$. It remains to show that the u_n's form a Δ-set map between L and Q. In other words, we need that $d_i^Q \circ u_{n+1} = u_n \circ d_i^L$. Let $[x] \in L_{n+1}$ for some $x \in H(X)_{n+1}$. Then

$$u_n \circ d_i^L([x]) = u_n([d_i^X(x)]).$$

On the other hand,

$$\begin{aligned}
d_i^Q \circ u_{n+1}([x]) &= d_i^Q \circ u_{n+1}(\phi_X^{n+1}(x)) \\
&= d_i^Q((\psi_X)_{n+1}(x)) \\
&= d_i^Q \circ (\psi_X)_{n+1}(x) \\
&= (\psi_X)_n \circ d_i^X(x) \\
&= u_n \circ \phi_X^n(d_i^X(x)) \\
&= u_n([d_i^X(x)]).
\end{aligned}$$

Hence u is well-defined, and unique by the uniqueness of the u_n's. ☐

The following theorem will give us a way of representing any Δ-set M by an associated diagram $H_M : I_M \to \Delta-$sets mapping only to Δ-simplices. We will make extensive usage of this way of representing Δ-sets.

Theorem 1.2.2. *Let M be a Δ-set. Then there is an induced connected diagram $H_M : I_M \to \Delta$-sets of which M is the colimit such that the objects $H_M(X)$ are Δ-simplices for each $X \in I_M$.*

Proof. Let the objects of the index category I_M be $\bigsqcup_{n=-1}^{\infty} M_n$. For every $f \in M_n$, let $d_i(f) \to f$ be a morphism in $\mathrm{Hom}(I_M)$ for each i. Let $\mathrm{Hom}(I_M)$ be generated by these, together with the identity morphisms, under composition. Define $H_M(f)$ to be the n-dimensional Δ-simplex Δ^n. For each morphism $d_i(f) \to f$, we define the map $H_M(d_i(f) \to f) : \Delta^{n-1} \to \Delta^n$ by sending $*_{n-1} \mapsto d_i(*_n)$, where $*_{n-1}$ and $*_n$ are the respective maximal faces. Extend H_M to $\mathrm{Hom}(I_M)$ by composing. I.e. H_M sends a composition of morphisms

$$f \leftarrow d_{i_1}(f) \leftarrow \cdots \leftarrow d_{i_k} d_{i_{k-1}} \ldots d_{i_1}(f)$$

to the composition of the maps

$$\Delta^n \leftarrow \Delta^{n-1} \leftarrow \cdots \leftarrow \Delta^{n-k}.$$

This defines the functor H_M, and we have a diagram $H_M : I_M \to \Delta$-sets.

Now, write $M_n = \{f_n^j\}_{j \in J_M}$, and define $\phi^{j,n} : H(f_n^j) = \Delta^n \to M$ as follows. We have that $\Delta_n^n = \{*_n\}$, so let $\phi_n^{j,n}(*_n) = f_n^j$. Define $\phi^{j,n}$ recursively: let $y \in \Delta_k^n$ be given. y is uniquely on the form $d_r(x)$ for some r and $x \in \Delta_{k+1}^n$. Define

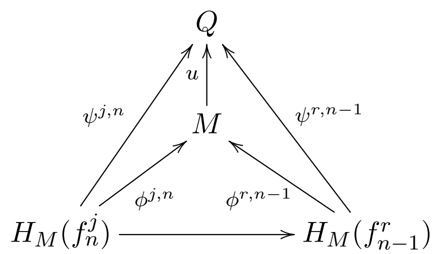
$$\phi_k^{j,n}(y) = d_r^M(\phi_{k+1}^{j,n}(x)).$$

Since $\phi_k^{j,n}(y) = \phi_k^{j,n}(d_r(x))$, this automatically makes $\phi^{j,n}$ a Δ-set map.

These maps clearly make M a node to the diagram. For any other node Q with maps $\psi^{j,n} : H_M(f_n^j) \to Q$, define the map $u : M \to Q$ by letting $u_n : M_n \to Q_n$ be defined by $f_n^j \to (\psi^{j,n})_n(*_n)$, where $*_n$ is the maximal face of $H_M(f_n^j) = \Delta^n$. We will prove that this is a Δ-set map. Write $d_i^M(f_{n+1}^j) = f_n^r$. Then

$$\begin{aligned} d_i^Q(u_{n+1}(f_{n+1}^j)) &= d_i^Q((\psi^{j,n+1})_{n+1}(*_{n+1})) = (\psi^{j,n+1})_n(d_i^{H(f_{n+1}^j)}(*_{n+1})) \\ &= (\psi^{j,n})_n(H_M(f_n^r \to f_{n+1}^j)_n(*_n)) = (\psi^{r,n})_n(*_n) \\ &= u_n(f_n^r) = u_n(d_i^M(f_{n+1}^j)). \end{aligned}$$

These maps makes the diagram

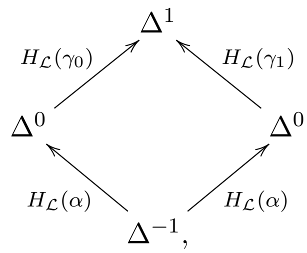


commute by definition, and therefore Q factors through M. u is clearly unique, since the maps $\psi^{j,n}$ determine what it must give for each element f_n^j. Thus M is the colimit of the diagram H_M. H_M is clearly a connected diagram. $\square$

Example 1.2.3. For the loop $\mathcal{L}$, the diagram $H_{\mathcal{L}} : I_{\mathcal{L}} \to \Delta$-sets is defined by letting the objects of the index category $I_{\mathcal{L}}$ be the set $\{\emptyset, v, e\}$, and the three generating morphisms $\emptyset \to v \rightrightarrows e$ be $\alpha : d_0(v) \to v, \gamma_0 : d_0(e) \to e$ and $\gamma_1 : d_1(e) \to e$. $H_{\mathcal{L}}$ sends $\emptyset$ to the empty simplex $\Delta^{-1} = \{\emptyset\}$, v to the simplex $\Delta^0 = \{\emptyset, 0\}$, and e to the simplex $\Delta^1 = \{\emptyset, 0, 1, \{0, 1\}\}$. The morphisms are sent to

$$\begin{aligned} H_{\mathcal{L}}(\alpha) :&\Delta^{-1} \subseteq \Delta^0, \\ H_{\mathcal{L}}(\gamma_0) :&\Delta^0 \to \Delta^1 \text{ such that } 0 \mapsto 0, \\ H_{\mathcal{L}}(\gamma_1) :&\Delta^0 \to \Delta^1 \text{ such that } 0 \mapsto 1. \end{aligned}$$

We may picture the diagram as such:



and $\mathcal{L}$ will be the colimit.

Remark 1.2.4. In the category of simplicial complexes maps need not respect dimension. For example, a 1-dimensional edge may be sent to a 0-dimensional point. In fact, in the category of simplicial complexes, the colimit of the diagram $H_{\mathcal{L}}$ in the above example is simply a point.

1.3 The Δ-face ring

Consider the category of Δ-simplices, a subcategory of the category of Δ-sets. We will define a contravariant functor $R : \Delta\text{-simplices} \to \text{Graded } k\text{-algebras}$. If M is an n-dimensional Δ-simplex, let $R(M) = k[x_\tau | \tau \in M_0]$ with each indeterminate x_τ of degree 1. For a map $\phi : M \to N$ between Δ-simplices, define $R(\phi) : F(N) \to F(M)$ by the following:

$$\begin{aligned} x_{\phi_0(\tau)} &\mapsto x_\tau, \\ x_s &\mapsto 0 \text{ for } s \in N_0 - \operatorname{im}(\phi_0). \end{aligned}$$

This clearly makes R a functor. Now, let M be a simplicial complex. Recall the induced diagram $H_M : I_M \to \Delta\text{-sets}$. For any $X \in I_M$, $H_M(X)$ is a Δ-simplex, so we may consider the functor defined by composing H_M with R, $R_M = R \circ H_M : I_M \to \text{Graded } k\text{-algebras}$, and the limit $F(M) = \varprojlim R_M$. Later in this section, we will prove that for a simplicial complex M, $F(M)$ is the Stanley-Reisner ring of M (see Theorem 1.3.7). See [MS05] for more on Stanley-Reisner rings. We want to generalize the Stanley-Reisner ring construction to Δ-sets. A natural choice is thus the following.

Definition 1.3.1. *For a Δ-set M, let $F(M) = \varprojlim R_M$ be its Δ-face ring.*

We will call this the naive definition of the Δ-face ring of M. However, as we will see, it is possible to define F as a contravariant functor $F : \Delta\text{-sets} \to \text{Graded } k\text{-algebras}$ such that for any connected diagram

$H : I \to \Delta$-sets with colimit M, we have $F(M) = \varprojlim F \circ H$ (see Theorem 1.3.2). In order to prove these things, we will go a slightly different route and rather define F as a functor by recursion on dimension in what follows. This will for now only define F on the subcategory of finite-dimensional Δ-sets, and we will from now on refer to this subcategory as the category of Δ-sets. Then, using Theorem 1.3.2, we will see that our new definition of $F(M)$ for a Δ-set M coincides with the naive definition of $F(M)$ as $\varprojlim R_M$.

First, we let F restricted to the subcategory of Δ-simplices to coincide with the functor R : Δ-simplices $\to$ Graded k-algebras as defined above. This defines F as a functor on the subcategory of Δ-simplices. We proceed to define F on general finite-dimensional Δ-sets inductively. Let M be an n-dimensional Δ set. Assume that F is well-defined on the full subcategory of $n-1$ dimensional (or lower) Δ-sets together with all Δ-simplices of every dimension (call this subcategory K_n). F is already defined on K_0. We will at each step extend this category by adding all n-dimensional Δ-sets. Let M be an n-dimensional Δ-set, and let $M_n = \{f_j\}_{j \in J_M}$. For each face f_j, assign an n-simplex (n-dimensional Δ-simplex) Δ^j with face maps $d_i^{\Delta^j}$. Let M' be the $n-1$-dimensional Δ-set defined by removing the n-dimensional faces. I.e. $M'_k = \emptyset$ for $k \geq n$, and $M'_k = M_k$ for $k \leq n-1$. We have a natural inclusion $\tau : M' \to M$. Let $\Delta^{j,i}$ be $n-1$-dimensional Δ-simplices for $i = 0, 1, \ldots, n$, with inclusion maps $\tau^{j,i} : \Delta^{j,i} \to \Delta^j$, such that $(\tau^{j,i})_n(*)$ is the i-th face of Δ^j_n, $d_i^{\Delta_j}(*)$ ($*$ simply denotes the maximal faces of the respective Δ-simplices).

Now, define the maps $g^j : \Delta^j \to M$ recursively as before by letting $g^j_n(*) = f_j$. Note that for each i, $g^{j,i} : \Delta^{j,i} \to M'$ is induced, and we have a commutative diagram

$$\begin{array}{ccc} \Delta^j & \xrightarrow{g^j} & M \\ {\scriptstyle \tau^{j,i}} \uparrow & & \uparrow {\scriptstyle \tau} \\ \Delta^{j,i} & \xrightarrow{g^{j,i}} & M' \end{array}$$

for each i, j. With these maps M is a node of the diagram. Note however that

$$M_k = \bigsqcup_{j \in J_M} (\Delta^j_k \sqcup_{i=0}^n \Delta^{j,i}_k) \sqcup M'_k / \sim$$

where $\sim$ is defined by the maps $g^{j,i}_k$ and $\tau^{j,i}_k$ (the expression to the right is clearly just M'_k for $k \leq n-1$, and for $k = n$, the only thing added are the maximal faces of Δ^j, forming $\{f_j\}_{j \in J_M}$). This makes M the colimit of the diagram formed by including all i's and j's. We can write this as

$$\begin{CD}
\{\Delta^j\}_{j\in J_M} \\
@A{\{\tau^{j,i}\}^{0\le i\le n}_{j\in J_M}}AA \\
\{\Delta^{j,i}\}^{0\le i\le n}_{j\in J_M} @>{\{g^{j,i}\}^{0\le i\le n}_{j\in J_M}}>> M',
\end{CD}$$

or for easier notation:

$$\begin{CD}
\{\Delta^j\} \\
@A{\tau^{j,i}}AA \\
\{\Delta^{j,i}\} @>{g^{j,i}}>> M'.
\end{CD}$$

Denote this last diagram by $\{M\}$, and call it the n-diagram of M (we will later define maps $\{\phi\}$ between such diagrams arising from maps between Δ-sets). Thus we have the commutative diagram

$$\begin{CD}
\{\Delta^j\} @>{g^j}>> M \\
@A{\tau^{j,i}}AA @A{\tau}AA \\
\{\Delta^{j,i}\} @>{g^{j,i}}>> M'.
\end{CD}$$

We will now define $F(M)$ as the limit of the following diagram which we will call $F\{M\}$:

$$\begin{CD}
\{F(\Delta^j)\} \\
@V{F(\tau^{j,i})}VV \\
\{F(\Delta^{j,i})\} @<{F(g^{j,i})}<< F(M').
\end{CD}$$

This constructs the maps $F(g^j)$ and $F(\tau)$ as well, and we have the diagram

$$\begin{CD}
\{F(\Delta^j)\} @<{F(g^j)}<< F(M) \\
@V{F(\tau^{j,i})}VV @VV{F(\tau)}V \\
\{F(\Delta^{j,i})\} @<{F(g^{j,i})}<< F(M').
\end{CD}$$

In order to fully define F as a functor from K_{n+1} (which is the inductive step), it remains to construct $F(\phi : M \to N)$ for any map $\phi : M \to N$ for the following two cases:

- M and N are n-dimensional Δ-sets,
- M is in K_n and N is n-dimensional,

and to show that this definition is compatible with the construction of $F(g^j)$ and $F(\tau)$. In addition, we need to verify that for a sequence of maps $M \to^{\phi} N \to^{\psi} P$, we have $F(\psi \circ \phi) = F(\phi) \circ F(\psi)$, and that $F(\mathrm{id}_M) = \mathrm{id}_{F(M)}$.

For maximally n-dimensional Δ-sets M and N, we let $\{f_M^j\}_{j \in J_M}$ and $\{f_N^j\}_{j \in J_N}$ denote the n-dimensional maximal faces of M and N respectively (we allow the index sets J_M and J_N to be empty). We have the diagrams $\{M\}$ and $\{N\}$:

$$\begin{array}{ccc} \{\Delta_M^j\} & & \\ \uparrow & & \\ \{\Delta_M^{j,i}\} & \longrightarrow & M', \end{array} \qquad\qquad \begin{array}{ccc} \{\Delta_N^j\} & & \\ \uparrow & & \\ \{\Delta_N^{j,i}\} & \longrightarrow & N'. \end{array}$$

Given a map $\phi : M \to N$, then $\phi' : M' \to N'$ is induced. Furthermore, we have a map $\psi : J_M \to J_N$ defined by $\phi_n(f_M^j) = f_N^{\psi(j)}$. Define
$\phi^j : \Delta_M^j \to \Delta_N^{\psi(j)}$ recursively similarly to before: $(\phi^j)_n(*) = *$, and

$$(\phi^j)_k(d_i^{\Delta_M^j}(x)) = d_i^{\Delta_N^{\psi(j)}}((\phi^j)_{k+1}(x)).$$

This induces maps $\phi^{j,i} : \Delta_M^{j,i} \to \Delta_N^{\psi(j),i}$. So we have a map $\{\phi\} : \{M\} \to \{N\}$ between diagrams:

$$\begin{array}{ccc} \{\Delta_M^j\} & \xrightarrow{\phi^j} & \{\Delta_N^{\psi(j)}\} \\ \uparrow & & \uparrow \\ \{\Delta_M^{j,i}\} & \xrightarrow{\phi^{j,i}} & \{\Delta_N^{\psi(j),i}\} \\ \downarrow & & \downarrow \\ M' & \xrightarrow{\phi'} & N', \end{array}$$

which clearly makes the above diagram commute. Thus we have an induced map $F\{\phi\} : F\{N\} \to F\{M\}$ given by the maps

$$\begin{array}{ccc}
\{F(\Delta_M^j)\} & \xleftarrow{F(\phi^j)} & \{F(\Delta_N^{\psi(j)})\} \\
\downarrow & & \downarrow \\
\{F(\Delta_M^{j,i})\} & \xleftarrow{F(\phi^{j,i})} & \{F(\Delta_N^{\psi(j),i})\} \\
\uparrow & & \uparrow \\
F(M') & \xleftarrow{F(\phi')} & F(N').
\end{array}$$

Note that not every object of the diagram $F\{N\}$ necessarily has a map from itself. However, every object of $F\{M\}$ is hit by some object of $F\{N\}$, which gives us a map $F(N) \to F\{M\}$. This induces a unique map $F(N) \to F(M)$ which we will call $F(\phi)$.

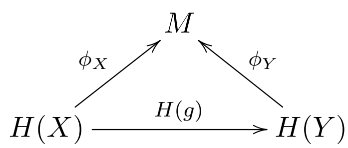
Now, given a sequence of Δ-set maps $M \to^{\phi} N \to^{\gamma} P$, we get a sequence of maps $F(M) \leftarrow^{F(\phi)} F(N) \leftarrow^{F(\gamma)} F(P)$, induced by the maps $F\{M\} \leftarrow^{F\{\phi\}} F\{N\} \leftarrow^{F\{\gamma\}} F\{P\}$. $\{\gamma\} \circ \{\phi\}$ may be defined in an obvious way by moving from one object in the diagram to the next one, and it is clear that $\{\gamma\} \circ \{\phi\} = \{\gamma \circ \phi\}$. Since the objects of the diagrams are elements of K_n, we may use the inductive hypothesis on F, and conclude that $F\{\phi\} \circ F\{\gamma\} = F\{\gamma \circ \phi\}$ by considering each map between the diagrams. Therefore, by the uniqueness of our construction we have that $F(\phi) \circ F(\gamma) = F(\gamma \circ \phi)$ as required. Note that this proof worked for the general case.

To show that $F(\mathrm{id}_M) = \mathrm{id}_{F(M)}$, we only need to check that our construction of $F\{\mathrm{id}_M\}$ does not change the maps from $F(M)$ to $F\{M\}$. But this is clear since each individual map of $\{\mathrm{id}_M\}$ between objects of the diagram $\{M\}$ is the identity, and by using the inductive hypothesis the unique induced map $F(M) \to F(M)$ must be the identity. Finally, that our definition is compatible with the construction of $F(g^j)$ and $F(\tau)$ is clear by looking at the maps $\{g^j\}$ and $\{\tau\}$, since the $n-$diagrams $\{\Delta^j\}$ and $\{M'\}$ are trivial.

Now that we have defined the functor F we will show that our construction of the Δ-face ring $F(M)$ is natural in a certain sense. It follows that $F(M)$ coincides with the naive definition of the Δ-face ring of M.

Theorem 1.3.2. *If $M = \varinjlim H$, where $H : I \to \Delta$-sets is a connected diagram, then $F(M) = \varprojlim F \circ H$.*

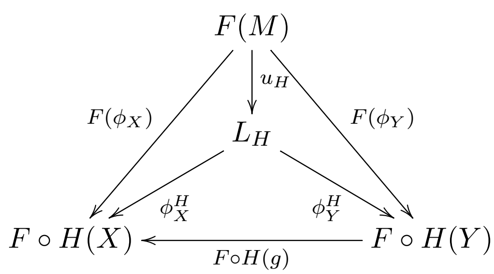
Proof. We proceed by induction on the dimension n of M. The initial case $(n = -1)$ for which $M_k = \emptyset$ for all $k \geq 0$ is trivial, since the diagram is connected. Assume the hypothesis is true for all Δ-sets of dimension lower than n. Let M be n-dimensional. We have the diagram



for each morphism $g : X \to Y$. This gives morphisms

$$F(\phi_X) : F(M) \to F \circ H(X).$$

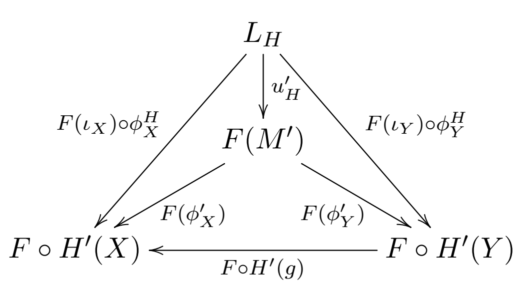
Let $L_H = \varprojlim F \circ H$ together with maps $\phi_X^H : L_H \to F \circ H(X)$. Since $F(M)$ is a node to the diagram H, we have a unique induced map
$u_H : F(M) \to L_H$ such that the diagram



commutes for all morphisms $g : X \to Y$.

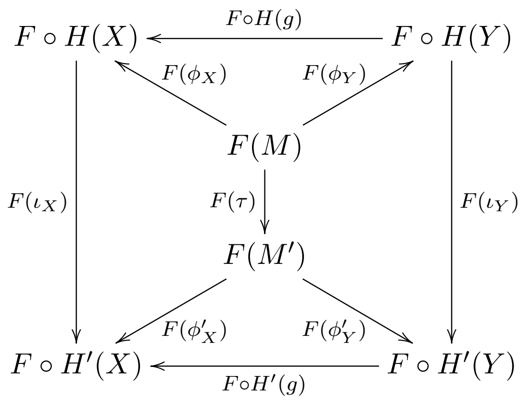
Now, $H(X)$ is maximally n-dimensional for $X \in I$. Define a new diagram $H' : I \to \Delta$-sets by $H'(X) = H(X)'$, where we remove the n-dimensional faces. The morphisms are induced. Now, clearly $M' = \varinjlim H'$ (we are simply removing the n-dimensional elements) where the maps $\phi'_X : H'(X) \to M'$ are induced. By the inductive hypothesis, we know that $F(M') = \varprojlim F \circ H'$ with maps $F(\phi'_X) : F(M') \to F \circ H'(X)$.

L_H, being a node to $F \circ H$, makes it into a node to $F \circ H'$ as well, given the natural injections $\iota_X : H'(X) \to H(X)$. This induces a unique map u'_H such that the following diagram commutes:



for morphisms $g : X \to Y$.

We have the inclusion $\tau : M' \to M$, and $F(\tau) : F(M) \to F(M')$. Now, M is clearly a node to H', where the morphisms are induced by the factorization through M'. This makes $F(M)$ a node to $F \circ H'$, where $F(\tau)$ is the unique map such that the diagram



commutes for morphisms $g : X \to Y$. So we have the following for $X \in I$.

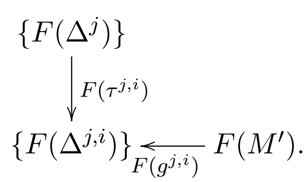
$$F(\phi_X) = \phi_X^H \circ u_H,$$
$$F(\iota_X) \circ \phi_X^H = F(\phi'_X) \circ u'_H,$$
$$F(\iota_X) \circ F(\phi_X) = F(\phi'_X) \circ F(\tau).$$

Therefore, for $u'_H \circ u_H$, we have

$$F(\phi'_X) \circ (u'_H \circ u_H) = F(\iota_X) \circ \phi_X^H \circ u_H = F(\iota_X) \circ F(\phi_X).$$

But $F(\tau)$ is the unique such morphism, which means that $F(\tau) = u'_H \circ u_H$.

Now, consider the diagram $F\{M\}$:



We want to make L_H a node to this diagram. We already have $u'_H : L_H \to F(M')$. We will construct a map $\gamma^j : L_H \to F(\Delta^j)$, such that $F(\tau^{j,i}) \circ \gamma^j = F(g^{j,i}) \circ u'_H$. This will make L_H a node.

Take $f^j \in M_n$, and an $X \in I$ such that there is an element $f_X^j \in H(X)_n$ with $(\phi_X)_n(f_X^j) = f^j$. Choose one such f_X^j, and define $\pi_X^j : \Delta^j \to H(X)$

by $* \mapsto f_X^j$ on the n-th level, and recursively as before for the other levels. This gives us the map $F(\pi_X^j) : F \circ H(X) \to F(\Delta^j)$, hence a map $\gamma^j : L_H \to F(\Delta^j)$, where $\gamma^j = F(\pi_X^j) \circ \phi_X^H$. We will show that this is well-defined. Given any $Y \in I$, with $f_Y^j \in H(Y)$, and π_Y^j defined as above, we need to show that

$$F(\pi_X^j) \circ \phi_X^H = F(\pi_Y^j) \circ \phi_Y^H.$$

By the set-theoretic construction of the colimit, we have that since $f_X^j \sim f_Y^j$ in M_n, there is a finite sequence

$$f_X^j = f_0 \sim f_1 \sim \ldots \sim f_r = f_Y^j,$$

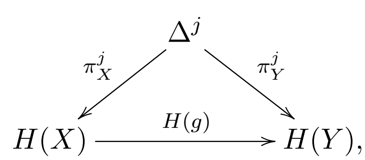
where $f_i \in H(X_i)$, $X_0 = X, X_r = Y$, and there is a morphism $g_i : X_i \to X_{i+1}$ such that $H(g_i)(f_i) = f_{i+1}$, or a morphism $h_i : X_{i+1} \to X_i$ such that $H(h_i)(f_{i+1}) = f_i$. We proceed by induction on r. If $r = 0$, there is nothing to show. For the inductive step, we may assume that

$$F(\pi_{X_0}^j) \circ \phi_{X_0}^H = F(\pi_{X_{r-1}}^j) \circ \phi_{X_{r-1}}^H,$$

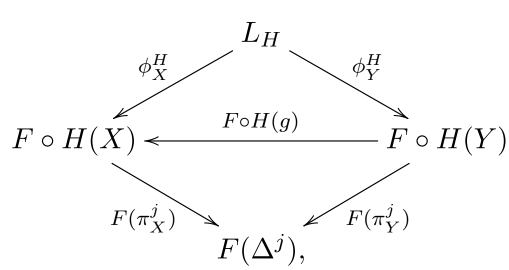
so it remains to show that

$$F(\pi_{X_{r-1}}^j) \circ \phi_{X_{r-1}}^H = F(\pi_{X_r}^j) \circ \phi_{X_r}^H.$$

So we reduce the proof of the inductive step to the proof of the statement for $r = 1$. Without loss of generality, we may assume that we have a morphism $g : X \to Y$ such that $H(g)(f_X^j) = f_Y^j$. So we have the following commutative diagram:



and thus the commutative diagram



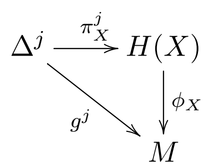
which proves that γ^j is well-defined. This also shows that γ^j is independent of our choice of f_X^j. It remains to prove that

$$F(\tau^{j,i}) \circ \gamma^j = F(g^{j,i}) \circ u'_H.$$

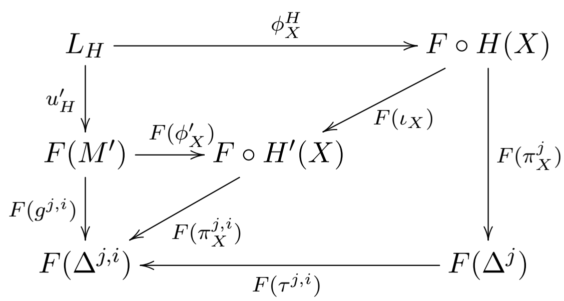
Pick an X for which $\gamma^j = F(\pi_X^j) \circ \phi_X^H$. We need to show the equality

$$F(\tau^{j,i}) \circ F(\pi_X^j) \circ \phi_X^H = F(g^{j,i}) \circ u'_H.$$

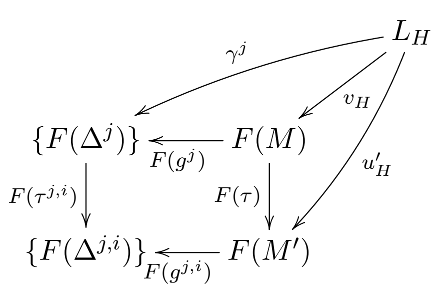
Note that for any $\pi_X^j : \Delta^j \to H(X)$,



commutes. Note further that the morphism $\pi_X^j : \Delta^j \to H(X)$ induces a morphism $\pi^{j,i} : \Delta^{j,i} \to H'(X)$. This gives us the diagram



which we can see commutes, and the equality above therefore holds. This makes L_H a node to the diagram $F\{M\}$, and induces a unique map $v_H : L_H \to F(M)$ such that



commutes.

Now, we have the following equality:

$$F(\pi_X^j) \circ F(\phi_X) = F(g^j).$$

This gives us the following for the map $v_H \circ u_H : F(M) \to F(M)$:

$$F(g^j) \circ v_H \circ u_H = \gamma^j \circ u_H = \pi_X^j \circ \phi_X^H \circ u_H = \pi_X^j \circ F(\phi_X) = F(g^j),$$
$$F(\tau) \circ v_H \circ u_H = u'_H \circ u_H = F(\tau).$$

But $\mathrm{id}_{F(M)}$ is the unique such map, which means that $v_H \circ u_H = \mathrm{id}_{F(M)}$. This means u_H is an isomorphism, which makes $F(M)$ with the maps $F(\phi_X) : F(M) \to F \circ H(X)$ the limit to the diagram $F \circ H$. □

Remark 1.3.3. It is vital that the diagram above is connected, or else the inductive hypothesis would not be satisfied.

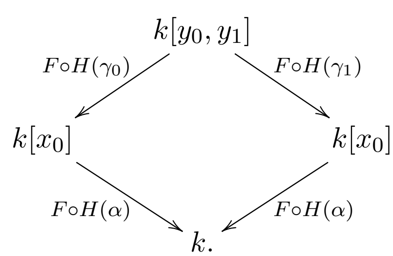
Example 1.3.4. We will compute the Δ-face ring $F(\mathcal{L})$ for the loop $\mathcal{L}$. Consider the diagram functor $H_{\mathcal{L}} : I_{\mathcal{L}} \to \Delta$-sets. We need to find the limit of the diagram $F \circ H_{\mathcal{L}}$. The objects of $I_{\mathcal{L}}$ are $\emptyset$, v and e, and we have

$$F \circ H_{\mathcal{L}}(\emptyset) = k,$$
$$F \circ H_{\mathcal{L}}(v) = k[x_0],$$
$$F \circ H_{\mathcal{L}}(e) = k[y_0, y_1].$$

The morphisms α, γ_0 and γ_1 are sent to

$$F \circ H_{\mathcal{L}}(\alpha) : F(\Delta^0) \to F(\Delta^{-1}) \text{ such that } x_0 \mapsto 0,$$
$$F \circ H_{\mathcal{L}}(\gamma_0) : F(\Delta^1) \to F(\Delta^0) \text{ such that } y_0 \mapsto x_0 \text{ and } y_1 \mapsto 0,$$
$$F \circ H_{\mathcal{L}}(\gamma_1) : F(\Delta^1) \to F(\Delta^0) \text{ such that } y_0 \mapsto 0 \text{ and } y_1 \mapsto x_0.$$

The diagram may be pictured as



The limit may be computed as

$$F(\mathcal{L}) = \{f(y_0, y_1) \in k[y_0, y_1] | f(x_0, 0) = f(0, x_0)\},$$

which is generated by the elements $y_0 + y_1, y_0y_1$ and $y_0^2y_1$ over k. We introduce the graded coordinate ring $P = k[t, u, v]$ with t of degree 1, u of degree 2 and v of degree 3 and the k-algebra homomorphism $P \to F(\mathcal{L})$ defined by

$$\begin{aligned} t \mapsto & y_0 + y_1, \\ u \mapsto & y_0 y_1, \\ v \mapsto & y_0^2 y_1, \end{aligned}$$

and find that the kernel is the ideal $(tuv - u^3 - v^2)$. Thus

$$F(\mathcal{L}) = k[t, u, v]/(tuv - u^3 - v^2).$$

Remark 1.3.5. This theorem does not apply to the category of simplicial complexes. As in Remark 1.2.4, the colimit of $H_{\mathcal{L}}$ is a point if we regard it as a functor to the category of simplicial complexes. Thus the identification of rings $F(\varinjlim H) = \varprojlim F \circ H$ fails for $H = H_{\mathcal{L}}$. The former is the ring $k[x]$, while the latter is the ring described in the above example. Thus this theorem is one benefit of restricting the possible maps between simplicial complexes.

Theorem 1.3.6. *If M is a Δ-set, then its Δ-face ring $F(M)$ has no nilpotents.*

Proof. M is the colimit of $H_M : I_M \to \Delta$-sets as in Theorem 1.2.2, and $F(M) = \varprojlim F \circ H_M$ by Theorem 1.3.2. Let $f \in F(M)$, and $X \in I_M$. Let $\overline{f}$ be the image of f in $F \circ H_M(X)$. If $f^n = 0$ for some n, then $\overline{f}^n = 0$. But $F \circ H_M(X)$ is an integral domain, so $\overline{f} = 0$. This applies to all $X \in I_M$, and it follows that $f = 0$ in $F(M)$. □

The following theorem shows that construction of the Δ-face rings is a natural generalization of the Stanley-Reisner ring construction.

Theorem 1.3.7. *If a Δ-set M is a simplicial complex, then its Δ-face ring $F(M)$ is the Stanley-Reisner ring of M.*

Proof. Consider the diagram $H = H_M : I_M \to \Delta$-sets where the objects are $I_M = \bigsqcup_{n=-1}^{\infty} M_n$, and the Δ-sets $H(X)$ are Δ-simplices from Theorem 1.2.2. M is the colimit, with maps $\phi_X : H(X) \to M$. We may consider $H(X)_0$ as elements of the simplicial complex M.

Since M is a simplicial complex, we may write $M_0 = \{0, 1, 2, \ldots, n\}$. Let Δ^n be the n-Δ-simplex with vertices $\{0, 1, 2, \ldots, n\}$. Then we have natural inclusions $h_X : H(X) \subseteq \Delta^n$ making Δ^n a node to the diagram H. Let

$S = F(\Delta^n) = k[x_0, \ldots, x_n]$. We have the maps $F(h_X) : F(\Delta^n) \to F(H(X))$, where $x_i \mapsto x_i$ if $i \in H(X)_0$, and $x_i \mapsto 0$ if $i \notin H(X)_0$ with corresponding splitting maps $\sigma_X : F(H(X)) \to F(\Delta^n)$ such that $F(h_X) \circ \sigma_X = \mathrm{id}_{F(H(X))}$. Note that σ_X is a k-algebra map, making $F(H(X))$ into subrings of S.

Now, considering M as a simplicial complex, we let

$$a_M = \{x_{i_1} x_{i_2} \ldots x_{i_r} | \{i_1, \ldots, i_r\} \notin M\} \subseteq S$$

for $0 \leq i_1 < i_2 < \ldots < i_r \leq n$ be the Stanley-Reisner ideal of M. $J \in M$ means that $J = H(X)_0 = (\mathrm{im}\ h_X)_0$ for some $X \in I_M$.

Now, consider a generator element $x_{i_1} x_{i_2} \cdots x_{i_r} \in a_M$. Since we have that $\{i_1, \ldots, i_r\} \notin M$, we know that $\{i_1, \ldots, i_r\}$ is not contained in any $H(X)_0$ either. Thus $F(h_X)(x_{i_1} x_{i_2} \cdots x_{i_r}) = 0$ for all $X \in I_M$, since at least some $x_{i_j} \mapsto 0$ in $F(H(X))$. Hence $a_M \subseteq \ker F(h_X)$, and $F(h_X)$ induces maps $\psi_X : S/a_M \to F(H(X))$. The splitting maps σ_X of $F(h_X)$ induces k-algebra splittings $\overline{\sigma_X}$ of ψ_X as well. $F(H(X))$ may be considered subrings of S/a_M. Since $F(M)$ is the limit of $F \circ H$, we have induced a unique map $\psi : S/a_M \to F(M)$ such that $F(\phi_X) \circ \psi = \psi_X$ for all $X \in I_M$.

The morphisms $g : X \to Y$ corresponds to the inclusion maps $H(g) : H(X) \subseteq H(Y)$. For each such inclusion $H(g)$, we have a surjective map $F(H(g)) : F(H(Y)) \to F(H(X))$. Let γ_{XY} denote this map. We shall write $X \cap Y$ for the unique element $Z \in I_M$ for which $H(Z) = H(X) \cap H(Y)$. Now, define a sequence of integers $(b_X)_{X \in I_M}$ recursively (on the size of $H(X)_0$) by $b_X = 1 - \sum_{H(Y) \supset H(X)} b_Y$. Then $\sum_{H(X) \supseteq H(Y)} b_X = 1$ for every $Y \in I_M$. Define $v : F(M) \to S/a_M$ by $v = \sum_{X \in I_M} b_X \overline{\sigma_X} \circ F(\phi_X)$. v is clearly a k-module homomorphism. Now,

$$\psi_Y \circ v = \psi_Y(\sum_{X \in I_M} b_X \overline{\sigma_X} \circ F(\phi_X)) = \sum_{X \in I_M} b_X \psi_Y(\overline{\sigma_X} \circ F(\phi_X)).$$

For a set $a \subseteq H(X)_0$, write $a = (a_1, \ldots, a_r)$. Introduce the following notation: $x_a^{\vec{e}} = x_{a_1}^{e_1} x_{a_2}^{e_2} \cdots x_{a_r}^{e_r}$, where $\vec{e} = (e_1, \ldots, e_r)$ has positive integer components. For an $f \in F(M)$, we may write

$$F(\phi_X)(f) = \overline{\sigma_X} \circ F(\phi_X)(f) = \sum_{\substack{a \subseteq H(X)_0 \\ i=0}}^{m} b_{ai} x_a^{\vec{e_i}},$$

where $b_{ai} \in k$. It is clear that $\psi_Y(x_a^{\vec{e}}) = x_a^{\vec{e}}$ if $a \subseteq H(Y)_0$, and 0 if $a \not\subseteq H(Y)_0$.

So

$$\psi_Y(\overline{\sigma_X} \circ F(\phi_X)(f)) = \sum_{\substack{a \subseteq H(X)_0 \cap H(Y)_0 \\ i=0}}^{m} b_{ai} x_a^{\vec{e_i}} = \gamma_{X\cap Y\ X}(F(\phi_X)(f))$$
$$= F(\phi_{X\cap Y})(f) = \gamma_{X\cap Y\ Y}(F(\phi_Y)(f)).$$

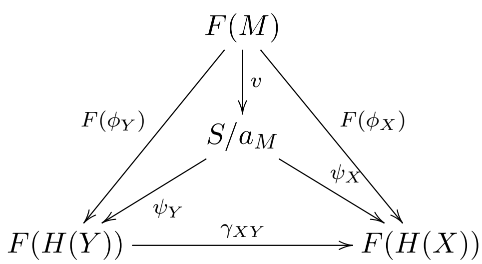
Hence

$$\psi_Y \circ v(f) = \sum_{X\in I_M} b_X \gamma_{X\cap Y\ Y}(F(\phi_Y)(f)) = \left(\sum_{X\in I_M} b_X \gamma_{X\cap Y\ Y}\right)(F(\phi_Y)(f)).$$

Now, let $a \subseteq H(Y)_0$. Then $\gamma_{X\cap Y\ Y}(x_a^{\vec{e}}) = x_a^{\vec{e}}$ if $a \subseteq H(X \cap Y)_0$, and 0 if $a \not\subseteq H(X\cap Y)_0$. So

$$\left(\sum_{X\in I_M} b_X \gamma_{X\cap Y\ Y}\right)(x_a^{\vec{e}}) = \left(\sum_{\substack{X\in I_M \\ a\subseteq H(X)_0\cap H(Y)_0}} b_X \gamma_{X\cap Y\ Y}\right)(x_a^{\vec{e}})$$
$$= \left(\sum_{H(X)_0 \supseteq a} b_X\right)(x_a^{\vec{e}}) = 1 \cdot x_a^{\vec{e}}.$$

The last equality follows from the fact that $a = H(Z)_0$ for some $Z \in I_M$. Thus $\sum_{X\in I_M} b_X \gamma_{X\cap Y\ Y} = \mathrm{id}_{F(H(Y))}$, and $\psi_Y \circ v = F(\phi_Y)$. This means that the diagram



commutes.

It remains to show that v is a ring homomorphism. Let $f_1, f_2 \in F(M)$ and define $h = v(f_1 f_2) - v(f_1)v(f_2) \in S/a_M$. Then, for every $X \in I_M$,

$$\begin{aligned}\psi_X(h) &= \psi_X(v(f_1f_2)) - \psi_X(v(f_1)v(f_2)) \\ &= (\psi_X \circ v)(f_1f_2) - \psi_X(v(f_1))\psi_X(v(f_2)) \\ &= (\psi_X \circ v)(f_1f_2) - (\psi_X \circ v)(f_1)(\psi_X \circ v)(f_2) \\ &= F(\phi_X)(f_1f_2) - F(\phi_X)(f_1)F(\phi_X)(f_2) = 0.\end{aligned}$$

Write $h = \sum_{\substack{a\in M\\ i=0}}^{m} b_{ai}x_a^{\vec{e_i}} + a_M$. For every $X \in I_M$,

$$\psi_X(h) = \sum_{\substack{a\subseteq H(X)\\ i=0}}^{m} b_{ai}x_a^{\vec{e_i}} = 0.$$

This means that for every $X \in I_M$, $b_{Xi} = 0$ for all $0 \leq i \leq m$. In other words, $b_{ai} = 0$ for all $a \in M$, meaning that $h = 0$ in S/a_M. This means that $v(f_1 f_2) = v(f_1)v(f_2)$, and v is a ring homomorphism. Suppose now that $v' : F(M) \to S/a_M$ is another factorization of $F(M)$ through S/a_M. Since $\psi_X \circ v'(f) = F(\phi_X)(f) = \psi_X \circ v(f)$ for all X, $\psi_X(v'(f) - v(f)) = 0$ for all $X \in I_M$. By what we just have shown for h, we know that this implies that $v'(f) = v(f)$, hence v is unique. Thus $F(M)$ factors uniquely through S/a_M, which means that S/a_M is the limit itself. □

Theorem 1.3.8. *An injective map $\phi : M \to N$ induces a k-module splitting $\sigma : F(M) \to F(N)$ of $F(\phi) : F(N) \to F(M)$, in other words a map such that $F(\phi) \circ \sigma = id_{F(M)}$.*

Proof. Consider the diagram $\{M\}$:

$$\begin{array}{ccc} \{\Delta_M^j\} & & \\ \uparrow & & \\ \{\Delta_M^{j,i}\} & \longrightarrow & M'. \end{array}$$

Note that M is the colimit of the following diagram as well, which we will call $\{\Delta_M\}$:

$$\begin{array}{ccc} \{\Delta_M^j\} & & \\ \uparrow & & \\ \{\Delta_M^{j\ \prime}\} & \xrightarrow{g_M^{j\ \prime}} & M', \end{array}$$

where the map $g_M^{j\ \prime} : \Delta_M^{j\ \prime} \to M'$ is induced by the map $g^j : \Delta_M^j \to M$. Thus $F(M)$ is the limit of the diagram $F\{\Delta_M\}$:

$$\begin{array}{ccc} \{F(\Delta_M^j)\} & & \\ \downarrow & & \\ \{F(\Delta_M^{j\ \prime})\} & \xleftarrow{F(g_M^{j\ \prime})} & F(M'), \end{array}$$

by Theorem 1.3.2. Similarly we have the diagram $F\{\Delta_N\}$

$$\begin{array}{ccc}
\{F(\Delta_N^j)\} & & \\
\downarrow & & \\
\{F({\Delta_N^j}')\} & \xleftarrow{F({g_N^j}')} & F(M')
\end{array}$$

of which $F(N)$ is the limit.

Consider the map $F\{\phi\} : F\{N\} \to F\{M\}$:

$$\begin{array}{ccc}
F(\Delta_M^j) & \xleftarrow{F(\phi^j)} & F(\Delta_N^{\psi(j)}) \\
\downarrow & & \downarrow \\
F(\Delta_M^{j,i}) & \xleftarrow{F(\phi^{j,i})} & F(\Delta_N^{\psi(j),i}) \\
\uparrow & & \uparrow \\
F(M') & \xleftarrow{F(\phi')} & F(N').
\end{array}$$

It induces a map $F\{\Delta_N\} \to F\{\Delta_M\}$:

$$\begin{array}{ccc}
F(\Delta_M^j) & \xleftarrow{F(\phi^j)} & F(\Delta_N^{\psi(j)}) \\
\downarrow & & \downarrow \\
F({\Delta_M^j}') & \xleftarrow{F({\phi^j}')} & F({\Delta_N^{\psi(j)}}') \\
\uparrow & & \uparrow \\
F(M') & \xleftarrow{F(\phi')} & F(N').
\end{array}$$

We proceed by induction. The initial case is trivial. By the inductive hypothesis, we have a splitting map $\sigma' : F(M') \to F(N')$ of $F(\phi')$. Thus we have a map $\{\sigma\} : F\{\Delta_M\} \to F\{\Delta_N\}$:

$$\begin{array}{ccc}
F(\Delta_M^j) & \xrightarrow{F(\phi^j)^{-1}} & F(\Delta_N^{\psi(j)}) \\
\downarrow & & \downarrow \\
F({\Delta_M^j}') & \xrightarrow{F({\phi^j}')^{-1}} & F({\Delta_N^{\psi(j)}}') \\
\uparrow & & \uparrow \\
F(M') & \xrightarrow{\sigma'} & F(N').
\end{array}$$

This diagram commutes, since

$$\begin{aligned}
&F(g_N^{\psi(j)'}) \circ \sigma' = F(\phi^{j'})^{-1} \circ F(g_M^{j}{}') \\
&\Leftrightarrow F(\phi^{j'}) \circ F(g_N^{\psi(j)'}) \circ \sigma' = F(g_M^{j}{}') \\
&\Leftrightarrow F(g_M^{j}{}') \circ F(\phi') \circ \sigma' = F(g_M^{j}{}'),
\end{aligned}$$

where the last equality follows by the inductive hypothesis. Note that we do not have a map from any element of the diagram $F\{\Delta_M\}$ to $F(\Delta_N^k)$ or $F(\Delta_N^k{}')$ for $k \notin \mathrm{im}(\psi)$. We may however construct such maps. By Theorem 1.3.7,

$$F(\Delta_N^k{}') = k[x_0, \ldots, x_n]/(x_1 \cdots x_n),$$

where $F(\Delta_N^k) = k[x_0, \ldots, x_n]$. There is a natural k-module splitting

$$\sigma^k : F(\Delta_N^k{}') = k[x_0, \ldots, x_n]/(x_0 \cdots x_n) \to k[x_0, \ldots, x_n] = F(\Delta_N^k)$$

of $F(\iota^k) : F(\Delta_N^k) \to F(\Delta_N^k{}')$, where $\iota^k : \Delta_N^k{}' \to \Delta_N^k$ is the inclusion, such that $F(\iota^k) \circ \sigma^k = \mathrm{id}_{F(\Delta_N^k{}')}$. Consider the k-module homomorphism $F(M') \to^{\sigma'} F(N') \to^{g_N^j{}'} F(\Delta_N^k{}') \to^{\sigma^k} F(\Delta_N^k)$. Including these homomorphisms (which trivially makes the entire diagram commute) gives us an induced k-module homomorphism $\sigma : F(M) \to F(N)$. Consider the map $\iota = F(\phi) \circ \sigma : F(M) \to F(M)$. We have the following:

$$\begin{aligned}
F(g_M^j) \circ \iota &= (F(g_M^j) \circ F(\phi)) \circ \sigma = (F(\phi^j) \circ F(g_N^{\psi(j)})) \circ \sigma \\
&= F(\phi^j) \circ (F(g_N^{\psi(j)}) \circ \sigma) = F(\phi^j) \circ (F(\phi^j)^{-1} \circ F(g_M^j)) = F(g_M^j), \\
F(\tau_M) \circ \iota &= (F(\tau_M) \circ F(\phi)) \circ \sigma = (F(\phi') \circ F(\tau_N)) \circ \sigma \\
&= F(\phi') \circ (F(\tau_N) \circ \sigma) = F(\phi') \circ (\sigma' \circ F(\tau_M)) \\
&= (F(\phi') \circ \sigma') \circ F(\tau_M) = \mathrm{id}_{F(M')} \circ F(\tau_M) = F(\tau_M).
\end{aligned}$$

But $\mathrm{id}_{F(M)}$ is the unique k-module homomorphism for which this holds, so $\iota = \mathrm{id}_{F(M)}$, and therefore σ is a splitting map. □

Remark 1.3.9. The splitting map σ in the above theorem is in general not a k-algebra homomorphism. Note that in the above proof we have used that the k-module limit construction is identical to the k-algebra limit construction considered a k-module.

Corollary 1.3.10. *If $\phi : M \to N$ is an injective map of Δ-sets, then the map $F(\phi) : F(N) \to F(M)$ is surjective.*

Proof. This follows from the fact that we have a k-module splitting map $\sigma : F(M) \to F(N)$ such that $F(\phi) \circ \sigma = \mathrm{id}_{F(M)}$ by Theorem 1.3.8. □

Chapter 2

The $\mathbb{P}$-functor and Δ-face schemes

2.1 Definition

Any homomorphism of graded rings $\psi : S \to T$ induces a morphism of schemes $f : U \to \operatorname{Proj} S$, where $U = \{p \in \operatorname{Proj} T | p \not\supseteq \psi(S_+)\}$. For a Δ-set M, let $\mathbb{P}(M) = \operatorname{Proj} F(M)$. We will call this its Δ-face scheme. For any homomorphism $F(\phi) : F(N) \to F(M)$ induced by a map $\phi : M \to N$ of Δ-sets, we will show that for any $p \in \operatorname{Proj} F(M)$, $p \not\supseteq F(\phi)(F(N)_+)$. Thus $F(\phi)$ induces a morphism $\mathbb{P}(\phi) : \mathbb{P}(M) \to \mathbb{P}(N)$. Hence we have defined a covariant functor $\mathbb{P} : \Delta\text{-sets} \to \mathrm{Sch}_k$. We will work exclusively in Sch_k, the category of schemes over k.

Example 2.1.1. We will realize the Δ-face scheme $\mathbb{P}(\mathcal{L})$ as a closed projective subspace of $\mathbb{P}^2$ for the loop $\mathcal{L}$. The following basic fact will be used: If R is any graded ring, then

$$R^{[d]} = \bigoplus_{n \in \mathbb{Z}} R_{dn} \subseteq R$$

for any non-zero integer d has the property that $\operatorname{Proj} R^{[d]} = \operatorname{Proj} R$ via the inclusion map. We usually rename the graded components of $R^{[d]}$ and let R_{dn} be the n-degree elements of $R^{[d]}$. See [Rei02]. Now, recall that

$$F(\mathcal{L}) = k[t, u, v]/(tuv - u^3 - v^2).$$

Consider the graded k-algebra homomorphism $k[x, y, z] \to F(\mathcal{L})$ defined by

$$x \mapsto t^3,$$
$$y \mapsto tu,$$
$$z \mapsto v.$$

It is easily seen that this homomorphism is surjective on $F(\mathcal{L})^{[3]}$. The kernel is computed to be the ideal $(y^3 - xyz + xz^2)$. Thus $\mathbb{P}(\mathcal{L})$ may be viewed as the nodal cubic curve $V(y^3 - xyz + xz^2) \subseteq \mathbb{P}^2$.

2.2 Some properties of $\mathbb{P}$

Now we will show some properties of the functor $\mathbb{P}$.

Theorem 2.2.1. *If $\phi : M \to N$ is an injective map of Δ-sets, then the morphism $\mathbb{P}(\phi) : \mathbb{P}(M) \to \mathbb{P}(N)$ is a closed immersion.*

Proof. By Corollary 1.3.10, the map $F(\phi) : F(N) \to F(M)$ is surjective. Hence $F(\phi)(F(N)_+) = F(M)_+$, and the morphism $\mathbb{P}(\phi) : \mathbb{P}(M) \to \mathbb{P}(N)$ is well-defined and surjective. □

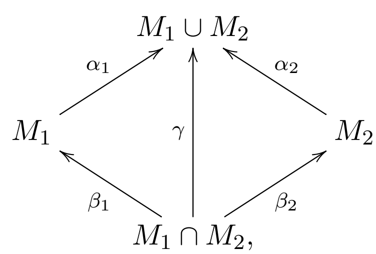
Theorem 2.2.2. *If $M_1, M_2 \subseteq M$, then*

$$\mathbb{P}(M_1 \cap M_2) = \mathbb{P}(M_1) \cap \mathbb{P}(M_2) \subseteq \mathbb{P}(M),$$

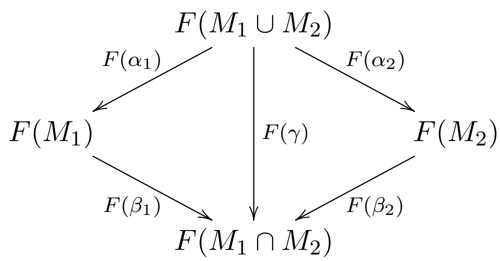
and

$$\mathbb{P}(M_1 \cup M_2) = \mathbb{P}(M_1) \cup \mathbb{P}(M_2) \subseteq \mathbb{P}(M).$$

Proof. We have the colimit diagram for $M_1 \cup M_2$:



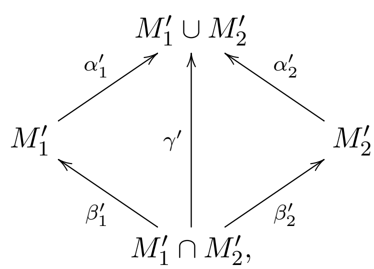
and the associated limit diagram for $F(M_1 \cup M_2)$:



by Theorem 1.3.2. To prove the theorem, we will show that

$$\ker F(\alpha_1) \cap \ker F(\alpha_2) = 0 \text{ and } \ker F(\alpha_1) + \ker F(\alpha_2) = \ker F(\gamma).$$

Note first that $F(M_1 \cup M_2)$ factors through $F(M_1 \cup M_2)/\ker F(\alpha_1) \cap F(\alpha_2)$ in the diagram, forcing $\ker F(\alpha_1) \cap F(\alpha_2)$ to be 0.
Let the maximal dimension of M_1 and M_2 be n. Let $M_1', M_2', (M_1 \cap M_2)'$ be the Δ-sets arising from M_1, M_2 and $M_1 \cap M_2$ by removing the n-dimensional faces. Note that $(M_1 \cup M_2)' = M_1' \cup M_2'$ is the colimit in the diagram



where the arrows are induced. Write $F(M_1 \cup M_2)$, $F(M_1)$, $F(M_2)$ and $F(M_1 \cup M_2)$ in the following way:

$F(M_1 \cup M_2) =$
$$\{((a^j), b) \in \prod_{j \in J} F(\Delta^j_{M_1 \cup M_2}) \times F(M_1' \cup M_2') | F(\tau^{j,i}_{M_1 \cup M_2})(a^j) = F(g^{j,i}_{M_1 \cup M_2})(b)\},$$

$$F(M_1) = \{((x^j), z) \in \prod_{j \in J_1} F(\Delta^j_{M_1}) \times F(M_1') | F(\tau^{j,i}_{M_1})(x^j) = F(g^{j,i}_{M_1})(z)\},$$

$$F(M_2) = \{((y^j), w) \in \prod_{j \in J_2} F(\Delta^j_{M_2}) \times F(M_2') | F(\tau^{j,i}_{M_2})(y^j) = F(g^{j,i}_{M_1})(w)\},$$

$F(M_1 \cap M_2) =$
$$\{((a^j), b) \in \prod_{j \in J_1 \cap J_2} F(\Delta^j_{M_1 \cap M_2}) \times F(M_1' \cap M_2') | F(\tau^{j,i}_{M_1 \cap M_2})(a^j) = F(g^{j,i}_{M_1 \cap M_2})(b)\},$$

where $J = J_{M_1 \cup M_2}$, $J_1 = J_{M_1}$, $J_2 = J_{M_2}$ and we treat J_1 and J_2 as subsets of J. Then,

$$\ker F(\alpha_k) = \{((a^j), b) \in \prod_{j \in J - J_k} F(\Delta^j_{M_1 \cup M_2}) \times \prod_{j \in J_k} 0 \times \ker F(\alpha'_k) | \\ F(\tau^{j,i}_{M_1 \cup M_2})(a^j) = F(g^{j,i}_{M_1 \cup M_2})(b)\}.$$

We proceed by induction on the dimension of M (the initial case is trivial).

$$\begin{aligned} &\ker F(\alpha_1) + \ker F(\alpha_2) \\ &= \{((a^j), b) \in \prod_{j \in J - J_1 \cap J_2} F(\Delta^j_{M_1 \cup M_2}) \times \prod_{j \in J_1 \cap J_2} 0 \times (\ker F(\alpha'_1) + \ker F(\alpha'_2)) | \\ &\qquad F(\tau^{j,i}_{M_1 \cup M_2})(a^j) = F(g^{j,i}_{M_1 \cup M_2})(b)\} \\ &= \{((a^j), b) \in \prod_{j \in J - J_1 \cap J_2} F(\Delta^j_{M_1 \cup M_2}) \times \prod_{j \in J_1 \cap J_2} 0 \times \ker F(\gamma') | \\ &\qquad F(\tau^{j,i}_{M_1 \cup M_2})(a^j) = F(g^{j,i}_{M_1 \cup M_2})(b)\} \\ &= \ker F(\gamma), \end{aligned}$$

and we are done. □

Remark 2.2.3. It is easily seen that the above argument can be applied for arbitrary unions and intersections, even infinite ones.

Lemma 2.2.4. *If $\phi : M \to N$ is a surjective map of Δ-sets, then the map $F(\phi) : F(N) \to F(M)$ is injective.*

Proof. Consider the Δ-set diagram $H_N : I_N \to \Delta$-sets to N from Theorem 1.2.2. $F \circ H_N$ is a diagram of which $F(N)$ is the limit. $F(N)$ can be written as a subset of $\prod_{j \in I_N} F(\Delta^j_N)$. The same applies for the diagram $H_M : I_M \to \Delta$-sets for M. $\phi : M \to N$ being surjective means that the induced map $\psi : I_M \to I_N$ is surjective. Consider the commutative diagram

$$\begin{array}{ccc} F(N) & \longrightarrow & \prod_{k \in I_N} F(\Delta^k_N) \\ \downarrow & & \downarrow \\ F(M) & \longrightarrow & \prod_{j \in I_M} F(\Delta^j_M) \end{array}$$

where the vertical map to the right is generated by the isomorphisms $F(\Delta^{\psi(j)}_N) \to F(\Delta^j_M)$. This means that this map is injective. Since both horizontal maps are injections too, we have that the vertical map to the left, $F(\phi) : F(N) \to F(M)$, is injective as well. □

Lemma 2.2.5. *If M is a Δ-set with a single maximal face f, then $F(M)$ is an integral domain. Furthermore, if $\phi : M \to N$ is a map between Δ-sets with single maximal faces f_M and f_N such that $\phi(f_M) = f_N$, then the injective map $F(\phi) : F(N) \to F(M)$ makes $F(M)$ integral over $F(N)$.*

Proof. Let M be of dimension n. The unique map $\Delta^n \to M$ where Δ^n is an n-dimensional Δ-simplex is surjective. This means that $F(M)$ may be written as a subset of $R = F(\Delta^n) = k[x_0, \ldots, x_n]$ by Theorem 2.2.4, so $F(M)$ is an integral domain.

Now, consider a map $\phi : M \to N$ as described above. Since $f_M \mapsto f_N$, M and N must be of equal dimension, say n. By Theorem 2.2.4 again, we may write $F(N) \subseteq F(M) \subseteq R$. Let $H = H_N : I_N \to \Delta$-sets be the functor from Theorem 1.2.2. Using the diagram $F \circ H$, we know that

$$F(N) = \{f \in R | F(H(g))(f) = F(H(h))(f)\ \forall g, h : r \to f_N\},$$

where r is any element in I_N. Let f be a symmetrical polynomial in R. Consider any two morphisms $g, h : r \to f_N$. We have that the homomorphism $F(H(g)) : R \to F(H(r))$ is defined by $x_{H(g)_0(i)} \mapsto x_i$, and $x_j \mapsto 0$ if $j \notin \text{im } H(g)_0$. $F(H(h))$ is defined similarly. Note that all maps in the diagram H are injective. This means in particular that $H(g)$ and $H(h)$ are injective, which means that the cardinalities of im $H(g)_0$ and im $H(h)_0$ are the same. This means that there exists a permutation σ of $\Delta^n_0 = \{0, 1, 2, \ldots, n\}$ such that $\sigma \circ H(g)_0 = H(h)_0$. Thus $F(H(h))$ is defined by $x_{\sigma(H(g)_0(i))} \to x_i$, and $x_{\sigma(j)} \to 0$ if $j \notin \text{im } H(g)_0$. f being symmetrical therefore implies that $F(H(g))$ and $F(H(h))$ sends f to the same element in $F(H(r))$. Thus the symmetric polynomials of R are contained in $F(N)$.

The identity

$$\prod_{j=0}^{n}(\gamma - x_j) = \gamma^{n+1} - \gamma^n e_1 + \gamma^{n-1} e_2 + \ldots + (-1)^{n+1} e_{n+1},$$

where $e_i \in R$ are the elementary symmetric polynomials (Example 1.1 [Eis95]) is easily obtained. By substituting γ with x_i in this formula we have shown that x_i are integral elements over $F(N)$ for any i, and thus that $k[x_0, \ldots, x_n]$ and in particular $F(M)$ is integral over $F(N)$ (Corollary 5.3 [AM69]). □

Theorem 2.2.6. *If $\phi : M \to N$ is a surjective map of Δ-sets, then the morhism $\mathbb{P}(\phi) : \mathbb{P}(M) \to \mathbb{P}(N)$ is surjective.*

Proof. For any maximal face $f_M^j \in M_n$, for some n, let $M^j \subseteq M$ be the Δ-set with the single maximal face $f_M^j \in M_n^j$. Note that $M = \bigcup M^j$. Thus $\mathbb{P}(M) = \bigcup_j \mathbb{P}(M^j)$ by Theorem 2.2.2 and Remark 2.2.3. If $p \in \mathbb{P}(M)$, then $p \in \mathbb{P}(M^j)$ for some j. Thus if $F(g_M^j) : F(M) \to F(M^j)$ is the surjective map corresponding to the injection $g_M^j : M^j \to M$, $p = F(g_M^j)^{-1}(q)$ for some prime $q \in \mathbb{P}(M^j)$. We may similarly define $N^{\psi(j)} \subseteq N$, for the surjective map $\psi : J_M \to J_N$. Consider the surjective map $\phi^j : M^j \to N^{\psi(j)}$. This induces an injective map $F(\phi^j) : F(N^{\psi(j)}) \to F(M^j)$ by Lemma 2.2.4.

Now, suppose $p \supseteq F(\phi)(F(N)_+)$. Then $q \supseteq F(\phi^j)(F(N^{\psi(j)})_+)$. By Lemma 2.2.5 $F(N^{\psi(j)})$ and $F(M^j)$ are integral domains, and $F(M^j)$ is integral over $F(N^{\psi(j)})$. So for any homogeneous element $x \in F(M^j)_+$, we have

$$x^n + a_{n-1}x^{n-1} + ... + a_0 = 0$$

for some homogeneous $a_i \in F(N^{\psi(j)})_+$. Thus $x^n \in (F(\phi^j)(F(N^{\psi(j)})_+)) \subseteq q$. So $x \in q$, and therefore $q \supseteq F(M^j)_+$, which is impossible. Hence we conclude that $p \not\supseteq F(\phi)(F(N)_+)$, and that the morphism $\mathbb{P}(\phi) : \mathbb{P}(M) \to \mathbb{P}(N)$ is well-defined.

Also, the integral extension satisfies the lying-over property (Theorem 5.10 [AM69]). So any prime ideal of $F(M^j)$ lies under some prime of $F(N^{\psi(j)})$, which means that $\mathbb{P}(M^j) \to \mathbb{P}(N^{\psi(j)})$ is surjective and therefore $\mathbb{P}(M) \to \mathbb{P}(N)$ as well. □

Theorem 2.2.7. *If M is a Δ-set, then $\mathbb{P}(M)$ is a reduced scheme.*

Proof. We have the open cover of $D_+(f) = \operatorname{Spec} F(M)_{(f)}$ for homogeneous $f \in F(M)_+$. $F(M)$ has no nilpotents by Theorem 1.3.6, so $F(M)_{(f)}$ has none either. It follows that $\mathbb{P}(M)$ is reduced. □

Lemma 2.2.8. *If $\phi : M \to N$ is a Δ-set map, then for any Δ-set $Q \subseteq M$, $\phi(Q) \subseteq N$ is a Δ-set.*

Proof. Consider $\phi(Q)_n \subseteq N_n$ for any n. Note that $f \in \phi(Q)_n$ if and only if there exists an $h \in Q_n$ such that $\phi_n(h) = f$. Hence $\phi(Q)_n = \phi_n(Q_n)$. We need to show that for any $f \in \phi_n(Q_n)$, $d_i(f) \in \phi_{n-1}(Q_{n-1})$. But

$$d_i(f) = d_i(\phi_n(h)) = \phi_{n-1}(d_i(h)) \in \phi_{n-1}(Q_{n-1}),$$

which completes the proof. □

Corollary 2.2.9. *If $\phi : M \to N$ is a Δ-set map, then for any Δ-set $Q \subseteq M$, $\mathbb{P}(\phi)(\mathbb{P}(Q)) = \mathbb{P}(\phi(Q))$.*

Proof. ϕ restricts to a surjective map $Q \to \phi(Q)$, so $\mathbb{P}(\phi)$ restricts to a surjective morphism $\mathbb{P}(Q) \to \mathbb{P}(\phi(Q))$. Thus $\mathbb{P}(\phi)(\mathbb{P}(Q)) = \mathbb{P}(\phi(Q))$. □

We are now ready to conclude that $\mathbb{P} : \Delta\text{-sets} \to \mathrm{Sch}_k$ indeed is a functor.

Corollary 2.2.10. *If $\phi : M \to N$ is a Δ-set map, $\mathbb{P}(\phi) : \mathbb{P}(M) \to \mathbb{P}(N)$ is a well-defined morphism.*

Proof. Since ϕ factors as a surjective map followed by an injective map as $M \to \phi(M) \to N$, we have induced a well-defined morphism of Δ-face schemes $\mathbb{P}(\phi) : \mathbb{P}(M) \to \mathbb{P}(N)$. □

Lemma 2.2.11. *If $\phi : M \to N$ is a Δ-set map, then for any Δ-set $Q \subseteq N$, $\phi^{-1}(Q) \subseteq M$ is a Δ-set.*

Proof. Consider $\phi^{-1}(Q)_n \subseteq M_n$ for any n. Note that $f \in \phi^{-1}(Q)_n$ if and only if $\phi_n(f) \subseteq Q_n$. Hence $\phi^{-1}(Q)_n = \phi_n^{-1}(Q_n)$. We need to show that for any $f \in \phi_n^{-1}(Q_n)$ for some n, $d_i(f) \in \phi_{n-1}^{-1}(Q_{n-1})$. But we know that $\phi_{n-1}(d_i(f)) = d_i(\phi_n(f)) \in Q_{n-1}$, which completes the proof. □

Corollary 2.2.12. *If $\phi : M \to N$ is a Δ-set map, then for any Δ-set $Q \subseteq N$, $\mathbb{P}(\phi)^{-1}(\mathbb{P}(Q)) = \mathbb{P}(\phi^{-1}(Q)) \subseteq \mathbb{P}(M)$.*

Proof. ϕ restricts to a map $\phi^{-1}(Q) \to Q$, so $\mathbb{P}(\phi)$ restricts to a morphism $\mathbb{P}(\phi^{-1}(Q)) \to \mathbb{P}(Q)$. Thus $\mathbb{P}(\phi)^{-1}(\mathbb{P}(Q)) = \mathbb{P}(\phi^{-1}(Q))$. □

It is possible that the next theorem may follow from more general results in category theory, considering that $F(M)$ is a finite limit of finitely generated k-algebras. I have not been able to find it in literature, so I will provide a proof of it here.

Theorem 2.2.13. *If M is a finite Δ-set, then $F(M)$ is a finitely generated k-algebra.*

Proof. We proceed by induction on the dimension of M. The initial case is trivial. Let M be n-dimensional. Then we may assume $F(M')$ is finitely generated as a k-algebra. Let $\mu : k[d_1, \ldots, d_m] \to F(M')$ be a surjective k-algebra homomorphism. Consider the map $F(\tau) : F(M) \to F(M')$ which is surjective by Corollary 1.3.10. Choose liftings $\tilde{c_i} \in F(M)$ of $c_i = \mu(d_i)$ in $F(M')$. The k-module splitting $\sigma : F(M') \to F(M)$ of $F(\tau)$ allows us to write $F(M) = F(M') \oplus \ker F(\tau)$ as k-modules. So we may assume $\tilde{c_i} \in F(M') \subseteq F(M)$. Now, by looking at the diagram $F\{M\}$ it is clear that $\ker F(\tau) = \prod_{j \in J_M} (x_0^j \cdots x_n^j)$, where $(x_0^j \cdots x_n^j) \subseteq k[x_0^j, \ldots, x_n^j] = F(\Delta^j)$ is the ideal considered a $F(M)$-module. This is because any polynomial in $F(\Delta^j)$ in $\ker F(\tau)$ must be sent to 0 in $F(\Delta^{j,i})$, and anything in the direct product of these ideals is clearly present in the limit.

Recall from the proof of Theorem 2.2.6 that the symmetric polynomials and in particular e_i^j is in $F(M^j)$ for every j where e_i^j are the elementary symmetric polynomials in $k[x_0^j, \ldots, x_n^j]$. By Corollary 1.3.10, $F(M) \to F(M^j)$ is surjective, and we may choose liftings of e_i^j to $\widetilde{e_i^j} \in F(M)$. Thus

$$x_0^j \cdots x_n^j \widetilde{e_i^j} = x_0^j \cdots x_n^j e_i^j$$

in $F(M)$ by moving to $F(M^j) \subseteq k[x_0^j, \ldots, x_n^j]$.

Now, consider the k-algebra map

$$\phi : k[d_1, \ldots, d_m, \{y_j^{a_0 \ldots a_n}\}, \{z_i^j\}] \to F(M)$$

for $j \in J_M$, $1 \leq i \leq n+1$, $a_k \geq 1$, and $\sum_k a_k \leq (n+1)(n+2)$, such that $d_i \mapsto \tilde{c}_i$, $y_j^{a_0 \ldots a_n} \mapsto (x_0^j)^{a_0} \cdots (x_n^j)^{a_n}$ and $z_i^j \mapsto \widetilde{e_i^j}$. ϕ is clearly surjective on the $F(M')$-part of $F(M)$. Pick any monomial of the form $(x_0^j)^{b_0} \cdots (x_n^j)^{b_n}$ in $F(M)$ where $b_i \geq 1$. We will show by induction on the degree that this is in the image of ϕ. Assume that they are in the image of ϕ for degrees $\leq m$. We may assume $m \geq (n+1)(n+2)$. Let $(x_0^j)^{b_0} \cdots (x_n^j)^{b_n}$ have degree $m+1$. Recall the equations

$$(x_i^j)^{n+1} - (x_i^j)^n e_1^j + (x_i^j)^{n-1} e_2^j + \ldots + (-1)^{n+1} e_{n+1}^j = 0$$

from the proof of Theorem 2.2.6. Since $\sum b_k > (n+1)(n+2)$, we have $b_k > n+2$ for some k. Thus

$$\begin{aligned}
&(x_0^j)^{b_0} \cdots (x_n^j)^{b_n} \\
&= (x_0^j)^{b_0} \cdots (x_k^j)^{b_k-(n+1)} \cdots (x_n^j)^{b_n} ((x_k^j)^n e_1^j - (x_k^j)^{n-1} e_2^j + \ldots + (-1)^n e_{n+1}^j) \\
&= (x_0^j)^{b_0} \cdots (x_k^j)^{b_k-(n+1)} \cdots (x_n^j)^{b_n} ((x_k^j)^n \widetilde{e_1^j} - (x_k^j)^{n-1} \widetilde{e_2^j} + \ldots + (-1)^n \widetilde{e_{n+1}^j}).
\end{aligned}$$

So we only need to prove that $(x_0^j)^{b_0} \cdots (x_k^j)^{b_k-r} \cdots (x_n^j)^{b_n} e_i^j$ is in the image of ϕ for $r = 1, \ldots, n+1$. However this follows from our assumption. We conclude that ϕ is surjective, and therefore that $F(M)$ is a finitely generated k-algebra. □

Corollary 2.2.14. *If M is a finite Δ-set, then $\mathbb{P}(M)$ is a scheme of finite type over k.*

Proof. $\mathbb{P}(M)$ may be covered by a finite number of open affine subschemes $\operatorname{Spec} F(M)_{(f)}$ for homogeneous $f \in F(M)_+$. By Lemma 10.53.9 in [Sta], $F(M)_{(f)}$ is a finitely generated k-algebra. Thus $\mathbb{P}(M)$ is of finite type over k. $\square$

In the following theorem, the restrictive nature of Δ-set maps is crucial. We do not have a corresponding result for the category of ordered simplicial complexes.

Theorem 2.2.15. *If $\phi : M \to N$ is a surjective Δ-set map of finite Δ-sets, then the map $F(\phi) : F(N) \to F(M)$ makes $F(M)$ a finitely generated $F(N)$-module.*

Proof. We proceed by induction on the dimension of M. The initial case is trivial. Let M be n-dimensional. Then N is n-dimensional. The map $\phi' : M' \to N'$ where we remove the n-dimensional faces is induced. By the inductive hypothesis $F(M')$ is a finitely generated $F(N')$-module. Consider the commutative diagram

$$\begin{array}{ccc} F(N) & \xrightarrow{F(\phi)} & F(M) \\ {\scriptstyle F(\tau_N)}\downarrow & & \downarrow{\scriptstyle F(\tau_M)} \\ F(N') & \xrightarrow{F(\phi')} & F(M'). \end{array}$$

Since $F(\tau_N) : F(N) \to F(N')$ is surjective by Theorem 1.3.10, $F(M')$ is a finitely generated $F(N)$-module as well. Let $c_1, \ldots, c_m$ be generators for $F(M')$ over $F(N)$ and $d_i = \tilde{c}_i$ a lifting to an element in $F(M)$ such that $d_i \in F(M') \subseteq F(M') \oplus \ker F(\tau_M) = F(M)$ as in the proof of Theorem 2.2.13. Consider the additional generator elements

$$y_j^{a_0 \ldots a_n} = (x_0^j)^{a_0} \cdots (x_n^j)^{a_n} \text{ for } j \in J_M, a_k \geq 1, \sum_k a_k \leq (n+1)(n+2).$$

We will show that the d_i's and the $y_j^{a_0 \ldots a_n}$'s generate $F(M)$ over $F(N)$. The d_i's generate $F(M')$, so we may restrict our attention to the sub-k-module $\ker F(\tau_M) = \prod_{j \in J_M} (x_0^j \cdots x_n^j)$. Choose any monomial $(x_0^j)^{b_0} \cdots (x_n^j)^{b_n}$ in $F(M)$ where $b_k \geq 1$. We will show by induction degree that these monomials are generated. Assume they are for degrees $\leq m$. We may assume that $m \geq (n+1)(n+2)$. Let $(x_0^j)^{b_0} \cdots (x_n^j)^{b_n}$ have degree $m+1$. Then $b_k > n+2$ for some k. As before, we have the equation

$$\begin{aligned} &(x_0^j)^{b_0} \cdots (x_n^j)^{b_n} \\ &= (x_0^j)^{b_0} \cdots (x_k^j)^{b_k-(n+1)} \cdots (x_n^j)^{b_n} ((x_k^j)^n e_1^j - (x_k^j)^{n-1} e_2^j + \ldots + (-1)^n e_{n+1}^j). \end{aligned}$$

By the inductive hypothesis, if we can show that

$$(x_0^j)^{b_0}\cdots(x_k^j)^{b_k-r}\cdots(x_n^j)^{b_n}e_i^j=(x_0^j)^{b_0}\cdots(x_k^j)^{b_k-r}\cdots(x_n^j)^{b_n}F(\phi)(e_i^{j\prime})$$

for some $e_i^{j\prime}\in F(N)$ and for $r=1,\ldots n+1$ we are done.

Recall the surjective index set map $\psi:J_M\to J_N$. Let $e_i^{\psi(j)}$ be the i-th elementary symmetric polynomial in $k[x_0^{\psi(j)},\ldots,x_n^{\psi(j)}]=F(\Delta_N^{\psi(j)})$. It exists in $F(N^{\psi(j)})$ as in the proof of Theorem 2.2.13, and similary as before lifts to $e_i^{\widetilde{\psi(j)}}$ in $F(N)$. Recall the construction of $F(\phi)$. We easily see that

$$(x_0^j)^{b_0}\cdots(x_k^j)^{b_k-r}\cdots(x_n^j)^{b_n}F(\phi)(e_i^{\widetilde{\psi(j)}})=(x_0^j)^{b_0}\cdots(x_k^j)^{b_k-r}\cdots(x_n^j)^{b_n}e_i^j$$

because $(x_0^j)^{b_0}\cdots(x_k^j)^{b_k-r}\cdots(x_n^j)^{b_n}F(\phi)(e_i^{\widetilde{\psi(j)}})\subseteq(x_0^j\cdots x_n^j)\subseteq\ker F(\tau)$ and therefore determined by what $F(\phi)$ sends to $F(\Delta_M^j)$, i.e. what the map $F(\phi^j):F(\Delta_N^{\psi(j)})\to F(\Delta_M^j)$ sends to $F(\Delta_M^j)$. And since $F(\phi^j)(e_i^{\psi(j)})=e_i^j$, we are done. □

Lemma 2.2.16. *If $\phi:M\to N$ is a Δ-set map, then the inverse image $\mathbb{P}(\phi)^{-1}(D_+(f))=D_+(F(\phi)(f))$ for any homogeneous $f\in F(N)_+$.*

Proof. Let $f\in F(N)_+$ be homogeneous, and $p\in\mathbb{P}(\phi)^{-1}(D_+(f))$. This means that $\mathbb{P}(\phi)(p)=F(\phi)^{-1}(p)\in D_+(f)$, i.e. $f\notin F(\phi)^{-1}(p)$. This means that $F(\phi)(f)\notin F(\phi)(F(\phi)^{-1}(p))\subseteq p$. Hence $p\in D_+(F(\phi)(f))$. Conversely, if $p\in D_+(F(\phi)(f))$, then $F(\phi)(f)\notin p$. If $f\in F(\phi)^{-1}(p)$, then we have that $F(\phi)(f)\in F(\phi)(F(\phi)^{-1}(p))\subseteq p$, which is impossible. This implies that $f\notin F(\phi)^{-1}(p)$, which means that $\mathbb{P}(\phi)(p)\in D_+(f)$. Thus we may conclude that $\mathbb{P}(\phi)^{-1}(D_+(f))=D_+(F(\phi)(f))$. □

Theorem 2.2.17. *If $\phi:M\to N$ is a surjective map of Δ-sets, then the sheaf morphism $\mathbb{P}(\phi)^\sharp:\mathcal{O}_{\mathbb{P}(N)}\to\mathbb{P}(\phi)_*\mathcal{O}_{\mathbb{P}(M)}$ is injective.*

Proof. We know that the map $F(\phi):F(N)\to F(M)$ is injective by Lemma 2.2.4, so for any homogenous $f\in F(N)_+$, $F(\phi)_{(f)}:F(N)_{(f)}\to F(M)_{(F(\phi)(f))}$ is injective too. Thus by Lemma 2.2.16

$$\mathbb{P}(\phi)^\sharp(D_+(f)):\mathcal{O}_{\mathbb{P}(N)}(D_+(f))\to\mathcal{O}_{\mathbb{P}(M)}(D_+(F(\phi)(f)))=\mathbb{P}(\phi)_*\mathcal{O}_{\mathbb{P}(M)}(D_+(f))$$

is injective as well. This means that $\ker\mathbb{P}(\phi)^\sharp$ is 0 on an affine cover of $\mathbb{P}(N)$, which means that $\mathbb{P}(\phi)^\sharp$ is an injective sheaf morphism. □

Theorem 2.2.18. *If $\phi:M\to N$ is a surjective Δ-set map of finite Δ-sets, then $\mathbb{P}(\phi):\mathbb{P}(M)\to\mathbb{P}(N)$ is a finite morphism.*

Proof. Consider any distinguished open set $D_+(f) \subseteq \mathbb{P}(N)$ for some homogeneous $f \in F(N)_+$. For the affine open cover of distinguished open sets $D_+(f) = \operatorname{Spec} F(N)_{(f)}$, we have $\mathbb{P}(\phi)^{-1}(D_+(f)) = \operatorname{Spec} F(M)_{(F(\phi)(f))}$ by Lemma 2.2.16. We need to show that the induced homomorphism $F(\phi)_{(f)} : F(N)_{(f)} \to F(M)_{(F(\phi)(f))}$ makes $F(M)_{(F(\phi)(f))}$ a finitely generated $F(N)_{(f)}$-module. $F(M)$ is a finitely generated $F(N)$-module by Theorem 2.2.13, and both $F(N)$ and $F(M)$ are finitely generated k-algebras by Theorem 2.2.15.

Now, let $A = F(N)$, $B = F(M)$, and $F(\phi)(f) = f'$. Let A be generated by homogeneous elements $y_1, \dots, y_r$ of positive degrees as a k-algebra, and B generated by homogeneous elements $x_1, \dots, x_n$ as an A-module. Let T be a positive integer. Let f have degree d. Consider the $A_{(f)}$-module

$$C_T = A_{(f)} \cdot \frac{B_{dT}}{(f')^T} \subseteq B_{(f')}.$$

Let m be the smallest integer for which there is an element $\frac{b}{(f')^m} \notin C_T$ for some $b \in B$ with degree dm. If $m \leq T$, then

$$\frac{b}{(f')^m} = \frac{b(f')^{T-m}}{(f')^T} \in C_T.$$

Thus $m > T$. Write $b = \sum_i^n a_i x_i$, for $a_i \in A$. a_i has degree

$$dm - \deg x_i \geq dT - \max_i\{\deg x_i\}.$$

Now, write $a_i = h_i(y_1, \dots, y_r)$ for some polynomial h_i. We may write this as

$$a_i = \sum d_i^{c_1 \dots c_r} {y_1}^{c_1} \cdots {y_r}^{c_r}$$

where $\sum_i^r c_i \deg y_i = \deg a_i$, and $d_i^{c_1 \dots c_r} \in k$. Choose T sufficiently large such that $T \geq \max_i\{\deg y_i\}$ and there is at least one $c_j \geq d$ for any sequence $(c_1, \dots, c_r)$ in the sum above. Then we may write

$$d_i^{c_1 \dots c_r} {y_1}^{c_1} \cdots {y_r}^{c_r} = d_i^{c_1 \dots c_r} {y_1}^{c_1} \cdots {y_j}^{c_j - d} \cdots {y_r}^{c_r} \cdot {y_j}^d,$$

and

$$\frac{d_i^{c_1 \dots c_r} {y_1}^{c_1} \cdots {y_r}^{c_r}}{(f')^m} = \frac{{y_j}^d}{f^{\deg y_j}} \cdot \frac{x_i d_i^{c_1 \dots c_r} {y_1}^{c_1} \cdots {y_j}^{c_j - d} \cdots {y_r}^{c_r}}{(f')^{m - \deg y_j}},$$

where $\frac{{y_j}^d}{f^{\deg y_j}} \in A_{(f)}$ and $\frac{x_i d_i^{c_1 \dots c_r} y_1^{c_1} \cdots {y_j}^{c_j - d} \cdots {y_r}^{c_r}}{(f')^{m - \deg y_j}} \in C_T$ by the hypothesis on m. Summing over all such sequences $(c_1, \dots, c_r)$, and then for all i, we conclude that $\frac{b}{(f')^m} \in C_T$ for T chosen sufficiently large, and thus that $C_T = B_{(f')}$.

Now, B_{dT} is a finite dimensional vector space over k since B is a finitely generated k-algebra of homogeneous elements of positive degree. Thus $B_{(f')}$ is finitely generated over $A_{(f)}$. We conclude that $\mathbb{P}(\phi)$ is a finite morphism. □

Corollary 2.2.19. *If $\phi : M \to N$ is a Δ-set map of Δ-sets where M is finite, then $\mathbb{P}(\phi) : \mathbb{P}(M) \to \mathbb{P}(N)$ is a finite morphism.*

Proof. ϕ is the composition of the maps $M \to \phi(M) \to N$, so $\mathbb{P}(\phi)$ is the composition of the maps $\mathbb{P}(M) \to \mathbb{P}(\phi(M)) \to \mathbb{P}(N)$ where the last one is a closed immersion, hence finite. The first one is finite since $M \to \phi(M)$ is surjective by Theorem 2.2.18. Thus the composition $\mathbb{P}(\phi)$ is finite as well. □

Theorem 2.2.20. *If M is an n-dimensional Δ-set with a single maximal face, then $\mathbb{P}(M)$ is n-dimensional and irreducible.*

Proof. By Lemma 2.2.5, $F(M)$ is an integral domain, and by the proof of the lemma we have the inclusion $F(M) \subseteq k[x_0, x_1, \ldots, x_n]$. So $\mathcal{O}_{\mathbb{P}(M)}$ is integral on an affine cover, which means that $\mathbb{P}(M)$ is integral and therefore irreducible. Since $k[x_0, x_1, \ldots, x_n]$ is an integral extension of $F(M)$, it satisfies the incomparability property (Corollary 5.9 [AM69]), which is that if q and q' are distinct prime ideals of $k[x_0, x_1, \ldots, x_n]$ such that $F(M) \cap q = F(M) \cap q'$, then $q \not\subseteq q'$ and $q' \not\subseteq q$. This means that if we pick a chain of $n+1$ prime ideals

$$p_1 \subset p_2 \subset \ldots \subset p_{n+1} \subseteq k[x_0, x_1, \ldots, x_n],$$

we have the corresponding chain of prime ideals

$$p_1 \cap F(M) \subset p_2 \cap F(M) \subset \ldots \subset p_{n+1} \cap F(M) \subseteq F(M).$$

This means that $F(M)$ is at least $n+1$-dimensional. The extension satisfies the lying-over property as well, which means that given any chain of prime ideals

$$q_1 \subset q_2 \subset \ldots \subset q_r \subseteq F(M),$$

there exists a chain of prime ideals

$$p_1 \subseteq p_2 \subseteq \ldots \subseteq p_r \subseteq k[x_0, x_1, \ldots, x_n]$$

for which p_i lies over q_i. Due to the incomparability property, the inclusions are proper. Hence $F(M)$ is maximally $n+1$-dimensional and therefore exactly $n+1$-dimensional, which means that $\mathbb{P}(M)$ is n-dimensional. □

For each maximal face f^j of M, let $M^j \subseteq M$ be the Δ-set with the single maximal face f^j. Then $M = \bigcup_{j \in J_M} M^j$.

Corollary 2.2.21. *The irreducible components of* $\mathbb{P}(M)$ *are* $\mathbb{P}(M^j)$.

Proof. This follows from Theorem 2.2.20 and Theorem 2.2.2. □

Corollary 2.2.22. *If* M *is* n*-dimensional, then* $\mathbb{P}(M)$ *is* n*-dimensional.*

Lemma 2.2.23. *Let* $H : I \to \Delta$*-sets be a finite connected diagram with colimit* M*, with maps* $\phi_X : H(X) \to M$*. For any homogeneous* $g \in F(M)_+$*, define a new diagram* $(F \circ H)_{(g)}$ *by* $(F \circ H)_{(g)}(X) = (F \circ H(X))_{(F(\phi_X)(g))}$*, and if* $F(\phi_X)(g) = 0$*, we define* $(F \circ H)_{(g)}(X) = 0$*, with maps induced from* $F \circ H(X) \to F \circ H(Y)$*. Then* $\varprojlim (F \circ H)_{(g)} = F(M)_{(g)}$*.*

Proof. Pick any homogeneous element $g = (g_X)_{X \in I} \in F(M)_+$. Then we have the diagram

$$\begin{array}{ccc} & F(M)_{(g)} & \\ {}^{F(\phi_X)_{(g)}}\swarrow & & \searrow^{F(\phi_Y)_{(g)}} \\ F \circ H(X)_{(F(\phi_X)(g))} & \xrightarrow{F \circ H(f)_{(g)}} & F \circ H(Y)_{(F(\phi_Y)(g))}. \end{array}$$

This makes $F(M)_{(g)}$ a node to the diagram $(F \circ H)_{(g)}$. Pick any element $(\frac{a_X}{F(\phi_X)(g)^{n_X}})_{X \in I}$ in the limit. If $F(\phi_X)(g) = 0$, we merely treat the element $\frac{a_X}{F(\phi_X)(g)^{n_X}}$ as 0. Then for any $f : Y \to X$,

$$(F \circ H(f))_{(g)}(\frac{a_X}{F(\phi_X)(g)^{n_X}}) = \frac{F(H(f))(a_X)}{F(\phi_Y)(g)^{n_X}} = \frac{a_Y}{F(\phi_Y)(g)^{n_Y}},$$

which means there exists an integer n_f such that

$$F(\phi_Y)(g)^{n_f}(F(H(f))(a_X)F(\phi_Y)(g)^{n_Y} - a_Y F(\phi_Y)(g)^{n_X}) = 0$$

in $F \circ H(Y)$.

We may define $n = \max_{X \in I, f:Y \to X}\{n_X, n_f\}$, and let $a = \frac{(a_X F(\phi_X)(g)^{3n-n_X})_{X \in I}}{((g_X)^{3n})_{X \in I}}$. a is in $F(M)_{(g)}$, since the numerator $(a_X F(\phi_X)(g)^{3n-n_X})_{X \in I}$ is in $F(M)$:

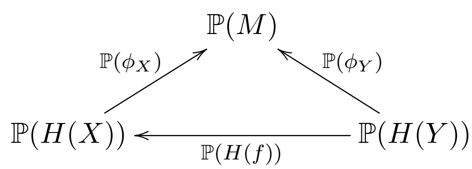
$$\begin{aligned} & F(H(f))(a_X F(\phi_X)(g)^{3n-n_X}) = F(H(f))(a_X)F(\phi_Y)(g)^{3n-n_X} \\ & = F(\phi_Y)(g)^{3n-n_X-n_Y-n_f}(F(\phi_Y)(g)^{n_f}(F(H(f))(a_X)F(\phi_Y)(g)^{n_Y})) \\ & = F(\phi_Y)(g)^{3n-n_X-n_Y-n_f}(F(\phi_Y)(g)^{n_f}(a_Y F(\phi_Y)(g)^{n_X})) \\ & = a_Y F(\phi_Y)(g)^{3n-n_Y}. \end{aligned}$$

$F(\phi_X)$ clearly sends a to $\frac{a_X}{F(\phi_X(g))^{n_X}}$. This gives us a well-defined map from the limit to $F(M)_{(g)}$ which factors the maps from the limit to $F(H(X))_{(F(\phi_X)(g))}$ through $F(M)_{(g)}$, making $F(M)_{(g)}$ isomorphic to the limit by the universal property. $\square$

The following proof will mimic the proof of Theorem 1.3.2 at some points.

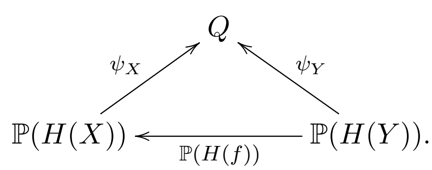
Theorem 2.2.24. *If a Δ-set $M = \varinjlim H$ is the colimit of a finite connected diagram $H : I \to \Delta$-sets, then $\mathbb{P}(M) = \varinjlim \mathbb{P} \circ H$.*

Proof. M, with maps $\phi_X : H(X) \to M$, being a node to H makes $\mathbb{P}(M)$ a node to $\mathbb{P} \circ H$, and we have the commutative diagram:



for every $f : X \to Y$.

Now, let Q be a node to the diagram $\mathbb{P} \circ H$:



We proceed by induction on the dimension of M. The initial case is trivial. Let M be n-dimensional. Recall the definition of the diagram H', which has colimit M', the maps $\iota_X : H'(X) \to H(X)$, and the construction of the maps $\pi_X^j : \Delta^j \to H(X)$ for each $f^j \in M_n$ with X such that there is an element $f_X^j \in H(X)_n$ such that $(\phi_X)_n(f_X^j) = f^j$ from the proof of Theorem 1.3.2. This gives us maps

$$\mathbb{P}(\Delta^j) \to^{\mathbb{P}(\pi_X^j)} \mathbb{P}(H(X)) \to^{\psi_X} Q.$$

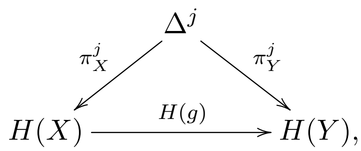
We will show that the composition $\gamma^j = \psi_X \circ \mathbb{P}(\pi_X^j)$ is independent of X. Given any $Y \in I$, with $f_Y^j \in H(Y)$, and π_Y^j defined as before, we need $\psi_X \circ \mathbb{P}(\pi_X^j) = \psi_Y \circ \mathbb{P}(\pi_Y^j)$. By the set-theoretic construction of the colimit, we have that since $f_X^j \sim f_Y^j$ in M_n, there is a finite sequence

$$f_X^j = f_0 \sim f_1 \sim \ldots \sim f_r = f_Y^j,$$

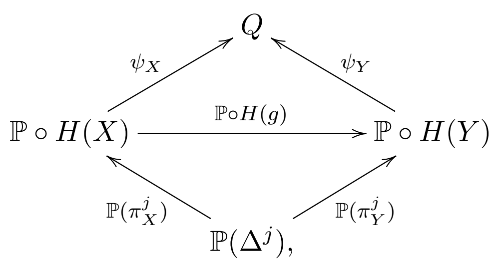
where $f_i \in H(X_i)$, $X_0 = X$, $X_r = Y$. In addition, there exists a morphism $g_i : X_i \to X_{i+1}$ such that $H(g_i)(f_i) = f_{i+1}$ or a morphism $h_i : X_{i+1} \to X_i$ such that $H(h_i)(f_{i+1}) = f_i$. We proceed by induction on r. If $r = 0$, there is nothing to show. For the inductive step, we may assume that

$$\psi_X \circ \mathbb{P}(\pi^j_X) = \psi_{X_{r-1}} \circ \mathbb{P}(\pi^j_{X_{r-1}}),$$

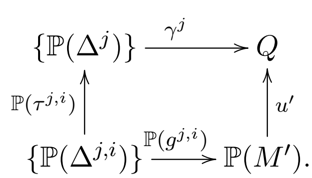
so it remains to show that $\psi_{X_{r-1}} \circ \mathbb{P}(\pi^j_{X_{r-1}}) = \psi_{X_r} \circ \mathbb{P}(\pi^j_{X_r})$, so we reduce the proof of the inductive step to the proof of the statement where $r = 1$. Without loss of generality, we may assume that we have a morphism $g : X \to Y$ such that $H(g)(f^j_X) = f^j_Y$. So we have the following commutative diagram:



and thus the commutative diagram



which proves that γ^j is well-defined, and independent of our choice of f^j_X. Q being a node to $\mathbb{P} \circ H$, makes it a node to $\mathbb{P} \circ H'$ as well with the unique morphism $u' : \mathbb{P}(M') \to Q$. This gives us a map from the diagram $\mathbb{P}\{M\}$ to Q:



We need to show that this commutes, making Q into a node.

Recall the induced maps $\pi^{j,i}_X : \Delta^{j,i} \to H'(X)$, and let $\psi'_X = \psi_X \circ \mathbb{P}(\iota_X)$. It remains to prove that

$$u' \circ \mathbb{P}(g^{j,i}) = \gamma^j \circ \mathbb{P}(\tau^{j,i}).$$

Pick an X for which $\gamma^j = \psi_X \circ \mathbb{P}(\pi_X^j)$. We need to show the equality

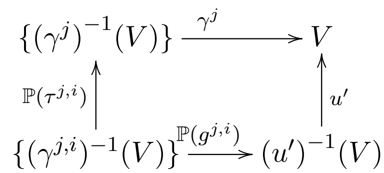
$$u' \circ \mathbb{P}(g^{j,i}) = \psi_X \circ \pi_X^j \circ \mathbb{P}(\tau^{j,i}).$$

We have that

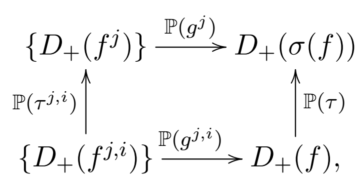
$$\begin{aligned}\psi_X \circ \pi_X^j \circ \mathbb{P}(\tau^{j,i}) &= \psi_X \circ \mathbb{P}(\iota_X) \circ \mathbb{P}(\pi_X^{j,i}) \\ &= \psi'_X \circ \mathbb{P}(\pi_X^{j,i}) \\ &= u' \circ \mathbb{P}(\phi'_X) \circ \mathbb{P}(\pi_X^{j,i}).\end{aligned}$$

But $\phi'_X \circ \pi_X^{j,i} = g^{j,i}$, so $\mathbb{P}(\phi'_X) \circ \mathbb{P}(\pi_X^{j,i}) = \mathbb{P}(g^{j,i})$, so the equality holds.

Now, pick an open affine $V = \operatorname{Spec} B \subseteq Q$. Taking inverse images, we get the diagram

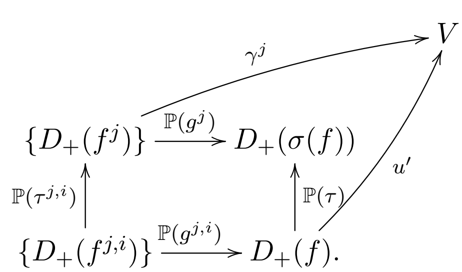


where $\gamma^{j,i} = \gamma^j \circ \mathbb{P}(\tau^{j,i})$. Pick some homogenous $f \in F(M')_+$ such that the distinguished open set $D_+(f) \subseteq (u')^{-1}(V)$. Let $f^j = F(g^j)(\sigma(f))$, where σ is the splitting map from Theorem 1.3.8, and $f^{j,i} = F(\tau^{j,i})(f^j)$. We have the diagram

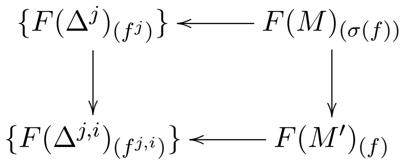


where $D_+(f^j) \subseteq (\gamma^j)^{-1}(V)$, $D_+(f^j) \subseteq (\gamma^{j,i})^{-1}(V)$ and $D_+(\sigma(f)) \subseteq \mathbb{P}(M)$ by Theorem 2.2.16.

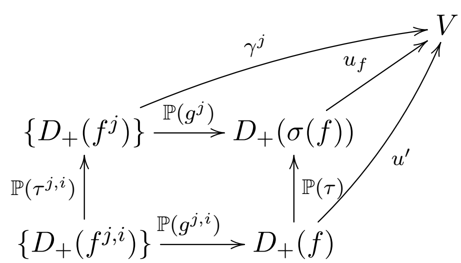
Thus we have the diagram



The morphisms of this diagram corresponds to homomorphisms between k-algebras, and $F(M)_{(\sigma(f))}$ being the limit in the corresponding diagram



by Lemma 2.2.23, there exists a unique induced k-algebra homomorphism $B \to F(M)_{(\sigma(f))}$, giving us a unique morphism $u_f : D_+(\sigma(f)) \to V$ such that

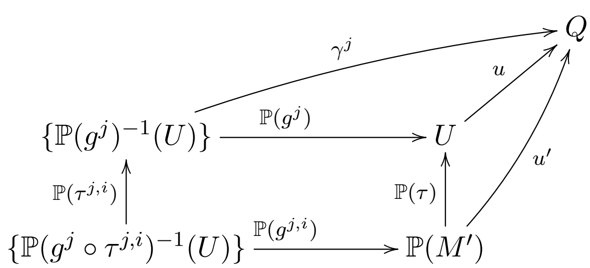


commutes.

Taking another distinguished open set $D_+(g)$ in $(u')^{-1}(V)$, we get a new morphism $u_g : D_+(\sigma(g)) \to V$. It is clear that u_f and u_g agrees on $D_+(\sigma(f)\sigma(g))$. By composing with the inclusion $V \subseteq Q$, we have the maps $u_f^V : D_+(\sigma(f)) \to Q$ gluing to a map $u^V : \bigcup_{D_+(f) \subseteq (u')^{-1}(V)} D_+(\sigma(f)) \to Q$. It is easily seen that the maps u_f are independent of the affine $V \subseteq Q$ as long as $D_+(f) \subseteq (u')^{-1}(V)$, meaning that we may glue the maps u^{V_i} for an open affine cover $\{V_i\}$ of Q, yielding a map $u : U \to Q$ where

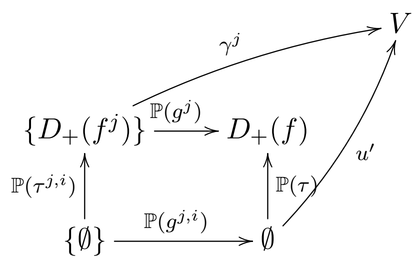
$$U = \bigcup_i \bigcup_{D_+(f) \subseteq (u')^{-1}(V_i)} D_+(\sigma(f)).$$

Since $\mathbb{P}(\tau)$ is a closed immersion, U is an open subscheme of $\mathbb{P}(M)$ containing $\mathbb{P}(M')$. The morphism u is the unique map such that

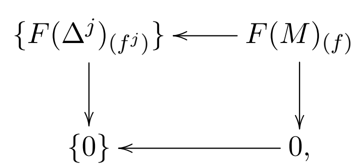


commutes.

We will extend the domain of u to $\mathbb{P}(M)$. Let $W = \mathbb{P}(M) - \mathbb{P}(M')$, and let $D_+(f) \subseteq W$ be a distinguished open set for some homogeneous $f \in F(M)_+$. Now, $\mathbb{P}(\tau)^{-1}(W) = \emptyset$, which means we have the diagram



where $f^j = F(g^j)(f)$. However in this case we have that $F(M)_{(f)}$ is the limit in the diagram



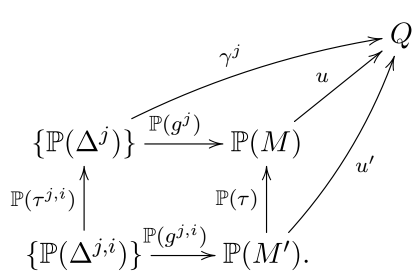
or simply

$$\{F(\Delta^j)_{(f^j)}\} \longleftarrow F(M)_{(f)}$$

by Theorem 2.2.23, which means that $F(M)_{(f)} = \bigoplus_j F(\Delta^j)_{(f^j)}$. Thus

$$D_+(f) = \operatorname{Spec} F(M)_{(f)} = \bigsqcup_j \operatorname{Spec} F(\Delta^j)_{(f^j)} = \bigsqcup_j D_+(f^j).$$

So a morphism $D_+(f) \to Q$ is induced by the morphisms $\gamma^j : D_+(f^j) \to Q$. This morphism clearly agrees with u on on $U \cap W$ considering γ^j factors through u. Thus by gluing we can extend u to $\mathbb{P}(M)$ and we get the commutative diagram



In order show that $\mathbb{P}(M)$ is the colimit of the diagram $\varinjlim \mathbb{P} \circ H$, it remains to show that $u \circ \mathbb{P}(\phi_X) = \psi_X$ for all $X \in I$.

For each X, consider the Δ-set $K(X) = H'(X) \sqcup \bigsqcup_{r \in H(X)_n} \Delta^{j_r}$ (where the Δ^{j_r}'s are n-dimensional Δ-simplices) with the surjective map of Δ-sets $h = \iota_X \sqcup \bigsqcup_r \pi_X^{j_r} : K(X) \to H(X)$. As topological spaces, $\mathbb{P}(H'(X))$ and the $\mathbb{P}(\Delta^{j_r})$'s live as a disjoint union $\mathbb{P}(H'(X)) \sqcup \bigsqcup_{r \in H(X)_n} \mathbb{P}(\Delta^{j_r})$ in $\mathbb{P}(K(X))$, so the surjective morphism $\mathbb{P}(h) : \mathbb{P}(K(X)) \to \mathbb{P}(H(X))$ as a map of topological spaces is just $\mathbb{P}(\iota_X) \sqcup \bigsqcup_r \mathbb{P}(\pi_X^{j_r})$.

Now, pick $P \in \mathbb{P}(H(X))$. Then $P = \mathbb{P}(h)(Q)$ for some $Q \in \mathbb{P}(K(X))$, meaning that either $P = \mathbb{P}(\iota_X)(Q)$ for some $Q \in \mathbb{P}(H'(X))$, or $P = \mathbb{P}(\pi_X^{j_r})(Q)$ for some $Q \in \mathbb{P}(\Delta^{j_r})$.

So either:

$$\begin{aligned} \psi_X(P) &= \psi_X \circ \mathbb{P}(\iota_X)(Q) = u' \circ \mathbb{P}(\phi'_X)(Q) \\ &= u \circ \mathbb{P}(\tau) \circ \mathbb{P}(\phi'_X)(Q) = u \circ \mathbb{P}(\phi_X) \circ \mathbb{P}(\iota_X)(Q) \\ &= u \circ \mathbb{P}(\phi_X)(P), \end{aligned}$$

or:

$$\begin{aligned} \psi_X(P) &= \psi_X \circ \mathbb{P}(\pi_X^{j_r})(Q) \\ &= \gamma^j(Q) = u \circ \mathbb{P}(g^j)(Q) \\ &= u \circ \mathbb{P}(\phi_X) \circ \mathbb{P}(\pi_X^{j_r})(Q) = u \circ \mathbb{P}(\phi_X)(P), \end{aligned}$$

which proves that $u \circ \mathbb{P}(\phi_X) = \psi_X$ on the topological space. Now, consider the sheaf morphism

$$\mathbb{P}(h)^\sharp : \mathcal{O}_{\mathbb{P}(H(X))} \to \mathbb{P}(h)_* \mathcal{O}_{\mathbb{P}(K(X))}.$$

The following diagram commutes:

$$\begin{array}{ccc} \mathbb{P}(h)_* \mathcal{O}_{\mathbb{P}(K(X))} & \longrightarrow & \mathbb{P}(\iota_X)_* \mathcal{O}_{\mathbb{P}(H'(X))} \oplus \prod \mathbb{P}(\pi_X^{j_r})_* \mathcal{O}_{\mathbb{P}(\Delta^{j_r})} \\ \uparrow {\scriptstyle \mathbb{P}(h)^\sharp} & \nearrow {\scriptstyle \mathbb{P}(\iota_X)^\sharp \oplus \prod \mathbb{P}(\pi_X^{j_r})^\sharp} & \\ \mathcal{O}_{\mathbb{P}(H(X))}. & & \end{array}$$

The vertical arrow is injective by Theorem 2.2.17, and the horizontal arrow is injective by Theorem 1.3.2 and Theorem 2.2.23. This means that we have $\ker \mathbb{P}(\iota_X)^\sharp \cap \bigcap_r \ker \mathbb{P}(\pi_X^{j_r})^\sharp = 0$.

Now, in our calculations above we established the following equations:

$$\psi_X \circ \mathbb{P}(\iota_X) = u \circ \mathbb{P}(\phi_X) \circ \mathbb{P}(\iota_X),$$
$$\psi_X \circ \mathbb{P}(\pi_X^{j_r}) = u \circ \mathbb{P}(\phi_X) \circ \mathbb{P}(\pi_X^{j_r}),$$

for all r as scheme morphisms. This gives us the equations

$$\mathbb{P}(\iota_X)^\sharp \circ \psi_X^\sharp = \mathbb{P}(\iota_X)^\sharp \circ (u \circ \mathbb{P}(\phi_X))^\sharp,$$
$$\mathbb{P}(\pi_X^{j_r})^\sharp \circ \psi_X^\sharp = \mathbb{P}(\pi_X^{j_r})^\sharp \circ (u \circ \mathbb{P}(\phi_X))^\sharp.$$

i.e.

$$\mathbb{P}(\iota_X)^\sharp \circ (\psi_X^\sharp - (u \circ \mathbb{P}(\phi_X))^\sharp) = 0,$$
$$\mathbb{P}(\pi_X^{j_r})^\sharp \circ (\psi_X^\sharp - (u \circ \mathbb{P}(\phi_X))^\sharp) = 0.$$

Thus

$$\text{im } (\psi_X^\sharp - (u \circ \mathbb{P}(\phi_X))^\sharp) \subseteq \ker \mathbb{P}(\iota_X)^\sharp \cap \bigcap_r \ker \mathbb{P}(\pi_X^{j_r})^\sharp = 0,$$

which means that $\psi_X^\sharp = (u \circ \mathbb{P}(\phi_X))^\sharp$. We conclude that $\psi_X = u \circ \mathbb{P}(\phi_X)$ as scheme morphisms.

□

Chapter 3

Group actions on Δ-face rings and Δ-face schemes

3.1 Group actions

Definition 3.1.1. *A left-action of a group G on an object M of a category C is a homomorphism of groups from G to the group of automorphisms of M, i.e. a map $\mu : G \to \mathrm{Aut}(M)$ such that*

- *If $e \in G$ is the identity element, then $\mu(e) : M \to M$ is the identity morphism,*
- *$\mu(g_1) \circ \mu(g_2) = \mu(g_1 g_2)$ for all $g_1, g_2 \in G$.*

For a right-action we have $\mu(g_1) \circ \mu(g_2) = \mu(g_2 g_1)$ for all $g_1, g_2 \in G$ instead. If M has an underlying set, the group action is free if $\mu(g) : M \to M$ has no fixed points for any $g \in G$ not equal to the identity.

Definition 3.1.2. *A categorical quotient of an object M in a category C by a group G acting on M by a group action $\mu : G \to \mathrm{Aut}(M)$ is a morphism $\pi : M \to N$, N written as M/G, satisfying the following properties:*

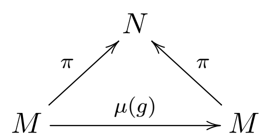
- *π is G-invariant, i.e. $\pi \circ \mu(g) = \pi \circ \mu(e)$ for all $g \in G$,*
- *Any G-invariant morphism $\gamma : M \to P$ factors uniquely through π.*

A categorical quotient is unique up to unique isomorphism.

Lemma 3.1.3. *Let M be an object of a category C, and G a group acting on M with group action $\mu : G \to \mathrm{Aut}(M)$. If the colimit N, with the map*

$\pi : M \to N$, exists of the diagram $H : I \to C$ defined as follows: I is a one-point set $\{X\}$ for which $H(X) = M$, $\mathrm{Hom}(I) = G$ *where any element $g \in G$ corresponds to the map $\mu(g) : M \to M$, then $\pi : M \to N$ is the categorical quotient of M by the group G.*

Proof. N being the colimit, means that the diagram



commutes for every $g \in G$. Thus π is G-invariant. Furthermore, if we have a G-invariant map $\gamma : M \to Q$, γ makes Q a node to the diagram H, meaning that we have a unique factorization of γ through π. □

Corollary 3.1.4. *The categorical $\pi : M \to N$ of a Δ-set M of a group G in the category of Δ-sets exists, and the elements of N are in one-to-one correspondence with orbits of elements of M.*

Proof. This follows from Theorem 1.2.1. □

Example 3.1.5. We consider the triangle T as a simplicial complex defined by

$$T_0 = \{p_1, p_2, p_3\} = \{0, 1, 2\},$$
$$T_1 = \{e_1, e_2, e_3\} = \{\{0, 1\}, \{1, 2\}, \{0, 2\}\}.$$

Consider the cyclic group G of order 3, $G = \{e, g, g^2\}$. We define a group action μ of G on the Δ-set T by letting $\mu(g) : T \to T$ be the map cycling the vertices:

$$0 \mapsto 1,$$
$$1 \mapsto 2,$$
$$2 \mapsto 0.$$

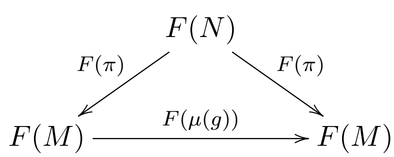
We have a (unique) map from T to the loop $\mathcal{L}$. It is easily seen that this map is G-invariant, and in fact that $\mathcal{L}$ is the group quotient of T by G.

Theorem 3.1.6. *If $\pi : M \to N$ is a categorical quotient of a finite group G acting on a Δ-set M, then $\mathbb{P}(\pi) : \mathbb{P}(M) \to \mathbb{P}(N)$ is a categorical quotient of G where the group action $\mathbb{P}(\mu) : G \to \mathrm{Aut}(\mathbb{P}(M))$ is induced by the group action $\mu : G \to \mathrm{Aut}(M)$.*

Proof. Using Lemma 3.1.3, N with map $\pi : M \to N$ is the colimit of the diagram $H : I \to \Delta$-sets. G is finite, so the diagram H is finite, and we have that $\varinjlim \mathbb{P} \circ H = \mathbb{P}(M)$ by Theorem 2.2.24, since H is connected. Note that the diagram $\mathbb{P} \circ H$ corresponds to the group action $\mathbb{P}(\mu) : G \to \mathrm{Aut}(\mathbb{P}(M))$, and thus $\mathbb{P}(\pi) : \mathbb{P}(M) \to \mathbb{P}(N)$ is the categorical quotient of $\mathbb{P}(M)$ by the group G. □

Theorem 3.1.7. *If $\pi : M \to N$ is a categorical quotient of a group G acting on a Δ-set M, then $F(N) = F(M)^G$, the ring of invariants (as a subring) under the group (right-)action $F(\mu) : G \to \mathrm{Aut}(F(M))$ induced by the group action $\mu : G \to \mathrm{Aut}(M)$.*

Proof. Using Lemma 3.1.3 again, N with the map $\pi : M \to N$ is the colimit of the diagram $H : I \to \Delta$-sets. Now, $F(N)$, with the map of Δ-face rings $F(\pi) : F(M) \to F(N)$ is the limit of the connected diagram $F \circ H$ by Theorem 1.3.2. So the diagram



commutes for every $g \in G$. The group action of G on $f \in F(M)$ is $g(f) = F(\mu(g))(f)$. By the construction of the limit,

$$F(N) = \{f \in F(M) | g(f) = h(f)\ \forall g, h \in G\} = \{f \in F(M) | g(f) = f\ \forall g \in G\}$$

which is just the G-invariants $F(M)^G$ of $F(M)$. □

Theorem 3.1.8. *Let G be a group acting on a Δ-set M. If the group action $\mu : G \to \mathrm{Aut}(M)$ is free, then the corresponding group action on $\mathbb{P}(M)$, $\mathbb{P}(\mu) : G \to \mathrm{Aut}(\mathbb{P}(M))$, is free.*

Proof. Assume there is a point $x \in \mathbb{P}(M)$ and an element $g \in G$ not equal to the identity element such that $\mu(g)(x) = x$, i.e. x is a fixed point of the morphism $\mu(g) : \mathbb{P}(M) \to \mathbb{P}(M)$. For each element $f \in M_n$ for any n, we consider the Δ-set $\Delta^f \subseteq M$ with the single maximal face f. The irreducible closed subschemes $\mathbb{P}(\Delta^f)$ cover $\mathbb{P}(M)$ in the sense that every point of $\mathbb{P}(M)$ is in the image of the closed immersion from $\mathbb{P}(\Delta^f)$ for some f. Let m be the least integer for which there exists a $\mathbb{P}(\Delta^f)$ of dimension m which contains $x = \mu(g)(x)$. The action of G on M is free, so $g(f) = h$ for some $h \neq f$. This means that $\mu(g) : M \to M$ restricts to a map $g_f : \Delta^f \to \Delta^h$, which in turn induces a restriction $\mathbb{P}(g_f) : \mathbb{P}(\Delta^f) \to \mathbb{P}(\Delta^h)$ of the morphism $\mathbb{P}(\mu(g)) : \mathbb{P}(M) \to \mathbb{P}(M)$. But this means that

$$x \in \mathbb{P}(\Delta^f) \cap \mathbb{P}(\Delta^h) = \mathbb{P}(\Delta^f \cap \Delta^h).$$

The Δ-set $\Delta^f \cap \Delta^h$ can be covered by Δ-sets $\Delta^r \subseteq M$ corresponding to elements $r \in (\Delta^f \cap \Delta^h)_n$ for various n. This means there exists an r such that $x \in \mathbb{P}(\Delta^r)$. However, the dimensions of Δ^f and Δ^h are both m, and $f \neq h$, so the dimension of Δ^r is less than m, which contradicts the minimality of m. Thus we conclude that the group action is free on $\mathbb{P}(M)$. □

3.2 Group schemes

We will now introduce group schemes. The goal is to utilize a special theorem in [Mum70] which will grant us additional information regarding the quotient map $\mathbb{P}(\pi) : \mathbb{P}(M) \to \mathbb{P}(N)$ (see Theorem 3.2.5). The following definitions are taken from [Mum70] Chapter 3.

Definition 3.2.1. *A group scheme over k is a scheme G over $\operatorname{Spec} k$ together with a multiplication morphism $m : G \times G \to G$, an identity morphism $e : \operatorname{Spec} k \to G$, and an inverse morphism $i : G \to G$ such that the following axioms hold.*

- *(Associativity). The diagram*

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{m \times 1_G} & G \times G \\ \Big\downarrow{\scriptstyle 1_G \times m} & & \Big\downarrow{\scriptstyle m} \\ G \times G & \xrightarrow{m} & G \end{array}$$

  is commutative.

- *(Identity). The diagram*

$$\begin{array}{ccc} G \times \operatorname{Spec} k & \xrightarrow{1_G \times e} & G \times G \\ \Big|{\scriptstyle \cong} & & \Big\downarrow{\scriptstyle m} \\ G & \xrightarrow{1_G} & G \\ \Big|{\scriptstyle \cong} & & \Big\uparrow{\scriptstyle m} \\ \operatorname{Spec} k \times G & \xrightarrow{e \times 1_G} & G \times G \end{array}$$

  is commutative.

- *(Inverse). The diagram*

$$\begin{array}{ccccc} & & G \times G & & \\ & \overset{(1_G,i)}{\nearrow} & & \overset{m}{\searrow} & \\ G & \longrightarrow & \operatorname{Spec} k & \xrightarrow{e} & G \\ & \underset{(i,1_G)}{\searrow} & & \underset{m}{\nearrow} & \\ & & G \times G & & \end{array}$$

is commutative.

Definition 3.2.2. *An action of a group scheme G on a scheme X is a morphism:*

$$\mu : G \times X \to X$$

such that

- *the composite*

$$X \cong \operatorname{Spec} k \times X \xrightarrow{e \times 1_X} G \times X \xrightarrow{\mu} X$$

is the identity;

- *the diagram*

$$\begin{CD} G \times G \times X @>{m \times 1_X}>> G \times X \\ @V{1_G \times \mu}VV @VV{\mu}V \\ G \times X @>{\mu}>> X \end{CD}$$

is commutative.

Definition 3.2.3. *Given an action $\mu : G \times X \to X$ of a group scheme G on a scheme X, a morphism $f : X \to Y$ is said to be G-invariant if the diagram*

$$\begin{CD} G \times X @>{\mu}>> X \\ @V{pr_2}VV @VV{f}V \\ X @>{f}>> Y \end{CD}$$

is commutative. Taking $Y = \mathbb{A}^1$ defines the notion of G-invariant functions.

Definition 3.2.4. *An action $\mu : G \times X \to X$ of a group scheme G on a scheme X is free if the morphism*

$$(\mu, pr_2) : G \times X \to X \times X$$

is a closed immersion.

See [Mum70] Chapter 3 for additional details regarding this. We want to utilize the following theorem.

Theorem 3.2.5. *(A) Let G be a finite group scheme acting on a scheme X of finite type such that the orbit of any point is contained in an affine open subset of X. Then there is a pair (Y, η), where Y is a scheme and $\eta : X \to Y$ is a morphism, satisfying the following conditions:*

- *as a topological space, (Y, η) is the quotient of X for the action of the underlying finite group;*
- *the morphism $\eta : X \to Y$ is G-invariant, and if $\eta_*(\mathcal{O}_X)^G$ denotes the subsheaf of $\eta_*(\mathcal{O}_X)$ of G-invariant functions, the natural homomorphism $\mathcal{O}_Y \to \eta_*(\mathcal{O}_X)^G$ is an isomorphism.*

The pair (Y, η) is uniquely determined up to isomorphism by these conditions. The morphism η is finite and surjective. Y will be denoted X/G, and it has the functorial property: $\forall$ G-invariant morphisms $f : X \to Z$, $\exists$ a unique morphism $g : Y \to Z$ such that $f = g \circ \eta$.
(B) Suppose further that the action of G is free and $G = \operatorname{Spec} R$, $n = \dim_k R$. Then η is a flat morphism of degree n, i.e. $\eta_*(\mathcal{O}_X)$ is a locally free $\mathcal{O}_Y$-module of rank n, and the subscheme of $X \times X$ defined by the closed immersion

$$(\mu, pr_2) : G \times X \to X \times X$$

is equal to the subscheme $X \times_Y X \subseteq X \times X$.

Proof. This is a shortened version of Theorem 1 in [Mum70] p.111. The finite type condition on X is assumed in the introduction of the chapter. □

3.3 From groups to group schemes

In order to use this theorem, we will need a way of going from group actions on a scheme to equivalent group scheme actions on a scheme.

For a group G we may define the corresponding group scheme G_k, the constant group scheme of G over k. Consider G as a Δ-set with $G_0 = \{g\}_{g \in G}$, and $G_n = 0$ for $n > 0$. The identity element $e_0 \in G$ may be considered a one-point Δ-set, and we have the inclusion $e : e_0 \to G$. We also have the map $i : G \to G$ defined by $g \mapsto g^{-1}$. Consider the Δ-set $G \times G$ defined by $(G \times G)_n = G_n \times G_n$ for all n. We have the map $m : G \times G \to G$ defined by $(g, h) \mapsto gh$. m restricts to isomorphisms $m_g : \{g\} \times G \to G$.

We get the isomorphisms $\mathbb{P}(m_g) : \mathbb{P}(\{g\} \times G) \to \mathbb{P}(G)$. The isomorphism $p_g : G \to \{g\} \times G$ defined by $h \mapsto (g, h)$ for all $h \in G$ induces an isomorphism $\mathbb{P}(p_g) : \mathbb{P}(G) \to \mathbb{P}(\{g\} \times G)$, hence a natural isomorphism

$$\mathbb{P}(p_g) : \operatorname{Spec} k \times \mathbb{P}(G) = \mathbb{P}(\{g\}) \times \mathbb{P}(G) \to \mathbb{P}(\{g\} \times G).$$

The compositions

$$\mathbb{P}(m_g) \circ \mathbb{P}(p_g) : \mathbb{P}(\{g\}) \times \mathbb{P}(G) \to \mathbb{P}(G)$$

trivially glue to a morphism $m_k : \mathbb{P}(G) \times \mathbb{P}(G) \to \mathbb{P}(G)$, since $\mathbb{P}(\{g\}) \subseteq \mathbb{P}(G)$ are disjoint open subschemes for every $g \in G$.

Thus we define $G_k = \mathbb{P}(G)$. We have an identity morphism

$$e_k = \mathbb{P}(e) : \operatorname{Spec} k = \mathbb{P}(e_0) \to \mathbb{P}(G),$$

an inverse morphism $i_k = \mathbb{P}(i) : \mathbb{P}(G) \to \mathbb{P}(G)$, and a multiplication morphism $m_k : \mathbb{P}(G) \times \mathbb{P}(G) \to \mathbb{P}(G)$. It is straightforward to check that these maps satisfy the axioms in Definition 3.2.1.

For a group action $\mu : G \to \operatorname{Aut}(X)$ of a group G on a scheme X we have a naturally induced group action $\mu_k : G_k \times X \to X$ by what follows. We may view $\mu(g) : X \to X$ as a morphism from

$$X \cong \operatorname{Spec} k \times X \cong \mathbb{P}(\{g\}) \times X \subseteq G_k \times X.$$

We glue these morphisms to a morphism $\mu_k : G_k \times X \to X$. It is straightforward to check that this map satisfy the conditions in Definition 3.2.2.

Theorem 3.3.1. *Let G be a group acting on a scheme X via a group action $\mu : G \to \operatorname{Aut}(X)$. Then a morphism $f : X \to Y$ is G-invariant if and only if it is G_k-invariant.*

Proof. Let $f : X \to Y$ be a morphism. The induced group action morphism $\mu_k : G_k \times X \to X$ is formed by gluing the morphisms $\mu(g) : \mathbb{P}(\{g\}) \times X \to X$. We immediately see that f is G_k-invariant if and only if it is G-invariant looking at the diagram in Definition 3.2.3. □

The following theorem gives us an equivalence of two different kinds of group quotients.

Theorem 3.3.2. *If the map $\pi : M \to N$ is a categorical quotient of a finite group G acting on a finite Δ-set M with group action $\mu : G \to \operatorname{Aut}(M)$, then the categorical quotient $\mathbb{P}(\pi) : \mathbb{P}(M) \to \mathbb{P}(N)$ (from Theorem 3.1.6) is the quotient of $\mathbb{P}(M)$ by the group scheme G_k via the induced group action $\mu_k : G_k \times \mathbb{P}(M) \to \mathbb{P}(M)$.*

Proof. By Corollary 2.2.14, $\mathbb{P}(M)$ is of finite type over k. To see that the orbit of any point is contained in an affine open subset, let $x \in \mathbb{P}(M)$, and $D_+(f) \subseteq \mathbb{P}(N)$ contain $\mathbb{P}(\pi)(x)$ for some homogeneous $f \in F(M)_+$. Then by Lemma 2.2.16,

$$\mathbb{P}(\pi)^{-1}(D_+(f)) = D_+(F(\pi)(f)) \subseteq \mathbb{P}(M)$$

contains the orbit of x, since $\mu(g)(x) \in \mathbb{P}(\pi)^{-1}(D_+(f))$ for every $g \in G$ as $\mathbb{P}(\pi)$ is G-invariant. Thus we may apply Theorem 3.2.5. The group action $\mu_k : G_k \times \mathbb{P}(M) \to \mathbb{P}(M)$ produces a group quotient $\eta : \mathbb{P}(M) \to Y$ satisfying the universal property in Definition 3.1.2 by Theorem 3.3.1, and is therefore unique up to unique isomorphism, hence the pair (Y, η) is just $(\mathbb{P}(N), \mathbb{P}(\pi))$, by Theorem 3.1.6. □

Corollary 3.3.3. *If the map of Δ-sets $\pi : M \to N$ is a categorical quotient of a finite group G acting on a finite Δ-set M via a free group action $\mu : G \to \mathrm{Aut}(M)$, then the categorical quotient $\mathbb{P}(\pi) : \mathbb{P}(M) \to \mathbb{P}(N)$ is flat.*

Proof. Apply Theorem 3.3.2 and Theorem 3.2.5. □

We take the following definition among other equivalent ones from [Sta14b] Lemma 36.14:

Definition 3.3.4. *A locally finite morphism $f : X \to Y$ of schemes is unramified if for every $x \in X$ the $\mathcal{O}_X$-module $\Omega_{X/Y,x} = 0$.*

Definition 3.3.5. *A morphism $f : X \to Y$ is étale if it is flat and unramified.*

Lemma 3.3.6. *If S is a finitely generated A-algebra, then the induced morphism of schemes $\mathrm{Proj}\, S \to \mathrm{Proj}\, A$ is separated.*

Proof. This is Proposition 1 in [Mur06]. □

Theorem 3.3.7. *If M is a finite Δ-set, then $\mathbb{P}(M)$ is separated over k.*

Proof. $\mathbb{P}(M)$ is Proj of a finitely generated k-algebra by Theorem 2.2.13. The theorem follows from Lemma 3.3.6. □

Theorem 3.3.8. *If G is group acting on a separated scheme X over k via a group action $\mu : G \to \mathrm{Aut}(X)$, then the action of G on X is free implies that the action $\mu_k : G_k \times X \to X$ of G_k on X is free.*

Proof. That μ is a free action means that $\mu(g)(x) = \mu(h)(x) \Rightarrow g = h$ for any point $x \in X$. We have to show that $(\mu_k, pr_2) : G_k \times X \to X \times X$ is a closed immersion. Since X is separated over k, the diagonal morphism $\Delta : X \to X \times X$ is a closed immersion. $(\mu_k, pr_2) : G_k \times X \to X \times X$ may be formed by gluing together $\phi_g = (\mu(g), pr_2) : \mathbb{P}(\{g\}) \times X \to X \times X$ for each $g \in G$.

We will show that the images of the morphisms ϕ_g are disjoint. Let $x, y \in X$. x may be considered an element of $\mathbb{P}(\{g\}) \times X$, and y an element of $\mathbb{P}(\{h\}) \times X$. If $\phi_g(x) = \phi_h(y)$, then $pr_2(\phi_g(x)) = pr_2(\phi_h(y)) \Rightarrow x = y$. So $\phi_g(x) = \phi_h(x)$, but this means that

$$pr_1(\phi_g(x)) = pr_1(\phi_h(x)) \Rightarrow \mu(g)(x) = \mu(h)(x).$$

The action is free so this implies that $g = h$.

We will now show that the morphisms ϕ_g are closed immersions. This is clear, since ϕ_g is the composition of the morphisms

$$\mathbb{P}(\{g\}) \times X = X \to^{\Delta} X \times X \to^{(\mu(g)\times \mathrm{id}_X, \mathrm{id}_X \times \mathrm{id}_X)} X \times X,$$

where $\Delta : X \to X \times X$ is a closed immersion and

$$(\mu(g) \times \mathrm{id}_X, \mathrm{id}_X \times \mathrm{id}_X) : X \times X \to X \times X$$

is an isomorphism.

Now, let $V_g = \mathrm{im}(\phi_g) \subseteq X \times X$. Consider the open subschemes

$$U_g = X \times X - \bigcup_{\substack{h \in G \\ h \neq g}} V_h \subseteq X \times X.$$

The restriction morphisms $\mu_k|_{\mu_k^{-1}(U_g)} : \mu_k^{-1}(U_g) \to U_g$ are just the closed immersions ϕ_g by restricting their range to the open subschemes U_g. Thus μ_k is locally a closed immersion, hence a closed immersion. □

We will need the following lemma.

Lemma 3.3.9. *A separated morphism $f : X \to Y$ of finite type is unramified if and only if the diagonal map $\Delta : X \to X \times_Y X$ is an open immersion.*

Proof. This is Proposition 10.4 in [Tsu]. □

Theorem 3.3.10. *If the map $\pi : M \to N$ is a categorical quotient of a finite group G acting on a finite Δ-set M via a free group action $\mu : G \to \mathrm{Aut}(M)$, then the categorical quotient morphism $\mathbb{P}(\pi) : \mathbb{P}(M) \to \mathbb{P}(N)$ is étale.*

Proof. We have to show that $\mathbb{P}(\pi) : \mathbb{P}(M) \to \mathbb{P}(N)$ is flat and unramified. It is flat by Corollary 3.3.3. The action μ is free, so the induced action $\mathbb{P}(\mu) : G \to \mathrm{Aut}(\mathbb{P}(M))$ is free by Theorem 3.1.8. By Theorem 3.3.7, $\mathbb{P}(M)$ is separated over k. Hence by Theorem 3.3.8, the induced group scheme action $\mu_k : G_k \times \mathbb{P}(M) \to \mathbb{P}(M)$ is free as well. Applying Theorem 3.3.2 and Theorem 3.2.5, we have that the closed subscheme defined by the closed immersion $(\mu_k, pr_2) : G_k \times \mathbb{P}(M) \to \mathbb{P}(M) \times \mathbb{P}(M)$ is $\mathbb{P}(M) \times_{\mathbb{P}(\mathbb{N})} \mathbb{P}(M)$. $\mathbb{P}(\pi)$ is a finite morphism by Corollary 2.2.18, hence separated, so by Lemma 3.3.9, it is sufficient to show that $\mathbb{P}(M) \to G_k \times \mathbb{P}(M)$ induced by the natural morphism $\mathbb{P}(M) \to \mathbb{P}(e_0) \times \mathbb{P}(M) \subseteq G_k \times \mathbb{P}(M)$ is an open immersion. This follows immediately since $\mathbb{P}(M) \to \mathbb{P}(e_0) \times \mathbb{P}(M)$ is an isomorphism and $\mathbb{P}(e_0) \times \mathbb{P}(M)$ is an open subscheme of $G_k \times \mathbb{P}(M)$. □

Example 3.3.11. For the triangle T, the Δ-face ring $F(T)$ will be the Stanley-Reisner ring $k[x_0, x_1, x_2]/(x_0x_1x_2)$. The map $F(\mathcal{L}) \to F(T)$ is given by

$$\begin{aligned} t &\mapsto x_0 + x_1 + x_2, \\ u &\mapsto x_0x_1 + x_1x_2 + x_2x_0, \\ v &\mapsto x_0^2x_1 + x_1^2x_2 + x_2^2x_0. \end{aligned}$$

Note that the elements in $F(T)$ above generate the group invariant elements of $F(T)$ by the induced action of G on $F(T)$. In order to compute the morphism $\mathbb{P}(T) \to \mathbb{P}(\mathcal{L})$, we will use the realization of $\mathbb{P}(\mathcal{L})$ as the closed subscheme $V(y^3 - xyz + xz^2) \subseteq \mathbb{P}^2$. The induced homomorphism

$$k[x, y, z]/(y^3 - xyz + xz^2) \to F(T)$$

is defined by

$$\begin{aligned} x &\mapsto (x_0 + x_1 + x_2)^3, \\ y &\mapsto (x_0 + x_1 + x_2)(x_0x_1 + x_1x_2 + x_2x_0), \\ z &\mapsto x_0^2x_1 + x_1^2x_2 + x_2^2x_0. \end{aligned}$$

The morphism $\mathbb{P}(T) \to \mathbb{P}(\mathcal{L})$ corresponds to these coordinate transformations. Here $\mathbb{P}(\mathcal{L})$ is the quotient of $\mathbb{P}(T)$ by the induced action of G on $\mathbb{P}(T)$.

3.4 Summary

We summarize the results of this and the previous chapters. In what follows, let M and N be Δ-sets, and G a finite group acting on M.

- If $\phi : M \to N$ is injective, then $\mathbb{P}(\phi) : \mathbb{P}(M) \to \mathbb{P}(N)$ is a closed immersion.
- If $\phi : M \to N$ is surjective, then $\mathbb{P}(\phi) : \mathbb{P}(M) \to \mathbb{P}(N)$ is a surjective morphism.
- If $M_1, M_2 \subseteq M$, then $\mathbb{P}(M_1 \cap M_2) = \mathbb{P}(M_1) \cap \mathbb{P}(M_2) \subseteq \mathbb{P}(M)$.
- If $M_1, M_2 \subseteq M$, then $\mathbb{P}(M_1 \cup M_2) = \mathbb{P}(M_1) \cup \mathbb{P}(M_2) \subseteq \mathbb{P}(M)$.
- If M is a Δ-set with a single maximal face, then $\mathbb{P}(M)$ is irreducible.
- If M is n-dimensional, then $\mathbb{P}(M)$ is reduced and n-dimensional.
- If M is a simplicial complex, then $F(M)$ is the Stanley-Reisner ring of M.
- If M is finite, then $\mathbb{P}(M)$ is of finite type and separated over k.
- If M is finite and $\phi : M \to N$ is any map, then $\mathbb{P}(\phi) : \mathbb{P}(M) \to \mathbb{P}(N)$ is a finite morphism.
- If $H : I \to \Delta$-sets is a finite diagram, then $\mathbb{P}(\varinjlim H) = \varinjlim \mathbb{P} \circ H$.
- If $\pi : M \to N$ is a categorical quotient by the action of a finite group G on M, then $\mathbb{P}(\pi) : \mathbb{P}(M) \to \mathbb{P}(N)$ is a categorical quotient by the induced action of G on $\mathbb{P}(M)$. Furthermore, $\mathcal{O}_{\mathbb{P}(N)} = (\mathbb{P}(\pi)_* \mathcal{O}_{\mathbb{P}(M)})^G$, the subsheaf of $\mathbb{P}(\pi)_* \mathcal{O}_{\mathbb{P}(M)}$ of G-invariant functions.
- If the action of G is free, the morphism $\mathbb{P}(\pi) : \mathbb{P}(M) \to \mathbb{P}(N)$ as above is étale, and the structure sheaf $\mathbb{P}(\pi)_* \mathcal{O}_{\mathbb{P}(M)}$ is a locally free sheaf of $\mathcal{O}_{\mathbb{P}(N)}$-modules of rank $|G|$.

Chapter 4

Cotangent cohomology for Δ-face schemes

4.1 Introduction

We will introduce the cotangent complex of sheaves for a morphism of schemes $f : X \to Y$ along the lines of [And74]. We start by considering an A-algebra B. For an integer m, let $A[m]$ be the A-algebra generated by m independent indeterminates. For each n, we define $E_n(A, B)$ to be the set of $n+1$ A-algebra morphisms $(\alpha_0, \ldots, \alpha_n)$ of a sequence of A-algebras

$$A[i_n] \to^{\alpha_n} A[i_{n-1}] \to^{\alpha_{n-1}} \cdots \to A[i_0] \to^{\alpha_0} B.$$

for non-negative integers $i_0, \ldots, i_n$. Note that these homomorphisms gives B a natural $A[i_n]$-module structure. For each such sequence, we define the B-module

$$C(\alpha_0, \ldots, \alpha_n) = \Omega_{A[i_n]/A} \otimes_{A[i_n]} B.$$

For each n, we define the B-module

$$C_n(A, B) = \bigoplus_{(\alpha_0, \ldots, \alpha_n) \in E_n(A,B)} C(\alpha_0, \ldots, \alpha_n).$$

We will now define B-module homomorphisms $\delta_n^i : C_n(A, B) \to C_{n-1}(A, B)$ for each i. If i and n are different, δ_n^i sends the component $C(\alpha_0, \ldots, \alpha_n)$ of the direct sum $C_n(A, B)$ to $C(\alpha_0, \ldots, \alpha_i \circ \alpha_{i+1}, \ldots, \alpha_n)$ in $C_{n-1}(A, B)$ via the identity map. If $i = n$, the map

$$\delta_n^i : C(\alpha_0, \ldots, \alpha_n) = \Omega_{A[i_n]/A} \otimes_{A[i_n]} B \to C(\alpha_0, \ldots, \alpha_{n-1}) = \Omega_{A[i_{n-1}]/A} \otimes_{A[i_{n-1}]} B$$

may be defined. These maps satisfy the following equation:

$$\delta_n^i \circ \delta_{n+1}^j = \delta_n^{j-1} \circ \delta_{n+1}^i \text{ for } 0 \leq i < j \leq n+1.$$

This allows us to define the boundary maps $\delta_n = \sum_{i=0}^n (-1)^i \delta_n^i$ for $n \geq 1$ forming the cotangent complex $C_\bullet(A, B)$ of B over A:

$$C_0(A,B) \leftarrow^\delta C_1(A,B) \leftarrow^\delta C_2(A,B) \leftarrow^\delta \cdots \leftarrow^\delta C_n(A,B) \leftarrow^\delta \cdots .$$

The terms of this complex are free B-modules. The complex $C_\bullet(A, B)$ may be extended. Define

$$C_{-1}(A,B) = \Omega_{B/A}.$$

There is an induced morphism $\delta_0 : C_0 \to C_{-1}$ extending the complex. Now, let C be a B-algebra, and M a C-module.

Definition 4.1.1. *The n-th homology module $H_n(A, B, M)$ is the B-module equal to the n-th homology module of the cocomplex*

$$C_\bullet(A,B,M) = C_\bullet(A,B) \otimes_B M$$

of C-modules.

Definition 4.1.2. *The n-th cohomology module $H^n(A, B, M)$ is the B-module equal to the n-th cohomology module of the cocomplex*

$$C^\bullet(A,B,M) = \mathrm{Hom}_B(C_\bullet(A,B), M)$$

of C-modules.

Lemma 4.1.3. *Let B be an A-algebra, M a B-module. Then*

$$H_0(A,B,M) = \Omega_{B/A} \otimes_B M,$$

and

$$H^0(A,B,M) = \mathrm{Hom}_B(\Omega_{B/A}, M).$$

Lemma 4.1.4. *Let B be an A-algebra, C a B-algebra and M a C-module. Then there is a natural exact sequence of C-modules*

$$\begin{aligned} \cdots \to & H_n(A,B,M) \to H_n(A,C,M) \to H_n(B,C,M) \to \\ \to & H_{n-1}(A,B,M) \to \cdots \to H_0(B,C,M) \to 0, \end{aligned}$$

and a natural exact sequence of C-modules

$$\begin{aligned} & 0 \to H^0(B,C,M) \to \cdots \to H^{n-1}(A,B,M) \to \\ & \to H^n(B,C,M) \to H^n(A,C,M) \to H^n(A,B,M) \to \cdots . \end{aligned}$$

Lemma 4.1.5. *Let B be an A-algebra, and the ring map $A \to B$ étale. Then $C_n(A, B) = 0$ for all $n \geq 0$.*

Proof. This is Lemma 8.4 in [Sta14a]. □

4.2 Double complexes

The theories of double complexes and double cocomplexes are similar. We will only be interested in double cocomplexes. We will from here on only use the term complex for both. Notation will also differ slightly from that in [And74].

Definition 4.2.1. *A double complex of A-modules is a set of A-modules $K^{p,q}$ for $p,q \geq 0$, and a set of A-module homomorphisms*

$$d_0^{p,q} : K^{p,q} \to K^{p+1,q} \text{ and } d_1^{p,q} : K^{p,q} \to K^{p,q+1}$$

such that the following equations are satisfied for all $p \geq 0, q \geq 0$:

- *(1)* $d_0^{p+1,q} \circ d_0^{p,q} = 0$,
- *(2)* $d_1^{p,q+1} \circ d_1^{p,q} = 0$,
- *(3)* $d_0^{p+1,q+1} \circ d_1^{p,q+1} = d_1^{p+1,q+1} \circ d_0^{p+1,q}$.

We will denote the double complex by $K^{\bullet,\bullet}$.

For each integer q, using equation (1) we may consider the single complex $K^{\bullet,q}$

$$K^{0,q} \to^{d_0^{0,q}} \cdots \to K^{p,q} \to^{d_0^{p,q}} K^{p+1,q} \to \cdots .$$

We define p-th cohomology module $H_0^{p,q}(K^{\bullet,\bullet}) = H^p(K^{\bullet,q})$. Similarly for each integer p, using equation (2) we may consider the single complex $K^{p,\bullet}$

$$K^{p,0} \to^{d_1^{p,0}} \cdots \to K^{p,q-1} \to K^{p,q} \to^{d_1^{p,q}} K^{p,q+1} \to \cdots ,$$

and the q-th cohomology module $H_1^{p,q}(K^{\bullet,\bullet}) = H^q(K^{p,\bullet})$. Equation (3) allows us to define homomorphisms of complexes

$$d_0^{\bullet,q} : K^{\bullet,q} \to K^{\bullet,q+1},$$

and homomorphisms

$$d_1^{p,\bullet} : K^{p,\bullet} \to K^{p+1,\bullet}.$$

This induces natural homomorphisms of the cohomology modules

$$H_0^{p,q}(K^{\bullet,\bullet}) \to H_0^{p,q+1}(K^{\bullet,\bullet}),$$

and homomorphisms

$$H_1^{p,q}(K^{\bullet,\bullet}) \to H_1^{p+1,q}(K^{\bullet,\bullet}),$$

yielding the complexes of cohomology modules $H_0^{p,\bullet}(K^{\bullet,\bullet})$ and $H_1^{\bullet,q}(K^{\bullet,\bullet})$.

Definition 4.2.2. *Let* $\hat{H}_0^{p,q}(K^{\bullet,\bullet})$ *be the* q*-th cohomology module*

$$H^q(H_0^{p,\bullet}(K^{\bullet,\bullet})),$$

and $\hat{H}_1^{p,q}(K^{\bullet,\bullet})$ *the* p*-th cohomology module*

$$H^p(H_1^{\bullet,q}(K^{\bullet,\bullet})).$$

Definition 4.2.3. *The associated single complex* $\hat{K}^\bullet$ *is defined as follows:*

$$\hat{K}^n = \bigoplus_{p+q=n} K^{p,q},$$

with coboundary maps defined by

$$d_n(x) = d_0^{p,q}(x) + (-1)^p d_1^{p,q}(x)$$

for $x \in K^{p,q}$*. The* n*-th cohomology module of this complex is denoted by*

$$\hat{H}^n(K^{\bullet,\bullet}) = H^n(\hat{K}^\bullet).$$

4.3 Globalizing to schemes

For a morphism of schemes $f : X \to Y$, we may define the sheaves of $\mathcal{O}_X$-modules $\mathcal{C}_n(X, Y)$ by the sheaf associated to the presheaf

$$U \mapsto C_n(f^*\mathcal{O}_Y(U), \mathcal{O}_X(U))$$

forming the cotangent complex $\mathcal{C}_\bullet(X, Y)$ of X over Y. We define

$$\mathcal{C}^n(X, Y, \mathcal{F}) = \mathrm{Hom}_X(\mathcal{C}_n(f^*\mathcal{O}_Y, \mathcal{O}_X), \mathcal{F})$$

for any quasi-coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules. We have the complex of sheaves $\mathcal{C}^\bullet(X, Y, \mathcal{F})$ of X over Y:

$$\mathcal{C}^0(X, Y, \mathcal{F}) \to \mathcal{C}^1(X, Y, \mathcal{F}) \to \mathcal{C}^2(X, Y, \mathcal{F}) \to \cdots.$$

Definition 4.3.1. *Taking the cohomology of this complex of sheaves, we get the* $\mathcal{T}^n_{X/Y}(-)$*-functors, namely*

$$\mathcal{T}^n_{X/Y}(\mathcal{F}) = \mathcal{H}^n(\mathcal{C}^\bullet(X, Y, \mathcal{F})).$$

We will write

$$\mathcal{T}^n_{X/Y} = \mathcal{T}^n_{X/Y}(\mathcal{O}_X).$$

Now, consider the double complex of k-modules formed by taking the Čech-complex in the vertical rows. For an affine cover $\mathcal{U} = \{U_i\}$ of X, consider

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
\oplus_{i<j<k}\mathcal{C}^0(X,Y,\mathcal{F})(U_{ijk}) & \longrightarrow & \oplus_{i<j<k}\mathcal{C}^1(X,Y,\mathcal{F})(U_{ijk}) & \longrightarrow & \oplus_{i<j<k}\mathcal{C}^2(X,Y,\mathcal{F})(U_{ijk}) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
\oplus_{i<j}\mathcal{C}^0(X,Y,\mathcal{F})(U_{ij}) & \longrightarrow & \oplus_{i<j}\mathcal{C}^1(X,Y,\mathcal{F})(U_{ij}) & \longrightarrow & \oplus_{i<j}\mathcal{C}^2(X,Y,\mathcal{F})(U_{ij}) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
\oplus_{i}\mathcal{C}^0(X,Y,\mathcal{F})(U_{i}) & \longrightarrow & \oplus_{i}\mathcal{C}^1(X,Y,\mathcal{F})(U_{i}) & \longrightarrow & \oplus_{i}\mathcal{C}^2(X,Y,\mathcal{F})(U_{i}) & \longrightarrow & \cdots .
\end{array}$$

We write this double complex as $\{K^{p,q}(X,Y,\mathcal{F})\}_{p,q\geq 0}$ for

$$K^{p,q}(X,Y,\mathcal{F}) = \bigoplus_{i_0<\ldots<i_q} C^p(X,Y,\mathcal{F})(U_{i_0\ldots i_q}).$$

The homomorphisms are

$$d_0^{p,q} : K^{p,q}(X,Y,\mathcal{F}) \to K^{p+1,q}(X,Y,\mathcal{F})$$

induced by the coboundary maps

$$d^p : \mathcal{C}^p(X,Y,\mathcal{F}) \to \mathcal{C}^{p+1}(X,Y,\mathcal{F})$$

of the complex $\mathcal{C}^\bullet(X,Y,\mathcal{F})$, and the standard Čech complex coboundary maps

$$d_1^{p,q} : K^{p,q}(X,Y,\mathcal{F}) \to K^{p,q+1}(X,Y,\mathcal{F}).$$

We will write $d_0^{p,q}(U_{i_0\ldots i_q})$ for the map

$$C^p(X,Y,\mathcal{F})(U_{i_0\ldots i_q}) \to C^{p+1}(X,Y,\mathcal{F})(U_{i_0\ldots i_q}).$$

To show that the double complex $K^{\bullet,\bullet}(X,Y,\mathcal{F})$ is well-defined, we need to verify the three equations on the coboundary maps.

Theorem 4.3.2. *We have the following equations for $K^{\bullet,\bullet}(X,Y,\mathcal{F})$ for all $p \geq 0, q \geq 0$:*

- *(1)* $d_0^{p+1,q} \circ d_0^{p,q} = 0,$

- *(2)* $d_1^{p,q+1} \circ d_1^{p,q} = 0$,
- *(3)* $d_0^{p+1,q+1} \circ d_1^{p,q+1} = d_1^{p+1,q+1} \circ d_0^{p+1,q}$.

Proof. Equation (1) is clear, since the maps $d_0^{p,q}$ are induced by those of the complex $C^\bullet(X,Y,\mathcal{F})$. Equation (2) is also clear, since the maps $d_1^{p,q}$ are induced by those of the Čech complex. Equation (3) is verified by straight calculation. □

Theorem 4.3.3. *If X is noetherian separated, and $U_{i_0\dots i_q} = U_{i_0} \cap \dots \cap U_{i_q}$ is affine for all $i_0 < \dots < i_q$, then*

$$\hat{H}_0^{p,q}(K^{\bullet,\bullet}(X,Y,\mathcal{F})) = H^q(X, \mathcal{T}^p_{X/Y}(\mathcal{F})).$$

Proof. $\hat{H}_0^{p,q}(K^{\bullet,\bullet}(X,Y,\mathcal{F}))$ is constructed by first considering the cohomology modules

$$H_0^{p,q}(K^{\bullet,\bullet}(X,Y,\mathcal{F})) = H^p(K^{\bullet,q}(X,Y,\mathcal{F})) = H^p(\bigoplus_{i_0<\dots<i_q} C^\bullet(X,Y,\mathcal{F})(U_{i_0\dots i_q})).$$

Since $U_{i_0\dots i_q}$ is affine, and $C^\bullet(X,Y,\mathcal{F})$ a complex of quasi-coherent $\mathcal{O}_X$-modules, we have

$$H_0^{p,q}(K^{\bullet,\bullet}(X,Y,\mathcal{F})) = H^p(\bigoplus_{i_0<\dots<i_q} C^\bullet(X,Y,\mathcal{F}))(U_{i_0\dots i_q}) = \mathcal{T}^p_{X/Y}(\mathcal{F})(U_{i_0\dots i_q}).$$

Now, $\hat{H}_0^{p,q}(K^{\bullet,\bullet}(X,Y,\mathcal{F}))$ is just the q-th Čech cohomology module of the quasi-coherent sheaf $\mathcal{T}^p_{X/Y}(X,Y,\mathcal{F})$. Hence equal to $H^q(X,\mathcal{T}^p_{X/Y}(\mathcal{F}))$, since X is noetherian separated. □

We may define the associated single complex

$$\hat{K}^n(X,Y,\mathcal{F}) = \bigoplus_{p+q=n} K^{p,q}(X,Y,\mathcal{F}),$$

with coboundary maps

$$d_n : \hat{K}^n(X,Y,\mathcal{F}) \to \hat{K}^{n+1}(X,Y,\mathcal{F})$$

defined by $d_n(x) = d_0^{p,q}(x) + (-1)^p d_1^{p,q}(x)$ for any element $x \in K^{p,q}(X,Y,\mathcal{F})$.

Definition 4.3.4. *Taking cohomology here, defines*

$$\mathbb{T}^n_{X/Y}(\mathcal{F}) = H^n(\hat{K}^\bullet(X,Y,\mathcal{F})) = \hat{H}^n(K^{\bullet,\bullet}(X,Y,\mathcal{F})).$$

In particular, we write

$$\mathbb{T}^n_{X/Y} = \mathbb{T}^n_{X/Y}(\mathcal{O}_X).$$

4.4 Deformations of schemes

The following definition is taken from [Ser06].

Definition 4.4.1. *A deformation of a scheme X over a scheme S is a commutative diagram*

$$\eta: \quad \begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \big\downarrow & & \big\downarrow \pi \\ \operatorname{Spec} k & \xrightarrow{\ s\ } & S \end{array}$$

such that $\mathcal{X} \times_S k \cong X$, π is flat and surjective and S connected. S is called the parameter scheme, and $\mathcal{X}$ the total scheme of the deformation, also called a deformation of X. When $S = \operatorname{Spec} D$, where $D = k[\epsilon]/(\epsilon^2)$ is the ring of dual numbers, such a deformation is called a first-order deformation.

A first order deformation of X determines an element of $H^0(X, \mathcal{T}^1_{X/k})$, and $H^1(X, \mathcal{T}^0_{X/k})$ classifies the locally trivial deformations of X. $\mathbb{T}^1_{X/k}$ classifies all first-order deformations of X up to equivalence. The following theorem is well-known:

Theorem 4.4.2. *We have an exact sequence*

$$0 \to H^1(X, \mathcal{T}^0_{X/k}) \to \mathbb{T}^1_{X/k} \to H^0(X, \mathcal{T}^1_{X/k}) \to H^2(X, \mathcal{T}^0_{X/k}).$$

Let X be a scheme with together with an invertible ample sheaf $\mathcal{L}_X$. We will define a functor $\operatorname{Def}_{(X,\mathcal{L}_X)}$: Artin rings $\to$ Set called the deformation functor of X together with an invertible sheaf $\mathcal{L}_X$ on X. Given an artinian ring A (with residue field k), we let $\operatorname{Def}_{(X,\mathcal{L}_X)}(A)$ be the set of isomorphism classes of pairs $(\mathcal{X}, \mathcal{L}_\mathcal{X})$ where $\mathcal{X}$ is a deformation of X over $\operatorname{Spec} A$, and $\mathcal{L}_\mathcal{X}$ an invertible sheaf of $\mathcal{X}$ such that $\mathcal{L}_\mathcal{X} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{O}_X \cong \mathcal{L}_X$. Two such pairs $(\mathcal{X}, \mathcal{L}_\mathcal{X})$ and $(\mathcal{Y}, \mathcal{L}_\mathcal{Y})$ are equivalent if there is an isomorphism of schemes $f : \mathcal{X} \to \mathcal{Y}$ such that the diagram

$$\begin{array}{ccccc} & \swarrow & X & \searrow & \\ \mathcal{X} & & \xrightarrow{\quad f \quad} & & \mathcal{Y} \\ & \searrow & & \swarrow & \\ & & \operatorname{Spec} A & & \end{array}$$

commutes, and $\mathcal{L}_\mathcal{X} \cong f_* \mathcal{L}_\mathcal{Y}$. Not requiring f to be an isomorphism defines a morphism of such deformations.

In particular, it is a versal formal family of this functor we are interested in. See [Har10] for a complete definition of formal families. In short, given a local complete noetherian k-algebra R with maximal ideal m and residue field k, we may consider the limit of sets $\varprojlim \mathrm{Def}_{(X,\mathcal{L}_X)}(R/m^n)$. Given an R, and an element $\eta = \{\eta_n\}$ in this limit, the pair (R, η) is called a formal family for $\mathrm{Def}_{(X,\mathcal{L}_X)}$. Such elements are in natural one-to-one correspondence with natural transformations of functors $h_R \to \mathrm{Def}_{(X,\mathcal{L}_X)}$, where h_R : Artin rings $\to$ Set is the functor such that $h_R(A) = \mathrm{Hom}(R, A)$, the set of k-algebra homomorphisms from R to A.

A formal family (R, η) for $\mathrm{Def}_{(X,\mathcal{L}_X)}$ corresponds to a family of deformations $\{(\mathcal{X}_n, \mathcal{L}_{\mathcal{X}_n})\}$ over X:

$$\eta_n : \quad \begin{array}{ccc} X & \xrightarrow{\phi_n} & \mathcal{X}_n \\ \downarrow & & \downarrow \pi \\ \mathrm{Spec}\, k & \longrightarrow & \mathrm{Spec}\, R/m^n, \end{array}$$

where $\mathcal{L}_{\mathcal{X}}$ is an invertible sheaf such that $\phi_n^* \mathcal{L}_{\mathcal{X}} \cong \mathcal{L}_X$. Note that this implies that $\mathcal{L}_{\mathcal{X}}$ is ample. The map $h_R \to \mathrm{Def}_{(X,\mathcal{L}_X)}$ is defined as follows: for any given artin ring A, a homomorphism $f : R \to A$ factors through R/m^n for some n, and f is sent to the deformation $\mathcal{X}_n \times_{\mathrm{Spec}\, R/m^n} \mathrm{Spec}\, A$ of X together with the line bundle $\mathcal{L}_{\mathcal{X}} \otimes \mathcal{O}_{\mathcal{X}_n \times \mathrm{Spec}\, A}$.

If such a transformation $h_R \to \mathrm{Def}_{(X,\mathcal{L}_X)}$ is strongly surjective, we say that (R, η) is a versal formal family. That it is strongly surjective means that $h_R(A) \to \mathrm{Def}_{(X,\mathcal{L}_X)}(A)$ is surjective for all artinian rings A, and for every surjection $B \to A$,

$$h_R(B) \to h_R(A) \times_{\mathrm{Def}_{(X,\mathcal{L}_X)}(A)} \mathrm{Def}_{(X,\mathcal{L}_X)}(B)$$

is surjective. If in addition $h_R(D) \to \mathrm{Def}_{(X,\mathcal{L}_X)}(D)$ is bijective (where D is the ring of dual numbers), then (R, η) is a miniversal formal family. If $h_R \to \mathrm{Def}_{(X,\mathcal{L}_X)}$ is an equivalence of functors, then we say $\mathrm{Def}_{(X,\mathcal{L}_X)}$ is pro-presentable, and (R, η) is a universal formal family.

In our case, strongly surjective means the following: Let a homomorphism $f : R \to A$ inducing a deformation $(\mathcal{X}, \mathcal{L}_{\mathcal{X}}) \in \mathrm{Def}_{(X,\mathcal{L}_X)}(A)$ be given, in addition to a surjection $\phi : B \to A$ of artin rings. Let $(\mathcal{Y}, \mathcal{L}_{\mathcal{Y}}) \in \mathrm{Def}_{(X,\mathcal{L}_X)}(B)$. Then f can be lifted to a homomorphism $f' : R \to B$ such that $\phi \circ f' = f$, and f' induces $(\mathcal{Y}, \mathcal{L}_{\mathcal{Y}})$.

If we have a versal formal family $\{(\mathcal{X}_n, \mathcal{L}_{\mathcal{X}_n})\}$ of deformations of X, we can construct a versal formal deformation $(\mathcal{X}, \mathcal{L}_{\mathcal{X}})$, where $\mathcal{X}$ is a noetherian formal scheme that is flat over $\operatorname{Spf} R$, the formal spectrum of R, such that $\mathcal{X} \times_R \operatorname{Spec} R/m^n \cong \mathcal{X}_n$, and $\mathcal{L}_{\mathcal{X}} \otimes \mathcal{O}_{\mathcal{X}_n} \cong \mathcal{L}_{\mathcal{X}_n}$. It is defined as a ringed space as such: the topological space of $\mathcal{X}$ is X, $\mathcal{O}_{\mathcal{X}} = \varprojlim \mathcal{O}_{\mathcal{X}_n}$, and $\mathcal{L}_{\mathcal{X}} = \varprojlim \mathcal{L}_{\mathcal{X}_n}$. Here we call $\operatorname{Spec} R$ the versal base space.

4.5 Our situation

Our situation is the following: We have an affine, finite, étale and surjective morphism $f : X \to Y$ of separated schemes that are of finite type over k and Proj of finitely generated graded k-algebras, such that $f_*\mathcal{O}_X$ is a locally free sheaf of $\mathcal{O}_Y$-modules. Furthermore, G is a finite group acting on X via a free group action $\mu : G \to \operatorname{Aut}(X)$, and Y is the categorical quotient of X by G, such that $\mathcal{O}_Y = (f_*\mathcal{O}_X)^G$, the sheaf of G-invariant functions. I.e.

$$\mathcal{O}_Y(U) = (f_*\mathcal{O}_X)(U)^G = \{t : f^{-1}(U) \to \mathbb{A}^1 | t \text{ is } G\text{-invariant}\}.$$

The identification is as follows: Given a morphism $t : f^{-1}(U) \to \mathbb{A}^1$, the k-algebra map of global sections

$$t^\sharp(\mathbb{A}^1) : k[x] \to \Gamma(\mathbb{A}^1, t_*\mathcal{O}_X|_{f^{-1}(U)}) = f_*\mathcal{O}_X(U)$$

defines an element $t^\sharp(\mathbb{A}^1)(x) \in f_*\mathcal{O}_X(U)$. For each $g \in G$, the morphism $\mu(g) : X \to X$ restricts to morphisms $\mu(g) : f^{-1}(U) \to f^{-1}(U)$. The $\mathcal{O}_Y(U)$-module maps $\mu(g)^\sharp(U) : f_*\mathcal{O}_X(U) \to f_*\mathcal{O}_X(U)$ gives us a group action of G on $f_*\mathcal{O}_X(U)$. Choosing U to be affine, it is clear that t is a G-invariant morphism if and only if $t^\sharp(\mathbb{A}^1)(x)$ is a G-invariant element of $f_*\mathcal{O}_X(U)$. We sum these remarks up in a theorem.

Theorem 4.5.1. *Let $f : X \to Y$ be as above. We have an induced group action of G on the sheaf $f_*\mathcal{O}_X$, of which the G-invariants is the structure sheaf $\mathcal{O}_Y$ of Y.*

Now, suppose we have a right-G-action on a quasi-coherent sheaf $\mathcal{F}$ of $\mathcal{O}_Y$-modules, i.e. a right-action $G \to \operatorname{Aut}_{\mathcal{O}_Y}(\mathcal{F})$. For such sheaves, we may consider the functor $(-)^G$ sending $\mathcal{F}$ to $\mathcal{F}^G$. We will use the notation $(-)^G$ for the functor on modules as well as on sheaves.

Lemma 4.5.2. *Let A be a k-algebra, and G a finite group acting on A-modules M_0, M_1, M_2 via group actions $\mu_i : G \to \operatorname{Aut}(M_i)$. If we have an exact sequence*

$$0 \to M_0 \to^\psi M_1 \to^\phi M_2 \to 0$$

such that the action of G commutes with the homomorphisms of the sequence, i.e.

$$\psi \circ \mu_0 = \mu_1 \circ \psi \text{ and } \phi \circ \mu_1 = \mu_2 \circ \phi,$$

then, taking G-invariants, the sequence

$$0 \to M_0^G \to^{\psi} M_1^G \to^{\phi} M_2^G \to 0$$

is exact. We say that the functor of A-modules $(-)^G$ is exact.

Proof. Consider the Reynolds-operators

$$D_i = \frac{\sum_{g\in G} \mu_i(g)}{|G|} : M_i \to M_i.$$

The sequence

$$0 \to M_0^G \to^{\psi} M_1^G \to^{\phi} M_2^G \to 0$$

is clearly left-exact. To show that it is right-exact, we need to show that the map $M_1^G \to^{\phi} M_2^G$ is surjective. Let $y \in M_2^G$, and choose $x \in M_1$ such that $\phi(x) = y$. Then $\phi(D_1(x)) = D_2(\phi(x)) = y$, where $D_1(x) \in M_2^G$. It is easily seen to be exact in the middle. □

Remark 4.5.3. The proof of the exactness of $(-)^G$ above relies on the assumption that the ground field k is of characteristic 0, since we are utilizing the Reyonolds-operator which requires $|G| \in k$ to be invertible.

Theorem 4.5.4. *If G is finite, then the functor $(-)^G$ on sheaves of $\mathcal{O}_Y$-modules is exact. I.e. if we have an exact sequence of sheaves of $\mathcal{O}_Y$-modules*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

such that the action of G commutes with the morphisms in the sequence, then

$$0 \to (\mathcal{F}')^G \to (\mathcal{F})^G \to (\mathcal{F}'')^G \to 0$$

is exact.

Proof. We have an induced G-action on $\mathcal{F}_P$ for any $P \in Y$. We will show that $(\mathcal{F}_P)^G = (\mathcal{F}^G)_P$. It is clear that $(\mathcal{F}_P)^G \supseteq (\mathcal{F}^G)_P$. Now take an element $(s, V) \in (\mathcal{F}_P)^G$. Then for every $g \in G$,

$$\mu(g)(s, V) = (\mu(g)(s), V) = (s, V) \Rightarrow \mu(g)(s)|_{W_g} = s|_{W_g}$$

for some $W_g \subseteq V$. Take $W = \cap W_g$. Then $(s, V) = (s|_W, W) \in (\mathcal{F}^G)_P$. This applies to all sheaves of $\mathcal{O}_Y$-modules with G-action. Since the sequence of $\mathcal{O}_{Y,P}$-modules

$$0 \to \mathcal{F}'_P \to \mathcal{F}_P \to \mathcal{F}''_P \to 0$$

is exact,

$$0 \to (\mathcal{F}'_P)^G \to (\mathcal{F}_P)^G \to (\mathcal{F}''_P)^G \to 0$$

is exact by Lemma 4.5.2. Thus

$$0 \to (\mathcal{F}')^G_P \to (\mathcal{F})^G_P \to (\mathcal{F}'')^G_P \to 0$$

is exact, and we are done. □

Theorem 4.5.5. *We have an induced G-action on the n-th cohomology module $H^n(Y, \mathcal{F})$ for each n, and $H^n(Y, \mathcal{F})^G = H^n(Y, \mathcal{F}^G)$.*

Proof. Let $\mathcal{U} = \{U_i\}$ be an affine cover of Y. Consider the Čech-complex of $\mathcal{F}$:

$$\bigoplus_i \mathcal{F}(U_i) \to \bigoplus_{i<j} \mathcal{F}(U_{ij}) \to \cdots .$$

The terms of this complex has an induced G-action from that on $\mathcal{F}$ commuting with the boundary maps. Taking G-invariants, we get the Čech-complex of $\mathcal{F}^G$:

$$\bigoplus_i \mathcal{F}^G(U_i) \to \bigoplus_{i<j} \mathcal{F}^G(U_{ij}) \to \cdots .$$

By Lemma 4.5.2, $(-)^G$ is exact. It follows that the Čech-cohomology modules $\check{H}^n(\mathcal{U}, \mathcal{F})^G = \check{H}^n(\mathcal{U}, \mathcal{F}^G)$. Since Y is noetherian separated, and $\mathcal{F}$ quasi-coherent, the theorem follows. □

The following remarks will be useful. We have an induced action of G on $\mathrm{Hom}_Y(\mathcal{G}, \mathcal{F})$ for any sheaf $\mathcal{G}$ of $\mathcal{O}_Y$-modules. It is clear that the invariants $\mathrm{Hom}_Y(\mathcal{G}, \mathcal{F})^G = \mathrm{Hom}_Y(\mathcal{G}, \mathcal{F}^G)$, where we take G to act trivially of $\mathcal{G}$. Suppose now we have a G-action on (every sheaf in) a complex of quasi-coherent sheaves $\mathcal{F}^\bullet$ of $\mathcal{O}_Y$-modules such that the action commutes with the coboundary maps, i.e. $\mu(g) \circ d_n = d_n \circ \mu(g)$ for $d_n : \mathcal{F}^n \to \mathcal{F}^{n+1}$. Then we have an induced G-action on the cohomology of sheaves, $\mathcal{H}^n(F^\bullet)$. Since the functor $(-)^G$ is exact, we have clearly that $\mathcal{H}^n(F^\bullet)^G = \mathcal{H}^n((F^G)^\bullet)$.

Lemma 4.5.6. *Let B be an A-algebra, and M a B-module. Then*

$$\mathcal{H}^n(\mathcal{C}^\bullet(\mathrm{Spec}\, B, \mathrm{Spec}\, A, \tilde{M})),$$

where $\tilde{M}$ is the associated sheaf on $\mathrm{Spec}\, B$ to the module M, is isomorphic to the quasi-coherent $\mathcal{O}_{\mathrm{Spec}\, B}$-module associated to $H^n(A, B, M)$.

Proof. This is Proposition 96 p.315 in [And74]. □

Theorem 4.5.7. *Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. We have that $\mathcal{T}^n_{X/Y}(\mathcal{F}) = 0$ for all $n \geq 0$.*

Proof. By definition, $\mathcal{T}^n_{X/Y}(\mathcal{F}) = \mathcal{H}^n(\mathcal{C}^\bullet(X, Y, \mathcal{F}))$. Now, let $V = \operatorname{Spec} A$ be an affine subspace of Y. Let $U = f^{-1}(V) = \operatorname{Spec} B$, and $\mathcal{F}|_U = \tilde{M}$ for a B-module M. Then $\mathcal{H}^n(\mathcal{C}^\bullet(X, Y, \mathcal{F}))(U) = H^n(A, B, M)$ by Lemma 4.5.6. Thus $\mathcal{T}^n_{X/Y}(\mathcal{F})(U) = H^n(A, B, M)$. Since f is étale, it follows that $H^n(A, B, M) = 0$ for all $n \geq 0$ by Lemma 4.1.5. This can be done for an affine covering of Y. We conclude that $\mathcal{T}^n_{X/Y}(\mathcal{F}) = 0$ for all $n \geq 0$. □

Theorem 4.5.8. *Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. We have that $f_*\mathcal{T}^n_{X/k}(\mathcal{F}) = \mathcal{T}^n_{Y/k}(f_*\mathcal{F})$ for all $n \geq 0$.*

Proof. Let $V = \operatorname{Spec} A$ be an open affine subscheme of Y. Define U to be $f^{-1}(V) = \operatorname{Spec} B$, and let $\mathcal{F}|_U = \tilde{M}$ for a B-module M. M has a natural A-module structure. Consider the sequence $k \to A \to B$. By Lemma 4.5.6 we have the identifications:

$$\begin{aligned} &\mathcal{T}^n_{X/Y}(\mathcal{F})(U) = H^n(A, B, M), \\ &\mathcal{T}^n_{X/k}(\mathcal{F})(U) = H^n(k, B, M), \\ &\mathcal{T}^n_{Y/k}(f_*\mathcal{F})(V) = H^n(k, A, M). \end{aligned}$$

Using Lemma 4.1.4, we have an exact sequence

$$\begin{aligned} 0 \to H^0(A, B, M) \to \cdots \to H^{n-1}(k, A, M) \to \\ \to H^n(A, B, M) \to H^n(k, B, M) \to H^n(k, A, M) \to \cdots . \end{aligned}$$

I.e. an exact sequence

$$\begin{aligned} 0 \to f_*\mathcal{T}^0_{X/Y}(\mathcal{F})(V) \to \cdots \to \mathcal{T}^{n-1}_{Y/k}(f_*\mathcal{F})(V) \to \\ \to f_*\mathcal{T}^n_{X/Y}(\mathcal{F})(V) \to f_*\mathcal{T}^n_{X/k}(\mathcal{F})(V) \to \mathcal{T}^n_{Y/k}(f_*\mathcal{F})(V) \to \cdots . \end{aligned}$$

By Theorem 4.5.7, $f_*\mathcal{T}^n_{X/Y}(\mathcal{F})(V) = 0$ for all $n \geq 0$. Hence we have that $f_*\mathcal{T}^n_{X/k}(\mathcal{F})(V) = \mathcal{T}^n_{Y/k}(f_*\mathcal{F})(V)$. This applies to all affine open V, hence on an open affine cover of Y. We conclude that $f_*\mathcal{T}^n_{X/k}(\mathcal{F}) = \mathcal{T}^n_{Y/k}(f_*\mathcal{F})$. □

Corollary 4.5.9. *We have an induced G-action on $f_*\mathcal{T}^n_{X/k}$, and*

$$(f_*\mathcal{T}^n_{X/k})^G = \mathcal{T}^n_{Y/k}.$$

Proof. $f_*\mathcal{O}_X$ is clearly a quasi-coherent $\mathcal{O}_Y$-module, so $f_*\mathcal{T}^n_{X/k} = \mathcal{T}^n_{Y/k}(f_*\mathcal{O}_X)$ by Theorem 4.5.8, and

$$\mathcal{T}^n_{Y/k}(f_*\mathcal{O}_X) = \mathcal{H}^n(\mathcal{C}^\bullet(k, Y, f_*\mathcal{O}_X)) = \mathcal{H}^n(\mathrm{Hom}_Y(\mathcal{C}_\bullet(k, Y), f_*\mathcal{O}_X)).$$

Now we will use our remarks about the functor $(-)^G$. The G-action on $f_*\mathcal{O}_X$ induces a G-action on $\mathrm{Hom}_Y(\mathcal{C}_n(k, Y), f_*\mathcal{O}_X)$ for each $n \geq 0$, and

$$\mathrm{Hom}_Y(\mathcal{C}_n(k, Y), f_*\mathcal{O}_X)^G = \mathrm{Hom}_Y(\mathcal{C}_n(k, Y), (f_*\mathcal{O}_X)^G).$$

This in turn induces a G-action on complex $\mathrm{Hom}_Y(\mathcal{C}_\bullet(k, Y), f_*\mathcal{O}_X)$ such that action commutes with the coboundary maps. Thus we have an induced G-action on the cohomology sheaves $\mathcal{H}^n(\mathrm{Hom}_Y(\mathcal{C}_\bullet(k, Y), f_*\mathcal{O}_X))$, and

$$\begin{aligned}\mathcal{H}^n(\mathrm{Hom}_Y(\mathcal{C}_\bullet(k, Y), f_*\mathcal{O}_X))^G &= \mathcal{H}^n(\mathrm{Hom}_Y(\mathcal{C}_\bullet(k, Y), f_*\mathcal{O}_X)^G) \\ &= \mathcal{H}^n(\mathrm{Hom}_Y(\mathcal{C}_\bullet(k, Y), (f_*\mathcal{O}_X)^G)).\end{aligned}$$

Since $(f_*\mathcal{O}_X)^G = \mathcal{O}_Y$, we conclude that $(f_*\mathcal{T}^n_{X/k})^G = \mathcal{T}^n_{Y/k}$. □

Theorem 4.5.10. *$f_*\mathcal{T}^n_{X/k}$ is locally isomorphic to a direct sum of $|G|$ copies of $\mathcal{T}^n_{Y/k}$.*

Proof. Let $V = \operatorname{Spec} A$ be an affine subspace of Y such that $f_*\mathcal{O}_X(V)$ is a free $\mathcal{O}_Y(V)$-module. Let $U = f^{-1}(V) = \operatorname{Spec} B$. By Lemma 4.5.6 and Theorem 4.5.8

$$f_*\mathcal{T}^n_{X/k}(V) = \mathcal{T}^n_{Y/k}(f_*\mathcal{O}_X)(V) = H^n(k, A, B).$$

The result follows since the functor $H^n(k, A, -)$ commutes with direct sums. □

Since X and Y are noetherian separated and Proj of graded k-algebras, we may choose distinguished open subschemes as coverings such that any finite intersection is also affine. This means that the results of the previous sections applies to X over k, and Y over k (in particular Theorem 4.3.3).

4.6 The T^i-functors

Consider the situation where we have B a graded k-algebra, with B a quotient of $P = k[y_1, \dots y_n]$ with ideal I. Then we have the exact sequence

$$0 \to I \to P \to B \to 0.$$

Consider the conormal sequence ([Har10] p.31)

$$0 \leftarrow \Omega_{B/k} \leftarrow \Omega_{P/A} \otimes_P B \leftarrow I/I^2.$$

Let M be a B-module. Dualizing, we get the exact sequence

$$0 \rightarrow \mathrm{Hom}_B(\Omega_{B/k}, M) \rightarrow \mathrm{Hom}_B(\Omega_{P/k} \otimes_P B, M) \rightarrow \mathrm{Hom}_B(I/I^2, M).$$

In this setting, we will write

$$H^i(A, B, M) = T^i_{B/k}(M)$$

for $i = 0, 1, 2$. For $M = B$, we will simply denote it by $T^i_{B/k}$. We have the following identifications:

- $T^0_{B/k}(M) = \mathrm{Hom}_B(\Omega_{B/k}, M) = \mathrm{Der}_k(B, M)$, and
- $T^1_{B/k}(M) = \mathrm{coker}(\mathrm{Hom}_B(\Omega_{P/k} \otimes_P B, M) \rightarrow \mathrm{Hom}_B(I/I^2, M))$.

Suppose we have an action of a group G on B. For every $g \in G$, the homomorphism $\mu(g) : B \rightarrow B$ can be lifted to $\mu_P(g) : P \rightarrow P$. This induces a homomorphism $\mu_I(g) : I/I^2 \rightarrow I/I^2$. For any B-module homomorphism $\phi : I/I^2 \rightarrow B$, consider the B-module homomorphism $\mu_H(g)$ defined by

$$\mu_H(g)(\phi) = \mu(g) \circ \phi \circ \mu_I(g^{-1}) : I/I^2 \rightarrow B.$$

If ϕ is a homomorphism coming from $\mathrm{Hom}_B(\Omega_{P/k} \otimes_P B, B)$ (i.e. if ϕ is a B-module derivation), then we easily see that $\mu_H(\phi)$ also is a B-module derivation, thus coming from $\mathrm{Hom}_B(\Omega_{P/k} \otimes_P B, B)$. Now, we made a choice when lifting $\mu(g)$ to $\mu_P(g)$, but if $\mu_H(g)'$ comes from any other lifting $\mu_P(g)'$, then it is easily seen that for any homomorphism $\phi : I/I^2 \rightarrow B$, the homomorphism $\mu_H(g)'(\phi) - \mu_H(g)(\phi)$ is a derivation. Thus $\mu(g)$ induces a unique homomorphism $\mu_0(g) : T^1_{B/k}(M) \rightarrow T^1_{B/k}(M)$, after dividing out by the image of $\mathrm{Hom}_B(\Omega_{P/k} \otimes_P B, M)$ in $\mathrm{Hom}_B(I/I^2, M)$.

To see that this defines an action μ_0 of G on $T^1_{B/k}(M)$, we note that if $\mu_P(g^{-1})$ is a lifting of $\mu(g^{-1})$, and $\mu_P(h^{-1})$ a lifting of $\mu(h^{-1})$, then $\mu_P(h^{-1}) \circ \mu_P(g^{-1})$ is a lifting of $\mu(h^{-1}) \circ \mu(g^{-1}) = \mu(g^{-1}h^{-1})$. Thus by the uniqueness of the homomorphisms $\mu_0(g)$ for each $g \in G$, we can verify that $\mu_0(g) \circ \mu_0(h) = \mu_0(hg)$ as required.

Chapter 5

Stanley-Reisner schemes and geometric realization

5.1 Definitions

The definitions and results of this chapter are taken from [AC10]. Let M be a simplicial complex. Let the set of vertices be $M_0 = \{0, 1, \dots, n\}$, which we will denote by $[n]$. Any element of M can be written as a subset of $[n]$. We have the Stanley-Reisner ring

$$F(M) = k[x_0, ..., x_n]/I_M$$

of M, where I_M is the Stanley-Reisner ideal. $F(M)$ is $\mathbb{Z}^{n+1}$-graded; for any $\mathbf{a} = (a_0, a_1, \dots, a_n) \in \mathbb{Z}^{n+1}$, we have the monomial $x^{\mathbf{a}} = x_0^{a_0} \cdots x_n^{a_n} \in F(M)$ of degree $\mathbf{a}$. If $p = \{p_1, \dots, p_k\} \in M$, we will write $x_p = x_{p_1} \cdots x_{p_n}$. Since $F(M)$ is the Stanley-Reisner ring of M, the Stanley-Reisner scheme of M is $\mathbb{P}(M)$. We define the support of $\mathbf{a}$ as

$$a = \{i \in \{0, 1, \dots, n\} | \mathbf{a}_i \neq 0\}.$$

The link (which is a simplicial complex itself) of an element $a \in M$ is defined as

$$\text{lk}(a, M) = \{b \in M | b \cap a = \emptyset \text{ and } b \cup a \in M\}.$$

If M and N are two simplicial complexes, we define the join of M and N to be

$$M * N = \{f \sqcup g | f \in M, g \in N\}.$$

The geometric realization of a simplicial complex M, denoted $|M|$, is defined as

$$|M| = \{\alpha : [n] \to [0,1] | \{i | \alpha(i) \neq 0\} \in M \text{ and } \sum_i \alpha(i) = 1\}.$$

For a description of the geometric realization of Δ-sets, consider the geometric realization of Δ^n:

$$|\Delta^n| = \{\alpha : [n] \to [0,1] | \sum_i \alpha(i) = 1\}.$$

We have inclusion maps $\partial_i : |\Delta^{n-1}| \to |\Delta^n|$ for $0 \leq i \leq n$ defined by mapping a function $\alpha : [n-1] \to [0,1]$ to $\partial_i(\alpha) : [n] \to [0,1]$ defined by

$$\begin{aligned}
&\partial_i(\alpha)(j) = \alpha(j) \text{ for } 0 \leq j < i, \\
&\partial_i(\alpha)(i) = 0, \\
&\partial_i(\alpha)(j) = \alpha(j-1) \text{ for } i < j \leq n-1.
\end{aligned}$$

∂_i maps $|\Delta^{n-1}|$ to the face of $|\Delta^n|$ opposite to the i-th vertex. We define the geometric realization of a Δ-set M to be

$$|M| = \left(\coprod_{n=0}^{\infty} |\Delta^n| \times M_n \right) / \sim,$$

where $(\partial_i(\alpha), f) \sim (\alpha, d_i^M(f))$ for $\alpha \in |\Delta^{n-1}|$ and $f \in M_n$. Our definition differs slightly but is equivalent to that found in [Ran92]. It agrees with the original definition of the geometric realization of a simplicial complex.

Example 5.1.1. Consider the loop $\mathcal{L} = \{\{v\}, \{e\}\}$. Its geometric realization is computed as

$$|\mathcal{L}| = \left(\left(|\Delta^0| \times \mathcal{L}_0 \right) \times \left(|\Delta^1| \times \mathcal{L}_1 \right) \right) / \sim,$$

where for any $\alpha \in |\Delta^0|$, $(\partial_i(\alpha), e) \sim (\alpha, v)$. Note that $|\Delta^0|$, $\mathcal{L}_0$ and $\mathcal{L}_1$ are one-point sets, so we may write

$$|\mathcal{L}| = |\Delta^1| / \sim .$$

Representing α by the point $(x, y) = (\alpha(0), \alpha(1)) \in [0,1]^2$, the equivalence relation is given by $(1,0) \sim (0,1)$. Thus $|\mathcal{L}|$ is topologically homeomorphic to the circle S^1.

For definitions and results in PL-topology we refer to [Hud69]. A combinatorial n-sphere is a simplicial complex M that is PL-homeomorphic to $|\partial\Delta^{n+1}|$. A simplicial complex M is a combinatorial n-manifold if for all non-empty faces $f \in M, |\operatorname{lk}(f,M)|$ is a combinatorial sphere of dimension $n - \dim f - 1$. If we allow $|\operatorname{lk}(f,M)|$ to be a ball of dimension $n - \dim f - 1$, then M is a combinatorial manifold with boundary. We denote the boundary by

$$\partial M = \{f \in M || \operatorname{lk}(f,M)| \text{ is a ball}\}.$$

We call M a manifold if it is a combinatorial manifold without boundary. Similar notions may be found for the geometric realizations of Δ-sets in [Ran92], but that will not be relevant for us for now.

We also have the sets $\mathcal{B}(M)$ consisting of those $b \subseteq [n], |b| \geq 2$, with the properties

- $M = L * \partial b$ where $|L|$ is a $(n - |b| + 1)$-sphere if $b \notin M$,
- $M = L * \partial b \cup \partial L * b$ where $|L|$ is a $(n - |b| + 1)$-ball if $b \in M$.

For a simplicial complex M, consider free abelian group S_n over k generated by M_n for each n, called the simplicial n-chain of M. We may define boundary maps $\partial^n : S_n \to S_{n-1}$ by $f \mapsto \sum_{i=0}^{n}(-1)^i d_i(f)$. This forms a complex, and we may consider the homology groups $H^p(S_\bullet)$, denoted by $H^p(M;k)$.

Theorem 5.1.2. *If M is a simplicial complex, then*

$$H^p(\mathbb{P}(M), \mathcal{O}_{\mathbb{P}(M)}) \cong H^p(M;k),$$

where $H^p(M;k)$ is the p-th simplicial homology group.

Proof. This is Theorem 2.2 in [AC10]. □

Remark 5.1.3. Note that simplicial homology groups may be generalized to finite Δ-sets via the corresponding definitions. It is clear that if we have a quotient map $\pi : M \to N$ of a finite group G acting on M, then we have the equality $H^p(N;k) = H^p(M;k)^G$.

5.2 T^i-functors for Stanley-Reisner schemes

The i-th cohomology modules of the cotangent complex of $F(M)$ over k are $T^i_{F(M)/k}$. We will denote these by T^i_M. These modules inherit the grading of $F(M)$, and we may consider the $\mathbf{c} = \mathbf{a} - \mathbf{b}$ graded elements $T^i_{M,\mathbf{c}}$.

Theorem 5.2.1. *The homogeneous pieces in degree* $\mathbf{c} = \mathbf{a} - \mathbf{b}$ *(with disjoint supports* a *and* b*) of the cotangent cohomology of the Stanley-Reisner ring* $F(M)$ *vanish unless* $a \in M$*,* $\mathbf{b} \in \{0,1\}^{n+1}$*,* $b \subseteq [\mathrm{lk}(a)]$ *and* $b \neq \emptyset$*.*

Thus $T^i_{M,\mathbf{c}}$ only depend on the supports a and b, and we will denote it by $T^i_{M,a-b}$.

Theorem 5.2.2. *If* M *is a manifold and* $\mathbf{c} = \mathbf{a} - \mathbf{b}$ *(with disjoint supports* a *and* b*), then*

$$\dim_k T^1_{M,\mathbf{c}} = \begin{cases} 1 & \text{if } a \in M \text{ and } b \in \mathcal{B}(\mathrm{lk}(a,M)), \\ 0 & \text{otherwise.} \end{cases}$$

A basis for T^1_M *may be explicitly described: if* $\phi \in T^1_{M,\mathbf{c}} \neq 0$ *and* $x_p \in I_M$*, then* $\phi(x_p) = x^{\mathbf{a}} x_{p/b}$ *if* $b \subseteq p$ *and* 0 *otherwise.*

Theorem 5.2.3. *If* M *is a manifold, then*

- $H^0(\mathbb{P}(M), \mathcal{T}^1_{\mathbb{P}(M)/k}) = T^1_{M,0}$ *in the induced* $\mathbb{Z}$*-grading of* T^1_M*, and*
- $H^1(\mathbb{P}(M), \mathcal{T}^1_{\mathbb{P}(M)/k}) = 0$*.*

Chapter 6

Equivelar Δ-face abelian surfaces

6.1 Definition

Consider the tesselation $\{3, 6\}$ of $\mathbb{R}^2$. This naturally defines a Δ-set K by letting K_0 be the vertices, K_1 the edges and K_2 the triangular faces of the tesselation. The geometric realization $|K|$ is formed by gluing the triangles $|\Delta^2|$ together along the contacting edges in the tesselation, thus forming the plane $\mathbb{R}^2$. Let T be the translation group of $\{3, 6\}$, i.e. $\{(n, \frac{\sqrt{3}}{2}m) | (n, m) \in \mathbb{Z}^2\}$. T is in one-to-one correspondence with the vertices of K. Let Γ be a subgroup of T of finite index, i.e. such that the group quotient T/Γ is a finite group. The group Γ acts on K by a free action $\mu : \Gamma \to \mathrm{Aut}(K)$, and we may consider the categorical group quotient $\eta : K \to N$ of K by Γ. In this case we have an equivelar triangulation N of the torus.

If we let $v_1 = (1, 0)$, and $v_2 = (0, \frac{\sqrt{3}}{2})$, the generators of Γ may be written as $av_1 + bv_2$ and $cv_1 + dv_2$ (which we will denote by (a, b) and (c, d)) for integers a, b, c and d such that the determinant of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is non-zero. Now, let p be the least non-negative integer for which $(p, 0) \in \Gamma$, and q the least non-negative integer for which there exists a non-negative integer $r \leq p$ such that $(r, q) \in \Gamma$. We may then write the generators of Γ on the form $(p, 0)$ and (r, q). This defines a fundamental domain of K relative to Γ, which is a sub-Δ-set K' of K for which the restriction of $\eta' : K' \to N$ is surjective. See Figure 6.1. We see that N is a simplicial complex if and only if $p, q + r, \max(p - r, q) \geq 3$. In this case we say the triangulation is polyhedral. In both the polyhedral and the non-polyhedral case the geometric

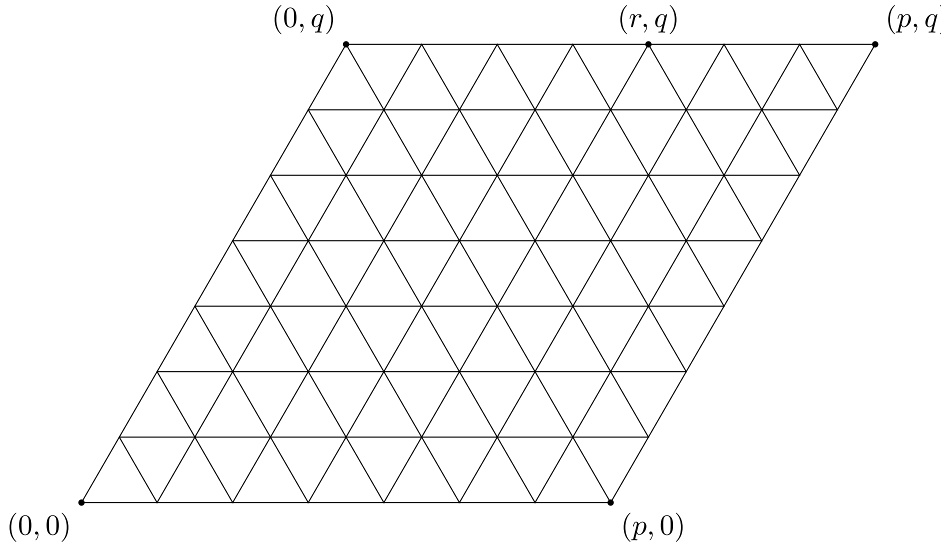


Figure 6.1: The fundamental domain K' of N. Points $(0, 0)$, $(p, 0)$ and (r, q) are identified. If $r = 0$ then all four corners are identified. Here N is a simplicial complex.

realization $|N|$ will be a topological torus. As a topological space, $|N|$ may be formed by dividing out the plane $|K|$ by the group action of the lattice Γ.

We will call the Δ-face schemes $\mathbb{P}(N)$ equivelar Δ-face abelian surfaces. We wish to study the deformations of $\mathbb{P}(N)$ for various subgroups Γ of finite index. For most choices of p, q and r, N will be a simplicial complex. In these cases, the deformations of the associated Stanley-Reisner scheme $\mathbb{P}(N)$ are known by the work of Christophersen (see [Chr10]). For the other cases (the non-polyhedral case), it will be helpful to define an intermediate group $\Gamma_0 \subseteq \Gamma$ of finite index, for which the associated Δ-set quotient M given by the quotient map $\tau : K \to M$ is a simplicial complex. Such a group Γ_0 is always easily found.

We have a naturally induced free group action of the finite group $G = \Gamma/\Gamma_0$ on M, such that $\pi : M \to N$ is the associated Δ-set quotient map. The benefit of this is that the morphism $\mathbb{P}(\pi) : \mathbb{P}(M) \to \mathbb{P}(N)$ is étale, where the deformations of $\mathbb{P}(M)$ are known. Here M is a combinatorial manifold.

For a simplicial complex M with Stanley-Reisner ring $F(M) = S/I_M$ in this setting, Christophersen describes $T^1_{M,0}$ explicitly. First we note that the link of an edge $\{a, b\} \in M$ must be the set of two vertices $\{\{i\}, \{j\}\}$. Note that the vertices a, b, i, j form a parallellogram. $T^1_{M,0}$ is generated by the homomorphisms $\phi_{a,b} : I_M/I_M^2 \to F(M)$ for each edge $\{a, b\}$ defined by

$$\phi_{a,b}(x_m) = \begin{cases} \frac{x_m x_a x_b}{x_i x_j} & \text{if } \{i, j\} \subseteq m, \\ 0 & \text{otherwise,} \end{cases}$$

where $x_m \in I_M$.

6.2 Computing the T^1-space

Our goal here is to find the T^1-space of $\mathbb{P}(N)$ for Δ-sets N. One problem is that the Δ-face ring $F(N)$ may not be a quotient of a polynomial ring with indeterminants of degree 1. A solution will be to consider subrings

$$F(N)^{[d]} = \bigoplus_{n=0} F(N)_{dn}$$

for integers d, for which $\operatorname{Proj} F(N)^{[d]} = \operatorname{Proj} F(N)$. For correctly chosen d, $F(N)^{[d]}$ may be written as a quotient of a graded ring of indeterminants of degree d. I.e. $\mathbb{P}(N)$ may be embedded into a smooth projective space $\mathbb{P}^n$ for some n. Note that this makes $\mathbb{P}(N)$ a proper scheme. We may use this presentation to compute $T^1_{F(N)^{[d]}/k}$.

Consider the sequence of k-algebra homomorphisms

$$k \to F(N)^{[d]} \to F(N) \to F(M)$$

with the graded rings $F(N)^{[d]}, F(N)$ and $F(M)$ respectively being quotients of polynomial rings

$$\begin{aligned} R_N &= k[z_1, ..., z_p], \\ P_N &= k[y_1, ..., y_m], \\ P_M &= k[x_1, ..., x_n], \end{aligned}$$

where the x_i's and z_i's are indeterminates of degree 1 (the degrees of the y_i's may vary). By Lemma 4.5.6 we have that for each distinguished open set $D_+(f)$ for some homogeneous $f \in F(M)_+$,

$$\mathcal{T}^1_{\mathbb{P}(M)/k}(U_i) = T^1_{F(M)_{(f)}/k}.$$

Take the finite open affine covering $\{U_i\}$ of $\mathbb{P}(M)$, where $U_i = D_+(x_i)$. An element of $H^0(\mathbb{P}(M), \mathcal{T}^1_{\mathbb{P}(M)/k})$ is represented by a cocycle

$$(\alpha_i) \in \bigoplus_i \mathcal{T}^1_{\mathbb{P}(M)/k}(U_i) = \bigoplus_i T^1_{F(M)_{(x_i)}/k}.$$

Now, suppose we have an element $\alpha \in T^1_{F(M)/k}$ of degree 0. α is represented by some element $\overline{\alpha} \in \mathrm{Hom}_{F(M)}(I/I^2, F(M))_0$. Then $\overline{\alpha}$ induces a homomorphism

$$\overline{\alpha_i} \in \mathrm{Hom}_{F(M)}(I_{(x_i)}/I^2_{(x_i)}, F(M)_{(x_i)})$$

for each i. Considering the exact sequence

$$0 \to I_{(x_i)} \to P_{M(x_i)} \to F(M)_{(x_i)} \to 0$$

(where $P_{M(x_i)}$ is a polynomial ring over k), we see that $\overline{\alpha_i}$ induces an element

$$\begin{aligned} \alpha_i \in \mathrm{coker}(&\mathrm{Hom}_{F(M)_{(x_i)}}(\Omega_{P_{M(x_i)}/k} \otimes_{P_{M(x_i)}} F(M)_{(x_i)}, F(M)_{(x_i)}) \\ &\to \mathrm{Hom}_{F(M)_{(x_i)}}(I_{(x_i)}/I^2_{(x_i)}, F(M)_{(x_i)})) = T^1_{F(M)_{(x_i)}/k}. \end{aligned}$$

Thus we obtain a homomorphism

$$T^i_{F(M)/k,0} \to H^0(\mathbb{P}(M), \mathcal{T}^1_{\mathbb{P}(M)/k}).$$

Now, we have that $\mathbb{P}(N) = \mathrm{Proj}\, F(N)^{[d]} = \mathrm{Proj}\, F(N)$, and the corresponding morphism $f = \mathbb{P}(\pi) : \mathbb{P}(M) \to \mathbb{P}(N)$. A similar line of argument as above gives us a homomorphism

$$T^i_{\mathrm{Proj}\, F(N)^{[d]}/k}(F(M))_0 \to H^0(\mathbb{P}(N), \mathcal{T}^1_{\mathbb{P}(N)/k}(f_* \mathcal{O}_{\mathbb{P}(M)})).$$

We have the exact sequences

$$\begin{aligned} 0 \to &J_N \to R_N \to F(N)^{[d]} \to 0, \\ 0 \to &I_N \to P_N \to F(N) \to 0, \\ 0 \to &I_M \to P_M \to F(M) \to 0. \end{aligned}$$

The homomorphism of k-algebras $F(N) \to F(M)$ lifts to a homomorphism of k-algebras $P_N \to P_M$, and restricts to a homomorphism $I_N \to I_M$. Similarly the homomorphism $F(N)^{[d]} \to F(N)$ lifts to a homomorphism $R_N \to P_N$; in total giving us a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J_N & \longrightarrow & R_N & \longrightarrow & F(N)^{[d]} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_N & \longrightarrow & P_N & \longrightarrow & F(N) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_M & \longrightarrow & P_M & \longrightarrow & F(M) & \longrightarrow & 0. \end{array}$$

This clearly gives us homomorphisms between T^1-modules of degree 0:

$$T^1_{F(M)/k,0} \to T^1_{F(N)/k}(F(M))_0 \to T^1_{F(N)^{[d]}/k}(F(M))_0,$$

and in fact a commutative diagram

$$\begin{array}{ccc} T^1_{F(N)^{[d]}/k}(F(M))_0 & \longrightarrow & H^0(\mathbb{P}(N), \mathcal{T}^1_{\mathbb{P}(N)/k}(f_*\mathcal{O}_{\mathbb{P}(M)})) \\ \uparrow & & \\ T^1_{F(N)/k}(F(M))_0 & & \uparrow \\ \uparrow & & \\ T^1_{F(M)/k,0} & \longrightarrow & H^0(\mathbb{P}(M), \mathcal{T}^1_{\mathbb{P}(M)/k}). \end{array}$$

Since $F(M)$ is the Stanley-Reisner ring of M, we know that the bottom horizontal arrow in the diagram above is an isomorphism. Since we have that $H^0(\mathbb{P}(M), \mathcal{T}^1_{\mathbb{P}(M)/k}) = H^0(\mathbb{P}(N), \mathcal{T}^1_{\mathbb{P}(N)/k}(f_*\mathcal{O}_{\mathbb{P}(M)}))$, we arrive at the commutative diagram

$$\begin{array}{ccc} T^1_{F(N)/k}(F(M))_0 & \longrightarrow & H^0(\mathbb{P}(N), \mathcal{T}^1_{\mathbb{P}(N)/k}(f_*\mathcal{O}_{\mathbb{P}(M)})) \\ \uparrow & & \uparrow \cong \\ T^1_{F(M)/k,0} & \xrightarrow{\cong} & H^0(\mathbb{P}(M), \mathcal{T}^1_{\mathbb{P}(M)/k}). \end{array}$$

Recall that $F(M)^G = F(N)$, so taking G-invariants we get a commutative diagram

$$\begin{array}{ccc} T^1_{F(N)/k,0} & \longrightarrow & H^0(\mathbb{P}(N), \mathcal{T}^1_{\mathbb{P}(N)/k}) \\ \uparrow & & \uparrow \cong \\ \left(T^1_{F(M)/k,0}\right)^G & \xrightarrow{\cong} & H^0(\mathbb{P}(M), \mathcal{T}^1_{\mathbb{P}(M)/k})^G. \end{array}$$

Recall how the action of G on $T^1_{F(M)/k,0}$ is defined in the section on the T^i-functors. We see that the map

$$T^1_{F(N)/k,0} \to H^0(\mathbb{P}(N), \mathcal{T}^1_{\mathbb{P}(N)/k})$$

is surjective. We sum these results up in a theorem.

Theorem 6.2.1. *Let M and N be finite Δ-sets, with M a simplicial complex, and $\pi : M \to N$ a categorical quotient of a finite group G acting freely on M. Then we may compute $H^0(\mathbb{P}(N), \mathcal{T}^1_{\mathbb{P}(N)/k})$ as the image of the composite of the two maps*

$$\left(T^1_{F(M)/k,0}\right)^G \to T^1_{F(N)/k,0} \to H^0(\mathbb{P}(N), \mathcal{T}^1_{\mathbb{P}(N)/k}).$$

In fact, the map

$$\left(T^1_{F(M)/k,0}\right)^G \to T^1_{F(N)/k,0}$$

allows us to compute first-order deformation rings of $F(N)$ yielding the first-order deformation schemes of $\mathbb{P}(N)$.

Example 6.2.2. We will compute $H^0(\mathbb{P}(\mathcal{L}), \mathcal{T}^1_{\mathbb{P}(\mathcal{L})/k})$ for the loop $\mathcal{L}$. We have the group quotient morphism $\pi : \mathbb{P}(T) \to \mathbb{P}(\mathcal{L})$. Hence we may compute it as the image of the composite of the maps

$$\left(T^1_{F(T)/k,0}\right)^G \to T^1_{F(\mathcal{L})/k,0} \to H^0(\mathbb{P}(\mathcal{L}), \mathcal{T}^1_{\mathbb{P}(\mathcal{L})/k}).$$

Now,

$$F(T) = k[x_0, x_1, x_2]/(x_0x_1x_2) = S/I_T,$$

and

$$F(\mathcal{L}) = k[t, u, v]/(tuv - u^3 - v^2) = P/I.$$

We have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & P & \longrightarrow & F(\mathcal{L}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I_T & \longrightarrow & S & \longrightarrow & F(T) & \longrightarrow & 0.
\end{array}$$

$\mathrm{Hom}_{F(T)}(I_T/I_T^2, F(T))$ is easily computed, it is simply isomorphic to $F(T)$ since I_T is generated by a single element $x_0x_1x_2$ of degree 3. Considering $\mathrm{Hom}_{F(T)}(I_T/I_T^2, F(T))_0$, we see that this is equal to the module of homomorphisms sending $x_0x_1x_2$ to an element of $F(T)$ of degree 3. Hence it is generated by the homomorphisms sending $x_0x_1x_2$ to either of

$$x_0^3, x_0^2x_1, x_0x_1^2, x_1^3, x_1^2x_2, x_1x_2^2, x_2^3, x_2^2x_0, x_2x_0^2.$$

However, the image of $\mathrm{Hom}_{F(T)}(\Omega_{S/k} \otimes_S F(T), F(T))$ in $\mathrm{Hom}_{F(T)}(I_T/I_T^2, F(T))$ is generated by the differentials $\frac{\partial}{\partial x_i}$ for $i = 0, 1, 2$. Since

$$\frac{\partial}{\partial x_0} x_0 x_1 x_2 = x_1 x_2,$$
$$\frac{\partial}{\partial x_1} x_0 x_1 x_2 = x_0 x_2,$$
$$\frac{\partial}{\partial x_2} x_0 x_1 x_2 = x_0 x_1,$$

we are left with the homomorphisms

$$\phi_i : I_T/I_T^2 \to F(T) \text{ such that } x_0 x_1 x_2 \mapsto x_i^3 \text{ for } i = 0, 1, 2$$

in $T^1_{F(T)/k,0}$. The G-invariants here are generated by the the single homomorphism

$$\phi = \phi_0 + \phi_1 + \phi_2 \text{ such that } x_0 x_1 x_2 \mapsto x_0^3 + x_1^3 + x_2^3$$

(we are applying the Reyonolds-operator here, ignoring the factor $\frac{1}{|G|}$). Considering $F(\mathcal{L}) \subseteq F(T)$, we may write

$$x_0^3 + x_1^3 + x_2^3 = t^3 - 3tu.$$

Considering $P \subseteq S$, we may write

$$tuv - u^3 - v^2 = x_0 x_1 x_2 (t^3 - 3v - 3ut) - 3x_0^2 x_1^2 x_2^2.$$

Hence the induced map $\overline{\phi} \in T^1_{F(\mathcal{L})/k,0}$ is the one such that

$$tuv - u^3 - v^2 \mapsto (t^3 - 3tu)(t^3 - 3v - 3ut) = t(t^2 - 3u)(t^3 - 3v - 3ut).$$

This induces the single generating element of $H^0(\mathbb{P}(\mathcal{L}), \mathcal{T}^1_{\mathbb{P}(\mathcal{L})/k})$. The first-order deformation rings are thus

$$F(\mathcal{L})_a = k[t, u, v][\epsilon]/(tuv - u^3 - v^2 + a\epsilon t(t^2 - 3u)(t^3 - 3v - 3ut))$$

where $k[\epsilon]/(\epsilon^2)$ is the ring of dual numbers, and $a \in k$. For the corresponding first-order deformation schemes we may look at the representation of $\mathbb{P}(\mathcal{L})$ as

$$V(y^3 - xyz + xz^2) \subseteq \mathbb{P}^2,$$

and we find that it corresponds to the graded rings

$$k[x, y, z, \epsilon]/(y^3 - xyz + xz^2 - a\epsilon x(x - 3y)(x - 3y - 3z)).$$

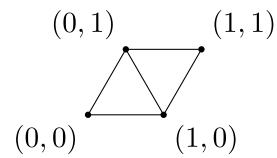


Figure 6.2: The fundamental domain K' of $\mathcal{N}$. Points $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$ are identified. Parallell edges are also identified. $\mathcal{N}$ has one vertex, three edges and two faces.

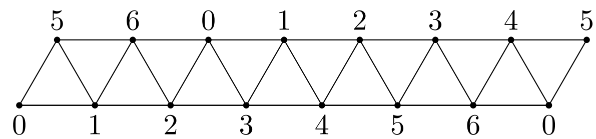


Figure 6.3: The fundamental domain of M. Points with identical labels are identified. M has 7 vertices, 21 edges and 14 faces.

6.3 An example computation

Our example will be the Δ-set $\mathcal{N}$ defined as the quotient of K by the entire group $\Gamma = T$. We choose our intermediate group Γ_0 to be the subgroup spanned by the vectors $(7,0)$ and $(1,2)$. $M = K/\Gamma_0$ will be a simplicial complex, and the minimal trianglulation of the torus. Fundamental domains for $\mathcal{N}$ and M are shown in figures 6.2 and 6.3.

We have the categorical quotient $\pi : M \to \mathcal{N}$ of the finite group $G = \Gamma/\Gamma_0$ by a free action on M. Since M is a simplicial complex, $F(M)$ will be the Stanley-Reisner ring

$$F(M) = k[x_0, ..., x_6]/I_M,$$

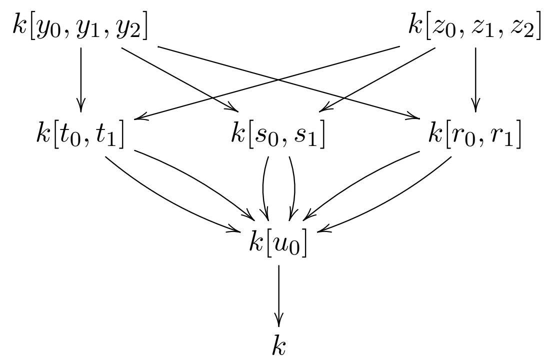
where

$$\begin{aligned} I_M = (&x_0x_1x_2,\ x_0x_1x_4,\ x_0x_1x_6.\ x_0x_2x_4,\ x_0x_2x_5,\ x_0x_3x_4,\\ &x_0x_3x_5,\ x_0x_3x_6,\ x_0x_5x_6,\ x_1x_2x_3,\ x_1x_2x_5,\ x_1x_3x_5,\\ &x_1x_3x_6,\ x_1x_4x_5,\ x_1x_4x_6,\ x_2x_3x_4,\ x_2x_3x_6,\ x_2x_4x_6,\\ &x_2x_5x_6,\ x_3x_4x_5,\ x_4x_5x_6). \end{aligned}$$

The fundamental domain K' of $\mathcal{N}$ is the parallellogram as shown in the figure. We label the vertices $(0,0), (1,0), (0,1)$ and $(1,1)$ by x, y, z and w. We have

$$F(K') = k[x, y, z, w]/(xw).$$

Now, $\mathcal{N}_0 = \{p\}$, $\mathcal{N}_1 = \{e_1, e_2, e_3\}$, and $\mathcal{N}_2 = \{f_1, f_2\}$. Since $K' \to \mathcal{N}$ is surjective, we have $F(\mathcal{N}) \subseteq k[x, y, z, w]/(xw)$. $F(\mathcal{N})$ may be computed as the limit of the diagram



corresponding to the diagram of Δ-simplices $H_{\mathcal{N}} : I_{\mathcal{N}} \to \Delta$-sets for $\mathcal{N}$. Thus an element of $F(\mathcal{N})$ is on the form $(f_1(y_0, y_1, y_2), f_2(z_0, z_1, z_2))$ for certain conditions on the polynomials f_1 and f_2. Interpreting these conditions for polynomials $f(x, y, z, w) \in k[x, y, z, w]/(xw)$, we have the following criteria for f to be an element of $F(\mathcal{N})$:

$$
\begin{aligned}
&f(x, y, 0, 0) = f(0, 0, x, y), \\
&f(x, 0, y, 0) = f(0, x, 0, y), \\
&f(0, x, y, 0) = f(0, x, y, 0), \\
&f(x, 0, 0, 0) = f(0, x, 0, 0) = f(0, 0, x, 0) = f(0, 0, 0, x).
\end{aligned}
$$

By direct computation we end up with the subring of $k[x, y, z, w]/(xw)$ generated by the 11 elements:

$$
\begin{aligned}
t &= x + y + z + w, \\
u_1 &= xy + zw, \ v_1 = x^2y + z^2w, \\
u_2 &= yz, \ v_2 = y^2z, \\
u_3 &= xz + yw, \ v_3 = x^2z + y^2w, \\
s_0 &= xyz, \ s_1 = yzw, \ s_2 = x^2yz, \ s_3 = xy^2z.
\end{aligned}
$$

(see the Appendix) The kernel I of the homomorphism

$$P = k[t, u_1, v_1, u_2, v_2, u_3, v_3, s_0, s_1, s_2, s_3] \to k[x, y, z, w]/(xw).$$

may be computed, but is too large to include here. It is generated by 37 elements. We have an injection $K' \to M$ sending the vertices labeled x, y, z, w

to the vertices labeled $0, 1, 5, 6$ respectively. Consider the commutative diagram

$$\begin{array}{ccc} K' & \longrightarrow & M \\ \downarrow & \swarrow & \\ \mathcal{N}. & & \end{array}$$

This induces a commutative diagram

$$\begin{array}{ccc} F(K') & \longleftarrow & F(M) \\ \uparrow & \nearrow & \\ F(\mathcal{N}). & & \end{array}$$

Now consider the homomorphism

$$F(\mathcal{N}) = P/I \to F(M).$$

The induced action of G on the index set of the vertices, $M_0 = \{0, 1, ..., 6\}$, is such that $F(\mu(g))(x_i) = x_{g^{-1}i}$ for every $g \in G$. In order for the diagram to commute, it is easy to see that the homomorphism is given by

$$t \mapsto \sum_{g \in G} x_{g0} = x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6,$$

$$u_1 \mapsto \sum_{g \in G} x_{g0}x_{g1} = x_0x_1 + x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_6 + x_6x_0,$$

$$v_1 \mapsto \sum_{g \in G} x_{g0}^2x_{g1} = x_0^2x_1 + x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_5 + x_5^2x_6 + x_6^2x_0,$$

$$u_2 \mapsto \sum_{g \in G} x_{g1}x_{g5} = x_1x_5 + x_2x_6 + x_3x_0 + x_4x_1 + x_5x_2 + x_6x_3 + x_0x_4,$$

$$v_2 \mapsto \sum_{g \in G} x_{g1}^2x_{g5} = x_1^2x_5 + x_2^2x_6 + x_3^2x_0 + x_4^2x_1 + x_5^2x_2 + x_6^2x_3 + x_0^2x_4,$$

$$u_3 \mapsto \sum_{g \in G} x_{g0}x_{g5} = x_0x_5 + x_1x_6 + x_2x_0 + x_3x_1 + x_4x_2 + x_5x_3 + x_6x_4,$$

$$v_3 \mapsto \sum_{g \in G} x_{g0}^2x_{g5} = x_0^2x_5 + x_1^2x_6 + x_2^2x_0 + x_3^2x_1 + x_4^2x_2 + x_5^2x_3 + x_6^2x_4,$$

$$s_0 \mapsto \sum_{g \in G} x_{g0}x_{g1}x_{g5} = x_0x_1x_5 + x_1x_2x_6 + x_2x_3x_0 + x_3x_4x_1 + x_4x_5x_2 + x_5x_6x_3 + x_6x_0x_4,$$

$$s_1 \mapsto \sum_{g \in G} x_{g1}x_{g5}x_{g6} = x_1x_5x_6 + x_2x_6x_0 + x_3x_0x_1 + x_4x_1x_2 + x_5x_2x_3 + x_6x_3x_4 + x_0x_4x_5,$$

$$s_2 \mapsto \sum_{g\in G} x_{g0}^2 x_{g1} x_{g5} = x_0^2 x_1 x_5 + x_1^2 x_2 x_6 + x_2^2 x_3 x_0 + x_3^2 x_4 x_1 + x_4^2 x_5 x_2 + x_5^2 x_6 x_3 + x_6^2 x_0 x_4,$$
$$s_3 \mapsto \sum_{g\in G} x_{g0} x_{g1}^2 x_{g5} = x_0 x_1^2 x_5 + x_1 x_2^2 x_6 + x_2 x_3^2 x_0 + x_3 x_4^2 x_1 + x_4 x_5^2 x_2 + x_5 x_6^2 x_3 + x_6 x_0^2 x_4.$$

$T^1_{M,0}$ is generated by the homomorphisms $\phi_{a,b}$ for edges $\{a,b\}$ in M as described above. Let $g \in G$. The action of g on $\phi_{a,b}$ is

$$g\phi_{a,b} = F(\mu(g)) \circ \phi_{a,b} \circ \mu_{I_M}(g^{-1}).$$

So

$$g\phi_{a,b}(x_m) = F(\mu(g))(\frac{x_{gm}x_a x_b}{x_i x_j}) = \frac{x_m x_{g^{-1}a} x_{g^{-1}b}}{x_{g^{-1}i} x_{g^{-1}j}}$$

when $\{g^{-1}i, g^{-1}j\} \subseteq m$, and 0 otherwise. Hence

$$g\phi_{a,b} = \phi_{g^{-1}a, g^{-1}b}.$$

Note that $\{i,j\} \subseteq m \Leftrightarrow \{gi, gj\} \subseteq gm$. Applying the Reyonolds-operator, we get a homomorphism such that

$$x_m \mapsto \sum_{g\in G} \phi_{ga,gb}(x_m) = \sum_{\substack{g\in G \\ \{gi,gj\}\subseteq m}} \frac{x_m x_{ga} x_{gb}}{x_{gi} x_{gj}}.$$

This yields three isomorphism classes, one for each equivalence class of edges $\{a,b\}$ where $\{a,b\} \sim \{ga, gb\}$, which corresponds to the edges $\{e_1, e_2, e_3\}$ of $\mathcal{N}$. Thus, in $T^1_{F(\mathcal{N})/k,0}$, we are left with the images $\psi_{e_1}, \psi_{e_2}, \psi_{e_3}$ of the three homomorphisms

$$\phi_{e_1} = \sum_{g\in G} g\phi_{0,1} : I_M/I_M^2 \to F(M),$$
$$\phi_{e_2} = \sum_{g\in G} g\phi_{0,5} : I_M/I_M^2 \to F(M),$$
$$\phi_{e_3} = \sum_{g\in G} g\phi_{1,5} : I_M/I_M^2 \to F(M).$$

Given an element $f \in I$, in order to compute, say $\psi_{e_1}(f)$, we will have to look at its image

$$\overline{f} = \sum_{j=0}^{k} r_{m_j} x_{m_j}$$

in I_M via the chosen lifting of $F(\mathcal{N}) \to F(M)$. Here x_{m_j} are the generators of I_M. Note that $\text{lk}(\{0,1\}) = \{\{5\},\{3\}\}$. Thus

$$\begin{aligned}
\psi_{e_1}(f) = \phi_{e_1}(\overline{f}) &= \sum_{g \in G} g\phi_{0,1}(\overline{f}) \\
&= \sum_{g \in G} g\phi_{0,1}(\sum_{j=0}^{k} r_{m_j} x_{m_j}) \\
&= \sum_{g \in G} \sum_{\substack{j=0 \\ \{g5,g3\} \subseteq m_j}}^{k} r_{m_j} \frac{x_{m_j} x_{g0} x_{g1}}{x_{g5} x_{g3}} \in F(\mathcal{N}) \subseteq F(M).
\end{aligned}$$

Likewise we may compute $\psi_{e_2}(f)$ and $\psi_{e_3}(f)$.

6.4 The general case

The above example elucidates the general case which we will explain here. Consider any Δ-set N defined as the quotient of K by a group $\Gamma \subseteq T$ of finite index. We choose an intermediate group $\Gamma_0 \subseteq \Gamma$ such that $M = K/\Gamma_0$ is a simplicial complex. Let $F(M) = S/I_M$ be the Stanley-Reisner ring (where $S = k[x_0, \ldots, x_n]$), and $F(N) = P/I$ for a polynomial ring P. $\pi : M \to N$ is the quotient map of the group $G = \Gamma/\Gamma_0$ acting on M. We have that $T^1_{F(M)/k} = H^0(\mathbb{P}(M), \mathcal{T}^1_{\mathbb{P}(M)/k})$ is generated by the homomorphisms $\phi_e : I_M/I_M^2 \to F(M)$ for each edge $e \in M_1$. G acts on $T^1_{F(M)/k}$ such that $g\phi_e = \phi_{g^{-1}e}$, and the images of the homomorphisms

$$\phi_{e_N} = \sum_{e \in \pi^{-1}(e_N)} \phi_e$$

for each edge $e_N \in N_1$ will generate $H^0(\mathbb{P}(N), \mathcal{T}^1_{\mathbb{P}(N)/k})$. The diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & P & \longrightarrow & F(N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I_M & \longrightarrow & S & \longrightarrow & F(M) & \longrightarrow & 0
\end{array}$$

(for a chosen lifting of the map $F(N) \to F(N)$) allows us to compute the images $\psi_{e_N} \in T^1_{F(N)/k}$ of the homomorphisms ϕ_{e_N}.

Write $I_M = (x_{m_j})_{j=0}^k$. For any given $f \in I$, compute its image

$$\overline{f} = \sum_{j=0}^{k} r_{m_j} x_{m_j}$$

in I_M. Write $\text{lk}(e) = \{\{l_1^e\}, \{l_2^e\}\}$ for each edge $e \in M_1$. Then

$$\begin{aligned}
\psi_{e_N}(f) = \phi_{e_N}(\overline{f}) &= \phi_{e_N}(\sum_{j=0}^{k} r_{m_j} x_{m_j}) \\
&= \sum_{e \in \pi^{-1}(e_N)} \sum_{j=0}^{k} r_{m_j} \phi_e(x_{m_j}) \\
&= \sum_{e \in \pi^{-1}(e_N)} \sum_{\substack{j=0 \\ \{l_1^e, l_2^e\} \subseteq m_j}}^{k} r_{m_j} \frac{x_{m_j} x_e}{x_{l_1^e} x_{l_2^e}}.
\end{aligned}$$

Note that if we pick M large enough, I_M will be generated by elements on the form $x_a x_b$ for vertices a, b. In particular, we will for every edge $e \in M_1$ have the special generator $x_{l_1^e} x_{l_2^e}$ where $\text{lk}(e) = \{\{l_1^e\}, \{l_2^e\}\}$. If x_{m_j} is a generator of I_M, we see that $\{l_1^e, l_2^e\} \subseteq m_j \Leftrightarrow \{l_1^e, l_2^e\} = m_j$. Thus we arrive at the formula

$$\psi_{e_N}(f) = \sum_{e \in \pi^{-1}(e_N)} r_{\{l_1^e, l_2^e\}} x_e.$$

6.5 The deformation functor

Let M be a polyhedral triangulation of the torus. $\mathbb{P}(M)$ comes equipped with the very ample line bundle $\mathcal{L}_M = \mathcal{O}_{\mathbb{P}(M)}(1)$. The deformation functor $\text{Def}_{(\mathbb{P}(M), \mathcal{L}_M)}$ was studied in [Chr10]. A versal deformation of X was found. Let

$$R = k[t_e | e \in M_1],$$

and I be the ideal generated by the 2×2 minors of the matrices

$$\begin{bmatrix} t_{p,\tau_1} & t_{p,\tau_2} & t_{p,\tau_3} \\ t_{p,-\tau_1} & t_{p,-\tau_2} & t_{p,-\tau_3} \end{bmatrix}$$

for each $p \in M_0$. Here $(p, \pm\tau_i)$ corresponds to the six different edges in M with a vertex in p: $\tau_1 = (1, 0)$, $\tau_2 = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$, and $\tau_3 = (-\frac{1}{2}, -\frac{\sqrt{3}}{2})$.

Taking the I-adic completion $\hat{R}$ of R defines a versal base space $\operatorname{Spec} \hat{R}$ for the deformation functor.

Now, let N be the quotient of K by a group $\Gamma \subseteq T$ of finite index, and let $\pi : M \to N$ be a group quotient map by the group $G = \Gamma/\Gamma_0$ acting on a simplicial complex M, and $f = \mathbb{P}(\pi) : \mathbb{P}(M) \to \mathbb{P}(N)$ the corresponding group quotient morphism. We refer to the Appendix for the fact that $\mathcal{L}_N = \mathcal{O}_{\mathbb{P}(N)}(1)$ is an invertible sheaf of $\mathcal{O}_{\mathbb{P}(N)}$-modules. It is clear from the construction that we can find an open cover $D_+(f_i)$ of $\mathbb{P}(N)$ for homogeneous $f_i \in F(N)_+$ such that the induced map $f^*\mathcal{L}_N \to \mathcal{L}_M$ is an isomorphism on the open cover $f^{-1}(D_+(f_i)) = D_+(F(\pi)(f_i))$ of $\mathbb{P}(M)$. Since f is finite and surjective, we conclude that $\mathcal{L}_N$ is an ample invertible sheaf (Ex. 5.7 Chapter III in [Har77]). Furthermore, it is easily seen that $\mathcal{L}_N = (f_*\mathcal{L}_M)^G$ via the induced action of G on $f_*\mathcal{L}_M$. Now, let

$$F(M) = S/I_M = k[x_p | p \in M_0]/I_M$$

be the Stanley-Reisner ring of M. The induced action $F(\mu) : G \to \operatorname{Aut}(F(M))$ of G on $F(M)$ permutes the variables, such that $F(\mu(g))(x_p) = x_{g^{-1}p}$. It is clear that the global sections of $\mathcal{L}_M$ is represented by the variables x_p for each $p \in M_0$, so the induced action of G on $H^0(\mathbb{P}(M), \mathcal{L}_M) = H^0(\mathbb{P}(N), f_*\mathcal{L}_M)$ similarly permutes the variables, hence

$$H^0(\mathbb{P}(N), \mathcal{L}_N) = H^0(\mathbb{P}(M), (f_*\mathcal{L}_M)^G) = H^0(\mathbb{P}(M), f_*\mathcal{L}_M)^G$$

is the vector space spanned by the global sections $\sum_{g \in G} x_{gp}$. Thus the global sections of $\mathcal{L}_N$ are in one-to-one correspondence with the vertices of N. We sum this up in a theorem.

Theorem 6.5.1. *If N is the quotient of K by some subgroup $\Gamma \subseteq T$ of finite index, then $\mathcal{L}_N$ is an ample invertible sheaf such that $\dim_k H^0(\mathbb{P}(N), \mathcal{L}_N)$ is equal to the number of vertices of N.*

The object of interest for us is thus the deformation functor $\operatorname{Def}_{(\mathbb{P}(N), \mathcal{L}_N)}$.

6.6 Further study

- The next step is to consider the question of finding a versal formal deformation $(\mathcal{X}, \mathcal{L}_{\mathcal{X}})$ of $(\mathbb{P}(N), \mathcal{L}_N)$. The hope is that the versal base space will have the same structure as for polyhedral deformations. The definition makes sense for Δ-sets as well. For example, for $\mathcal{N}$, we will have $R = k[t_{e_1}, t_{e_2}, t_{e_3}]$, with the ideal I generated by the 2×2 minors of the matrix

$$\begin{bmatrix} t_{p,\tau_1} & t_{p,\tau_2} & t_{p,\tau_3} \\ t_{p,-\tau_1} & t_{p,-\tau_2} & t_{p,-\tau_3} \end{bmatrix}$$

for the single point p. The matrix is identified as

$$\begin{bmatrix} t_{e_1} & t_{e_2} & t_{e_3} \\ t_{e_1} & t_{e_2} & t_{e_3} \end{bmatrix}.$$

So $I = 0$, and the versal base space should be the completion of $\operatorname{Spec} R = \mathbb{A}^3$ along the point $0 \in \mathbb{A}^3$.

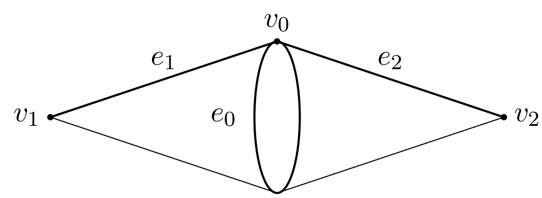
We note the following. Suppose that $\pi : M \to N$ is a group quotient as before, where both M and N are polyhedral triangulations. Let I_1 be the ideal in $R_1 = k[t_e | e \in M_1]$, and I_2 the ideal in $R_2 = k[t_e | e \in N_1]$ defining the versal base space for the deformations of $\mathbb{P}(M)$ and $\mathbb{P}(N)$ respectively. Consider the ideal $J = I_1 + (\{t_e - t_{e'} | \pi(e) = \pi(e')\})$. Then we have $R_1/J \cong R_2/I_2$. It is a reasonable assumption the same would apply for the case where N is a non-polyhedral triangulation, thus extending the result to this case as well.

- The fact that the line bundle $\mathcal{L}_N = \mathcal{O}_{\mathbb{P}(N)}(1)$ is ample for every triangulation N of the torus, suggests that the $\mathbb{P}(N)$'s corresponds to interesting points on the boundary of the moduli space of polarized abelian surfaces. The ample, but not necessarily very ample line bundle $\mathcal{L}_N$ should have the necessary properties that corresponds to a polarization. The question of whether $\mathcal{O}_{\mathbb{P}(N)}(1)$ is ample for any finite Δ-set N should be considered as well.

- Another question is what the homology modules $H^p(N; k)$ as described in Remark 5.1.3 represents. We note that with a categorical group quotient $\pi : M \to N$ of a finite group G acting freely on a finite simplicial complex M, we have $H^p(\mathbb{P}(M), \mathcal{O}_{\mathbb{P}(M)})^G = H^p(\mathbb{P}(N), \mathcal{O}_{\mathbb{P}(N)})$, and the equality $H^p(M; k)^G = H^p(N; k)$. Since $H^p(M; k) \cong H^p(\mathbb{P}(M), \mathcal{O}_{\mathbb{P}(M)})$, it is reasonable to suspect that we have $H^p(N; k) \cong H^p(\mathbb{P}(N), \mathcal{O}_{\mathbb{P}(N)})$ as well.

- An idea to create Δ-sets from simplicial complexes is to consider the notion of a universal covering space of a Δ-set. The tesselation $\{3, 6\}$ (with the accessible structure as an infinite ordered simplicial complex) is in a sense a covering space for the equivelar triangulations of the torus. One could hope to find a method of constructing covering spaces for any Δ-set symmetric enough for it to be possible to divide out with a group (preferably with a free group action) to end up with the Δ-set in question.

- We have considered triangulations of the torus, but there are other natural situations to consider, for example non-polyhedral triangulations of the sphere. One example is the following Δ-set, the double cone $\mathcal{D}$:



It is defined by $\mathcal{D}_0 = \{v_0, v_1, v_2\}$, $\mathcal{D}_1 = \{e_0, e_1, e_2\}$, and $\mathcal{D}_2 = \{f_0, f_1\}$, with edge maps illustrated by the figure. $\mathcal{D}$ has the geometric realization of the sphere S^2.

Chapter 7

Appendix

7.1 Explicit representations of the Δ-face rings

We will here give representations of the Δ-face rings $F(N)$ for all quotients $\pi : K \to N$ of K by a group $\Gamma \subseteq T$ of finite index. As before, we let Γ be generated by the elements $(p, 0)$ and (r, q), with an associated fundamental domain K' of N. The surjection $K' \to N$ makes $F(N)$ a subring of $F(K')$, which may be written as

$$F(K') = S/I = k[x_{ij} | 0 \leq i \leq p, 0 \leq j \leq q]/(\{x_{ij} | \{i, j\} \notin K'_1\}).$$

Let $f \in F(K')$. We may write

$$\begin{aligned} f =& a + \sum_{i=0}^{p} \sum_{j=0}^{q} x_{ij} f_{ij}(x_{ij}) + \sum_{i=0}^{p-1} \sum_{j=0}^{q} x_{ij} x_{i+1j} g_{ij}(x_{ij}, x_{i+1j}) \\ &+ \sum_{i=0}^{p} \sum_{j=0}^{q-1} x_{ij} x_{ij+1} h_{ij}(x_{ij}, x_{ij+1}) + \sum_{i=1}^{p} \sum_{j=0}^{q-1} x_{ij} x_{i-1j+1} s_{ij}(x_{ij}, x_{i-1j+1}) + H, \end{aligned}$$

where $f_{ij}, g_{ij}, h_{ij}, s_{ij}$ are polynomials, a a constant and H a polynomial in which every monomial term is a product of at least three different variables. As a convention we will consider an index (i, j) as $(i - p, j)$ if $p < i < 2p$, and $(i, j - q)$ if $q < j < 2q$. The conditions for $f \in F(N)$ are the following:

1) $f_{i0}(x) = f_{i+r,q}(x)$ for $0 \leq i \leq p$,
2) $f_{0j}(x) = f_{pj}(x)$ for $0 \leq j \leq q$,
3) $g_{i0}(x, y) = g_{i+r,q}(x, y)$ for $0 \leq i \leq p - 1$,
4) $h_{0j}(x, y) = h_{pj}(x, y)$ for $0 \leq j \leq q - 1$.

Now we split up in two cases.

7.1.1 Case 1 : $r = 0$

If f satisfy these four conditions, we may write

$$
\begin{aligned}
f =& a + \sum_{i=1}^{p-1} \sum_{j=1}^{q-1} x_{ij} f_{ij}(x_{ij}) + \sum_{i=0}^{p-1} \sum_{j=1}^{q-1} x_{ij} x_{i+1j} g_{ij}(x_{ij}, x_{i+1j}) \\
&+ \sum_{i=1}^{p-1} \sum_{j=0}^{q-1} x_{ij} x_{ij+1} h_{ij}(x_{ij}, x_{ij+1}) + \sum_{i=1}^{p} \sum_{j=0}^{q-1} x_{ij} x_{i-1j+1} s_{ij}(x_{ij}, x_{i-1j+1}) + H \\
&+ x_{00} f_{00}(x_{00}) + x_{p0} f_{00}(x_{p0}) + x_{0q} f_{00}(x_{0q}) + x_{pq} f_{00}(x_{pq}) \\
&+ \sum_{j=1}^{q-1} \left(x_{0j} f_{0j}(x_{0j}) + x_{pj} f_{0j}(x_{pj})\right) + \sum_{i=1}^{p-1} \left(x_{i0} f_{i0}(x_{i0}) + x_{iq} f_{i0}(x_{iq})\right) \\
&+ \sum_{i=0}^{p-1} \left(x_{i0} x_{i+10} g_{i0}(x_{i0}, x_{i+10}) + x_{iq} x_{i+1q} g_{i0}(x_{iq}, x_{i+1q})\right) \\
&+ \sum_{j=0}^{q-1} \left(x_{0j} x_{0j+1} h_{0j}(x_{0j}, x_{0j+1}) + x_{pj} x_{pj+1} h_{0j}(x_{pj}, x_{pj+1})\right).
\end{aligned}
$$

Assume now $p, q \geq 2$. Then

$$
\begin{aligned}
& x_{00} f_{00}(x_{00}) + x_{p0} f_{00}(x_{p0}) + x_{0q} f_{00}(x_{0q}) + x_{pq} f_{00}(x_{pq}) \\
& = (x_{00} + x_{p0} + x_{0q} + x_{pq}) f_{00}(x_{00} + x_{p0} + x_{0q} + x_{pq}), \\
& x_{0j} f_{0j}(x_{0j}) + x_{pj} f_{0j}(x_{pj}) = (x_{0j} + x_{pj}) f_{0j}(x_{0j} + x_{pj}), \\
& x_{i0} f_{i0}(x_{i0}) + x_{iq} f_{i0}(x_{iq}) = (x_{i0} + x_{iq}) f_{i0}(x_{i0} + x_{iq}).
\end{aligned}
$$

Furthermore we have

$$
\begin{aligned}
& x_{i0} x_{i+10} g_{i0}(x_{i0}, x_{i+10}) + x_{iq} x_{i+1q} g_{i0}(x_{iq}, x_{i+1q}) \\
& = (x_{i0} + x_{iq})(x_{i+10} + x_{i+1q}) g_{i0}(x_{i0} + x_{iq}, x_{i+10} + x_{i+1q}), \\
& x_{0j} x_{0j+1} h_{0j}(x_{0j}, x_{0j+1}) + x_{pj} x_{pj+1} h_{0j}(x_{pj}, x_{pj+1}) \\
& = (x_{0j} + x_{pj})(x_{0j+1} + x_{pj+1}) h_{0j}(x_{0j} + x_{pj}, x_{0j+1} + x_{pj+1}).
\end{aligned}
$$

This means that

$$
\begin{aligned}
F(N) =& k[x_{00} + x_{p0} + x_{0q} + x_{pq}] + \sum_{j=1}^{q-1} k[x_{0j} + x_{pj}] + \sum_{i=1}^{p-1} k[x_{i0} + x_{iq}] \\
&+ \sum_{i=0}^{p-1} (x_{i0} + x_{iq})(x_{i+10} + x_{i+1q}) k[x_{i0} + x_{iq}, x_{i+10} + x_{i+1q}] \\
&+ \sum_{j=0}^{q-1} (x_{0j} + x_{pj})(x_{0j+1} + x_{pj+1}) k[x_{0j} + x_{pj}, x_{0j+1} + x_{pj+1}] \\
&+ \sum_{i=1}^{p-1}\sum_{j=1}^{q-1} k[x_{ij}] + \sum_{i=0}^{p-1}\sum_{j=1}^{q-1} x_{ij}x_{i+1j} k[x_{ij}, x_{i+1j}] + \sum_{i=1}^{p-1}\sum_{j=0}^{q-1} x_{ij}x_{ij+1} k[x_{ij}, x_{ij+1}] \\
&+ \sum_{i=0}^{p-1}\sum_{j=0}^{q-1} x_{i+1j}x_{ij+1} k[x_{i+1j}, x_{ij+1}] + \sum_{i=0}^{p-1}\sum_{j=0}^{q-1} x_{ij}x_{i+1j}x_{ij+1} k[x_{ij}, x_{i+1j}, x_{ij+1}] \\
&+ \sum_{i=0}^{p-1}\sum_{j=0}^{q-1} x_{i+1j+1}x_{i+1j}x_{ij+1} k[x_{i+1j+1}, x_{i+1j}, x_{ij+1}] \bmod I,
\end{aligned}
$$

which we may rewrite as

$$
\begin{aligned}
F(N) =& k[x_{00} + x_{p0} + x_{0q} + x_{pq}] + \sum_{j=1}^{q-1} k[x_{0j} + x_{pj}] + \sum_{i=1}^{p-1} k[x_{i0} + x_{iq}] \\
&+ \sum_{i=1}^{p-1}\sum_{j=1}^{q-1} k[x_{ij}] + (x_{00} + x_{0q})(x_{10} + x_{1q}) k[x_{00} + x_{0q}, x_{10} + x_{1q}] \\
&+ (x_{p-10} + x_{p-1q})(x_{p0} + x_{pq}) k[x_{p-10} + x_{p-1q}, x_{p0} + x_{pq}] \\
&+ (x_{00} + x_{p0})(x_{01} + x_{p1}) k[x_{00} + x_{p0}, x_{01} + x_{p1}] \\
&+ (x_{0q-1} + x_{pq-1})(x_{0q} + x_{pq}) k[x_{0q-1} + x_{pq-1}, x_{0q} + x_{pq}] \\
&+ \sum_{i=0}^{p-1}\sum_{j=1}^{q-1} (x_{ij}x_{i+1j}) + \sum_{i=1}^{p-1}\sum_{j=0}^{q-1} (x_{ij}x_{ij+1}) + \sum_{i=0}^{p-1}\sum_{j=0}^{q-1} (x_{i+1j}x_{ij+1}) \bmod I.
\end{aligned}
$$

It is not hard to verify that $F(N)$ will be on the same form when $p = 1$ or $q = 1$.

7.1.2 Case 2 : $0 < r < p$

If f satisfies the four conditions, we may now write

$$
\begin{aligned}
f =& a + \sum_{i=1}^{p-1}\sum_{j=1}^{q-1} x_{ij} f_{ij}(x_{ij}) + \sum_{i=0}^{p-1}\sum_{j=1}^{q-1} x_{ij}x_{i+1j} g_{ij}(x_{ij}, x_{i+1j}) \\
&+ \sum_{i=1}^{p-1}\sum_{j=0}^{q-1} x_{ij}x_{ij+1} h_{ij}(x_{ij}, x_{ij+1}) + \sum_{i=1}^{p}\sum_{j=0}^{q-1} x_{ij}x_{i-1j+1} s_{ij}(x_{ij}, x_{i-1j+1}) + H \\
&+ x_{00} f_{00}(x_{00}) + x_{p0} f_{00}(x_{p0}) + x_{rq} f_{00}(x_{rq}) \\
&+ x_{0q} f_{0q}(x_{0q}) + x_{p-r,0} f_{0q}(x_{p-r,0}) + x_{pq} f_{0q}(x_{pq}) \\
&+ \sum_{j=1}^{q-1} x_{0j} f_{0j}(x_{0j}) + x_{pj} f_{0j}(x_{pj}) + \sum_{i=1}^{p-1} x_{i0} f_{i0}(x_{i0}) + x_{i+rq} f_{i0}(x_{i+rq}) \\
&+ \sum_{i=0}^{p-1} x_{i0}x_{i+10} g_{i0}(x_{i0}, x_{i+10}) + x_{i+rq}x_{i+r+1q} g_{i0}(x_{i+rq}, x_{i+r+1q}) \\
&+ \sum_{j=0}^{q-1} x_{0j}x_{0j+1} h_{0j}(x_{0j}, x_{0j+1}) + x_{pj}x_{pj+1} h_{0j}(x_{pj}, x_{pj+1}).
\end{aligned}
$$

Assume now that $\max(p-r, q) \geq 2$. By the same method as above, we now have

$$
\begin{aligned}
& x_{00} f_{00}(x_{00}) + x_{p0} f_{00}(x_{p0}) + x_{rq} f_{00}(x_{rq}) \\
& = (x_{00} + x_{p0} + x_{rq}) f_{00}(x_{00} + x_{p0} + x_{rq}), \\
& x_{0q} f_{0q}(x_{0q}) + x_{p-r,0} f_{0q}(x_{p-r,0}) + x_{pq} f_{0q}(x_{pq}) \\
& = (x_{0q} + x_{p-r0} + x_{0q}) f_{0q}(x_{0q} + x_{p-r0} + x_{pq}), \\
& x_{0j} f_{0j}(x_{0j}) + x_{pj} f_{0j}(x_{pj}) = (x_{0j} + x_{pj}) f_{0j}(x_{0j} + x_{pj}), \\
& x_{i0} f_{i0}(x_{i0}) + x_{i+rq} f_{i0}(x_{i+rq}) = (x_{i0} + x_{i+rq}) f_{i0}(x_{i0} + x_{i+rq}).
\end{aligned}
$$

Furthermore we have

$$
\begin{aligned}
& x_{i0}x_{i+10} g_{i0}(x_{i0}, x_{i+10}) + x_{i+rq}x_{i+r+1q} g_{i0}(x_{i+rq}, x_{i+r+1q}) \\
& = (x_{i0} + x_{i+rq})(x_{i+10} + x_{i+r+1q}) g_{i0}(x_{i0} + x_{i+rq}, x_{i+10} + x_{i+r+1q}), \\
& x_{0j}x_{0j+1} h_{0j}(x_{0j}, x_{0j+1}) + x_{pj}x_{pj+1} h_{0j}(x_{pj}, x_{pj+1}) \\
& = (x_{0j} + x_{pj})(x_{0j+1} + x_{pj+1}) h_{0j}(x_{0j} + x_{pj}, x_{0j+1} + x_{pj+1}).
\end{aligned}
$$

This means that

$$
\begin{aligned}
F(N) =& k[x_{00}+x_{p0}+x_{rq}] + k[x_{0q}+x_{p-r0}+x_{pq}] + \sum_{j=1}^{q-1} k[x_{0j}+x_{pj}] + \sum_{\substack{i=1\\ i\neq p-r}}^{p-1} k[x_{i0}+x_{i+rq}] \\
&+ \sum_{i=0}^{p-1} (x_{i0}+x_{i+rq})(x_{i+10}+x_{i+r+1q}) k[x_{i0}+x_{i+rq}, x_{i+10}+x_{i+r+1q}] \\
&+ \sum_{j=0}^{q-1} (x_{0j}+x_{pj})(x_{0j+1}+x_{pj+1}) k[x_{0j}+x_{pj}, x_{0j+1}+x_{pj+1}] \\
&+ \sum_{i=1}^{p-1}\sum_{j=1}^{q-1} k[x_{ij}] + \sum_{i=0}^{p-1}\sum_{j=1}^{q-1} x_{ij}x_{i+1j} k[x_{ij}, x_{i+1j}] + \sum_{i=1}^{p-1}\sum_{j=0}^{q-1} x_{ij}x_{ij+1} k[x_{ij}, x_{ij+1}] \\
&+ \sum_{i=0}^{p-1}\sum_{j=0}^{q-1} x_{i+1j}x_{ij+1} k[x_{i+1j}, x_{ij+1}] + \sum_{i=0}^{p-1}\sum_{j=0}^{q-1} x_{ij}x_{i+1j}x_{ij+1} k[x_{ij}, x_{i+1j}, x_{ij+1}] \\
&+ \sum_{i=0}^{p-1}\sum_{j=0}^{q-1} x_{i+1j+1}x_{i+1j}x_{ij+1} k[x_{i+1j+1}, x_{i+1j}, x_{ij+1}] \bmod I,
\end{aligned}
$$

which similarly may be rewritten as

$$
\begin{aligned}
F(N) =& k[x_{00}+x_{p0}+x_{rq}] + k[x_{0q}+x_{p-r0}+x_{pq}] + \sum_{j=1}^{q-1} k[x_{0j}+x_{pj}] + \sum_{\substack{i=1\\ i\neq p-r}}^{p-1} k[x_{i0}+x_{i+rq}] \\
&+ \sum_{i=1}^{p-1}\sum_{j=1}^{q-1} k[x_{ij}] + (x_{00}+x_{rq})(x_{10}+x_{r+1q}) k[x_{00}+x_{rq}, x_{10}+x_{r+1q}] \\
&+ (x_{p-10}+x_{r-1q})(x_{p0}+x_{rq}) k[x_{p-10}+x_{r-1q}, x_{p0}+x_{rq}] \\
&+ (x_{00}+x_{p0})(x_{01}+x_{p1}) k[x_{00}+x_{p0}, x_{01}+x_{p1}] \\
&+ (x_{0q-1}+x_{pq-1})(x_{0q}+x_{pq}) k[x_{0q-1}+x_{pq-1}, x_{0q}+x_{pq}] \\
&+ \sum_{i=0}^{p-1}\sum_{j=1}^{q-1} (x_{ij}x_{i+1j}) + \sum_{i=1}^{p-1}\sum_{j=0}^{q-1} (x_{ij}x_{ij+1}) + \sum_{i=0}^{p-1}\sum_{j=0}^{q-1} (x_{i+1j}x_{ij+1}) \bmod I.
\end{aligned}
$$

Also here it is not hard to verify that $F(N)$ will be on the same form when $\max(p-r, q) = 1$.

7.2 Generators of $F(N)$

We will now provide generators of $F(N)$ as a k-algebra. Note that some of them might be superfluous. Again we split into two cases.

7.2.1 Case 1: $r = 0$

First, assume that $p, q \geq 2$. Then it is easy to see that $F(N)$ will be generated by

$$\begin{aligned}
t_{00} &= x_{00} + x_{p0} + x_{0q} + x_{pq}, \\
t_{i0} &= x_{i0} + x_{iq}, \text{ for } 1 \leq i \leq p-1, \\
t_{0j} &= x_{0j} + x_{pj}, \text{ for } 1 \leq j \leq q-1, \\
t_{ij} &= x_{ij}, \text{ for } 1 \leq i \leq p-1, 1 \leq j \leq q-1, \\
u_0 &= x_{00}x_{10} + x_{0q}x_{1q}, \\
u_1 &= x_{p-10}x_{p0} + x_{p-1q}x_{pq}, \\
u_3 &= x_{00}x_{01} + x_{p0}x_{p1}, \\
u_4 &= x_{0q-1}x_{0q} + x_{pq-1}x_{pq}, \\
r_{ij} &= x_{ij}x_{i+1j} \text{ for } 0 \leq i \leq p-1, 1 \leq j \leq q-1, \\
l_{ij} &= x_{ij}x_{ij+1} \text{ for } 1 \leq i \leq p-1, 0 \leq j \leq q-1, \\
d_{ij} &= x_{i+1j}x_{ij+1} \text{ for } 0 \leq i \leq p-1, 0 \leq j \leq q-1.
\end{aligned}$$

Now, assume that $q = 1, p \geq 2$. Without much effort one can see by induction on degrees that $F(N)$ will be generated by

$$\begin{aligned}
t_0 &= x_{00} + x_{p0} + x_{01} + x_{p1}, \\
t_i &= x_{i0} + x_{i1}, \text{ for } 1 \leq i \leq p-1, \\
u &= x_{00}x_{01} + x_{p0}x_{p1}, \ v = x_{00}^2x_{01} + x_{p0}^2x_{p1}, \\
u_i &= x_{i0}x_{i+10} + x_{i1}x_{i+11}, \ v_i = x_{i0}^2x_{i+10} + x_{i1}^2x_{i+11}, \text{ for } 0 \leq i \leq p-1, \\
l_i &= x_{i0}x_{i1}, \ l'_i = x_{i0}^2x_{i1} \text{ for } 1 \leq i \leq p-1, \\
d_i &= x_{i+10}x_{i1}, \ d'_i = x_{i+10}^2x_{i1}, \text{ for } 0 \leq i \leq p-1.
\end{aligned}$$

Similarly, if $p = 1, q \geq 2$, the generators will be

$$
\begin{aligned}
&t_0 = x_{00} + x_{10} + x_{0q} + x_{1q},\\
&t_j = x_{0j} + x_{1j}, \text{ for } 1 \leq j \leq q-1,\\
&u = x_{00}x_{10} + x_{0q}x_{1q}, \ v = x_{00}^2x_{10} + x_{0q}^2x_{1q}\\
&u_i = x_{0j}x_{0j+1} + x_{1j}x_{1j+1}, \ v_i = x_{0j}^2x_{0j+1} + x_{1j}^2x_{1j+1} \text{ for } 0 \leq j \leq q-1,\\
&r_j = x_{0j}x_{1j}, \ r'_j = x_{0j}^2x_{1j} \text{ for } 1 \leq j \leq q-1,\\
&d_j = x_{1j}x_{0j+1}, \ d'_j = x_{1j}^2x_{0j+1} \text{ for } , 0 \leq j \leq q-1.
\end{aligned}
$$

Finally, if $p = q = 1$, the generators will be

$$
\begin{aligned}
&t_0 = x_{00} + x_{10} + x_{01} + x_{11},\\
&u_1 = x_{00}x_{10} + x_{01}x_{11}, \ v_1 = x_{00}^2x_{10} + x_{01}^2x_{11},\\
&u_2 = x_{10}x_{01}, \ v_2 = x_{10}^2x_{01},\\
&u_3 = x_{00}x_{01} + x_{10}x_{11}, \ v_3 = x_{00}^2x_{01} + x_{10}^2x_{11},\\
&s_0 = x_{00}x_{10}x_{01}, \ s_1 = x_{10}x_{01}x_{11}, \ s_2 = x_{00}^2x_{10}x_{01}, \ s_3 = x_{00}x_{10}^2x_{01}.
\end{aligned}
$$

7.2.2 Case 2: $0 < r < p$

First, assume that $q + r \geq 3, \max(p-r, q) \geq 2$. Then it is easy to see that $F(N)$ is generated by

$$
\begin{aligned}
t_{00} &= x_{00} + x_{p0} + x_{rq},\\
t_{0q} &= x_{0q} + x_{p-r0} + x_{pq},\\
t_{i0} &= x_{i0} + x_{i+rq} \text{ for } 1 \leq i \leq p-1, i \neq p-r,\\
t_{0j} &= x_{0j} + x_{pj} \text{ for } 1 \leq j \leq q-1,\\
t_{ij} &= x_{ij} \text{ for } 1 \leq i \leq p-1, 1 \leq j \leq q-1,\\
u_0 &= x_{00}x_{10} + x_{rq}x_{r+1q},\\
u_1 &= x_{p-10}x_{p0} + x_{r-1q}x_{rq},\\
u_2 &= x_{00}x_{01} + x_{p0}x_{p1},\\
u_4 &= x_{0q-1}x_{0q} + x_{pq-1}x_{pq},\\
r_{ij} &= x_{ij}x_{i+1j} \text{ for } 0 \leq i \leq p-1, 1 \leq j \leq q-1,\\
l_{ij} &= x_{ij}x_{ij+1} \text{ for } 1 \leq i \leq p-1, 0 \leq j \leq q-1,\\
d_{ij} &= x_{i+1j}x_{ij+1} \text{ for } 0 \leq i \leq p-1, 0 \leq j \leq q-1.
\end{aligned}
$$

Now, assume $q + r = 2, \max(p - r, q) \geq 2$. Then $q = r = 1$, and $p \geq 3$. We may prove by induction on degrees that $F(N)$ in this case is generated by

$$\begin{aligned}
t_{00} &= x_{00} + x_{p0} + x_{11}, \\
t_{0q} &= x_{01} + x_{p-10} + x_{p1}, \\
t_{i0} &= x_{i0} + x_{i+11} \text{ for } 1 \leq i \leq p-1, i \neq p-1, \\
u_0 &= x_{00}x_{10} + x_{11}x_{21}, \\
u_1 &= x_{p-10}x_{p0} + x_{01}x_{11}, \\
u_2 &= x_{00}x_{01} + x_{p0}x_{p1}, \\
r_{ij} &= x_{ij}x_{i+1j} \text{ for } 0 \leq i \leq p-1, 1 \leq j \leq q-1, \\
l_{ij} &= x_{ij}x_{ij+1} \text{ for } 1 \leq i \leq p-1, 0 \leq j \leq q-1, \\
d_{ij} &= x_{i+1j}x_{ij+1} \text{ for } 0 \leq i \leq p-1, 0 \leq j \leq q-1. \\
s_i &= x_{i0}x_{i+10}x_{i1} \text{ for } 0 \leq i \leq p-1.
\end{aligned}$$

Now, assume that $q + r \geq 3$, $\max(p - r, q) = 1$. Then $q = 1$, $r \geq 2$ and $r = p - 1$. We similarly see that $F(N)$ is generated by

$$\begin{aligned}
t_{00} &= x_{00} + x_{p0} + x_{p-11}, \\
t_{01} &= x_{01} + x_{10} + x_{p1}, \\
t_{i0} &= x_{i0} + x_{i+p-11} \text{ for } 2 \leq i \leq p-1, \\
u_0 &= x_{00}x_{10} + x_{p-11}x_{p1}, \ v_0 = x_{00}^2x_{10} + x_{p-11}^2x_{p1}, \\
u_1 &= x_{p-10}x_{p0} + x_{p-21}x_{p-11}, \ v_1 = x_{p-10}^2x_{p0} + x_{p-21}^2x_{p-11}, \\
u_2 &= x_{00}x_{01} + x_{p0}x_{p1}, \ v_2 = x_{00}^2x_{01} + x_{p0}^2x_{p1}, \\
l_i &= x_{i0}x_{i1}, \ l'_i = x_{i0}^2x_{i1} \text{ for } 1 \leq i \leq p-1, \\
d_i &= x_{i+10}x_{i1}, \ d'_i = x_{i+10}^2x_{i1} \text{ for } 0 \leq i \leq p-1.
\end{aligned}$$

Finally, assume that $q + r = 2$, $\max(p - r, q) = 1$. Then $q = r = 1$, and $p = 2$. The generators will be

$$
\begin{aligned}
t_{00} &= x_{00} + x_{20} + x_{11}, \\
t_{01} &= x_{01} + x_{10} + x_{21}, \\
u_0 &= x_{00}x_{10} + x_{11}x_{21},\ v_0 = x_{00}^2 x_{10} + x_{11}^2 x_{21}, \\
u_1 &= x_{10}x_{20} + x_{01}x_{11},\ v_1 = x_{10}^2 x_{20} + x_{01}^2 x_{11}, \\
u_2 &= x_{00}x_{01} + x_{20}x_{21},\ v_2 = x_{00}^2 x_{01} + x_{20}^2 x_{21}, \\
u_3 &= x_{10}x_{11},\ v_3 = x_{10}^2 x_{11}, \\
u_4 &= x_{10}x_{01},\ v_4 = x_{10}^2 x_{01}, \\
u_4 &= x_{20}x_{11},\ v_4 = x_{20}^2 x_{11}.
\end{aligned}
$$

7.3 $\mathcal{O}_{\mathbb{P}(N)}(1)$ is invertible

We will now show that the sheaf $\mathcal{L}_N = \mathcal{O}_{\mathbb{P}(N)}(1)$ of $\mathcal{O}_{\mathbb{P}(N)}$-modules is locally free of rank 1. It will be helpful to define the two elements $t = \sum_{i=0,j=0}^{p,q} x_{ij}$ and $s = \sum_{i=0,j=0}^{p-1,q-1} x_{ij}x_{i+1j}x_{ij+1} + x_{i+1j}x_{ij+1}x_{i+1j+1}$. Note that these are in $F(N)$ for any N. We split into two cases.

7.3.1 Case 1: $r = 0$

First, assume $p, q \geq 2$. Then $F(N)$ is generated by elements of degree 1 and 2. If g is a generator of degree 1, then clearly $\mathcal{L}_N(D_+(g)) \cong \mathcal{O}_{\mathbb{P}(N)}(D_+(g))$ via multiplication by g. Now, for each generator g of degree 2, introduce the new element g' of degree 3:

$$
\begin{aligned}
u_0' &= x_{00}^2 x_{10} + x_{0q}^2 x_{1q}, \\
u_1' &= x_{p-10}^2 x_{p0} + x_{p-1q}^2 x_{pq}, \\
u_3' &= x_{00}^2 x_{01} + x_{p0}^2 x_{p1}, \\
u_4' &= x_{0q-1}^2 x_{0q} + x_{pq-1}^2 x_{pq}, \\
r_{ij}' &= x_{ij}^2 x_{i+1j} \text{ for } 0 \leq i \leq p-1, 1 \leq j \leq q-1, \\
l_{ij}' &= x_{ij}^2 x_{ij+1} \text{ for } 1 \leq i \leq p-1, 0 \leq j \leq q-1, \\
d_{ij}' &= x_{i+1j}^2 x_{ij+1} \text{ for } 0 \leq i \leq p-1, 0 \leq j \leq q-1.
\end{aligned}
$$

Clearly these are present in $F(N)$. Moreover, the identity $g'(tg - g' - 2s) = g^3$ is easily verified. Hence if $a \in \mathcal{L}_N(D_+(g))$, then we have $a = \frac{g'}{g} \cdot \frac{a(tg-g'-2s)}{g^2}$, hence $\mathcal{L}_N(D_+(g))$ is generated by $\frac{g'}{g}$.

If $q = 1, p \geq 2$, or $p = 1, q \geq 2$, the same argument as above may be used to show that $\mathcal{L}_N(D_+(g)) \cong \mathcal{O}_{\mathbb{P}(N)}(D_+(g))$ for g of degree 1 or 2. In these two cases however we also have generators of degree 3. It will suffice to note that if g is a generator of degree 3, g^2 may always be written as a product where one of the factors is a generator degree 2. Thus the distinguished open sets given by elements of degree 1 and 2 cover $\mathbb{P}(N)$.

If $p = q = 1$, using the remarks above, it only remains to verify that $\mathcal{L}_N(D_+(s_i))$ is generated by a single element for $i = 0, 1, 2, 3$. We note that s_2^2 and s_3^2 may be written as a product where one of the factors is s_0, so we only need to consider $\mathcal{L}_N(D_+(s_i))$ for $i = 0, 1$. In fact, by symmetry we only need to consider it for $i = 0$. We note that $s_0^2 = s_2 u_1$, and therefore if $a \in \mathcal{L}_N(D_+(s_0))$, then $a = \frac{s_2}{s_0} \cdot \frac{a u_1}{s_0}$. Hence $\mathcal{L}_N(D_+(s_0))$ is generated by $\frac{s_2}{s_0}$.

7.3.2 Case 2: $0 < r < p$

The same arguments for Case 1 works here as well, they may be directly extended to Case 2. We conclude that $\mathcal{L}_N$ is an invertible sheaf of $\mathcal{O}_{\mathbb{P}(N)}$-modules.

Bibliography

[AC10] Klaus Altmann and Jan Arthur Christophersen. Deforming Stanley-Reisner schemes. *Math. Ann.*, 348(3):513–537, 2010.

[AM69] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[And74] Michel André. *Homologie des algèbres commutatives*. Springer-Verlag, Berlin-New York, 1974. Die Grundlehren der mathematischen Wissenschaften, Band 206.

[Chr10] Jan Arthur Christophersen. Deformations of equivelar stanley-reisner abelian surfaces, 2010.

[Eis95] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[Har10] Robin Hartshorne. *Deformation theory*, volume 257 of *Graduate Texts in Mathematics*. Springer, New York, 2010.

[Hud69] J. F. P. Hudson. *Piecewise linear topology*. University of Chicago Lecture Notes prepared with the assistance of J. L. Shaneson and J. Lees. W. A. Benjamin, Inc., New York-Amsterdam, 1969.

[MS05] Ezra Miller and Bernd Sturmfels. *Combinatorial commutative algebra*, volume 227 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005.

[Mum70] David Mumford. *Abelian varieties*. Tata Institute of Fundamental Research Studies in Mathematics, No. 5. Published for the Tata Institute of Fundamental Research, Bombay; Oxford University Press, London, 1970.

[Mur06] Daniel Murfet. The Proj Construction. `http://therisingsea.org/notes/TheProjConstruction.pdf`, 2006.

[Ran92] A. A. Ranicki. *Algebraic L-theory and topological manifolds*, volume 102 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1992.

[Rei02] Miles Reid. Graded rings and varieties in weighted projective space. `http://homepages.warwick.ac.uk/~masda/surf/more/grad.pdf`, 2002.

[Ser06] Edoardo Sernesi. *Deformations of algebraic schemes*, volume 334 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006.

[Sta] The Stacks Project Authors. Proj of a graded ring. `http://stacks.math.columbia.edu/tag/07Z2`.

[Sta14a] The Stacks Project Authors. The Cotangent Complex. `http://stacks.math.columbia.edu/download/cotangent.pdf`, 2014.

[Sta14b] The Stacks Project Authors. Morphisms of Schemes. `http://stacks.math.columbia.edu/download/morphisms.pdf`, 2014.

[Tsu] Yoshifumi Tsuchimoto. Topics in non commutative algebraic geometry and congruent zeta functions (part iii). supplementary results on commutative algebraic geometry. `http://www.math.kochi-u.ac.jp/docky/bourdoki/NAS/nas003/node28.html`.