SIAM J. MATH. ANAL.
Vol. 45, No. 4, pp. 2499–2522

A MAXIMUM PRINCIPLE FOR INFINITE HORIZON DELAY EQUATIONS*

N. AGRAM†, S. HAADEM‡, B. ØKSENDAL‡, AND F. PROSKE‡

Abstract. We prove a maximum principle of optimal control of stochastic delay equations on infinite horizon. We establish first and second sufficient stochastic maximum principles as well as necessary conditions for that problem. We illustrate our results with an application to the optimal consumption rate from an economic quantity.

Key words. infinite horizon, optimal control, stochastic delay equation, Lévy processes, maximum principle, Hamiltonian, adjoint process, partial information

AMS subject classifications. 93EXX, 93E20, 60J75, 34K50, 60H10, 60H20, 49J55

DOI. 10.1137/120882809

1. Introduction.

To solve the stochastic control problems, there are two approaches: the dynamic programming method (HJB equation) and the maximum principle.

In this paper, our system is governed by the stochastic differential delay equation (SDDE),

$$\begin{cases} dX(t) = b\left(t, X(t), Y(t), A(t), u(t)\right) dt \\ +\,\sigma\left(t, X(t), Y(t), A(t), u(t)\right) dB(t) \\ +\int_{\mathbb{R}_0} \theta\left(t, X(t), Y(t), A(t), u(t), z\right) \tilde{N}(dt, dz), & t \in [0, \infty)\,, \\ X(t) = X_0(t), & t \in [-\delta, 0], \\ Y(t) = X(t-\delta), & t \in [0, \infty), \\ A(t) = \int_{t-\delta}^{t} e^{-\lambda(t-r)} X(r) dr, & t \in [0, \infty), \end{cases} \tag{1.1}$$

with a corresponding performance functional,

$$J(u) = E\left[\int_0^{\infty} f\left(t, X(t), Y(t), A(t), u(t)\right)\ dt\right], \tag{1.2}$$

where $u(t)$ is the control process.

The SDDE is not Markovian so we cannot use the dynamic programming method. However, we will prove stochastic maximum principles for this problem. A sufficient maximum principle in infinite horizon with the trivial transversality conditions were treated by Haadem, Øksendal, and Proske [4]. The natural transversality condition in the infinite case would be a zero limit condition, meaning in the economic sense that one more unit of good at the limit gives no additional value. But this property is not necessarily verified. In fact Halkin [5] provides a counterexample for a natural

*Received by the editors June 28, 2012; accepted for publication (in revised form) June 13, 2013; published electronically August 22, 2013.
http://www.siam.org/journals/sima/45-4/88280.html

†Laboratory of Applied Mathematics, University Med Khider, Biskra 07000, Algeria (agramnacira@yahoo.fr).

‡Center of Mathematics for Applications (CMA), University of Oslo, Blindern, N-0316 Oslo, Norway (svenhaadem@gmail.com, bernt.oksendal@cma.uio.no, proske@math.uio.no). The third author received funding from the European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013)/ERC grant agreement [228087].

extension of the finite horizon transversality conditions. Thus some care is needed in the infinite horizon case. For the case of the natural transversality condition the discounted control problem was studied by Maslowski and Veverka [7].

In real life, delay occurs everywhere in our society. For example, this is the case in biology, where the population growth depends not only on the current population size but also on the size some time ago. The same situation may occur in many economic growth models.

The stochastic maximum principle with delay has been studied by many authors. For example, Elsanosi, Øksendal, and Sulem [3] proved a verification theorem of variational inequality. Øksendal and Sulem [9] established the sufficient maximum principle for a certain class of stochastic control systems with delay in the state variable. In Haadem, Øksendal, and Proske [4] an inifinite horizon system is studied, but without delay. In Chen and Wu [2], a finite horizon version of a stochastic maximum principle for a system with delay in both the state variable and the control variable is derived. In Øksendal, Sulem, and Zhang [11] a maximum principle for systems with delay is studied in the finite horizon case. However, to our knowledge, no one has studied the infinite horizon case for delay equations.

For backward differential equations see Situ [15] and Li and Peng [6]. For the infinite horizon backward SDE (BSDE) see Peng and Shi [13], Pardoux [12], Yin [16], Barles, Buckdahn, and Pardoux [1] and Royer [14]. For more details about jump diffusion markets see Øksendal and Sulem [10] and for background and details about stochastic fractional delay equations see Mohammed and Scheutzow [8].

In this work, we establish two sufficient maximum principles and one necessary for the stochastic delay systems on infinite horizon with jumps.

Our paper is organized as follows. In the second section, we formulate the problem. The third section is devoted to the first and second sufficient maximum principles with an application to the optimal consumption rate from an economic quantity described by a stochastic delay equation. In the fourth section, we formulate a necessary maximum principle, and we prove an existence and uniqueness of the advanced BSDEs on infinite horizon with jumps in the last section.

2. Formulation of the problem. Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, on which an $\mathbb{R}$-valued standard Brownian motion $B(.)$ and an independent compensated Poisson random measure $\tilde{N}(dt,dz) = N(dt,dz) - \nu(dz)dt$ are defined.

We consider the following stochastic control system with delay:

$$\begin{cases} dX(t) = b\left(t, X(t), Y(t), A(t), u(t)\right) dt \\ \qquad\qquad + \sigma\left(t, X(t), Y(t), A(t), u(t)\right) dB(t) \\ \qquad\qquad + \int_{\mathbb{R}_0} \theta\left(t, X(t), Y(t), A(t), u(t), z\right) \tilde{N}(dt,dz),\ t \in [0,\infty), \\ X(t) = X_0(t),\ t \in [-\delta, 0], \\ Y(t) = X(t-\delta),\ t \in [0,\infty), \\ A(t) = \int_{t-\delta}^{t} e^{-\lambda(t-r)} X(r) dr,\ t \in [0,\infty), \end{cases} \tag{2.1}$$

where $X_0(t)$ is a given continuous (deterministic) function, and

$$\begin{aligned} &\delta > 0,\ \lambda > 0 \text{ are given constants,} \\ &b : [0,\infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \Omega \to \mathbb{R}, \\ &\sigma : [0,\infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \Omega \to \mathbb{R}, \\ &\theta : [0,\infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \mathbb{R}_0 \times \Omega \to \mathbb{R} \end{aligned}$$

are given continuous functions such that for all t, $b(t,x,y,a,u)$, $\sigma(t,x,y,a,u)$, and $\theta(t,x,y,a,u,z)$ are $\mathcal{F}_t$-measurable for all $x \in \mathbb{R}$, $y \in \mathbb{R}$, $a \in \mathbb{R}$, $u \in \mathcal{U}$, and $z \in \mathbb{R}_0 := \mathbb{R} \setminus \{0\}$. We assume that b, σ, θ are C^1 (i.e., continuously differentiable/Fréchet differentiable) with respect to x, y, a, u, and z for all t and a.a. ω. Let $\mathcal{E}_t \subset \mathcal{F}_t$ be a given subfiltration, representing the information available to the controller at time t. Let $\mathcal{U}$ be a nonempty subset of $\mathbb{R}$. We let $\mathcal{A}_{\mathcal{E}}$ denote a given family of admissible $\mathcal{E}_t$-adapted control processes. An element of $\mathcal{A}_{\mathcal{E}}$ is called an admissible control. The corresponding performance functional is

$$J(u) = E\left[\int_0^\infty f\left(t, X(t), Y(t), A(t), u(t)\right)\, dt\right], u \in \mathcal{A}_{\mathcal{E}}, \tag{2.2}$$

where we assume that

$$E\int_0^\infty \left\{ \left| f\left(t, X(t), Y(t), A(t), u(t)\right) \right| + \left| \frac{\partial f}{\partial x_i}\left(t, X(t), Y(t), A(t), u(t)\right) \right|^2 \right\} dt < \infty. \tag{2.3}$$

We also assume that f is C^1 with respect to x, y, a, u for all t and a.a. ω. The value function Φ is defined as

$$\Phi(X_0) = \sup_{u \in \mathcal{A}_{\mathcal{E}}} J(u). \tag{2.4}$$

An admissible control u^* is called an optimal control for (2.1) if it attains the maximum of $J(u)$ over $\mathcal{A}_{\mathcal{E}}$. Equation (2.1) is called the state equation, and the solution $X^*(t)$ corresponding to u^* is called an optimal trajectory.

3. A sufficient maximum principle.

In this section our objective is to establish a sufficient maximum principle.

3.1. Hamiltonian and time-advanced BSDEs for adjoint equations.

We now introduce the adjoint equations and the Hamiltonian function for our problem. The Hamiltonian is defined by

$$\begin{aligned} H(t,x,y,a,u,p,q,r(\cdot)) &= f(t,x,y,a,u) + b(t,x,y,a,u)p + \sigma(t,x,y,a,u)q \\ &\quad + \int_{\mathbb{R}_0} \theta(t,x,y,a,u,z) r(z) \nu(dz), \end{aligned} \tag{3.1}$$

where

$$H : [0,\infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \mathbb{R} \times \mathbb{R} \times \Re \times \Omega \to \mathbb{R}$$

and $\Re$ is the set of functions r: $\mathbb{R}_0 \to \mathbb{R}$ such that the integral term in (3.1) converges and $\mathcal{U}$ is the set of possible control values.

We suppose that b, σ, and θ are C^1 functions with respect to (x,y,a,u) and that

$$\begin{aligned} E\Bigg[\int_0^\infty \Bigg\{ &\left| \frac{\partial b}{\partial x_i}\left(t, X(t), Y(t), A(t), u(t)\right) \right|^2 + \left| \frac{\partial \sigma}{\partial x_i}\left(t, X(t), Y(t), A(t), u(t)\right) \right|^2 \\ &+ \int_{\mathbb{R}_0} \left| \frac{\partial \theta}{\partial x_i}\left(t, X(t), Y(t), A(t), u(t)\right) \right|^2 \nu(dz) \Bigg\} dt \Bigg] < \infty \end{aligned} \tag{3.2}$$

for $x_i = x$, y, a, and u.

The adjoint processes $(p(t), q(t), r(t,\cdot))$, $t \in [0,\infty)$, $z \in \mathbb{R}_0$, are assumed to satisfy the equation

$$dp(t) = E\left[\mu(t) \mid \mathcal{F}_t\right] dt + q(t) dB(t) + \int_{\mathbb{R}_0} r(t,z)\tilde{N}(dt,dz),\ t \in [0,\infty)\,, \tag{3.3}$$

where

$$\begin{aligned} \mu(t) = &-\frac{\partial H}{\partial x}\left(t, X(t), Y(t), A(t), u(t), p(t), q(t), r(t,\cdot)\right) \\ &- \frac{\partial H}{\partial y}\left(t+\delta, X(t+\delta), Y(t+\delta), A(t+\delta), u(t+\delta), p(t+\delta), q(t+\delta), r(t+\delta,\cdot)\right) \\ &- e^{\lambda t}\left(\int_t^{t+\delta} \frac{\partial H}{\partial a}\left(s, X(s), Y(s), A(s), u(s), p(s), q(s), r(s,\cdot)\right) e^{-\lambda s} ds\right). \end{aligned} \tag{3.4}$$

Remark 3.1. Note that we do not require a priori that the solution of (3.3)–(3.4) is unique.

The following result is an infinite horizon version of Theorem 3.1 in [11].

3.2. A first sufficient maximum principle.

THEOREM 3.2. *Let $\hat{u} \in \mathcal{A}_{\mathcal{E}}$ with corresponding state processes $\hat{X}(t)$, $\hat{Y}(t)$, and $\hat{A}(t)$ and adjoint processes $\hat{p}(t)$, $\hat{q}(t)$, and $\hat{r}(t,\cdot)$ assumed to satisfy the advanced BSDE (ABSDE)* (3.3)–(3.4). *Suppose that the following assertions hold:*

(i)

$$\overline{\lim_{T\to\infty}} E\left[\ \hat{p}(T)(X(T) - \hat{X}(T))\right] \geq 0 \tag{3.5}$$

for all $u \in \mathcal{A}_{\mathcal{E}}$ with corresponding solution $X(t)$.

(ii) *The function*

$$(x,y,a,u) \to H(t,x,y,a,u,\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))$$

is concave for each $t \in [0,\infty)$ a.s.

(iii)

$$E\left[\int_0^T \left\{\hat{q}^2(t)\left(\sigma(t) - \hat{\sigma}(t)\right)^2 + \int_{\mathbb{R}} \hat{r}^2(t,z)(\theta(t,z) - \hat{\theta}(t,z))^2 \nu(dz)\right\} dt\right] < \infty \tag{3.6}$$

for all $T < \infty$.

(iv)

$$\begin{aligned} &\max_{v\in\mathcal{U}} E\left[H\left(t, \hat{X}(t), \hat{X}(t-\delta), \hat{A}(t), v, \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot)\right) \Big| \mathcal{E}_t\right] \\ &\quad = E\left[H\left(t, \hat{X}(t), \hat{X}(t-\delta), \hat{A}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot)\right) \Big| \mathcal{E}_t\right] \end{aligned}$$

for all $t \in [0,\infty)$ a.s.

Then $\hat{u}$ is an optimal control for the problem (2.4).

Proof. Choose an arbitrary $u \in \mathcal{A}_{\mathcal{E}}$, and consider

$$J(u) - J(\hat{u}) = I_1, \tag{3.7}$$

where

$$I_1 = E\left[\int_0^\infty \left\{ f\big(t, X(t), Y(t), A(t), u(t)\big) - f\big(t, \hat{X}(t), \hat{Y}(t), \hat{A}(t), \hat{u}(t)\big)\right\} dt\right]. \tag{3.8}$$

By the definition (3.1) of H and the concavity, we have

$$\begin{aligned} I_1 \le E\Bigg[\int_0^\infty \Bigg\{ &\frac{\partial H}{\partial x}(t)(X(t) - \hat{X}(t)) + \frac{\partial H}{\partial y}(t)(Y(t) - \hat{Y}(t)) + \frac{\partial H}{\partial a}(t)(A(t) - \hat{A}(t)) \\ &+ \frac{\partial H}{\partial u}(t)(u(t) - \hat{u}(t)) - (b(t) - \hat{b}(t))\hat{p}(t) - (\sigma(t) - \hat{\sigma}(t))\hat{q}(t) \\ &- \int_{\mathbb{R}_0} (\theta(t,z) - \hat{\theta}(t,z))\hat{r}(t,z)\nu(dz)\Bigg\} dt\Bigg], \end{aligned} \tag{3.9}$$

where we have used the simplified notation

$$\frac{\partial H}{\partial x}(t) = \frac{\partial H}{\partial x}\left(t, \hat{X}(t), \hat{Y}(t), \hat{A}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot)\right),$$

and similarly for b and σ.

Applying the Itô formula to $\hat{p}(t)(X(t) - \hat{X}(t))$ we get, by (3.5) and (3.6),

$$\begin{aligned} 0 &\le \varlimsup_{T\to\infty} E[\hat{p}(T)(X(T) - \hat{X}(T))] \\ &= \varlimsup_{T\to\infty} E\Bigg[\Bigg(\int_0^T (b(t) - \hat{b}(t))\hat{p}(t)dt + \int_0^T (X(t) - \hat{X}(t))E\left[\hat{\mu}(t) \mid \mathcal{F}_t\right] dt \\ &\qquad + \int_0^T (\sigma(t) - \hat{\sigma}(t))\hat{q}(t)dt + \int_0^T \int_{\mathbb{R}_0} (\theta(t,z) - \hat{\theta}(t,z))\hat{r}(t,z)\nu(dz)dt\Bigg)\Bigg] \\ &= \varlimsup_{T\to\infty} E\Bigg[\Bigg(\int_0^T (b(t) - \hat{b}(t))\hat{p}(t)dt + \int_0^T (X(t) - \hat{X}(t))\hat{\mu}(t)dt \\ &\qquad + \int_0^T (\sigma(t) - \hat{\sigma}(t))\hat{q}(t)dt + \int_0^T \int_{\mathbb{R}_0} (\theta(t,z) - \hat{\theta}(t,z))\hat{r}(t,z)\nu(dz)dt\Bigg)\Bigg]. \end{aligned} \tag{3.10}$$

Using the definition (3.4) of μ we see that

$$\begin{aligned} &\varlimsup_{T\to\infty} E\left[\left(\int_0^T (X(t) - \hat{X}(t))\hat{\mu}(t)dt\right)\right] \\ &= \varlimsup_{T\to\infty} E\left[\left(\int_\delta^{T+\delta} (X(t-\delta) - \hat{X}(t-\delta))\hat{\mu}(t-\delta)dt\right)\right] \\ &= \varlimsup_{T\to\infty} E\Bigg[\Bigg(-\int_\delta^{T+\delta} \frac{\partial H}{\partial x}(t-\delta)(X(t-\delta) - \hat{X}(t-\delta))dt \\ &\qquad - \int_\delta^{T+\delta} \frac{\partial H}{\partial y}(t)\left(Y(t) - \hat{Y}(t)\right)dt \\ &\qquad - \int_\delta^{T+\delta} \left(\int_{t-\delta}^t \frac{\partial H}{\partial a}(s)\, e^{-\lambda s} ds\right) e^{\lambda(t-\delta)}\left(X(t-\delta) - \hat{X}(t-\delta)\right)\Bigg)dt\Bigg]. \end{aligned} \tag{3.11}$$

Using Fubini and substituting $r = t - \delta$, we obtain

$$
\begin{aligned}
(3.12)\qquad & \int_0^T \frac{\partial H}{\partial a}(s)(A(s) - \hat{A}(s))ds \\
&= \int_0^T \frac{\partial H}{\partial a}(s) \int_{s-\delta}^{s} e^{-\lambda(s-r)}(X(r) - \hat{X}(r))dr\ ds \\
&= \int_0^T \left(\int_r^{r+\delta} \frac{\partial H}{\partial a}(s) e^{-\lambda s} ds \right) e^{\lambda r}(X(r) - \hat{X}(r))\ dr \\
&= \int_0^{T+\delta} \left(\int_{t-\delta}^{t} \frac{\partial H}{\partial a}(s)\, e^{-\lambda s} ds \right) e^{\lambda(t-\delta)}(X(t-\delta) - \hat{X}(t-\delta))dt.
\end{aligned}
$$

Combining (3.10), (3.11), and (3.12) we get

$$
\begin{aligned}
(3.13)\qquad 0 \le \overline{\lim_{T\to\infty}} E\left[\hat{p}(T)(X(T) - \hat{X}(T))\right] &= E\left[\left(\int_0^\infty (b(t) - \hat{b}(t))\hat{p}(t)dt\right.\right. \\
&- \int_0^\infty \frac{\partial H}{\partial x}(t)\,(X(t) - \hat{X}(t))dt - \int_\delta^\infty \frac{\partial H}{\partial y}(t)\,(Y(t) - \hat{Y}(t))dt \\
&- \int_\delta^\infty \frac{\partial H}{\partial a}(t)\,(A(t) - \hat{A}(t))dt + \int_0^\infty (\sigma(t) - \hat{\sigma}(t))\hat{q}(t)dt \\
&\left.\left. + \int_0^\infty \int_{\mathbb{R}_0} (\theta(t,z) - \hat{\theta}(t,z))\hat{r}(t,z)\nu(dz)dt \right)\right].
\end{aligned}
$$

Subtracting and adding $\int_0^\infty \frac{\partial H}{\partial u}(t)(u(t) - \hat{u}(t))dt$ in (3.12) we conclude

$$
\begin{aligned}
0 \le \overline{\lim_{T\to\infty}} E\left[\hat{p}(T)(X(T) - \hat{X}(T))\right] &= E\ \left[\left(\int_0^\infty (b(t) - \hat{b}(t))\hat{p}(t)dt\right.\right. \\
&- \int_0^\infty \frac{\partial H}{\partial x}(t)\,(X(t) - \hat{X}(t))dt - \int_\delta^\infty \frac{\partial H}{\partial y}(t)\,(Y(t) - \hat{Y}(t))dt \\
&- \int_\delta^\infty \frac{\partial H}{\partial a}(t)\,(A(t) - \hat{A}(t))dt + \int_0^\infty (\sigma(t) - \hat{\sigma}(t))\hat{q}(t)dt \\
&+ \int_0^\infty \int_{\mathbb{R}_0} (\theta(t,z) - \hat{\theta}(t,z))\hat{r}(t,z)\nu(dz)dt \\
&\left.\left. - \int_0^\infty \frac{\partial H}{\partial u}(t)(u(t) - \hat{u}(t))dt + \int_0^\infty \frac{\partial H}{\partial u}(t)(u(t) - \hat{u}(t))dt \right)\right] \\
&\le -I_1 + E\left[\int_0^\infty E\left[\frac{\partial H}{\partial u}(t)(u(t) - \hat{u}(t)) \mid \mathcal{E}_t\right] dt\right].
\end{aligned}
$$

Hence

$$
I_1 \le E\left[\int_0^\infty E\left[\frac{\partial H}{\partial u}(t) \mid \mathcal{E}_t\right](u(t) - \hat{u}(t))dt\right] \le 0.
$$

Since $u \in \mathcal{A}_{\mathcal{E}}$ was arbitrary, this proves Theorem 3.1. ∎

3.3. A second sufficient maximum principle. We extend the result in Øksendal and Sulem [9] to infinite horizon with jump diffusions.

Consider again the system

$$\begin{cases} dX(t) = b\left(t, X(t), Y(t), A(t), u(t)\right) dt \\ + \sigma\left(t, X(t), Y(t), A(t), u(t)\right) dB(t) \\ + \int_{\mathbb{R}_0} \theta\left(t, X(t), Y(t), A(t), u(t), z\right) \tilde{N}(dt, dz), & t \in [0, \infty)\,, \\ X(t) = X_0(t), & t \in [-\delta, 0], \\ Y(t) = X(t-\delta), & t \in [0, \infty), \\ A(t) = \int_{t-\delta}^{t} e^{-\lambda(t-r)} X(r) dr, & t \in [0, \infty). \end{cases}$$

We now give an Itô formula which is proved in [3] without jumps. Adding the jump parts is just an easy observation.

LEMMA 3.3 (the Itô formula for delayed system). *Consider a function*

$$G(t) = F(t, X(t), A(t)), \tag{3.14}$$

where F is a function in $C^{1,2,1}(\mathbb{R}^3)$. Note that

$$A(t) = \int_{-\delta}^{0} e^{\lambda s} X(t+s) ds.$$

Then

$$\begin{aligned} dG(t) = {} & (LF)(t, X(t), Y(t), A(t), u(t))dt \\ & + \sigma(t, X(t), Y(t), A(t), u(t)) \frac{\partial F}{\partial x}(t, X(t), A(t)) dB(t) \\ & + \int_{\mathbb{R}_0} \Bigg\{ F(t, X(t^-) + \theta(t, X(t), Y(t), A(t), u(t), z), A(t^-)) \\ & - F(t, X(t^-), A(t^-)) \\ & - \frac{\partial F}{\partial x}(t, X(t^-), A(t^-)) \theta(t, X(t), Y(t), A(t), u(t), z) \Bigg\} \nu(dz) dt \\ & + \int_{\mathbb{R}_0} \{ F(t, X(t^-) + \theta(t, X(t), Y(t), A(t), u(t), z), A(t^-)) \\ & - F(t, X(t^-), A(t^-)) \} \tilde{N}(dt, dz) \\ & + [X(t) - \lambda A(t) - e^{-\lambda\delta} Y(t)] \frac{\partial F}{\partial a}(t, X(t), A(t)) dt, \end{aligned} \tag{3.15}$$

where

$$LF = LF(t, x, y, a, u) = \frac{\partial F}{\partial t} + b(t, x, y, a, u) \frac{\partial F}{\partial x} + \frac{1}{2} \sigma^2(t, x, y, a, u) \frac{\partial^2 F}{\partial x^2}.$$

In particular, note that

$$dA(t) = X(t) - \lambda A(t) - e^{-\lambda\delta} Y(t), \; t \geq 0. \tag{3.16}$$

Now, define the Hamiltonian, $H^{'} : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \Re \to \mathbb{R}$, as

$$\begin{aligned} & H^{'}(t, x, y, a, u, p, q, r(\cdot)) \\ & \quad = f(t, x, y, a, u) + b(t, x, y, a, u) p_1 + (x - \lambda a - e^{-\lambda\delta} y) p_3 \\ & \qquad + \sigma(t, x, y, a, u) q_1 + \int_{\mathbb{R}_0} \theta(t, x, y, a, u, z) r(z) \nu(dz), \end{aligned} \tag{3.17}$$

where $p = (p_1, p_2, p_3)^T \in \mathbb{R}^3$ and $q = (q_1, q_2, q_3) \in \mathbb{R}^3$. For each $u \in \mathcal{A}_{\mathcal{E}}$ the associated adjoint equations are the following BSDEs in the unknown $\mathcal{F}_t$-adapted processes $(p(t), q(t), r(t, \cdot))$ given by

$$dp_1(t) = -\frac{\partial H^{'}}{\partial x}(t, X(t), Y(t), A(t), u(t), p(t), q(t), r(t, \cdot))dt + q_1(t)dB(t) + \int_{\mathbb{R}_0} r(t, z)\tilde{N}(dt, dz), \tag{3.18}$$

$$dp_2(t) = -\frac{\partial H^{'}}{\partial y}(t, X(t), Y(t), A(t), u(t), p(t), q(t), r(t, \cdot))dt, \tag{3.19}$$

$$dp_3(t) = -\frac{\partial H^{'}}{\partial a}(t, X(t), Y(t), A(t), u(t), p(t), q(t), r(t, \cdot))dt + q_3(t)dB(t). \tag{3.20}$$

THEOREM 3.4 (a second infinite horizon maximum principle for delay equations). *Suppose* $\hat{u} \in \mathcal{A}_{\mathcal{E}}$ *and let* $(\hat{X}(t), \hat{Y}(t), \hat{A}(t))$ *and* $(\hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot))$ *be corresponding solutions of* (3.18)–(3.20), *respectively. Suppose that*

$$(x, y, a, u) \mapsto H^{'}(t, x, y, a, u, \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot))$$

is concave for all $t \geq 0$ *a.s. and*

$$E\left[H^{'}(t, \hat{X}(t), \hat{Y}(t), \hat{A}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot))|\mathcal{E}_t\right] = \max_{u\in\mathcal{U}} E\left[H^{'}(t, \hat{X}(t), \hat{Y}(t), \hat{A}(t), u, \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot))|\mathcal{E}_t\right]. \tag{3.21}$$

Further, assume that

$$\overline{\lim_{T\to\infty}} E[\hat{p}_1(T)(X(T) - \hat{X}(T)) + \hat{p}_3(T)(A(T) - \hat{A}(T))] \geq 0. \tag{3.22}$$

In addition assume that

$$\hat{p}_2(t) = 0$$

for all t. *Then* $\hat{u}$ *is an optimal control for the control problem* (2.4).

Proof. To simplify notation we put

$$\zeta(t) = (X(t), Y(t), A(t))$$

and

$$\hat{\zeta}(t) = (\hat{X}(t), \hat{Y}(t), \hat{A}(t)).$$

Let

$$I := J(\hat{u}) - J(u) = E\left[\int_0^\infty (f(t, \hat{\zeta}(t), \hat{u}(t)) - f(t, \zeta(t), u(t)))dt\right].$$

Then we have that

(3.23)

$$
\begin{aligned}
I = \overline{\lim_{T\to\infty}} E\left[\int_0^T (H^{'}(t,\hat{\zeta}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot)) - H^{'}(t,\zeta(t),u(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))dt\right] \\
- E\left[\int_0^T (b(t,\hat{\zeta}(t),\hat{u}(t)) - b(t,\zeta(t),u(t))\hat{p}_1(t)dt\right] \\
- E\left[\int_0^T \{(\hat{X}(t)-\lambda\hat{A}(t)-e^{-\lambda\delta}\hat{Y}(t)) - (X(t)-\lambda A(t)-e^{-\lambda\delta}Y(t))\}\hat{p}_3(t)dt\right] \\
- E\left[\int_0^T \{\sigma(t,\hat{\zeta}(t),\hat{u}(t)) - \sigma(t,\zeta(t),u(t))\}\hat{q}_1(t)dt\right] \\
- E\left[\int_0^T \int_{\mathbb{R}_0} (\theta(t,\hat{\zeta}(t),\hat{u}(t),z) - \theta(t,\zeta,u(t),z))\hat{r}(t,z)\nu(dz)dt\right] \\
=: I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}
$$

Since $(\zeta, u) \to H^{'}(t,\zeta,u,\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))$ is concave, we have by (3.21) that

$$
\begin{aligned}
H^{'}(t,\zeta,u,\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot)) - H^{'}(t,\hat{\zeta},\hat{u},\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot)) \\
\leq \nabla_\zeta H^{'}(t,\hat{\zeta},\hat{u},\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))\cdot(\zeta-\hat{\zeta}) + \frac{\partial H^{'}}{\partial u}(t,\hat{\zeta},\hat{u},\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))\cdot(u-\hat{u}) \\
\leq \nabla_\zeta H^{'}(t,\hat{\zeta},\hat{u},\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))\cdot(\zeta-\hat{\zeta}),
\end{aligned}
$$

where $\nabla_\zeta H^{'} = (\frac{\partial H^{'}}{\partial x}, \frac{\partial H^{'}}{\partial y}, \frac{\partial H^{'}}{\partial a})$. From this we get that

$$
\begin{aligned}
I_1 \geq \overline{\lim_{T\to\infty}} E\left[\int_0^T -\nabla_\zeta H^{'}(t,\hat{\zeta}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))\cdot(\zeta(t)-\hat{\zeta}(t))dt\right] \\
= \overline{\lim_{T\to\infty}} E\left[\int_0^T (\zeta(t)-\hat{\zeta}(t))d\hat{p}(t)\right] \\
= E\left[\int_0^\infty (X(t)-\hat{X}(t))d\hat{p}_1(t) + \int_0^\infty (A(t)-\hat{A}(t))d\hat{p}_3(t)\right].
\end{aligned}
\tag{3.24}
$$

From (3.18), (3.19), and (3.20) we get that

$$
\begin{aligned}
0 \geq -\overline{\lim_{T\to\infty}} E[\hat{p}_1(T)(X(T)-\hat{X}(T)) + \hat{p}_3(T)(A(T)-\hat{A}(T))] \\
= -\overline{\lim_{T\to\infty}} E\left[\int_0^T (X(t)-\hat{X}(t))d\hat{p}_1(t) + \int_0^T \hat{p}_1(t)d(X(t)-\hat{X}(t))\right. \\
+ \int_0^T \left[\sigma(t,\zeta(t),u(t)) - \sigma(t,\hat{\zeta}(t),\hat{u}(t))\right]\hat{q}_1(t)dt \\
+ \int_0^T \int_{\mathbb{R}_0} (\theta(t,\zeta(t),u(t),z) - \theta(t,\hat{\zeta},\hat{u}(t),z))\hat{r}(t,z)\nu(dz)dt \\
\left. + \int_0^T (A(t)-\hat{A}(t))d\hat{p}_3(t) + \int_0^T \hat{p}_3(t)d(A(t)-\hat{A}(t))\right].
\end{aligned}
\tag{3.25}
$$

Combining this with (3.23) and (3.24) and using (3.16), we have that

$$-I = I_1 + I_2 + I_3 + I_4 + I_5 \le 0.$$

Hence $J(\hat{u}) - J(u) = I \ge 0$, and $\hat{u}$ is an optimal control for our problem. □

Example 3.5 (a nondelay infinite horizon example). Let us first consider a nondelay example. Assume we are given the performance functional

$$J(u) = E\left[\int_0^\infty e^{-\rho t}\frac{1}{\gamma}u^\gamma(t)dt\right] \tag{3.26}$$

and the state equation

$$\begin{cases} dX(t) = [X(t)\mu - u(t)]\,dt \\ \qquad\qquad + \sigma(t, X(t), u(t))dB(t); t \ge 0, \\ X(0) = X_0 > 0, \end{cases} \tag{3.27}$$

where $X_0 > 0$, $\gamma \in (0, 1)$, $\rho > 0$, and $\mu \in \mathbb{R}$ are given constants. We assume that

$$\mu\gamma < \rho. \tag{3.28}$$

Here $u(t) \ge 0$ is our control. It can be interpreted as the consumption rate from a cash flow $X(t)$. The performance $J(u)$ is the total expected discounted utility of the consumption. For u to be admissible we require that $E\big[X(t)\big] \ge 0$ for all $t \ge 0$.

In this case the Hamiltonian (3.17) takes the form

$$\begin{aligned} H^{'}(t, x, u, p, q) = e^{-\rho t}\frac{1}{\gamma}u^\gamma + [x\mu - u]p \\ + \sigma(t, x, u)q, \end{aligned} \tag{3.29}$$

so that we get the partial derivative

$$\frac{\partial H^{'}}{\partial u}(t, u, x, p, q) = e^{-\rho t}u^{\gamma-1} - p + \frac{\partial\sigma}{\partial u}q.$$

Therefore, if $\frac{\partial H^{'}}{\partial u} = 0$ we get

$$p(t) = e^{-\rho t}u^{\gamma-1}(t) + \frac{\partial\sigma}{\partial u}(t, X(t), u(t))q(t). \tag{3.30}$$

We now see that the adjoint equation is given by

$$dp(t) = -\left[\mu p(t) + \frac{\partial\sigma}{\partial x}(t, X(t), u(t))q(t)\right]dt + q(t)dB(t).$$

Now assume that

$$\sigma(t, x, u) = \sigma_0(t)x \tag{3.31}$$

for some bounded adapted process $\sigma_0(t)$. Let us try to choose $q = 0$. Then

$$dp(t) = -\mu p(t)dt,$$

which gives

$$p(t) = p(0)e^{-\mu t}$$

for some constant $p(0)$. Hence, by (3.30)

$$\hat{u}(t) = p^{\frac{1}{\gamma-1}}(0)e^{\frac{(\mu-\rho)t}{1-\gamma}} \tag{3.32}$$

for all $t > 0$. Inserting $\hat{u}(t)$ into the dynamics of $\hat{X}(t)$, we get that

$$d\hat{X}(t) = \left[\mu\hat{X}(t) - p^{\frac{1}{\gamma-1}}(0)e^{\frac{1}{\gamma-1}(\rho t - \mu t)}\right]dt + \sigma_0(t)\hat{X}(t)dB(t).$$

So

$$\hat{X}(t) = \left[\hat{X}(0)\Gamma(t) - p^{\frac{1}{\gamma-1}}(0)\int_0^t \frac{\Gamma(t)}{\Gamma(s)}\exp\left(\frac{(\mu-\rho)s}{1-\gamma}\right)ds\right], \tag{3.33}$$

where

$$\Gamma(t) = \exp\left(\int_0^t \sigma_0(s)dB(s) + \mu t - \frac{1}{2}\int_0^t \sigma_0^2(s)ds\right). \tag{3.34}$$

Hence

$$E\left[\hat{X}(t)\right] = e^{\mu t}\left[\hat{X}(0) - p^{\frac{1}{\gamma-1}}(0)\int_0^t \exp\left(\frac{(\mu\gamma-\rho)s}{1-\gamma}\right)ds\right].$$

Therefore, to ensure that $E\big[\hat{X}(t)\big]$ is nonnegative, we get the optimal $\hat{p}(0)$ as

$$\hat{p}(0) = \left[\frac{\hat{X}(0)}{\displaystyle\int_0^\infty \exp\left(\frac{(\mu\gamma-\rho)s}{1-\gamma}\right)ds}\right]^{\gamma-1}. \tag{3.35}$$

We now see that $\overline{\lim}_{T\to\infty} E[\hat{p}(T)\hat{X}(T)] = 0$, so that we have

$$\overline{\lim_{T\to\infty}}\, E[\hat{p}(T)(X(T) - \hat{X}(T))] \geq 0.$$

This tells us that $\hat{u}$ with $p(0) = \hat{p}(0)$ given by (3.35), the control $\hat{u}$ given by (3.32) is indeed an optimal control.

Example 3.6 (an infinite horizon example with delay). Now let us consider a case where we have delay. This is an infinite horizon version of Example 1 in Øksendal and Sulem [9]. Let

$$J(u) = E\left[\int_0^\infty e^{-\rho t}\frac{1}{\gamma}u(t)^\gamma dt\right], \tag{3.36}$$

and define

$$\begin{cases} dX(t) = dX^{(u)}(t) = [X(t)\mu + Y(t)\beta + \alpha A(t) - u(t)]dt \\ \qquad\qquad + \sigma(t, X(t), Y(t), A(t), u(t))dB(t),\ t \geq 0, \\ X(t) = X_0(t) > 0,\ t \in [-\delta, 0]. \end{cases} \tag{3.37}$$

We want to find a consumption rate $u^*(t)$ such that

$$J(u^*) = \sup\left\{J(u); E\big[X^{(u)}(t)\big] \geq 0 \text{ for all } t \geq 0\right\}. \tag{3.38}$$

Here $\gamma \in (0,1)$, $\rho, \delta \geq 0$, and $\beta \in \mathbb{R}$ are given constants.

In this case the Hamiltonian (3.17) takes the form

$$H^{'}(t,x,u,y,a,p,q) = e^{-\rho t}\frac{1}{\gamma}u^{\gamma} + [x\mu + \beta y + \alpha a - u]p_1 + [x - \lambda a - e^{-\lambda\delta}y]p_3 + \sigma(t,x,y,a,u)q_1, \tag{3.39}$$

so that we get the partial derivative

$$\frac{\partial H^{'}}{\partial u}(t,x,u,y,a,p,q) = e^{-\rho t}u^{\gamma-1} - p_1 + \frac{\partial\sigma}{\partial u}q_1.$$

This, together with the maximality condition, gives that

$$p_1(t) = e^{-\rho t}u(t)^{\gamma-1} + \frac{\partial\sigma}{\partial u}q_1.$$

We now see that the adjoint equations are given by

$$\begin{aligned} dp_1(t) &= -\left[\mu p_1(t) + p_3(t) + \frac{\partial\sigma}{\partial x}q_1(t)\right]dt + q_1(t)dB(t),\\ dp_2(t) &= -\left[\beta p_1(t) - e^{-\lambda\delta}p_3(t) + \frac{\partial\sigma}{\partial y}q_1(t)\right]dt,\\ dp_3(t) &= -\left[\alpha p_1(t) - \lambda p_3(t) + \frac{\partial\sigma}{\partial a}q_1(t)\right]dt + q_3(t)dB(t). \end{aligned}$$

Since the coefficents in front of p_1 and p_3 are deterministic we can choose $q_1 = q_3 = 0$. Since we want $p_2(t) = 0$, we then get

$$p_1(t) = \frac{e^{-\lambda\delta}}{\beta}p_3(t),$$

which gives us that

$$\begin{aligned} dp_1(t) &= -[\mu p_1(t) + \beta e^{\lambda\delta}p_1(t)]dt,\\ dp_3(t) &= -\left[\frac{\alpha}{\beta}e^{-\lambda\delta}p_3(t) - \lambda p_3(t)\right]dt, \end{aligned}$$

and

$$u(t) = e^{\frac{\rho t}{\gamma-1}}p_1^{\frac{1}{\gamma-1}}(t). \tag{3.40}$$

Hence, to ensure that

$$p_1(t) = \frac{e^{-\lambda\delta}}{\beta}p_3(t) \tag{3.41}$$

we need that

$$\alpha = \beta e^{\lambda\delta}(\mu + \lambda + \beta e^{\lambda\delta}). \tag{3.42}$$

So

$$p_1(t) = p_1(0)e^{-(\mu+\beta e^{\lambda\delta})t} \tag{3.43}$$

for some constant $p_1(0)$. Hence by (3.40) we get

$$u(t) = u_{p_1(0)} = p_1(0)^{\frac{1}{\gamma-1}} \exp\left(\frac{(\mu + \beta e^{\lambda\delta} - \rho)t}{1-\gamma}\right) \tag{3.44}$$

for all $t > 0$ and some $p_1(0)$. Now assume that

$$\alpha = 0, \text{ i.e. } \lambda + \beta e^{\lambda\delta} = -\mu \tag{3.45}$$

and that

$$\sigma(t, X(t), Y(t), A(t), u(t)) = \kappa A(t) \ (\kappa \text{ constant}). \tag{3.46}$$

Then (3.37) gets the form

$$\begin{cases} dX(t) = [\mu X(t) + \beta Y(t) - u(t)]dt + \kappa A(t)dB(t), \ t \geq 0, \\ X(t) = X_0(t), \ t \in [-\delta, 0], \end{cases} \tag{3.47}$$

and

$$p_1(t) = p_1(0)e^{\lambda t}. \tag{3.48}$$

Let θ be the unique solution of the equation

$$\mu + \theta + |\beta| e^{\theta\delta} = 0. \tag{3.49}$$

Then by Corollary 4.1 in Mohammed and Scheutzow [8] the top a.s. Lyapunov exponent λ_1 of the solution $X^{(0)}(t)$ of the stochastic delay equation (3.47) corresponding to $u = 0$ satisfies the inequality

$$\lambda_1 \leq -\theta + \frac{\kappa^2}{2|\beta|} e^{|\theta|\delta}. \tag{3.50}$$

Therefore we see that

$$\begin{aligned} \lim_{T\to\infty} \hat{p}_1(T)\hat{X}(T) &\leq \lim_{T\to\infty} \hat{p}_1(T)\hat{X}^{(0)}(T) \\ &\leq \text{ const. } \lim_{T\to\infty} \exp\left(-\left(-\lambda + \theta - \frac{\kappa^2}{2|\beta|} e^{|\theta|\delta}\right) T\right) = 0 \end{aligned}$$

if

$$\lambda + \frac{\kappa^2}{2|\beta|} e^{|\theta|\delta} < \theta. \tag{3.51}$$

By (3.41) condition (3.51) also implies that

$$\lim_{T\to\infty} \hat{p}_3(T)\hat{A}(T) = 0. \tag{3.52}$$

We conclude that (3.22) holds. It remains to determine the optimal value of $\hat{p}_1(0)$. To maximize the expected utility of the consumption (3.36), we choose $\hat{p}_1(0)$ as big as possible under the constraint that $E\big[\hat{X}(t)\big] \geq 0$ for all $t \geq 0$. Hence we put

$$\hat{p}_1(0) = \sup\left\{p_1(0); E\big[X^{(\hat{u})}(t)\big] \geq 0 \text{ for all } t \geq 0\right\}, \tag{3.53}$$

where

$$\begin{cases} dX^{(\hat{u})}(t) = \left[X^{(\hat{u})}(t)\mu + Y^{(\hat{u})}(t)\beta - p_1^{\frac{1}{\gamma-1}}(0)\exp\left(\frac{-(\lambda+\rho)t}{1-\gamma}\right)\right]dt \\ \qquad\qquad + \kappa A^{(\hat{u})}(t)dB(t),\ t \ge 0, \\ X^{(\hat{u})}(t) = X_0(t) > 0,\ t \in [-\delta, 0]. \end{cases} \tag{3.54}$$

In this case, however, in lack of a solution formula for $E\big[X^{(u)}(t)\big]$, we are not able to find an explicit expression for $\hat{p}_1(0)$, as we could in Example 3.5. We conclude that our candidate for the optimal control is given by

$$\hat{u}(t) = \hat{p}_1^{\frac{1}{\gamma-1}}(0)\exp\left(\frac{-(\lambda+\rho)t}{1-\gamma}\right).$$

4. A necessary maximum principle. In addition to the assumptions in sections 2 and 3.1, we now assume the following:

(A_1) For all $u \in \mathcal{A}_\mathcal{E}$ and all $\beta \in \mathcal{A}_\mathcal{E}$ bounded, there exists $\epsilon>0$ such that

$$u + s\beta \in \mathcal{A}_\mathcal{E} \qquad \text{for all } s \in (-\epsilon, \epsilon).$$

(A_2) For all t_0, h such that $0 \le t_0 < t_0 + h \le T$ and all bounded $\mathcal{E}_{t_0}$-measurable random variables α, the control process $\beta(t)$ defined by

$$\beta(t) = \alpha 1_{[t_0, t_0+h]}(t) \tag{4.1}$$

belongs to $\mathcal{A}_\mathcal{E}$.

(A_3) The derivative process

$$\xi(t) := \frac{d}{ds}X^{u+s\beta}(t)\ |_{s=0} \tag{4.2}$$

exists and belongs to $L^2(m \times P)$, where m denotes the Lebesgue measure on $\mathbb{R}$.

It follows from (2.1) that

$$\begin{aligned} d\xi(t) &= \left\{\frac{\partial b}{\partial x}(t)\xi(t) + \frac{\partial b}{\partial y}(t)\xi(t-\delta) + \frac{\partial b}{\partial a}(t)\int_{t-\delta}^{t} e^{-\lambda(t-r)}\xi(r)dr + \frac{\partial b}{\partial u}(t)\beta(t)\right\}dt \\ &\quad + \left\{\frac{\partial \sigma}{\partial x}(t)\xi(t) + \frac{\partial \sigma}{\partial y}(t)\xi(t-\delta) + \frac{\partial \sigma}{\partial a}(t)\int_{t-\delta}^{t} e^{-\lambda(t-r)}\xi(r)dr + \frac{\partial \sigma}{\partial u}(t)\beta(t)\right\}dB(t) \\ &\quad + \int_{\mathbb{R}_0}\left\{\frac{\partial \theta}{\partial x}(t,z)\xi(t) + \frac{\partial \theta}{\partial y}(t,z)\xi(t-\delta)\right. \\ &\quad \left. + \frac{\partial \theta}{\partial a}(t,z)\int_{t-\delta}^{t} e^{-\lambda(t-r)}\xi(r)dr + \frac{\partial \theta}{\partial u}(t,z)\beta(t)\right\}\tilde{N}(dt,dz), \end{aligned} \tag{4.3}$$

where, for simplicity of notation, we define

$$\frac{\partial}{\partial x}b(t) := \frac{\partial}{\partial x}b(t, X(t), X(t-\delta), A(t), u(t))$$

and use that

$$\frac{d}{ds}Y^{u+s\beta}(t)\ |_{s=0} = \frac{d}{ds}X^{u+s\beta}(t-\delta)\ |_{s=0} = \xi(t-\delta) \tag{4.4}$$

and

$$\frac{d}{ds}A^{u+s\beta}(t)\mid_{s=0} = \frac{d}{ds}\left(\int_{t-\delta}^{t} e^{-\lambda(t-r)}X^{u+s\beta}(r)dr\right)\mid_{s=0} = \left(\int_{t-\delta}^{t} e^{-\lambda(t-r)}\frac{d}{ds}X^{u+s\beta}(r)dr\right)\mid_{s=0} = \int_{t-\delta}^{t} e^{-\lambda(t-r)}\xi(r)dr. \tag{4.5}$$

Note that

$$\xi(t) = 0 \text{ for } t \in [-\delta, 0]. \tag{4.6}$$

THEOREM 4.1 (necessary maximum principle). *Suppose that* $\hat{u} \in \mathcal{A}_{\mathcal{E}}$ *with corresponding solutions* $\hat{X}(t)$ *of* (2.1)–(2.2) *and* $\hat{p}(t)$, $\hat{q}(t)$, *and* $\hat{r}(t,\cdot)$ *of* (3.2)–(3.3), *and corresponding derivative process* $\hat{\xi}(t)$ *given by* (4.2). *Assume that for all* $u \in \mathcal{A}_{\mathcal{E}}$ *with corresponding* $(X(t), p(t), q(t), r(t,\cdot))$ *the following hold:*

(4.7)

$$E\Bigg[\int_0^T \hat{p}^2(t)\Bigg\{\left(\frac{\partial\sigma}{\partial x}\right)^2(t)\hat{\xi}^2(t) + \left(\frac{\partial\sigma}{\partial y}\right)^2(t)\hat{\xi}^2(t-\delta) + \left(\frac{\partial\sigma}{\partial a}\right)^2(t)\left(\int_{t-\delta}^{t} e^{-\lambda(t-r)}\hat{\xi}(r)dr\right)^2 + \left(\frac{\partial\sigma}{\partial u}\right)^2(t) + \int_{\mathbb{R}_0}\Bigg\{\left(\frac{\partial\theta}{\partial x}\right)^2(t,z)\hat{\xi}^2(t) + \left(\frac{\partial\theta}{\partial y}\right)^2(t,z)\hat{\xi}^2(t-\delta) + \left(\frac{\partial\theta}{\partial a}\right)^2(t,z)\left(\int_{t-\delta}^{t} e^{-\lambda(t-r)}\hat{\xi}(r)dr\right)^2 + \left(\frac{\partial\theta}{\partial u}\right)^2(t,z)\Bigg\}\nu(dz)\Bigg\}dt + \int_0^T \hat{\xi}^2(t)\left\{\hat{q}^2(t) + \int_{\mathbb{R}_0}\hat{r}^2(t,z)\,\nu(dz)\right\}dt\Bigg]$$

$< \infty$ *for all* $T < \infty$

and

$$\lim_{T\to\infty} E\left[\hat{p}(T)\hat{\xi}(T)\right] = 0. \tag{4.8}$$

Then the following assertions are equivalent:

(i) *For all bounded* $\beta \in \mathcal{A}_{\mathcal{E}}$,

$$\frac{d}{ds}J(\hat{u} + s\beta)\mid_{s=0} = 0.$$

(ii) *For all* $t \in [0,\infty)$,

$$E\left[\frac{\partial H}{\partial u}\left(t,\ \hat{X}(t), \hat{Y}(t), \hat{A}(t), u, \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot)\right)\Big|\mathcal{E}_t\right]_{u=\hat{u}(t)} = 0 \ a.s.$$

Proof. Suppose that assertion (i) holds. Then

$$0 = \frac{d}{ds}J(\hat{u}+s\beta)\mid_{s=0} = \frac{d}{ds}E\left[\int_0^\infty f(t, X^{\hat{u}+s\beta}(t), Y^{\hat{u}+s\beta}(t), A^{\hat{u}+s\beta}(t), \hat{u}(t)+s\beta(t)dt\right]\Bigg|_{s=0} = E\Bigg[\int_0^\infty \Bigg\{\frac{\partial f}{\partial x}(t)\xi(t) + \frac{\partial f}{\partial y}(t)\xi(t-\delta) + \frac{\partial f}{\partial a}(t)\int_{t-\delta}^{t} e^{-\lambda(t-r)}\xi(r)dr + \frac{\partial f}{\partial u}(t)\beta(t)\Bigg\}dt\Bigg].$$

We know by the definition of H that

$$\frac{\partial f}{\partial x}(t) = \frac{\partial H}{\partial x}(t) - \frac{\partial b}{\partial x}(t)p(t) - \frac{\partial \sigma}{\partial x}(t)q(t) - \int_{\mathbb{R}_0} \frac{\partial \theta}{\partial x}(t,z)r(t,z)\nu(dz) \tag{4.9}$$

and the same for $\frac{\partial f}{\partial y}(t)$, $\frac{\partial f}{\partial a}(t)$, and $\frac{\partial f}{\partial u}(t)$.

Applying the Itô formula to $\hat{p}(t)\hat{\xi}(t)$, we obtain by (4.8) and (4.9)

$$\begin{aligned}
0 &= \lim_{T\to\infty} E\left[\hat{p}(T)\xi(T)\right] \\
&= E\Bigg[\int_0^\infty \hat{p}(t)\ \Big\{\frac{\partial b}{\partial x}(t)\xi(t) + \frac{\partial b}{\partial y}(t)\xi(t-\delta) \\
&\quad + \frac{\partial b}{\partial a}(t)\int_{t-\delta}^t e^{-\lambda(t-r)}\xi(r)dr + \frac{\partial b}{\partial u}(t)\beta(t)\Big\}\,dt + \int_0^\infty \xi(t)E\left(\mu(t)\mid \mathcal{F}_t\right)dt \\
&\quad + \int_0^\infty q(t)\left\{\frac{\partial \sigma}{\partial x}(t)\xi(t) + \frac{\partial \sigma}{\partial y}(t)\xi(t-\delta) + \frac{\partial \sigma}{\partial a}(t)\int_{t-\delta}^t e^{-\lambda(t-r)}\xi(r)dr + \frac{\partial \sigma}{\partial u}(t)\beta(t)\right\}dt \\
&\quad + \int_0^\infty \int_{\mathbb{R}_0} r(t,z)\Big\{\frac{\partial \theta}{\partial x}(t,z)\xi(t) + \frac{\partial \theta}{\partial y}(t,z)\xi(t-\delta) + \frac{\partial \theta}{\partial a}(t,z)\int_{t-\delta}^t e^{-\lambda(t-r)}\xi(r)dr \\
&\quad + \frac{\partial \theta}{\partial u}(t,z)\beta(t)\Big\}\nu(dz)dt\Bigg] \\
&= -\frac{d}{ds}J(\hat{u}+s\beta)\mid_{s=0} + E\left(\int_0^\infty \frac{\partial H}{\partial u}(t)\beta(t)dt\right).
\end{aligned}$$

Therefore

$$E\left(\int_0^\infty \frac{\partial H}{\partial u}(t)\beta(t)dt\right) = \frac{d}{ds}J(\hat{u}+s\beta)\Big|_{s=0}. \tag{4.10}$$

Now apply this to

$$\beta(t) = \alpha 1_{[t_0,t_0+h]}(t)\ ,$$

where α is bounded and $\mathcal{E}_{t_0}$-measurable, $0 \le t_0 < t_o + h \le T$. Then if (i) holds we get

$$E\left(\int_{t_0}^{t_0+h} \frac{\partial H}{\partial u}(t)dt\ \alpha\right) = 0.$$

Differentiating with respect to h at 0, we have

$$E\left(\frac{\partial H}{\partial u}(t_0)\ \alpha\right) = 0.$$

This holds for all $\mathcal{E}_{t_0}$-measurable α and hence we obtain that

$$E\left(\frac{\partial H}{\partial u}(t_0)\mid \mathcal{E}_{t_0}\right) = 0.$$

This proves that assertion (i) implies (ii).

To complete the proof, we need to prove the converse implication, which is obtained since every bounded $\beta \in \mathcal{A}_{\mathcal{E}}$ can be approximated by linear combinations of controls β of the form (4.1). ∎

5. Existence and uniqueness of the time-advanced BSDEs on infinite horizon. The main result in this section refers to the existence and uniqueness for (3.3)–(3.4) where the coefficients satisfy a Lipschitz condition.

Given a positive constant δ, denote by $D([0,\delta],\mathbb{R})$ the space of all càdlàg paths from $[0,\delta]$ into $\mathbb{R}$. For a path $X(\cdot):\mathbb{R}_+\to\mathbb{R}$, X_t will denote the function defined by $X_t(s)=X(t+s)$ for $s\in[0,\delta]$. Let $\mathcal{H}=L^2(\nu)$ be the set of all functions $r:\mathbb{R}_0\to\mathbb{R}$ such that $\int_{\mathbb{R}_0} r^2(z)\nu(dz)<\infty$. Consider the L^2 space $V_1:=L^2([0,\delta]\to\mathbb{R};ds)$ and $V_2:=L^2([0,\delta]\to\mathcal{H};ds)$. Let

$$F:\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}\times V_1\times\mathbb{R}\times\mathbb{R}\times V_1\times\mathcal{H}\times\mathcal{H}\times V_2\times\Omega\to\mathbb{R}$$

be a function satisfying the following Lipschitz condition: There exists a positive constant C such that

$$\begin{aligned}|F(t,p_1,p_2,p,q_1,q_2,q,r_1,r_2,r,\omega)-F(t,\bar{p}_1,\bar{p_2},\bar{p},\bar{q}_1,\bar{q}_2,\bar{q},\bar{r}_1,\bar{r}_2,\bar{r},\omega)|\\ \le C(|p_1-\bar{p}_1|+|p_2-\bar{p}_2|+|p-\bar{p}|_{V_1}+|q_1-\bar{q}_1|+|q_2-\bar{q}_2|+|q-\bar{q}|_{V_1}\\ +|r_1-\bar{r}_1|_{\mathcal{H}}+|r_2-\bar{r}_2|_{\mathcal{H}}+|r-\bar{r}|_{V_2})\text{ a.s.}\end{aligned} \tag{5.1}$$

Assume that $(t,\omega)\to F(t,p_1,p_2,p,q_1,q_2,q,r_1,r_2,r,\omega)$ is predictable for all $p_1,p_2,p,q_1,q_2,q,r_1,r_2,r$. Further we assume the following:

$$E\int_0^\infty e^{\lambda t}|F(t,0,0,0,0,0,0,0,0,0)|^2dt<\infty \tag{5.2}$$

for all $\lambda\in\mathbb{R}$. We now consider the following BSDE in the unknown $\mathcal{F}_t$-adapted, $\mathbb{R}\times\mathbb{R}\times\mathcal{H}$-valued process $(p(t),q(t),r(t)=r(t,\cdot))$:

$$\begin{aligned}dp(t)&=E[F(t,p(t),p(t+\delta),p_t,q(t),q(t+\delta),q_t,r(t),r(t+\delta),r_t)|\mathcal{F}_t]dt\\ &\quad+q(t)dB(t)+\int_{\mathbb{R}_0}r(t,z)\tilde{N}(dt,dz),\end{aligned} \tag{5.3}$$

where

$$E\left[\int_0^\infty e^{\lambda t}|p(t)|^2dt\right]<\infty \tag{5.4}$$

for all $\lambda\in\mathbb{R}$.

Theorem 5.1 (existence and uniqueness). *Assume the conditions* (5.1)–(5.2) *are fulfilled. Then the BSDE* (5.3)–(5.4) *admits a unique solution* $(p(t),q(t),r(t,\cdot))$ *such that*

$$E\left[\int_0^\infty e^{\lambda t}\left\{|p(t)|^2+|q(t)|^2+\int_{\mathbb{R}_0}|r(t,z)|^2\nu(dz)\right\}dt\right]<\infty$$

for all $\lambda>0$.

Proof.

Existence:

Step 1:

First, assume F is independent of its second, third, and fourth parameters.

Set $q^0(t):=0$, $r^0(t,\cdot):=0$. For $n\ge1$, define $(p^n(t),q^n(t),r^n(t,\cdot))$ to be the unique solution of the following BSDE:

(5.5)

$$dp^n(t) = E\left[F(t, q^{n-1}(t), q^{n-1}(t+\delta), q_t^{n-1}, r^{n-1}(t,\cdot), r^{n-1}(t+\delta,\cdot), r_t^{n-1}(\cdot)) \mid \mathcal{F}_t\right] dt + q^n(t)dB(t) + \int_{\mathbb{R}_0} r^n(t,z)\tilde{N}(dt,dz)$$

for $t \in [0,\infty)$ such that

$$E\left[\int_0^\infty e^{\lambda t}|p^n(t)|^2 dt\right] < \infty.$$

The triples $(p^n(t), q^n(t), r^n(t,\cdot))$ exist by Theorem 3.1 in Haadem, Øksendal, and Proske [4].

Our goal is to show that $(p^n(t), q^n(t), r^n(t,\cdot))$ forms a Cauchy sequence. By the Itô formula we get that

$$\begin{aligned}0 = E\Bigg[&e^{\lambda t}|p^{n+1}(t) - p^n(t)|^2 + \int_t^\infty \lambda e^{\lambda s}|p^{n+1}(s) - p^n(s)|^2 ds\\
&+ \int_t^\infty e^{\lambda s}|q^{n+1}(s) - q^n(s)|^2 ds\\
&+ \int_t^\infty e^{\lambda s}\int_{\mathbb{R}_0} |\left(r^{n+1}(s,z) - r^n(s,z)\right)|^2 ds\nu(dz)\\
&+ 2\int_t^\infty e^{\lambda s}\left(p^{n+1}(s) - p^n(s)\right) E\left[F^n(s) - F^{n-1}(s) \mid \mathcal{F}_s\right] ds\Bigg].\end{aligned}$$

Rearranging, using that for all $a, b \in \mathbb{R}$: $2ab \leq \frac{a^2}{\epsilon} + \epsilon b^2$, we have by the Lipschitz requirement (5.1)

$$\begin{aligned}&E[e^{\lambda t}|p^{n+1}(t) - p^n(t)|^2]\\
&\quad + E\left[\int_t^\infty \lambda e^{\lambda s}|p^{n+1}(s) - p^n(s)|^2 ds\right]\\
&\quad + E\left[\int_t^\infty e^{\lambda s}|q^{n+1}(s) - q^n(s)|^2 ds\right]\\
&\quad + E\left[\int_t^\infty \int_{\mathbb{R}_0} e^{\lambda s}|\left(r^{n+1}(s,z) - r^n(s,z)\right)|^2 \nu(dz)ds\right]\\
&\leq C_\epsilon E\left[\int_t^\infty e^{\lambda s}|p^{n+1}(s) - p^n(s)|^2 ds\right]\\
&\quad + \epsilon 6E\left[\int_t^\infty e^{\lambda s}|q^n(s) - q^{n-1}(s)|^2 ds\right]\\
&\quad + \epsilon 6E\left[\int_t^\infty e^{\lambda s}|q^n(s+\delta) - q^{n-1}(s+\delta)|^2 ds\right]\\
&\quad + \epsilon 6E\left[\int_t^\infty e^{\lambda s}\int_s^{s+\delta} |q^n(u) - q^{n-1}(u)|^2 du ds\right]\\
&\quad + \epsilon 6E\left[\int_t^\infty e^{\lambda s}|r^n(s) - r^{n-1}(s)|_{\mathcal{H}}^2 ds\right]\\
&\quad + \epsilon 6E\left[\int_t^\infty e^{\lambda s}|r^n(s+\delta) - r^{n-1}(s+\delta)|_{\mathcal{H}}^2 ds\right]\\
&\quad + \epsilon 6E\left[\int_t^\infty e^{\lambda s}\int_s^{s+\delta} |r^n(u) - r^{n-1}(u)|_{\mathcal{H}}^2 du ds\right],\end{aligned}$$

where $C_\epsilon = \frac{C^2}{\epsilon}$, and we used the abbreviation

$$F^n(t) := F(t, q^n(t), q^n(t+\delta), q^n_t, r^n(t,\cdot), r^n(t+\delta,\cdot), r^n_t(\cdot)).$$

Note that

$$\begin{aligned} &E\left[\int_t^\infty e^{\lambda s}|q^n(s+\delta) - q^{n-1}(s+\delta)|^2 ds\right] \\ &\quad \le e^{-\lambda\delta} E\left[\int_t^\infty e^{\lambda s}|q^n(s) - q^{n-1}(s)|^2 ds\right]. \end{aligned}$$

Using Fubini

$$\begin{aligned} &E\left[\int_t^\infty \int_s^{s+\delta} e^{\lambda s}|q^n(u) - q^{n-1}(u)|^2 du ds\right] \\ &\quad \le E\left[\int_t^\infty \int_{u-\delta}^{u} e^{\lambda s}|q^n(u) - q^{n-1}(u)|^2 ds du\right] \\ &\quad \le \frac{1}{\lambda}(1 - e^{-\lambda\delta}) E\left[\int_t^\infty e^{\lambda s}|q^n(s) - q^{n-1}(s)|^2 ds\right] \\ &\quad \le E\left[\int_t^\infty e^{\lambda s}|q^n(s) - q^{n-1}(s)|^2 ds\right]. \end{aligned}$$

Similarly for $r^n - r^{n-1}$. It now follows that

$$\begin{aligned} &E[e^{\lambda t}|p^{n+1}(t) - p^n(t)|^2] \\ &\quad + E\left[\int_t^\infty e^{\lambda s}|q^{n+1}(s) - q^n(s)|^2 ds\right] \\ &\quad + E\left[\int_t^\infty \int_{\mathbb{R}_0} e^{\lambda s}|\big(r^{n+1}(s,z) - r^n(s,z)\big)|^2 \nu(dz) ds\right] \\ &\le (C_\epsilon - \lambda) E\left[\int_t^\infty e^{\lambda s}|p^{n+1}(s) - p^n(s)|^2 ds\right] \\ &\quad + \epsilon 6(2 + e^{-\lambda\delta}) E\left[\int_t^\infty e^{\lambda s}|q^n(s) - q^{n-1}(s)|^2 ds\right] \\ &\quad + \epsilon 6(2 + e^{-\lambda\delta}) E\left[\int_t^\infty e^{\lambda s}|r^n(s) - r^{n-1}(s)|^2_{\mathcal{H}} ds\right]. \end{aligned}$$

Choosing $\epsilon = \frac{1}{12(2+e^{-\lambda\delta})}$ we get

(5.6)

$$\begin{aligned} &E[e^{\lambda t}|p^{n+1}(t) - p^n(t)|^2] \\ &\quad + E\left[\int_t^\infty e^{\lambda s}|q^{n+1}(s) - q^n(s)|^2 ds\right] \\ &\quad + E\left[\int_t^\infty \int_{\mathbb{R}_0} e^{\lambda s}|\big(r^{n+1}(s,z) - r^n(s,z)\big)|^2 \nu(dz) ds\right] \\ &\le (C_\epsilon - \lambda) E\left[\int_t^\infty e^{\lambda s}|p^{n+1}(s) - p^n(s)|^2 ds\right] + \frac{1}{2} E\left[\int_t^\infty e^{\lambda s}|q^n(s) - q^{n-1}(s)|^2 ds\right] \\ &\quad + \frac{1}{2} E\left[\int_t^\infty e^{\lambda s}|r^n(s) - r^{n-1}(s)|^2_{\mathcal{H}} ds\right]. \end{aligned}$$

From this we deduce that

$$
\begin{aligned}
&-\frac{d}{dt}\left(e^{(C_\epsilon-\lambda)t}E\left[\int_t^\infty e^{\lambda s}|p^{n+1}(s)-p^n(s)|^2ds\right]\right)\\
&\quad+e^{(C_\epsilon-\lambda)t}E\left[\int_t^\infty e^{\lambda s}|q^{n+1}(s)-q^n(s)|^2\right]ds\\
&\quad+e^{(C_\epsilon-\lambda)t}E\left[\int_t^\infty\int_{\mathbb{R}_0} e^{\lambda s}|\big(r^{n+1}(s,z)-r^n(s,z)\big)|^2\nu(dz)ds\right]\\
&\le\frac{1}{2}e^{(C_\epsilon-\lambda)t}E\left[\int_t^\infty e^{\lambda s}|q^n(s)-q^{n-1}(s)|^2ds\right]\\
&\quad+\frac{1}{2}e^{(C_\epsilon-\lambda)t}E\left[\int_t^\infty e^{\lambda s}|r^n(s)-r^{n-1}(s)|^2_{\mathcal{H}}ds\right].
\end{aligned}
$$

Integrating the last inequality we get that

$$
\begin{aligned}
&E\left[\int_0^\infty e^{\lambda t}|p^{n+1}(t)-p^n(t)|^2dt\right]\\
&\quad+\int_0^\infty e^{(C_\epsilon-\lambda)t}E\left[\int_t^\infty e^{\lambda s}|q^{n+1}(s)-q^n(s)|^2ds\right]dt\\
&\quad+\int_0^\infty e^{(C_\epsilon-\lambda)t}E\left[\int_t^\infty\int_{\mathbb{R}_0} e^{\lambda s}|\big(r^{n+1}(s,z)-r^n(s,z)\big)|^2\nu(dz)ds\right]dt\\
&\le\frac{1}{2}\int_0^\infty e^{(C_\epsilon-\lambda)t}E\left[\int_t^\infty e^{\lambda s}|q^n(s)-q^{n-1}(s)|^2ds\right]dt\\
&\quad+\frac{1}{2}\int_0^\infty e^{(C_\epsilon-\lambda)t}E\left[\int_t^\infty e^{\lambda s}|r^n(s)-r^{n-1}(s)|^2_{\mathcal{H}}ds\right]dt.
\end{aligned}
\tag{5.7}
$$

So that

$$
\begin{aligned}
&\int_0^\infty e^{(C_\epsilon-\lambda)t}E\left[\int_t^\infty e^{\lambda s}|q^{n+1}(s)-q^n(s)|^2ds\right]dt\\
&\quad+\int_0^\infty e^{(C_\epsilon-\lambda)t}E\left[\int_t^\infty\int_{\mathbb{R}_0} e^{\lambda s}|\big(r^{n+1}(s,z)-r^n(s,z)\big)|^2\nu(dz)ds\right]dt\\
&\le\frac{1}{2}\int_0^\infty e^{(C_\epsilon-\lambda)t}E\left[\int_t^\infty e^{\lambda s}|q^n(s)-q^{n-1}(s)|^2ds\right]dt\\
&\quad+\frac{1}{2}\int_0^\infty e^{(C_\epsilon-\lambda)t}E\left[\int_t^\infty e^{\lambda s}|r^n(s)-r^{n-1}(s)|^2_{\mathcal{H}}ds\right]dt.
\end{aligned}
$$

This gives that

$$
\begin{aligned}
&\int_0^\infty e^{(C_\epsilon-\lambda)t}E\left[\int_t^\infty e^{\lambda s}|q^{n+1}(s)-q^n(s)|^2ds\right]dt\\
&\quad+\int_0^\infty e^{(C_\epsilon-\lambda)t}E\left[\int_t^\infty\int_{\mathbb{R}_0} e^{\lambda s}|\big(r^{n+1}(s,z)-r^n(s,z)\big)|^2\nu(dz)ds\right]dt\\
&\le\frac{1}{2^n}C_3
\end{aligned}
$$

if $\lambda > \frac{C}{\epsilon}$. It then follows from (5.7) that

$$
E\left[\int_0^\infty e^{\lambda t}|p^{n+1}(t)-p^n(t)|^2dt\right]\le\frac{1}{2^n}C_3.
$$

From (5.6) and (5.7), we now get

$$E\left[\int_t^\infty \int_{\mathbb{R}_0} e^{\lambda s}|\left(r^{n+1}(s,z)-r^n(s,z)\right)|^2\nu(dz)ds\right] + E\left[\int_t^\infty e^{\lambda s}|q^{n+1}(s)-q^n(s)|^2 ds\right] \leq \frac{1}{2^n}C_3 n C_\epsilon.$$

From this we conclude that there exist progressively measurable processes $(p(t), q(t), r(t,\cdot))$, such that

$$\lim_{n\to\infty} E\left[e^{\lambda t}|p^n(t)-p(t)|^2 dt\right] = 0,$$

$$\lim_{n\to\infty} E\left[\int_0^\infty e^{\lambda t}|p^n(t)-p(t)|^2 dt\right] = 0,$$

$$\lim_{n\to\infty} E\left[\int_0^\infty e^{\lambda t}|q^n(t)-q(t)|^2 dt\right] = 0,$$

$$\lim_{n\to\infty} E\left[\int_t^\infty \int_{\mathbb{R}_0} e^{\lambda s}|r^n(s,z)-r(s,z)|^2\nu(dz)ds\right] = 0.$$

Letting $n\to\infty$ in (5.5) we see that $(p(t), q(t), r(t,\cdot))$ satisfies

$$dp(t) = E\big[F(t, q(t), q(t+\delta), q_t, r(t,\cdot), r(t+\delta,\cdot), r_t(\cdot)) \mid \mathcal{F}_t\big]dt + q(t)dB(t) + \int_{\mathbb{R}_0} r(t,z)\tilde{N}(dt,dz)$$

for all $t > 0$.

Step 2:

General F.

Let $p^0(t) = 0$. For $n \geq 1$ define $(p^n(t), q^n(t), r^n(t,\cdot))$ to be the unique solution to the following ABSDE:

$$dp^n(t) = E\left[F(t, p^{n-1}(t), p^{n-1}(t+\delta), p_t^{n-1}, q^n(t), q^n(t+\delta), q_t^n, r^n(t), r^n(t+\delta), r_t^n)|\mathcal{F}_t\right]dt + q^n(t)dB(t) + \int_{\mathbb{R}_0} r^n(t,z)\tilde{N}(dz,dt)$$

for $t\in[0,\infty)$. The existence of $(p^n(t), q^n(t), r^n(t,\cdot))$ was proved in Step 1.

By using the same arguments as above, we deduce that

$$\begin{aligned}
&E\left[e^{\lambda t}|p^{n+1}(t)-p^n(t)|^2\right] \\
&\quad + E\left[\int_t^\infty e^{\lambda s}|q^{n+1}(s)-q^n(s)|^2 ds\right] \\
&\quad + E\left[\int_t^\infty e^{\lambda s}\int_{\mathbb{R}_0}|\left(r^{n+1}(s,z)-r^n(s,z)\right)|^2\nu(dz)ds\right] \\
&\leq (C_\epsilon-\lambda)E\left[\int_t^\infty e^{\lambda s}|p^{n+1}(s)-p^n(s)|^2 ds\right] + \frac{1}{2}E\left[\int_t^\infty e^{\lambda s}|p^n(s)-p^{n-1}(s)|^2 ds\right].
\end{aligned}$$

This implies that

$$-\frac{d}{dt}\left(e^{(C_\epsilon-\lambda)t}E\left[\int_t^\infty e^{\lambda s}|p^{n+1}(s)-p^n(s)|^2 ds\right]\right) \leq \frac{1}{2}e^{(C_\epsilon-\lambda)t}E\left[\int_t^\infty e^{\lambda s}|p^n(s)-p^{n-1}(s)|^2 ds\right].$$

Integrating from 0 to ∞, we get

$$E\left[\int_0^\infty e^{\lambda s}|p^{n+1}(s)-p^n(s)|^2 ds\right] \leq \frac{1}{2}\int_0^\infty e^{(C_\epsilon-\lambda)t} E\left[\int_t^\infty e^{\lambda s}|p^n(s)-p^{n-1}(s)|^2 ds\right] dt.$$

So if $\lambda \geq C_\epsilon$ then by iteration we see that

$$E\left[\int_0^\infty e^{\lambda s}|p^{n+1}(s)-p^n(s)|^2 ds\right] \leq \frac{K}{2^n(\lambda - C_\epsilon)^n}$$

for some constant K.

Uniqueness:

In order to prove the uniqueness, we assume that there are two solution triples $\left(p^1(t), q^1(s), r^1(s,\cdot)\right)$ and $\left(p^2(t), q^2(s), r^2(s,\cdot)\right)$ to the ABSDE

$$dp(t) = E\left[F(t,p(t),p(t+\delta),p_t,q(t),q(t+\delta),q_t,r(t),r(t+\delta),r_t) \mid \mathcal{F}_t\right] dt + q(t)dB(t) + \int_{\mathbb{R}_0} r(t,z)\tilde{N}(dt,dz); t \in [0,\infty),$$

where

$$E\left[\int_0^\infty e^{\lambda\, t}\left|p(t)\right|^2\ dt\right] < \infty$$

and

$$\lambda \geq \frac{3C^2}{\epsilon} + \frac{1}{2}.$$

By the Itô formula, we have

$$\begin{aligned}
&E\left[e^{\lambda t}\left|p^1(t)-p^2(t)\right|^2\right] + E\left[\int_t^\infty \lambda\, e^{\lambda s}\left|p^1(s)-p^2(s)\right| ds\right] \\
&\quad + E\left[\int_t^\infty e^{\lambda s}\left|q^1(s)-q^2(s)\right|^2 ds\right] + E\left[\int_t^\infty e^{\lambda s}\int_{\mathbb{R}_0}\left|r^1(s,z)-r^2(s,z)\right|^2 ds\nu(dz)\right] \\
&= 2E\Bigg[\int_t^\infty e^{\lambda s}\Bigg[\left|p^1(s)-p^2(s)\right| \\
&\quad \times \Bigg(E\left[F(s,p^1(s),p^1(s+\delta),p^1_s,q^1(s),q^1(s+\delta),q^1_s,r^1(s),r^1(s+\delta),r^1_s) \mid \mathcal{F}_s\right] \\
&\quad - E\left[F(s,p^2(s),p^2(s+\delta),p^2_s,q^2(s),q^2(s+\delta),q^2_s,r^2(s),r^2(s+\delta),r^2_s) \mid \mathcal{F}_s\right]\Bigg)\Bigg] ds\Bigg] \\
&\leq 2E\Bigg[\int_t^\infty e^{\lambda s}\Bigg[\left|p^1(s)-p^2(s)\right| \\
&\quad \times C\Bigg(\left|p^1(s)-p^2(s)\right| + \left|p^1(s+\delta)-p^2(s+\delta)\right| + \int_s^{s+\delta}\left|p^1(u)-p^2(u)\right| du]
\end{aligned}$$

$$+ \left|q^1(s) - q^2(s)\right| + \left|q^1(s+\delta) - q^2(s+\delta)\right| + \int_s^{s+\delta} \left|q^1(u) - q^2(u)\right| du$$
$$+ \left|r^1(s) - r^2(s)\right|_{\mathcal{H}}^2 + \left|r^1(s+\delta) - r^2(s+\delta)\right|_{\mathcal{H}}^2 + \int_s^{s+\delta} \left|r^1(u) - r^2(u)\right|_{\mathcal{H}}^2 du \Big) \Big] ds \Big].$$

By the above inequalities for (p, q, r) and the fact that $2ab \leq \frac{a^2}{\epsilon} + \epsilon b^2$, we have that

$$E\left[e^{\lambda t}\left|p^1(t) - p^2(t)\right|^2\right] + E\left[\int_t^\infty e^{\lambda s}\left|q^1(s) - q^2(s)\right|^2 ds\right]$$
$$+ E\left[\int_t^\infty e^{\lambda s}\int_{\mathbb{R}_0}\left|r^1(s,z) - r^2(s,z)\right|^2 ds\nu(dz)\right]$$
$$\leq \left(\frac{3C^2}{\epsilon} - \lambda\right) E\left[\int_t^\infty e^{\lambda s}\left|p^1(s) - p^2(s)\right|^2 ds\right]$$
$$+ (2 + e^{-\lambda\,\delta})\,\epsilon E\left[\int_t^\infty e^{\lambda s}\left|p^1(s) - p^2(s)\right|^2 ds\right]$$
$$+ (2 + e^{-\lambda\,\delta})\,\epsilon\, E\left[\int_t^\infty e^{\lambda s}\left[\left|q^1(s) - q^2(s)\right|^2 + \left|r^1(s,z) - r^2(s,z)\right|_{\mathcal{H}}^2\right] ds\right].$$

Taking ϵ such that $(2 + e^{-\lambda\delta})\ \epsilon = \frac{1}{2}$

$$E\left[e^{\lambda t}\left|p^1(t) - p^2(t)\right|^2\right] + E\left[\int_t^\infty e^{\lambda s}\left|q^1(s) - q^2(s)\right|^2 ds\right]$$
$$+ E\left[\int_t^\infty e^{\lambda\, s}\int_{\mathbb{R}_0}\left|r^1(s,z) - r^2(s,z)\right|^2 ds\nu(dz)\right]$$
$$\leq \left(\frac{3C^2}{\epsilon} - \lambda + \frac{1}{2}\right) E\left[\int_t^\infty e^{\lambda s}\left|p^1(s) - p^2(s)\right|^2 ds\right]$$
$$+ \frac{1}{2}E\left[\int_t^\infty \left|q^1(s) - q^2(s)\right|^2 ds\right]$$
$$+ \frac{1}{2}E\left[\int_t^\infty \left|r^1(s,z) - r^2(s,z)\right|_{\mathcal{H}}^2 ds\right].$$

We get

$$E\left[e^{\lambda t}\left|p^1(t) - p^2(t)\right|^2\right] + \frac{1}{2}E\left[e^{\lambda s}\left|q^1(s) - q^2(s)\right|^2 ds\right]$$
$$+ \frac{1}{2}E\left[\int_t^\infty e^{\lambda s}\int_{\mathbb{R}_0}\left|r^1(s,z) - r^2(s,z)\right|^2 ds\nu(dz)\right]$$
$$\leq \left(\frac{3C^2}{\epsilon} - \lambda + \frac{1}{2}\right) E\left[\int_t^\infty e^{\lambda s}\left|p^1(s) - p^2(s)\right|^2 ds\right].$$

Using the fact that $\lambda \geq \frac{3C^2}{\epsilon} + \frac{1}{2}$, we obtain for all $t \in [0, \infty)$,

$$E\left[e^{\lambda t}\left|p^1(t) - p^2(t)\right|^2\right] = 0,$$

which proves that $p^1(t)$ and $p^2(t)$ are indistinguishable. ▯

Acknowledgment. We thank Salah-E.A. Mohammed for fruitful conversations.

References

[1] G. Barles, R. Buckdahn, and E. Pardoux, *Backward stochastic differential equations and integral-partial differential equations*, Stoch. Stoch. Rep., 60 (2009), pp. 57–83.

[2] L. Chen and Z. Wu, *Maximum principle for the stochastic optimal control problem with delay and application*, Automatica, 46 (2010), pp. 1074–1080.

[3] I. Elsanosi, B. Øksendal, and A. Sulem, *Some solvable stochastic control problems with delay*, Stoch. Stoch. Rep., 71 (2000), pp. 69–89.

[4] S. Haadem, B. Øksendal, and F. Proske, *Maximum principles for jump diffusion processes with infinite horizon*, Automatica, 49 (2013), pp. 2267–2275.

[5] H. Halkin, *Necessary conditions for optimal control problems with infinite horizons*, Econometrica, 42 (1974), pp. 267–272.

[6] J. Li and S. Peng, *Stochastic optimization theory of backward stochastic differential equations with jumps and viscosity solutions of Hamilton-Jacobi-Bellman equations*, Nonlinear Anal., 70 (2009), pp. 1779–1796.

[7] B. Maslowski and P. Veverka, *Sufficient Stochastic Maximum Principle for Discounted Control Problem*, arXiv:1105.4737v, 2011.

[8] S.-E. A. Mohammed and M. K. R. Scheutzow, *Lyapunov exponents of linear stochastic functional differential equations driven by semimartingales, Part* ii*: Examples and case studies*, Ann. Probab., 25 (1997), pp. 1210–1240.

[9] B. Øksendal and A. Sulem, *A maximum principle for optimal control of stochastic systems with delay, with applications to finance,* in Optimal Control and Partial Differential Equations: Innovations and Applications, J. M. Menaldi, E. Rofman, and A. Sulem, eds., IOS Press, Amsterdam, 2000.

[10] B. Øksendal and A. Sulem, *Applied Stochastic Control of Jump Diffusions*, 2nd ed., Springer, New York, 2007.

[11] B. Øksendal, A. Sulem, and T. Zhang, *Optimal control of stochastic delay equations and time-advanced backward stochastic differential equations*, Adv. Appl. Prob., 43 (2011), pp. 572–596.

[12] E. Pardoux, *BSDEs, weak convergence and homogenizations of semilinear PDEs*, in Nonlinear Analysis, Differential Equations and Control, F. H. Clark and R. J. Stern, eds., Kluwer Academic Publishers, Dordrecht, Netherlands, 1999, pp. 503–549.

[13] S. Peng and Y. Shi, *Infinite horizon forward-backward stochastic differential equations*, Stochastic Process. Appl., 85 (2000), pp. 75–92.

[14] M. Royer, *Backward stochastic differential equations with jumps and related non-linear expectations*, Stochastic Process. Appl., 116 (2006), pp. 1358–1376.

[15] R. Situ, *On solutions of backward stochastic differential equations with jumps and with non-Lipschitzian coefficients in Hilbert spaces and stochastic control*, Statist. Probab. Lett., 60 (2002), pp. 279–288.

[16] J. Yin, *On solutions of a class of infinite horizon fbsdes*, Statist. Probab. Lett., 78 (2008), pp. 2412–2419.