

OPTIMAL STOPPING AND STOCHASTIC CONTROL DIFFERENTIAL GAMES FOR JUMP DIFFUSIONS

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Abstract

We study stochastic differential games of jump diffusions driven by Brownian motions and compensated Poisson random measures, where one of the players can choose the stochastic control and the other player can decide when to stop the system. We prove a verification theorem for such games in terms of a Hamilton-Jacobi-Bellman variational inequality (HJBVI). We also prove that the value function of the game is a viscosity solution of this associated HJBVI.

The results are applied to study some specific examples, including optimal resource extraction in a worst case scenario, and risk minimizing optimal portfolio and stopping.

1 Introduction

Let $X(t) = X(t, \omega) \in [0, \infty) \times \Omega$ be a stochastic process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ representing the wealth of an investment at time t . The owner of the investment wants to find the optimal time for selling the investment. If we interpret “optimal” in the sense of “risk minimal”, then the problem is to find a stopping time $\tau = \tau(\omega)$ which minimizes $\rho(X(\tau))$, where ρ denotes a *risk measure*. If the risk measure ρ is chosen to be a convex risk measure in the sense of [10] and (or) [9], then it can be given the representation

$$\rho(X) = \sup_{Q \in \mathcal{N}} \{ \mathbb{E}_Q[-X] - \zeta(Q) \}, \quad (1)$$

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for some set \mathcal{N} of probability measures $\mathbb{Q} \ll \mathbb{P}$ and some convex “penalty” function $\zeta : \mathcal{N} \rightarrow \mathbb{R}$.

Using this representation the optimal stopping problem above gets the form

$$\inf_{\tau \in \mathcal{T}} \left(\sup_{\mathbb{Q} \in \mathcal{N}} \{ \mathbb{E}_{\mathbb{Q}}[-X(\tau)] - \zeta(\mathbb{Q}) \} \right) \quad (2)$$

where \mathcal{T} is a given family of admissible \mathcal{F}_t -stopping times. This may be regarded as an *optimal stopping-stochastic control differential game*.

In this paper we study this problem in a jump diffusion context. In Section 2 we formulate a general optimal stopping-stochastic control differential game problem in this context and we prove a general verification theorem for such games in terms of variational inequality-Hamilton-Jacobi-Bellman (VIHJB) equations. Then in Section 3 we apply the general results obtained in Section 2 to study the problem (2). By parametrizing the measures $\mathbb{Q} \in \mathcal{N}$ by a stochastic process $\theta(t, z) = (\theta_0(t), \theta_1(t, z))$ we may regard (2) as a special case of the general stochastic differential game in Section 2. We use this to solve the problem in some special cases.

2 General formulation

In this section we put the problem in the introduction into a general framework of optimal stopping and stochastic control differential game for jump diffusions and we prove a verification theorem for the value function of such a game. We refer to [16] for information about optimal stopping and stochastic control for jump diffusions. The following presentation follows [15] closely.

Suppose the state $Y(t) = Y^u(t) = Y_t^{y,u}$ at time t is given as the solution of a stochastic differential equation of the form

$$\begin{cases} dY(t) = b(Y(t), u_0(t)) dt + \sigma(Y(t), u_0(t)) dB(t) \\ + \int_{\mathbb{R}_0^k} \gamma(Y(t^-), u_1(t, z), z) \tilde{N}(dt, dz); \\ Y(0) = y \in \mathbb{R}^k. \end{cases} \quad (3)$$

Here $b : \mathbb{R}^k \times K \rightarrow \mathbb{R}^k$, $\sigma : \mathbb{R}^k \times K \rightarrow \mathbb{R}^{k \times k}$ and $\gamma : \mathbb{R}^k \times K \times \mathbb{R}^k \rightarrow \mathbb{R}^{k \times k}$ are given functions, $B(t)$ is a k -dimensional Brownian motion and $\tilde{N}(\cdot, \cdot) = \left(\tilde{N}_1(\cdot, \cdot), \dots, \tilde{N}_k(\cdot, \cdot) \right)$ are k independent compensated Poisson random measures independent of $B(\cdot)$, while K is a given subset of \mathbb{R}^p . For each $j = 1, \dots, k$ we have $\tilde{N}_j(dt, dz) = N_j(dt, dz) - \nu_j(dz)dt$, where ν_j is the Lévy measure (intensity measure) of the Poisson random measure $N_j(\cdot, \cdot)$.

We may regard $u(t, z) = (u_0(t), u_1(t, z))$ as our control process, assumed to be càdlàg, \mathcal{F}_t -adapted and with values in $K \times K$ for a.a. t, z, ω .

Thus $Y(t) = Y^{(u)}(t)$ is a *controlled jump diffusion*.

Let $f : \mathbb{R}^k \times K \rightarrow \mathbb{R}$ and $g : \mathbb{R}^k \rightarrow \mathbb{R}$ be given functions. Let \mathcal{A} be a given set of controls contained in the set of $u = (u_0, u_1)$ such that (3) has a unique strong solution and such that

$$\mathbb{E}^y \left[\int_0^{\tau_S} |f(Y(t), u(t))| dt \right] < \infty \quad (4)$$

(where \mathbb{E}^y denotes expectation when $Y(0) = y$) where

$$\tau_S = \inf \{t > 0; Y(t) \notin \mathcal{S}\} \quad (\text{the bankruptcy time}) \quad (5)$$

is the first exit time of a given open *solvency set* $\mathcal{S} \subset \mathbb{R}^k$. We let \mathcal{T} denote the set of all stopping times $\tau \leq \tau_S$. We assume that

$$\{g^-(X(\tau))\}_{\tau \in \mathcal{T}} \quad \text{is uniformly integrable.} \quad (6)$$

Note that $Y^{(u)}(t)$ is quasi-left continuous, in the sense that for each given $\tau \in \mathcal{T}$ we have

$$\lim_{t \rightarrow \tau^-} Y^{(u)}(t) = Y^{(u)}(\tau),$$

see [12], Proposition I. 2.26 and Proposition I. 3.27.

For $\tau \in \mathcal{T}$ and $u \in \mathcal{A}$ we define the *performance functional* $J^{\tau, u}(y)$ by

$$J^{\tau, u}(y) = \mathbb{E}^y \left[\int_0^\tau f(Y(t), u(t)) dt + g(Y(\tau)) \right] \quad (7)$$

(we interpret $g(Y(\tau))$ as 0 if $\tau = \infty$).

We regard τ as the ‘‘control’’ of player number 1 and u as the control of player number 2, and consider the *stochastic differential game* to find the value function Φ and an optimal pair $(\tau^*, u^*) \in \mathcal{T} \times \mathcal{A}$ such that

$$\Phi(y) = \inf_{u \in \mathcal{A}} \left(\sup_{\tau \in \mathcal{T}} J^{\tau, u}(y) \right) = J^{\tau^*, u^*}(y). \quad (8)$$

We restrict ourselves to Markov controls $u = (u_0, u_1)$, i.e. we assume that $u_0(t) = \bar{u}_0(Y(t))$ and $u_1(t) = \bar{u}_1(Y(t), z)$ for some functions $\bar{u}_0 : \mathbb{R}^k \rightarrow K$, $\bar{u}_1 : \mathbb{R}^k \times \mathbb{R}^k \rightarrow K$. For simplicity of notation we will in the following not distinguish between u_0 and \bar{u}_0 , u_1 and \bar{u}_1 .

When the control u is Markovian the corresponding process $Y^{(u)}(t)$ becomes a Markov process, with generator A^u given by

$$\begin{aligned} A^u \varphi(y) &= \sum_{i=1}^k b_i(y, u_0(y)) \frac{\partial \varphi}{\partial y_i}(y) \\ &+ \frac{1}{2} \sum_{i,j=1}^k (\sigma \sigma^t)_{ij}(y, u_0(y)) \frac{\partial^2 \varphi}{\partial y_i \partial y_j}(y) \\ &+ \sum_{j=1}^k \int_{\mathbb{R}} \{ \varphi(y + \gamma^{(j)}(y, u_1(y, z), z)) - \varphi(y) \\ &\quad - \nabla \varphi(y) \cdot \gamma^{(j)}(y, u_1(y, z), z) \} \nu_j(dz); \quad \varphi \in \mathcal{C}^2(\mathbb{R}^k). \end{aligned} \quad (9)$$

Here $\nabla \varphi = (\frac{\partial \varphi}{\partial y_1}, \dots, \frac{\partial \varphi}{\partial y_k})$ is the gradient of φ and $\gamma^{(j)}$ is column number j of the $k \times k$ matrix γ .

We can now formulate the main result of this section:

Theorem 2.1 (*Verification theorem for stopping-control games*)

Suppose there exists a function $\varphi : \bar{\mathcal{S}} \rightarrow \mathbb{R}$ such that

(i) $\varphi \in C^1(\mathcal{S}) \cap C(\bar{\mathcal{S}})$

(ii) $\varphi \geq g$ on \mathcal{S}

Define

$$D = \{y \in \mathcal{S}; \varphi(y) > g(y)\} \quad (\text{the continuation region}). \quad (10)$$

Suppose, with $Y(t) = Y^{(u)}(t)$,

(iii) $\mathbb{E}^y \left[\int_0^{\tau_S} \chi_{\partial D}(Y(t)) dt \right] = 0$ for all $u \in \mathcal{A}$

(iv) ∂D is a Lipschitz surface

(v) $\varphi \in C^2(\mathcal{S} \setminus \partial D)$, with locally bounded derivatives near ∂D

(vi) there exists $\hat{u} \in \mathcal{A}$ such that

$$A^{\hat{u}}\varphi(y) + f(y, \hat{u}(y)) = \inf_{u \in \mathcal{A}} \{A^u\varphi(y) + f(y, u(y))\} \begin{cases} = 0, & \text{for } y \in D, \\ \leq 0, & \text{for } y \in \mathcal{S} \setminus D. \end{cases}$$

(vii) $\mathbb{E}^y \left[|\varphi(Y(\tau))| + \int_0^\tau |A^u\varphi(Y(t))| dt \right] < \infty$, for all $\tau \in \mathcal{T}$ and all $u \in \mathcal{A}$.

For $u \in \mathcal{A}$ define

$$\tau_D = \tau_D^{(u)} = \inf\{t > 0; Y^{(u)}(t) \notin D\} \quad (11)$$

and, in particular,

$$\hat{\tau} = \tau_D^{(\hat{u})} = \inf\{t > 0; Y^{(\hat{u})}(t) \notin D\}.$$

(viii) Suppose that the family $\{\varphi(Y(\tau)); \tau \in \mathcal{T}, \tau \leq \tau_D\}$ is uniformly integrable, for each $u \in \mathcal{A}$, $y \in \mathcal{S}$.

Then $\varphi(y) = \Phi(y)$ and $(\hat{\tau}, \hat{u}) \in \mathcal{T} \times \mathcal{A}$ is an optimal pair, in the sense that

$$\Phi(y) = \inf_u \left(\sup_\tau J^{\tau, u}(y) \right) = \sup_\tau J^{\tau, \hat{u}}(y) = J^{\hat{\tau}, \hat{u}}(y) = \varphi(y) = \inf_u J^{\tau_D, u}(y) = \sup_\tau \left(\inf_u J^{\tau, u}(y) \right). \quad (12)$$

Proof. Choose $\tau \in \mathcal{T}$ and let $\hat{u} \in \mathcal{A}$ be as in (vi). By an approximation argument (see Theorem 3.1 in [16]) we may assume that $\varphi \in C^2(\mathcal{S})$. Then by the Dynkin formula (see Theorem 1.24 in [16]) and (vi) we have, with $\hat{Y} = Y^{(\hat{u})}$

$$\begin{aligned} \mathbb{E}^y \left[\varphi \left(\hat{Y}(\tau_m) \right) \right] &= \varphi(y) + \mathbb{E}^y \left[\int_0^{\tau_m} A^{\hat{u}}\varphi \left(\hat{Y}(t) \right) dt \right] \\ &\leq \varphi(y) - \mathbb{E}^y \left[\int_0^{\tau_m} f \left(\hat{Y}(t), \hat{u}(t) \right) dt \right], \end{aligned}$$

where $\tau_m = \tau \wedge m$; $m = 1, 2, \dots$.

Letting $m \rightarrow \infty$ this gives, by (4), (6), (vii), (i) and the Fatou Lemma,

$$\begin{aligned} \varphi(y) &\geq \liminf_{m \rightarrow \infty} \mathbb{E}^y \left[\int_0^{\tau_m} f \left(\hat{Y}(t), \hat{u}(t) \right) dt + \varphi \left(\hat{Y}(\tau_m) \right) \right] \\ &\geq \mathbb{E}^y \left[\int_0^\tau f \left(\hat{Y}(t), \hat{u}(t) \right) dt + g \left(\hat{Y}(\tau) \right) \chi_{\{\tau < \infty\}} \right] = J^{\tau, \hat{u}}(y). \end{aligned} \quad (13)$$

Since this holds for all τ we have

$$\varphi(y) \geq \sup_{\tau} J^{\tau, \hat{u}}(y) \geq \inf_u \left(\sup_{\tau} J^{\tau, u}(y) \right), \text{ for all } u \in \mathcal{A}. \quad (14)$$

Next, for given $u \in \mathcal{A}$ define, with $Y(t) = Y^u(t)$,

$$\tau_D = \tau_D^u = \inf\{t > 0; Y(t) \notin D\}.$$

Choose a sequence $\{D_m\}_{m=1}^{\infty}$ of open sets such that \bar{D}_m is compact, $\bar{D}_m \subset D_{m+1}$ and $D = \bigcup_{m=1}^{\infty} D_m$ and define

$$\tau_D(m) = m \wedge \inf\{t > 0; Y(t) \notin D_m\}.$$

By the Dynkin formula we have, by (vi), for $m = 1, 2, \dots$,

$$\begin{aligned} \varphi(y) &= \mathbb{E}^y \left[- \int_0^{\tau_D(m)} A^u \varphi(Y(t)) dt + \varphi(Y(\tau_D(m))) \right] \\ &\leq \mathbb{E}^y \left[\int_0^{\tau_D(m)} f(Y(t), u(t)) dt + \varphi(Y(\tau_D(m))) \right]. \end{aligned} \quad (15)$$

By the quasi-left continuity of $Y(\cdot)$ (see [12], Proposition I. 2. 26 and Proposition I. 3. 27), we get

$$Y(\tau_D(m)) \rightarrow Y(\tau_D) \text{ a.s. as } m \rightarrow \infty.$$

Therefore, if we let $m \rightarrow \infty$ in (15) we get

$$\varphi(y) \leq \mathbb{E}^y \left[\int_0^{\tau_D} f(Y(t), u(t)) dt + g(Y(\tau_D)) \right] = J^{\tau_D, u}(y).$$

Since this holds for all $u \in \mathcal{A}$ we get

$$\varphi(y) \leq \inf_u J^{\tau_D, u}(y) \leq \sup_{\tau} \left(\inf_u J^{\tau, u}(y) \right). \quad (16)$$

In particular, applying this to $u = \hat{u}$ we get equality, i.e.

$$\varphi(y) = J^{\hat{\tau}, \hat{u}}(y). \quad (17)$$

Combining (14), (16) and (17) we obtain

$$\begin{aligned} \inf_u \left(\sup_{\tau} J^{\tau, u}(y) \right) &\leq \sup_{\tau} J^{\tau, \hat{u}}(y) \leq \varphi(y) = J^{\hat{\tau}, \hat{u}}(y) = \varphi(y) \\ &\leq \inf_u J^{\tau_D, u}(y) \leq \sup_{\tau} \left(\inf_u J^{\tau, u}(y) \right) \leq \inf_u \left(\sup_{\tau} J^{\tau, u}(y) \right). \end{aligned} \quad (18)$$

Since we always have

$$\sup_{\tau} \left(\inf_u J^{\tau, u}(y) \right) \leq \inf_u \left(\sup_{\tau} J^{\tau, u}(y) \right) \quad (19)$$

we conclude that we have equality everywhere in (18) and the proof is complete. ■

Remark 2.1 *It is natural to ask if the value function in the above theorem is the unique viscosity solution of the corresponding HJB variational inequalities. This will be proved to be the case by some of us in a subsequent paper (work in progress).*

3 Viscosity solutions

Let the state, $Y(t) = Y^u(t)$, be given by equation (3), the performance functional by equation (7) and the value function by equation (8). In the following we will assume that the functions b, σ, γ, f, g are continuous with respect to (y, u) . Further, the following standard assumptions are adopted; there exists $C > 0, \alpha : \mathbb{R}^k \rightarrow \mathbb{R}^k$ with $\int \alpha^2(z)\nu(dz) < \infty$ such that for all $x, y \in \mathbb{R}^k, z \in \mathbb{R}^k$ and $u \in K$,

$$\text{A1. } |b(x, u) - b(y, u)| + |\sigma(x, u) - \sigma(y, u)| + |f(x, u) - f(y, u)| + |g(x) - g(y)| \leq C|x - y|,$$

$$\text{A2. } |b(y, u)| + |\sigma(y, u)| \leq C(1 + |y|),$$

$$\text{A3. } |f(y, u)| + |g(y)| \leq C(1 + |y|)^m,$$

$$\text{A4. } |\gamma(x, u_1, z) - \gamma(y, u_1, z)| \leq \alpha(z)|x - y|,$$

$$\text{A5. } |\gamma(x, u_1, z)| \leq \alpha(z)(1 + |x|) \text{ and } |\gamma(x, u_1, z)|\mathbf{1}_{|z|<1} \leq C_x, C_x \in \mathbb{R}.$$

Let us define a HJB variational inequality by

$$\max \left\{ \inf_{u \in K} [A^u \varphi(y) + f(y, u(y))], g(y) - \varphi(y) \right\} = 0, \quad (20)$$

and

$$\varphi = g \text{ on } \partial \mathcal{S}. \quad (21)$$

where $A^y \varphi(y)$ is defined by equation (9).

Definition 3.1 (Viscosity solutions) A locally bounded function $\varphi \in USC(\bar{\mathcal{S}})$ is called a viscosity subsolution of (20)-(21) in \mathcal{S} if (21) holds and for each $\psi \in C_0^2(\mathcal{S})$ and each $y_0 \in \mathcal{S}$ such that $\psi \geq \varphi$ on \mathcal{S} and $\psi(y_0) = \varphi(y_0)$, we have

$$\max \left\{ \inf_{u \in K} [A^u \psi(y_0) + f(y_0, u(y_0))], g(y_0) - \psi(y_0) \right\} \geq 0 \quad (22)$$

A function $\varphi \in LSC(\bar{\mathcal{S}})$ is called a viscosity supersolution of the (20)-(21) in \mathcal{S} if (21) holds and for each $\psi \in C_0^2(\mathcal{S})$ and each $y_0 \in \mathcal{S}$ such that $\psi \leq \varphi$ on \mathcal{S} and $\psi(y_0) = \varphi(y_0)$, we have

$$\max \left\{ \inf_{u \in K} [A^u \psi(y_0) + f(y_0, u(y_0))], g(y_0) - \psi(y_0) \right\} \leq 0 \quad (23)$$

Further, if $\varphi \in C([0, T] \times \mathbb{R}^n)$ is both a viscosity subsolution and a viscosity supersolution it is called a viscosity solution.

Proposition 3.2 (Dynamic programming principle) Let Φ be as in (8). Then we have

$$(i) \quad \forall h > 0, \forall y \in \mathbb{R}^k$$

$$\Phi(y) = \sup_{\tau \in \mathcal{T}} \inf_{u \in \mathcal{A}} E^y \left[\int_0^{\tau \wedge h} f(Y(s), u(s)) ds + g(Y_\tau) \mathbf{1}_{\tau < h} + \Phi(Y_h) \mathbf{1}_{h \leq \tau} \right].$$

(ii) Let $\varepsilon > 0$, $y \in \mathbb{R}^k$, $u \in \mathcal{A}$ and define the stopping time

$$\tau_{y,u}^\varepsilon = \inf\{0 \leq t \leq \tau_s; \Phi(Y_t^{y,u}) \leq g(Y_t^{y,u}) + \varepsilon\}.$$

Then, if $\tau_u \leq \tau_{y,u}^\varepsilon$, for all $u \in \mathcal{A}$, we have that:

$$\Phi(y) = \inf_{u \in \mathcal{A}} E^y \left[\int_0^{\tau_u} f(Y(s)) ds + \Phi(Y_{\tau_u}^y) \right].$$

Remark 3.1 Prop.3.2 (i) is a consequence of Prop. 3.2 (ii) as observed in Krylov [14] p135.

Proof. The demonstration being long is postponed for more brightness in Part 5. ■

Theorem 3.3 Under assumptions A1-A4, the value function Φ is a viscosity solution of (20)-(21).

Proof. Φ is continuous according to the estimates of the moments of the jump diffusion state process (see Lemma 3.1 p.9 in [18]) and from Lipschitz condition A2 on f and g we get that

$$\Phi(y) = g(y) \text{ on } \partial\mathcal{S}.$$

We now prove that Φ is a subsolution of (20)-(21). Let $\psi \in C_0^2(\mathcal{S})$ and $y_0 \in \mathcal{S}$ such that

$$0 = (\psi - \Phi)(y_0) = \min_y (\psi - \Phi). \quad (24)$$

Define

$$D = \{y \in \mathcal{S} | \Phi(y) > g(y)\}.$$

If $y_0 \notin D$ then $g(y_0) = \Phi(y_0)$ and hence (22) holds. Next suppose $y_0 \in D$. Then we have by Proposition 3.2 for $\hat{\tau} = \tau_D$ and $h > 0$ small enough:

$$\Phi(y_0) = \inf_{u \in \mathcal{A}} E^{y_0} \left[\int_0^h f(Y^{y_0}(t), u(t)) dt + \Phi(Y^{y_0}(h)) \right].$$

From (24) we get

$$0 \leq \inf_{u \in \mathcal{A}} E^{y_0} \left[\int_0^h f(Y^{y_0}(t), u(t)) dt + \psi(Y^{y_0}(h)) - \psi(y_0) \right].$$

By Itô 's formula we obtain that

$$0 \leq \inf_{u \in \mathcal{A}} \frac{1}{h} E^{y_0} \left[\int_0^h [A^u \psi(Y_t^{y_0}) + f(Y^{y_0}(t), u(t))] dt \right].$$

Using assumptions A1-A4 with estimates on the moments of a jump diffusion and by letting $h \rightarrow 0^+$, we have

$$\inf_{u \in \mathcal{K}} [A^u \psi(y_0) + f(y_0, u(y_0))] \geq 0,$$

and hence

$$\max \left\{ \inf_{u \in \mathcal{K}} [A^u \psi(y_0) + f(y_0, u(y_0))], g(y_0) - \psi(y_0) \right\} \geq 0.$$

This shows that Φ is a viscosity subsolution. The proof for supersolution is similar. ■

The problem of showing uniqueness of viscosity solution is not addressed in this paper but will be considered in a future article.

4 Examples

Let us look at some control problems where we include stopping times as one of the controls. We then apply the result of the previous section to find a solution. We will look at both a jump and a non-jump market.

Example 4.1 (Optimal Resource Extraction in a Worst Case Scenario) *Let*

$$dP(t) = P(t)[\alpha dt + \beta dB(t) + \int_{\mathbb{R}_0} \gamma(z) \tilde{N}(dt, dz)]; P(0) = y_1 > 0,$$

where α, β are constants and $\gamma(z)$ is a given function such that $\int_{\mathbb{R}_0} \gamma^2(z) \nu(dz) < \infty$. Let $Q(t)$ be the amount of remaining resources at time t , and let the dynamics be described by

$$dQ(t) = -u(t)Q(t)dt; Q(0) = y_2 \geq 0.$$

where $u(t)$ controls the consumption rate of the resource $Q(t)$, and m is the maximum extraction rate. We let

$$dY(t) = \begin{cases} dY_0(t) & = dt \\ dY_1(t) & = dP(t); P(0) = y_1 > 0, \\ dY_2(t) & = dQ(t); Q(0) = y_2 \geq 0, \\ dY_3(t) & = -Y_3(t) \left[\theta_0(t)dB(t) + \int_{\mathbb{R}_0} \theta_1(t, z) \tilde{N}(dt, dz) \right]; Y_3(0) = y_3 > 0. \end{cases} \quad (25)$$

Let the running cost be given by $K_0 + K_1 u_t$ ($K_0, K_1 \geq 0$, constants). Then we let our performance functional be given by, with $\theta = (\theta_0, \theta_1)$,

$$\begin{aligned} & J^{\tau, u, \theta}(s, y_1, y_2, y_3) \quad (26) \\ & = E^y \left[\int_0^\tau e^{-\delta(s+t)} (u(t)(P(t)Q(t) - K_1) - K_0) Y_3(t) dt + e^{-\delta(s+\tau)} (MP(\tau)Q(\tau) - a) Y_3(\tau) \right], \end{aligned}$$

where $\delta > 0$ is the discounting rate and $M > 0, a > 0$ are constants (a can be seen as a transaction cost). Our problem is to find $(\hat{\tau}, \hat{u}, \hat{\theta})$ in $\mathcal{T} \times U \times \Theta$ such that

$$\Phi(y) = \Phi(s, y_1, y_2, y_3) = \sup_u \left[\inf_\theta \left(\sup_\tau J^{\tau, u, \theta}(y) \right) \right] = J^{\hat{\tau}, \hat{u}, \hat{\theta}}(y). \quad (27)$$

Then the generator of $Y^{u, \theta}$ is given by;

$$\begin{aligned} A^{u, \theta} \varphi(y) & = A^{u, \theta} \varphi(s, y_1, y_2, y_3) = \frac{\partial \varphi}{\partial s} + y_1 \alpha \frac{\partial \varphi}{\partial y_1} - u y_2 \frac{\partial \varphi}{\partial y_2} + \frac{1}{2} y_1^2 \beta^2 \frac{\partial^2 \varphi}{\partial^2 y_1} + \frac{1}{2} y_3^2 \theta_0^2 \frac{\partial^2 \varphi}{\partial^2 y_3} \\ & \quad - y_1 y_3 \beta \theta_0 \frac{\partial^2 \varphi}{\partial y_1 \partial y_3} \\ & \quad + \int_{\mathbb{R}_0} \left\{ \varphi(s, y_1 + y_1 \gamma(z), y_2, y_3 - y_3 \theta_1(z)) - \varphi(s, y_1, y_2, y_3) - y_1 \gamma(z) \frac{\partial \varphi}{\partial y_1} + y_3 \theta_1(z) \frac{\partial \varphi}{\partial y_3} \right\} \nu(dz). \end{aligned}$$

We need to find a subset D of $\mathcal{S} = \mathbb{R}_+^4 = [0, \infty)^4$ and $\varphi(s, y_1, y_2, y_3)$ such that

$$\begin{aligned} \varphi(s, y_1, y_2, y_3) &= g(s, y_1, y_2, y_3) := e^{-\delta s}(My_1y_2 - a)y_3, \quad \forall (s, y_1, y_2, y_3) \notin D, \\ \varphi(s, y_1, y_2, y_3) &\geq e^{-\delta s}(My_1y_2 - a)y_3, \quad \forall (s, y_1, y_2, y_3) \in \mathcal{S}, \\ A^{u, \theta} \varphi(s, y_1, y_2, y_3) + f(s, y_1, y_2, y_3, u) &:= A^{u, \theta} \varphi(s, y_1, y_2, y_3) + e^{-\delta s}(u(y_1y_2 - K_1) - K_0)y_3 \\ &\leq 0, \quad \forall (s, y_1, y_2, y_3) \in \mathcal{S} \setminus D, \quad \forall u \in [0, m], \\ \sup_u \left[\inf_{\theta} \{A^{u, \theta} \varphi(s, y_1, y_2, y_3) + e^{-\delta s}(u(y_1y_2 - K_1) - K_0)y_3\} \right] &= 0, \quad \forall (s, y_1, y_2, y_3) \in D. \end{aligned}$$

Then

$$\hat{\theta}_0 = \frac{y_1}{y_3} \beta \frac{\varphi_{13}}{\varphi_{33}}, \quad (28)$$

is a minimizer of $\theta_0 \mapsto A^{u, \theta} \varphi(s, y_1, y_2, y_3)$ where we are using the notation

$$\varphi_{ij} = \frac{\partial^2 \varphi}{\partial y_j \partial y_i}.$$

Let $\hat{\theta}_1(z)$ be the minimizer of $\theta_1(z) \mapsto A^{u, \theta} \varphi(y)$ and let \hat{u} be the the maximizer of $u \mapsto A^{u, \theta} \varphi(y) + f(y, u)$ i.e.

$$u \mapsto A^{u, \theta} \varphi(s, y_1, y_2, y_3) + e^{-\delta s} u y_3 (y_1 y_2 - K_1) - u y_2 \varphi_2 - y_3 K_0. \quad (29)$$

Let us try a function on the form

$$\varphi(s, y_1, y_2, y_3) = e^{-\delta s} F(w), \quad \text{where } w = y_1 y_2 y_3. \quad (30)$$

Then

$$\hat{u} = \begin{cases} m, & \text{if } w F'(w) < w - y_3 K_1 \\ 0, & \text{otherwise,} \end{cases} \quad (31)$$

and

$$\hat{\theta} = \beta \left(1 + \frac{F'(w)}{F''(w)w} \right). \quad (32)$$

Further, the first order condition for $\hat{\theta}_1(z)$ is

$$\int_{\mathbb{R}_0} \left\{ (1 + \gamma(z)) F'(w(1 + \gamma(z))(1 - \hat{\theta}_1(z))) - F'(w) \right\} \nu(dz) = 0. \quad (33)$$

For $wF'(w) < w - y_3K_1$ we have

$$\begin{aligned}
A^{\hat{u}, \hat{\theta}} e^{-\delta s} F(y_1, y_2, y_3) &= -\delta e^{-\delta s} F(w) + w e^{-\delta s} \alpha F'(w) - m w e^{-\delta s} F'(w) \\
&+ \frac{1}{2} w^2 \beta^2 F''(w) e^{-\delta s} + \frac{1}{2} w^2 \beta^2 F''(w) e^{-\delta s} \left(1 + \left(\frac{F'(w)}{F''(w)w} \right)^2 + \frac{2F'(w)}{F''(w)w} \right) \\
&- w \beta^2 e^{-\delta s} \left(F'(w) + \frac{(F'(w))^2}{F''(w)w} + w F''(w) + F'(w) \right) \\
&+ e^{-\delta s} \int_{\mathbb{R}_0} \left\{ F(w(1 + \gamma(z))(1 - \hat{\theta}_1(z))) - F(w) - w \gamma(z) F'(w) + \hat{\theta}_1(z) w F'(w) \right\} \nu(dz) \\
&= -\delta e^{-\delta s} F(w) + w e^{-\delta s} \alpha F'(w) - m w e^{-\delta s} F'(w) \\
&+ \beta^2 e^{-\delta s} \left(-\frac{(F'(w))^2}{2F''(w)} - w F'(w) \right) \\
&+ e^{-\delta s} \int_{\mathbb{R}_0} \left\{ F(w(1 + \gamma(z))(1 - \hat{\theta}_1(z))) - F(w) - w \gamma(z) F'(w) + \hat{\theta}_1(z) w F'(w) \right\} \nu(dz).
\end{aligned} \tag{34}$$

We then need that if $wF'(w) < w - y_3K_1$, then

$$\begin{aligned}
A^{\hat{u}, \hat{\theta}} F(w) + (m(y_1 y_2 - K_1) - K_0) y_3 &= -\delta F(w) + w \alpha F'(w) - m w F'(w) \\
&- \beta^2 \left(\frac{(F'(w))^2}{2F''(w)} + w F'(w) \right) \\
&+ \int_{\mathbb{R}_0} \left\{ F(w(1 + \gamma(z))(1 - \hat{\theta}_1(z))) - F(w) - w \gamma(z) F'(w) + \hat{\theta}_1(z) w F'(w) \right\} \nu(dz) \\
&+ (m(y_1 y_2 - K_1) - K_0) y_3 = 0.
\end{aligned} \tag{35}$$

Similarly, if $wF'(w) \geq w - y_3K_1$, then $\hat{u} = 0$ and hence we must have

$$\begin{aligned}
A^{\hat{u}, \hat{\theta}} F(w) - K_0 y_3 &= -\delta F(w) + w \alpha F'(w) \\
&- \beta^2 \left(\frac{(F'(w))^2}{2F''(w)} + w F'(w) \right) \\
&+ \int_{\mathbb{R}_0} \left\{ F(w(1 + \gamma(z))(1 - \hat{\theta}_1(z))) - F(w) - w \gamma(z) F'(w) + \hat{\theta}_1(z) w F'(w) \right\} \nu(dz) \\
&- K_0 y_3 = 0.
\end{aligned} \tag{36}$$

The continuation region D gets the form

$$D = \{(s, y_1, y_2, y_3) : F(w) > (M y_1 y_2 - a) y_3\}$$

Therefore we get the requirement

$$F(w) = (M y_1 y_2 - a) y_3, \forall (s, y_1, y_2, y_3) \notin D. \tag{37}$$

In light of this requirement and in order for φ to be on the form (30) we see that we need K_0, K_1 and a to be zero. Hence we let $K_0 = K_1 = a = 0$ from now on. Then we need that F satisfies the variational inequality

$$\max\{A_0^{\hat{u}, \hat{\theta}} F(w) + \tilde{m}w, Mw - F(w)\} = 0, w > 0, \tag{38}$$

where

$$A_0^{\hat{u}, \hat{\theta}} F(w) = -\delta F(w) + w\alpha F'(w) - \tilde{m}wF'(w) - \beta^2 \left(\frac{(F'(w))^2}{2F''(w)} + wF'(w) \right) \quad (39)$$

$$+ \int_{\mathbb{R}_0} \left\{ F(w(1 + \gamma(z))(1 - \hat{\theta}_1(z))) - F(w) - w\gamma(z)F'(w) + \hat{\theta}_1(z)wF'(w) \right\} \nu(dz),$$

with

$$\tilde{m} := m\chi_{(-\infty, 1)}(F'(w)). \quad (40)$$

The variational inequality (38) - (40) is hard to solve analytically, but it may be accessible by numerical methods.

Exemple 4.2 (Worst case scenario optimal control and stopping in a Lévy -market) *Let our dynamics be given by*

$$\begin{aligned} dY_0(t) &= dt; & Y_0(0) &= s \in \mathbb{R}. \\ dY_1(t) &= (Y_1(t)\alpha(t) - u(t))dt + Y_1(t)\beta dB(t) \\ &\quad + Y_1(t^-) \int_{\mathbb{R}} \gamma(s, z) \tilde{N}(ds, dz); & Y_1(0) &= y_1 > 0. \\ dY_2(t) &= -Y_2(t)\theta_0(t)dB(t) - Y_2(t) \int_{\mathbb{R}} \theta_1(s, z) \tilde{N}(ds, dz); & Y_2(0) &= y_2 > 0. \end{aligned}$$

Solve

$$\Phi(s, x) = \sup_u \left[\sup_{\tau} \left(\inf_{\theta_0, \theta_1} J^{\theta, u, \tau}(s, x) \right) \right]$$

where

$$J^{\theta, u, \tau}(s, x) = E^x \left[\int_0^{\tau} e^{-\delta(s+t)} \frac{u^\lambda}{\lambda} Y_2(t) dt \right]$$

The interpretation of this problem is the following:

$Y_1(T)$ represents the size of the population (e.g. fish) when a harvesting strategy $u(t)$ is applied to it.

The process $Y_2(t)$ represents the Radon-Nikodym derivative of a measure Q with respect to P , i.e.

$$Y_2(t) = \frac{d(Q|\mathcal{F}_t)}{d(P|\mathcal{F}_t)} = E\left[\frac{dQ}{dP}|\mathcal{F}_t\right]; 0 \leq t \leq T.$$

This means that we can write

$$\begin{aligned} J^{\theta, y, \tau}(s, x) &= E^x \left[\int_0^{\infty} e^{-\delta(s+t)} \chi_{[0, \tau](t)} \frac{u^\lambda(t)}{\lambda} E\left[\frac{dQ}{dP}|\mathcal{F}_t\right] dt \right] \\ &= E_Q^x \left[\int_0^{\infty} e^{-\delta(s+t)} \frac{u^\lambda(t)}{\lambda} dt \right]. \end{aligned}$$

Hence $J^{\theta, y, \tau}$ represents the expected utility up to the stopping time τ , measured in terms of a scenario (probability measure Q) chosen by the market. Therefore our problem may be regarded as a worst case scenario optimal harvesting/stopping problem. Alternatively, the problem may be interpreted as a risk

minimizing optimal stopping and control problem. To see this, we use the following representation of a given convex risk measure ρ :

$$\rho(F) = \sup_{Q \in \mathcal{P}} \{E_Q[-F] - \varsigma(Q)\}; F \in L^\infty(P),$$

where \mathcal{P} is the set of all measures Q above and $\varsigma : \mathcal{P} \rightarrow \mathbb{R}$ is a given convex ‘‘penalty’’ function. If $\varsigma = 0$ as above, the risk measure ρ is called coherent. See [1], [9] and [10].

In this case our generator becomes

$$\begin{aligned} A^{u,\theta} \varphi(s, y_1, y_2) &= \frac{\partial \varphi}{\partial s} + (y_1 \alpha - u) \frac{\partial \varphi}{\partial y_1} + \frac{1}{2} y_1^2 \beta^2 \frac{\partial^2 \varphi}{\partial y_1^2} \\ &+ \frac{1}{2} y_2^2 \theta_0^2 \frac{\partial^2 \varphi}{\partial y_2^2} - y_1 y_2 \beta \theta_0 \frac{\partial^2 \varphi}{\partial y_1 \partial y_2} \\ &+ \int_{\mathbb{R}} \left[\varphi(s, y_1 + y_1 \gamma(s, z), y_2 - y_2 \theta_1(s, z)) - \varphi(s, y_1, y_2) - y_1 \gamma(s, z) \frac{\partial \varphi}{\partial y_1} + y_2 \theta_1(z) \frac{\partial \varphi}{\partial y_2} \right] \nu(dz). \end{aligned}$$

and hence

$$\begin{aligned} A^{u,\theta} \varphi(s, y_1, y_2) + f(s, y_1, y_2) &= \frac{\partial \varphi}{\partial s} + (y_1 \alpha - u) \frac{\partial \varphi}{\partial y_1} + \frac{1}{2} y_1^2 \beta^2 \frac{\partial^2 \varphi}{\partial y_1^2} \\ &+ \frac{1}{2} y_2^2 \theta_0^2 \frac{\partial^2 \varphi}{\partial y_2^2} - y_1 y_2 \beta \theta_0 \frac{\partial^2 \varphi}{\partial y_1 \partial y_2} \\ &+ \int_{\mathbb{R}} \left[\varphi(s, y_1 + y_1 \gamma(s, z), y_2 - y_2 \theta_1(s, z)) - \varphi(s, y_1, y_2) - y_1 \gamma(s, z) \frac{\partial \varphi}{\partial y_1} + y_2 \theta_1(z) \frac{\partial \varphi}{\partial y_2} \right] \nu(dz) \\ &+ e^{-\delta s} \frac{u^\lambda}{\lambda} y_2. \end{aligned}$$

Imposing the first-order condition we get the following equations for the optimal control processes $\hat{\theta}_0, \hat{\theta}_1$ and \hat{u} :

$$\hat{\theta}_0 = \frac{y_1 \beta \varphi_{12}}{y_2 \varphi_{22}},$$

$$\int_{\mathbb{R}} \{ \varphi_2(s, y_1 + y_1 \gamma(s, z), y_2 - y_2 \hat{\theta}_1(s, z)) - \varphi_2(s, y_1, y_2) \} \nu(dz) = 0,$$

and

$$\hat{u} = \left(\frac{e^{\delta s} \varphi_1}{y_2} \right)^{\frac{1}{\lambda-1}},$$

where $\varphi_i = \frac{\partial \varphi}{\partial y_i}$; $i = 1, 2$. This gives

$$\begin{aligned} A^{\hat{u}, \hat{\theta}} \varphi(s, y_1, y_2) + f(s, y_1, y_2, \hat{u}) &= \frac{\partial \varphi}{\partial s} + (y_1 \alpha - (e^{\delta s} \varphi_1)^{\frac{1}{\lambda-1}}) \varphi_1 + \frac{1}{2} y_1^2 \beta^2 \varphi_{11} - \frac{1}{2} y_1^2 \beta^2 \frac{\varphi_{12}^2}{\varphi_{22}} \quad (41) \\ &+ \int_{\mathbb{R}} \left[\varphi(s, y_1 + y_1 \gamma(s, z), y_2 - y_2 \hat{\theta}_1(s, z)) - \varphi(s, y_1, y_2) - y_1 \gamma(s, z) \frac{\partial \varphi}{\partial y_1} + y_2 \hat{\theta}_1(z) \frac{\partial \varphi}{\partial y_2} \right] \nu(dz) \\ &+ e^{-\delta s} \frac{\left(\frac{\varphi_1 e^{\delta s}}{y_2} \right)^{\frac{\lambda}{\lambda-1}}}{\lambda} y_2. \end{aligned}$$

Let us try a value function of the form

$$\varphi(s, y_1, y_2) = e^{-\delta s} y_1^\lambda F(y_2), \quad (42)$$

for some function F (to be determined). Then

$$\hat{\theta}_0 = \beta \frac{\lambda \beta F'(y_2)}{y_2 F''(y_2)}, \quad (43)$$

$$\int_{\mathbb{R}} \{(1 + \gamma(s, z))^\gamma F'(y_2 - y_2 \hat{\theta}_1(s, z)) - F'(y_2)\} \nu(dz) = 0, \quad (44)$$

and

$$\hat{u} = \left(\frac{F(y_2) \lambda}{y_2} \right)^{\frac{1}{\lambda-1}} y_1. \quad (45)$$

With $\hat{\theta}_1$ as in (44) put

$$\begin{aligned} A_0^{\hat{\theta}, \hat{u}} F(y_2) &= -\delta F(y_2) + \left(\alpha - \left(\frac{\lambda F(y_2)}{y_2} \right)^{\frac{1}{\lambda-1}} \right) \lambda F(y_2) + \frac{1}{2} \beta^2 \lambda (\lambda - 1) F(y_2) - \frac{1}{2} \beta^2 \frac{F'^2(y_2)}{F''(y_2)} \\ &+ \frac{y_2}{\lambda} \left(\frac{\lambda F(y_2)}{y_2} \right)^{\frac{\lambda}{\lambda-1}} + \int_{\mathbb{R}} \left[(1 + \gamma(z))^\lambda F(y_2 - y_2 \hat{\theta}_1(z)) \right. \\ &\left. - F(y_2) - \gamma(z) \lambda F(y_2) + y_2 \hat{\theta}_1(z) F'(y_2) \right] \nu(dz). \end{aligned} \quad (46)$$

Thus we see that the problem reduces to the problem of solving a non-linear variational-integro inequality as follows:

Suppose there exists a process $\hat{\theta}_1(s, z)$ satisfying (44) and a C^1 -function $F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that if we put

$$D = \{y_2 > 0; F(y_2) > 0\}$$

then $F \in C^2(D)$ and

$$A_0^{\hat{\theta}, \hat{u}} F(y_2) = 0 \text{ for } y_2 \in D.$$

Then the function φ given by (42) is the value function of the problem. The optimal control process are as in (43) - (45) and an optimal stopping time is

$$\tau^* = \inf\{t > 0; Y_2(t) \notin D\}.$$

Exemple 4.3 (Risk minimizing optimal portfolio and stopping)

$$dY_0(t) = dt; \quad Y_0(0) = s \in \mathbb{R}. \quad (47)$$

$$dY_1(t) = Y_1(t)[(r + (\alpha - r)\pi(t))dt + \beta\pi(t)dB(t)]; \quad Y_1(0) = y_1 > 0. \quad (48)$$

$$dY_2(t) = -Y_2(t)\theta(t)dB(t); \quad Y_2(0) = y_2 > 0, \quad (49)$$

where r, α and $\beta > 0$ are constants. Solve

$$\Phi(s, x) = \sup_{\pi} \left[\sup_{\tau} \left(\inf_{\theta} J^{\pi, \theta, \tau} \right) \right] \quad (50)$$

where

$$J^{\pi, \theta, \tau}(s, x) = E^x \left[e^{-\delta \tau} \lambda Y_1(\tau) Y_2(\tau) \right], \quad (51)$$

where $0 < \lambda \leq 1$ and $(1 - \lambda)$ is a percentage transaction cost. The generator is

$$\begin{aligned} A^{\theta, \pi} \varphi(s, y_1, y_2) + f(s, y_1, y_2) &= \frac{\partial \varphi}{\partial s} + y_1(r + (\alpha - r)\pi) \frac{\partial \varphi}{\partial y_1} + \frac{1}{2} y_1^2 \beta^2 \pi^2 \frac{\partial^2 \varphi}{\partial^2 y_1} \\ &+ \frac{1}{2} y_2^2 \theta^2 \frac{\partial^2 \varphi}{\partial^2 y_2} - y_1 y_2 \beta \theta \pi \frac{\partial^2 \varphi}{\partial y_1 \partial y_2}. \end{aligned}$$

From the first order conditions we get that

$$\hat{\pi} = \frac{(\alpha - r) \varphi_1 \varphi_{22}}{y_1 \beta^2 (\varphi_{12}^2 - \varphi_{11} \varphi_{22})},$$

and

$$\hat{\theta} = \frac{(\alpha - r) \varphi_1 \varphi_{12}}{\beta y_2 (\varphi_{12}^2 - \varphi_{11} \varphi_{22})}.$$

Let us try to put

$$\varphi(s, y_1, y_2) = e^{-\delta s} \lambda y_1 y_2. \quad (52)$$

Then we get

$$A^{\hat{\theta}, \hat{\pi}} \varphi(s, y_1, y_2) = y_1 y_2 (r - \delta),$$

$$\hat{\theta} = \frac{\alpha - r}{\beta}. \quad (53)$$

and

$$\hat{\pi} = 0. \quad (54)$$

So if

$$r - \delta \leq 0,$$

then $A^{\hat{\theta}, \hat{\pi}} \varphi \leq 0$ and the best is to stop immediately and $\varphi = \Phi$. If

$$r - \delta > 0,$$

then

$$D = [0, T] \times \mathbb{R}^k \times \mathbb{R}^k,$$

so $\hat{\tau} = T$.

Remark 4.1 Note that the optimal value given in (53) for $\hat{\theta}$ corresponds to choosing the measure Q defined by

$$dQ(\omega) = Y_2(T)dP(\omega)$$

to be an equivalent martingale measure for the underlying financial market $(S_0(t), S_1(t))$ defined by

$$\begin{aligned} dS_0(t) &= rdt; S_0(0) = 0, \\ dS_1(t) &= S_1(t)[\alpha dt + \beta dB(t)]; S_1(0) > 0. \end{aligned}$$

This illustrates that equivalent martingale measures often appear as solutions of stochastic differential games between the agent and the market. This was first proved in [17] and subsequent in a partial information context in [2] and [3].

5 Proof of the Dynamical Programming Principle (Prop. 3.2)

The Proposition 3.2 (ii) is proved by Krylov [14] for diffusion processes and mixed strategies whereas Pham [18] has mentioned how to generalize it to this context of jump diffusions. Let us explain how it may be adapted to our case of stochastic differential games with optimal stopping and stochastic control for jump diffusions. First we need a Bellman's Principle for stochastic differential games. We here refer to Fleming and Souganidis [8] Theorem 1.6 and Biswas [5] Theorem 2.2, whose proofs rest on the continuity of the value functions and the introduction of a restrictive class of admissible strategies. Next, to generalize the Dynamic Programming Principle to our optimal stopping and stochastic control differential games problem, we use the technique of randomized stopping developed by Krylov [14] p. 36. For greater generality, we establish a version of the Dynamical Programming Principle whose Prop. 3.2. is a particular case.

5.1 A general context

5.1.1 Dynamics

For a fixed positive constant T and $s \in [0, T)$, the state $Y(t) = Y_t^{s,y,u}$ where $0 \leq t \leq T - s$ is driven by

$$\begin{aligned} dY(t) &= b(s+t, Y(t), u(t)) dt + \sigma(s+t, Y(t), u(t)) dB(t) \\ &\quad + \int_{\mathbb{R}_0^k} \gamma(s+t, Y(t^-), u(t), z) \tilde{N}(dt, dz) \end{aligned} \quad (55)$$

with the initial condition

$$Y(s) = y \in S.$$

$b : [0, T] \times \mathbb{R}^k \times K \rightarrow \mathbb{R}^k$, $\sigma : [0, T] \times \mathbb{R}^k \times K \rightarrow \mathbb{R}^{k \times k}$ and $\gamma : [0, T] \times \mathbb{R}^k \times K \times \mathbb{R}^k \rightarrow \mathbb{R}^{k \times k}$ are given functions and verify the assumptions of regularity of the part 3 uniformly with respect to t and are lipschitz continuous in the variable t for all (y, u) . $B(t)$ and $\tilde{N}(\cdot, \cdot)$ are defined as in part 2.

$u(t)$ is the control process assumed to be predictable and with values in K for a.a. t, ω . The set of controls is denoted by $M(s)$.

$$\tau_S = \inf \{0 < t \leq T - s; Y(t) \notin \mathcal{S}\} \quad (56)$$

$\mathcal{T}_{s,T}$ denotes the set of all stopping times $\tau \leq \tau_S$

For $\tau \in \mathcal{T}_{s,T}$ and $u \in M(s)$ the performance functional is

$$J^{\tau,u}(s, y) = \mathbb{E}_{s,y}^u \left[\int_0^\tau f(s+t, Y(t), u(t)) e^{-\varphi t} dt + g(\tau, Y(\tau)) e^{-\varphi \tau} \right] \quad (57)$$

where $\varphi_t^{s,y,u} = \int_0^t c^{u_r}(s+r, Y_r^{s,y,u}) dr$, $c : [0, T] \times \mathbb{R}^k \times K \rightarrow \mathbb{R}^+$, $f : [0, T] \times \mathbb{R}^k \times K \rightarrow \mathbb{R}$ and $g : [0, T] \times \mathbb{R}^k \rightarrow \mathbb{R}$ are given functions.

The value function is defined as

$$\Phi(s, y) = \sup_{\tau \in \mathcal{T}_{s,T}} \inf_{u \in M(s)} J^{\tau,u}(s, y). \quad (58)$$

5.1.2 The canonical sample space

We work in a canonical Wiener-Poisson space, following [5], [11] and [6]. For a constant T and $0 \leq s < t \leq T$, let $\Omega_{s,t}^1$ be the standard Wiener space i.e. the set of all functions from $[s, t]$ to \mathbb{R}^k starting from 0 and topologized by the sup-norm. We denote the corresponding Borel σ -algebra by \mathcal{B}_1^0 and let $P_{s,t}^1$ be the Wiener measure on $(\Omega_{s,t}^1, \mathcal{B}_1^0)$.

In addition, upon denoting $\mathcal{Q}_{s,t}^* = [s, t] \times (\mathbb{R}^k \setminus \{0\})$, let $\Omega_{s,t}^2$ be the set of all $\mathbb{N} \cup \{\infty\}$ -valued measures on $(\mathcal{Q}_{s,t}^*, \mathcal{B}(\mathcal{Q}_{s,t}^*))$ where $\mathcal{B}(\mathcal{Q}_{s,t}^*)$ is the usual Borel σ -algebra of $\mathcal{Q}_{s,t}^*$. We denote \mathcal{B}_2^0 to be the smallest σ -algebra over $\Omega_{s,t}^2$ so that the mappings $q \in \Omega_{s,t}^2 \mapsto q(A) \in \mathbb{N} \cup \{\infty\}$ are measurable for all $A \in \mathcal{B}(\mathcal{Q}_{s,t}^*)$. Let the co-ordinate random measure $N_{s,t}$ be defined as $N_{s,t}(q, A) = q(A)$ for all $q \in \Omega_{s,t}^2$, $A \in \mathcal{B}(\mathcal{Q}_{s,t}^*)$ and denote $P_{s,t}^2$ to be the probability measure on $(\Omega_{s,t}^2, \mathcal{B}_2^0)$ under which $N_{s,t}$ is a Poisson random measure with Lévy measure ν satisfying

$$\int_{\mathbb{R} \setminus \{0\}} \min(|z|^2, 1) \nu(dz) < \infty.$$

Next, for very $0 \leq s < t \leq T$, we define $\Omega_{s,t} = \Omega_{s,t}^1 \times \Omega_{s,t}^2$, $P_{s,t} \equiv P_{s,t}^1 \otimes P_{s,t}^2$ and $\mathcal{B}_{s,t} \equiv \mathcal{B}_1^0 \times \mathcal{B}_2^0$ i.e. the completion of $\mathcal{B}_1^0 \times \mathcal{B}_2^0$ with respect to the probability measure $P_{s,t}$. We will follow the convention that $\Omega_{t,T} \equiv \Omega_t$ and $\mathcal{B}_{t,T} \equiv \mathcal{F}_t$. A generic element of Ω_t is denoted by $\omega = (\omega_1, \omega_2)$, where $\omega_i \in \Omega_{t,T}^i$ for $i \in \{1, 2\}$, and we define the coordinate functions

$$W_s^t(\omega) = \omega_1(s) \text{ and } N^t(\omega, A) = \omega_2(A)$$

for all $0 \leq t \leq s \leq T$, $\omega \in \Omega$, $A \in \mathcal{B}(\mathcal{Q}_{t,T}^*)$. The process W^t is a Brownian motion starting at t and N^t is a Poisson random measure on the probability space $(\Omega_t, \mathcal{F}_t, P_t)$, and they are independent. Also, for $t \in [0, T]$, the filtration $\mathcal{F}_{t,\cdot} = (\mathcal{F}_{t,s})_{s \in [t, T]}$ is defined as follows:

We make $\hat{\mathcal{F}}_{t,\cdot}$ to be right-continuous and denote it by $\mathcal{F}_{t,\cdot}^+$. Finally, we augment $\mathcal{F}_{t,\cdot}^+$ by P_t -null sets and call it $\mathcal{F}_{t,\cdot}$. As and when it necessitates, we extend the filtration $\mathcal{F}_{t,\cdot}$ for $s < t$ by choosing $\mathcal{F}_{t,s}$ as the trivial σ algebra augmented by P_t -null sets. We follow the convention that $\mathcal{F}_{t,T} = \mathcal{F}_t$. When the terminal time T is replaced by another time point, say τ , the filtration we have just described is denoted by $\mathcal{F}_{t,\cdot}^\tau$.

Finally, note that the space $\Omega_{s,t}$ is defined as the product of canonical Wiener space and Poisson space. Therefore, for any $\tau \in (t, T)$, we can identify the probability space $(\Omega_t, \mathcal{F}_{t,\cdot}, P_t)$ with $(\Omega_{t,\tau} \times \Omega_\tau, \mathcal{F}_{t,\cdot}^\tau \otimes \mathcal{F}_{\tau,\cdot}, P_{t,\tau} \otimes P_\tau)$ by the following bijection $\pi : \Omega_t \rightarrow \Omega_{t,\tau} \times \Omega_\tau$. For a generic element $\omega = (\omega_1, \omega_2) \in \Omega_t = \Omega_{t,T}^1 \times \Omega_{t,T}^2$, we define

$$\begin{aligned}\omega^{t,\tau} &= (\omega_1|_{[t,\tau]}, \omega_2|_{[t,\tau]}) \in \Omega_{t,\tau} \\ \omega^{\tau,T} &= ((\omega_1 - \omega_1(\tau))|_{[t,\tau]}, \omega_2|_{[t,\tau]}) \in \Omega_{\tau,T} \\ \pi(\omega) &= (\omega^{t,\tau}, \omega^{\tau,T})\end{aligned}$$

The description of the inverse map π^{-1} is also apparent from above.

5.1.3 The general DPP

Theorem 5.1 *Let $s \in [0, T]$, $y \in \mathbb{R}^k$ and let $\tau = \tau^u \in \mathcal{T}_{s,T}$ be defined for each $u \in \mathcal{M}(s)$. Then*

$$\begin{aligned}\Phi(s, y) &= \sup_{\gamma \in \mathcal{T}_{s,T}} \inf_{u \in \mathcal{M}(s)} \mathbb{E}_{s,y}^u \left[\int_0^{\tau \wedge \gamma} f^{u_t}(s+t, Y_t) e^{-\varphi t} dt \right. \\ &\quad \left. + g(s+\gamma, Y_\gamma) e^{-\varphi \gamma} \chi_{\gamma \leq \tau} + \Phi(s+\tau, Y_\tau) e^{-\varphi \tau} \chi_{\tau \leq \gamma} \right],\end{aligned}\tag{59}$$

where $f^{u_t}(s+t, Y_t) = f(s+t, Y_t, u_t)$.

This theorem is in fact a consequence of the more general following result. Let $\varepsilon > 0$, we define: $\tau_{s,y,u}^\varepsilon = \inf \{t \geq 0 : \Phi(s+t, Y_t^{s,y,u}) \leq g(s+t, Y_t^{s,y,u}) + \varepsilon\}$.

Theorem 5.2 *Let $s \in [0, T]$, $y \in \mathbb{R}^k$, $u \in \mathcal{M}(s)$ and $\tau^u \in \mathcal{T}_{s,T}$. We are given a nonnegative process r_t^u , progressively measurable and bounded. Then*

$$\begin{aligned}\Phi(s, y) &\geq \inf_{u \in \mathcal{M}(s)} \mathbb{E}_{s,y}^u \left[\int_0^{\tau^u} (f^{u_t} + r_t \Phi)(s+t, Y_t) e^{-\varphi t - \int_0^t r_p dp} dt \right. \\ &\quad \left. + \Phi(s+\tau^u, Y_{\tau^u}) e^{-\varphi \tau^u - \int_0^{\tau^u} r_p dp} \right].\end{aligned}\tag{60}$$

If $\tau^u \leq \tau_{y,u}^\varepsilon$ for some $\varepsilon > 0$ and all $u \in \mathcal{M}(s)$, we have equality in (60).

Proof. (Theorem (5.2) to Theorem 5.1)

We write the right side of (59) as $W_1(s, y)$ then

$$\begin{aligned}W_1(s, y) &\geq \inf_{u \in \mathcal{M}(s)} \mathbb{E}_{s,y}^u \left[\int_0^{\tau \wedge \tau^\varepsilon} f^{u_t}(s+t, Y_t) e^{-\varphi t} dt + g(s+\tau^\varepsilon, Y_{\tau^\varepsilon}) e^{-\varphi \tau^\varepsilon} \chi_{\tau^\varepsilon < \tau} \right. \\ &\quad \left. + \Phi(s+\tau, Y_\tau) e^{-\varphi \tau} \chi_{\tau < \tau^\varepsilon} \right].\end{aligned}\tag{61}$$

and from the inequality

$$g(s + \tau^\varepsilon, Y_{\tau^\varepsilon}) \geq \Phi(s + \tau^\varepsilon, Y_{\tau^\varepsilon}) - \varepsilon,$$

it follows that

$$W_1(s, y) \geq \inf_{u \in M(s)} \mathbb{E}_{s,y}^u \left[\int_0^{\tau \wedge \tau^\varepsilon} f^{u_t}(s+t, Y_t) e^{-\varphi t} dt + \Phi(s + \tau \wedge \tau^\varepsilon, Y_{\tau \wedge \tau^\varepsilon}) e^{-\varphi \tau \wedge \tau^\varepsilon} - \varepsilon \right]. \quad (62)$$

Since $\tau \wedge \tau^\varepsilon \leq \tau^\varepsilon$, by Theorem 5.2 for $r_t^u = 0$, we have that the last lowerbound is equal to $\Phi(s, y) - \varepsilon$. Now let ε tend to zero therefore $W_1(s, y) \geq \Phi(s, y)$.

On the other hand, $g(s, y) \leq \Phi(s, y)$ so that

$$W_1(s, y) \leq \sup_{\gamma} \inf_{u \in M(s)} \mathbb{E}_{s,y}^u \left[\int_0^{\tau \wedge \gamma} f^{u_t}(s+t, Y_t) e^{-\varphi t} dt + \Phi(s + \tau \wedge \gamma, Y_{\tau \wedge \gamma}) e^{-\varphi \tau \wedge \gamma} \right]. \quad (63)$$

Let assume in Theorem 5.2 that $r_t^u = 0$, we note that the last upper bound does not exceed $\Phi(s, y)$. Hence $W_1(s, y) \leq \Phi(s, y)$. ■

In order to investigate the proof of the Theorem 5.2 we introduce the case of stochastic differential games (see [5]).

5.2 The stochastic games

5.2.1 Context

We introduce a two-player zero-sum stochastic differential game where the state is governed by controlled jump-diffusions. For $t \in [0, T - s]$

$$\begin{aligned} dY(t) &= b(s+t, Y(t), u(t), v(t)) dt + \sigma(s+t, Y(t), u(t), v(t)) dB(t) \\ &\quad + \int_{\mathbb{R}_0^k} \gamma(s+t, Y(t^-), u(t), v(t), z) \tilde{N}(dt, dz); \\ Y(s) &= y \in \mathbb{R}^k. \end{aligned} \quad (64)$$

Remark 5.3 For us and on the rest of the paper $b(t, y, u, v) = b(t, y, u)$, $\sigma(t, y, u, v) = \sigma(t, y, u)$ and $\gamma(t, y, u, v, z) = \gamma(t, y, u, z)$ so that $Y_t^{s,y,u,v} = Y_t^{s,y,u} = Y(t)$.

For convenience, we shall use the superscripts u, v and the subscripts s, y on the expectation sign to indicate expectation of quantites which depend on s, y and strategies u, v . We introduce $f(t, y, u, v) = f^u(t, y) + v g(t, y)$; $c^{u,v}(t, y) = c^u(t, y) + v$ and $\varphi_t^{s,y,u,v} = \int_0^t c^{u_r, v_r}(s+r, Y_r) dr$. We use the notation $\psi = (u, v)$ and define

$$J_n^\psi(s, y) = \mathbb{E}_{s,y}^\psi \left[\int_0^{T-s} f(s+t, Y_t, u_t, v_t) e^{-\varphi t} dt + g(T-s, Y_{T-s}) e^{-\varphi(T-s)} \right], \quad (65)$$

for strategie $v = (v_t)$ with values in $[0, n]$, $n \in \mathbb{N}^*$.

5.2.2 Admissible controls and strategies

Definition 5.4 An admissible control process $u(\cdot)$ (resp. $v(\cdot)$) for player I (resp. player II) on $[s, T]$ is a K (resp. $[0, n]$)-valued process which is $\mathcal{F}_{s,\cdot}$ -predictable. The set of all admissible controls for player I (resp. II) is denoted by $\mathcal{M}(s)$ (resp. $\mathcal{N}(s)$).

We say the controls $u, \tilde{u} \in \mathcal{M}(s)$ are the same on $[s, t]$ and write $u \approx \tilde{u}$ on $[s, t]$ if $\mathbb{P}_t(u(r) = \tilde{u}(r)) = 1$ for a.e. $r \in [s, t]$. A similar convention is followed for members of $\mathcal{N}(s)$.

Definition 5.5 An admissible strategy α (resp. β) for player I (resp. II) is a mapping $\alpha : \mathcal{N}(s) \rightarrow \mathcal{M}(s)$ (resp. $\beta : \mathcal{M}(s) \rightarrow \mathcal{N}(s)$) such that if $v(\cdot) \approx \tilde{v}$ (resp. $u \approx \tilde{u}$) on $[s, t]$, then $\alpha[v] \approx \alpha[\tilde{v}]$ on $[s, t]$ for every $t \in [s, T]$ (resp. $\beta[u] \approx \beta[\tilde{u}]$). The set of admissible strategies for player I (resp. II) on $[s, T]$ is denoted by $\Gamma_n(s)$ (resp. $\Delta_n(s)$).

Remark 5.6 We will denote $\Gamma(s) := \bigcup_n \Gamma_n(s)$ and $\Delta(s) := \bigcup_n \Delta_n(s)$.

We set $B_n = K \times [0, n]$ and \mathcal{B}_n the set of strategies. Let \mathcal{R}_n be a set of nonnegative processes \bar{r}_t which are progressively measurable with respect to (\mathcal{F}_t) and such that $\bar{r}_t(\omega) \leq n$ for all (t, ω) , $\mathcal{B} = \bigcup_n \mathcal{B}_n$ and $\mathcal{R} = \bigcup_n \mathcal{R}_n$. Each strategy $\psi \in \mathcal{B}_n$ is obviously a pair of processes (u, \bar{v}) with $u = (u_t) \in \mathcal{M}(s)$, $\bar{v} = (\beta[u]_t) \in \Delta_n(s)$.

Definition 5.7 i) The lower value of the SDG (64- 65) with initial data (s, y) is given by

$$\underline{\Phi}_n(s, y) := \inf_{\alpha \in \Gamma_n(s)} \left(\sup_{v \in \mathcal{N}(s)} J_n(s, y, \alpha[v], v) \right) \quad (66)$$

ii) The upper value of the SDG (64 - 65) is

$$\overline{\Phi}_n(s, y) := \sup_{\beta \in \Delta_n(s)} \left(\inf_{u \in \mathcal{M}(s)} J_n(s, y, u, \beta[u]) \right). \quad (67)$$

5.2.3 DPP for stochastic games

Proposition 5.8 The upper and lower value functions are Lipschitz continuous in y , Hölder continuous in t and verify $|\underline{\Phi}_n(s, y)| + |\overline{\Phi}_n(s, y)| \leq C(1 + |y|)^m$.

Proposition 5.9 Let $s \in [0, T]$, $\tau \in \mathcal{T}_{s, T}$, for every $y \in \mathbb{R}^k$,

$$\begin{aligned} \overline{\Phi}_n(s, y) &= \sup_{\beta \in \Delta_n(s)} \inf_{u \in \mathcal{M}(s)} E_{s, y}^{u, \beta} \left[\int_0^\tau (f^{u_t} + \beta[u]_t g)(s + t, Y_t) e^{-\varphi t} dt \right. \\ &\quad \left. + \overline{\Phi}_n(s + \tau, Y_\tau) e^{-\varphi \tau} \right] \end{aligned} \quad (68)$$

$$\begin{aligned} \underline{\Phi}_n(s, y) &= \inf_{\alpha \in \Gamma_n(s)} \sup_{v \in \mathcal{N}(s)} E_{s, y}^{\alpha, v} \left[\int_0^\tau (f^{\alpha[v]_t} + v_t g)(s + t, Y_t) e^{-\varphi t} dt \right. \\ &\quad \left. + \underline{\Phi}_n(s + \tau, Y_\tau) e^{-\varphi \tau} \right]. \end{aligned} \quad (69)$$

Remark 5.10 (i) Thanks to comparison theorems for Isaacs equations and the inequality $H^+ \leq H^-$ where

$$H^+ = \inf_{u \in K} \sup_{v \in [0, n]} (A^u \varphi(t, y) + f(t, y, u, v)),$$

$$H^- = \sup_{v \in [0, n]} \inf_{u \in K} (A^u \varphi(t, y) + f(t, y, u, v)),$$

we have $\underline{\Phi}_n \leq \overline{\Phi}_n$.

(ii) Under Isaac's condition, that is $H^+ = H^-$, we have

$$\underline{\Phi}_n(s, y) = \overline{\Phi}_n(s, y).$$

This is true for instance:

- in the deterministic case, see [7]

- if the controls appear separated in both the dynamics and the payoff, see [4].

Proposition 5.11 (i) $\Phi(s, y)$ is Lipschitz continuous in (s, y) .

(ii) $\overline{\Phi}_n(s, y) \nearrow \Phi(s, y)$ uniformly on each set of the form $\{(s, y) : s \in [0, T], |y| \leq R\}$.

5.3 Proof of the Thm. 5.2

5.3.1 Intermediate results

For more lightness we give all the intermediate results whose proofs are postponed in the next subsection. For $\psi = (u, v) \in K \times [0, n]$, let define

$$G_{s,t}^\psi w(y) = \mathbb{E}_{s,y}^\psi \left[\int_0^{t-s} f^\psi(s+r, X_r) e^{-\varphi r} dr + w(X_{t-s}) e^{-\varphi(t-s)} \right]$$

and

$$G_{s,t} w(y) = \sup_{\psi} G_{s,t}^\psi w(y).$$

Lemma 5.12 Let $s_0 < s_1 < \dots < s_n = T$. Then

$$\overline{\Phi}_n(s_0, y) \geq G_{s_0, s_1} G_{s_1, s_2} \dots G_{s_{n-1}, s_n} g(y).$$

Theorem 5.13 Let $s_0 = s_0^i < s_1^i < \dots < s_{n(i)}^i = T$ ($i = 1, 2, \dots$), $\max_j (s_{j+1}^i - s_j^i) \rightarrow 0$ for $i \rightarrow \infty$. Then

$$\overline{\Phi}_n(s_0, y) = \lim_{i \rightarrow \infty} G_{s_0^i, s_1^i} G_{s_1^i, s_2^i} \dots G_{s_{n(i)-1}^i, s_{n(i)}^i} g(y) \quad (70)$$

$$= \sup_i G_{s_0^i, s_1^i} G_{s_1^i, s_2^i} \dots G_{s_{n(i)-1}^i, s_{n(i)}^i} g(y). \quad (71)$$

Lemma 5.14 (a) Let $s \in [0, T], y \in \mathbb{R}^k, \psi = (u, v) \in M(s) \times \Delta_n(s)$ with $v = \beta[u]$ then the processes

$$\delta_t^{\psi, s, y} \equiv \overline{\Phi}_n(s + t, Y_t^{u, s, y})e^{-\varphi_t^{\psi, s, y}} - \mathbb{E}_{s, y}^{\psi} \left[\int_t^{T-s} f^{\psi_r}(s + r, Y_r)e^{-\varphi_r} dr + g(Y_{T-s})e^{-\varphi_{T-s}} / F_t \right]$$

and

$$K_t^{\psi, s, y} \equiv \overline{\Phi}_n(s + t, Y_t^{u, s, y})e^{-\varphi_t^{\psi, s, y}} + \int_0^t f^{\psi_r}(s + r, Y_r)e^{-\varphi_r} dr,$$

defined for $t \in [0, T-s]$ are supermartingales with respect to $\{F_t\}$, the first process being nonnegative (a.s.).

(b) $G_{s, t} \overline{\Phi}_n(t, y) \leq \overline{\Phi}_n(s, y)$ for $y \in \mathbb{R}^k, 0 \leq s \leq t \leq T$.

Lemma 5.15 Let $s \in [0, T], y \in \mathbb{R}^k, \psi = (u, \beta) \in M(s) \times \Gamma_n(s)$ then the process

$$\overline{\Phi}_n(s + t, Y_t^{u, s, y})e^{-\varphi_t^{\psi, s, y}} + \int_0^t [f^{u_p} + \beta[u_p] \overline{\Phi}_n](s + p, Y_p^{u, s, y})e^{-\varphi_p^{\psi, s, y}} dp$$

defined for $t \in [0, T-s]$ is a continuous supermartingale.

Lemma 5.16 (a) Let $s \in [0, T], y \in \mathbb{R}^k$ and for each $u \in M(s)$ let $\tau^u \in \mathcal{T}_{s, T}, r_t^u \in \mathcal{R}$ be defined. Then

$$\overline{\Phi}_n(s, y) = \inf_{u \in \mathcal{M}(s)} \mathbb{E}_{s, y}^u \left[\int_0^{\tau} [f^{u_t} + n(g - \overline{\Phi}_n)_+] (s + t, Y_t)e^{-\varphi_t - \int_0^t r_p dp} dt + \overline{\Phi}_n(s + \tau, Y_{\tau})e^{-\varphi_{\tau} - \int_0^{\tau} r_p dp} \right]$$

(b) Let $g_n = g \wedge \overline{\Phi}_n$. Then

$$\overline{\Phi}_n(s, y) = \sup_{\tau \in \mathcal{T}_{s, T}} \inf_{u \in \mathcal{M}(s)} \mathbb{E}_{s, y}^u \left[\int_0^{\tau} f^{u_t}(s + t, Y_t)e^{-\varphi_t} dt + g_n(s + \tau, Y_{\tau})e^{-\varphi_{\tau}} \right]$$

Corollaire 5.17 Since $g_n \leq g, \overline{\Phi}_n \leq \Phi$.

Corollaire 5.18 (Consequence of Prop 5.9 and Lemma 5.15) Let $s \in [0, T], y \in \mathbb{R}^k, \psi = (u, \bar{v}) \in \mathcal{B}$ then the process

$$\rho_t^{\psi, s, y} \equiv \Phi(s + t, Y_t^{u, s, y})e^{-\varphi_t^{\psi, s, y}} + \int_0^t [f^{u_p} + \bar{v}_p \Phi](s + p, Y_p^{u, s, y})e^{-\varphi_p^{\psi, s, y}} dp$$

defined for $t \in [0, T-s]$ is a continuous supermartingale.

5.3.2 Proofs

The following result is given without demonstration, we refer to [14] Lemma 2.14 p148.

Lemma 5.19 Let $s \in [0, T], 0 \leq t_1 \leq t_2 \leq T-s$ and $\psi \in \mathcal{M}(s) \times \Gamma(s)$ such that $\psi_t = \psi_{t_1}$ for $t \in [t_1, t_2]$. Let the continuous function $w(y)$ satisfying $|w(y)| \leq N(1 + |y|)^m$. Then a.s.

$$\mathbb{E}_{s, y}^{\psi} \left[\int_{t_1}^{t_2} f^{\psi_t}(s + t, Y_t)e^{-\varphi_t} dt + w(Y_{t_2})e^{-\varphi_{t_2}} / \mathcal{F}_{t_1} \right] = e^{-\varphi_{t_1}^{\psi, s, y}} G_{s+t_1, s+t_2}^{\psi_{t_1}} w(Y_{t_1}^{\psi_{t_1}}).$$

Proof. (Lemme 5.15) By Lemma 5.14 (a),

$$K_t^{\psi,s,y} = \overline{\Phi}_n(s+t, Y_t^{\psi,s,y})e^{-\varphi_t^{\psi,s,y}} + \int_0^t f^{\psi_p}(s+p, Y_p)e^{-\varphi_p^{\psi,s,y}} dp$$

is a supermartingale. In particular for $\psi = (u, 0)$,

$$\overline{\Phi}_n(s+t, Y_t^{u,s,y})e^{-\varphi_t^{u,s,y}} + \int_0^t f^{u_p}(s+p, Y_p^{u,s,y})e^{-\varphi_p^{u,s,y}} dp$$

is a supermartingale. Applying the lemma from Appendix 2 in [14] with $\Phi_t = e^{-\int_0^t \beta[u]_s ds}$ thus completing the proof. ■

Proof. (Lemme 5.12)

We introduce $w_i(y) = G_{s_i, s_{i+1}} \dots G_{s_{n-1}, s_n} g(y)$, $i = 0, \dots, n-1$ and $w_n(y) = g(y)$. Let $\varepsilon > 0$, by the Theorem 2.2 (Chap. 3) in [14] and the assumption (A3), we deduce the continuity of w_{n-1} and that $|w_{n-1}(y)| \leq N(1 + |y|)^m$. Arguing in the same way, we convince ourselves that all the functions $w_i(y)$ are continuous. Furthermore $w_i(y) = G_{s_i, s_{i+1}} w_{i+1}(y) = \sup_{\psi} G_{s_i, s_{i+1}}^{\psi} w_{i+1}(y)$, $i = 0, \dots, n-1$. By the corollary 2.8 in [14] p145, the functions $G_{s_i, s_{i+1}}^{\psi} w_{i+1}(y)$ are continuous with respect to ψ and y . Then for all $y \in \mathbb{R}^k$ there exists a Borel function β_i^y such that

$$w_i(y) \leq G_{s_i, s_{i+1}}^{u, \beta_i^y} w_{i+1}(y) + \varepsilon, \quad \forall u \in \mathcal{M}(s). \quad (72)$$

We construct a strategy $\psi_t^\varepsilon = (u_t^\varepsilon, v_t^\varepsilon)$ by the following way: $u_t^\varepsilon = u_i^\varepsilon \in K$ and $v_t^\varepsilon = \beta[u_i^\varepsilon]$ for $t \in [s_i, s_{i+1})$. Then ψ_t^ε is admissible and by the lemma 5.19,

$$\mathbb{E}_{s_0, y}^{\psi^\varepsilon} \left[\int_{s_i - s_0}^{s_{i+1} - s_0} f^{\psi_t}(s_0 + t, Y_t) e^{-\varphi_t} dt + w_{i+1}(Y_{s_{i+1} - s_0}) e^{-\varphi_{s_{i+1} - s_0}} \right] = \mathbb{E}_{s_0, y}^{\psi^\varepsilon} e^{-\varphi_{s_i - s_0}} G_{s_i, s_{i+1}}^{\psi^\varepsilon} w_{i+1}(Y_{s_i - s_0}).$$

Thus this result combining with (72) yields

$$\mathbb{E}_{s_0, y}^{\psi^\varepsilon} \left[\int_{s_i - s_0}^{s_{i+1} - s_0} f^{\psi_t}(s_0 + t, Y_t) e^{-\varphi_t} dt + w_{i+1}(Y_{s_{i+1} - s_0}) e^{-\varphi_{s_{i+1} - s_0}} \right] \geq \mathbb{E}_{s_0, y}^{\psi^\varepsilon} (e^{-\varphi_{s_i - s_0}} w_i(Y_{s_i - s_0})) - \varepsilon.$$

Adding up all such inequalities and collecting like terms, we find

$$\mathbb{E}_{s_0, y}^{\psi^\varepsilon} \left[\int_0^{T-s} f^{\psi_t}(s_0 + t, Y_t) e^{-\varphi_t} dt + g(Y_{T-s}) e^{-\varphi_{T-s}} \right] = J_{n_0}^{\psi^\varepsilon}(s_0, y) \geq w_0(y) - n\varepsilon.$$

Then

$$\sup_{\beta \in \Gamma(s)} \inf_{u \in \mathcal{M}(s)} J_n^\psi(s_0, y) \geq w_0(y) - n\varepsilon$$

and letting ε tend to zero we prove

$$\overline{\Phi}_n(s_0, y) \geq w_0(y) = G_{s_0, s_1} G_{s_1, s_2} \dots G_{s_{n-1}, s_n} g(y).$$

■

Proof. (Thm 5.13) By lemma 5.12 we have

$$\overline{\Phi}_n(s_0, y) \geq G_{s_0^i, s_1^i} G_{s_1^i, s_2^i} \dots G_{s_{n(i)-1}^i, s_{n(i)}^i} g(y).$$

so that it remains to prove

$$\overline{\Phi}_n(s_0, y) \leq \liminf_{i \rightarrow \infty} G_{s_0^i, s_1^i} G_{s_1^i, s_2^i} \dots G_{s_{n(i)-1}^i, s_{n(i)}^i} g(y).$$

We give us step strategies $(u^i, v^i = \beta[u^i])$ such that $J_n^{\psi^i}(s_0, y) \rightarrow \overline{\Phi}_n(s_0, y)$ as $i \rightarrow \infty$ and $u_t^i = u_{s_j^i - s_0}^i$ for $t \in [s_j^i - s_0, s_{j+1}^i - s_0)$. Then if we introduce $w_{n(i)}^i(y) = g(y)$, $w_j^i(y) = G_{s_j^i, s_{j+1}^i} w_{j+1}^i(y)$, ($j = 0, \dots, n(i) - 1$), we get by the lemma 5.19 that

$$\mathbb{E}_{s_0, y}^{\psi^i} \left[\int_{s_j^i - s_0}^{s_{j+1}^i - s_0} f^{\psi_r}(s_0 + r, Y_r) e^{-\varphi r} dr + w_{j+1}^i(Y_{s_{j+1}^i - s_0}) e^{-\varphi_{s_{j+1}^i - s_0}} \right] \quad (73)$$

$$= \mathbb{E}_{s_0, y}^{\psi^i} \left[e^{-\varphi_{s_j^i - s_0}} G_{s_j^i, s_{j+1}^i}^{\psi_{s_j^i - s_0}^i} w_{j+1}^i(Y_{s_j^i - s_0}) \right] \quad (74)$$

$$\leq \mathbb{E}_{s_0, y}^{\psi^i} \left[e^{-\varphi_{s_j^i - s_0}} w_j^i(Y_{s_j^i - s_0}) \right]. \quad (75)$$

Adding up such inequalities with respect to j from $j = 0$ to $j = n(i) - 1$, and collecting like terms, we obtain:

$$\mathbb{E}_{s_0, y}^{\psi^i} \left[\int_0^{T-s_0} f^{\psi_r}(s_0 + r, Y_r) e^{-\varphi r} dr + w_{n(i)}^i(Y_{s_{n(i)}^i - s_0}) e^{-\varphi_{s_{n(i)}^i - s_0}} \right] \leq w_0^i(y),$$

that is $J_n(s_0, y, \psi^i) \leq w_0^i(x)$ and finally $\overline{\Phi}_n(s_0, y) \leq \liminf_{i \rightarrow \infty} w_0^i(y)$. ■

Proof. (Lemma 5.14) (a) Obviously

$$\delta_t^{\psi, s, y} - K_t^{\psi, s, y} = -\mathbb{E}_{s, y}^{\psi} \left[g(Y_{T-s}) e^{-\varphi_{T-s}} + \int_0^{T-s} f^{\psi_r}(s + r, Y_r) e^{-\varphi r} dr / \mathcal{F}_t \right]$$

where the right side is a martingale. Hence $\delta_t^{\psi, s, y}$ is a supermartingale if $K_t^{\psi, s, y}$ is a supermartingale. The nonnegativity of $\delta_t^{\psi, s, y}$ is a consequence of the definition of a supermartingale, $\delta_t^{\psi, s, y} \geq \mathbb{E}_{s, y}^{\psi} [\delta_{T-s} / \mathcal{F}_t]$ and $\delta_{T-s}^{\psi, s, y} \equiv 0$. Furthermore by the Prop 5.8, the function $\overline{\Phi}_n(s + t, y)$ is continuous with respect to y and verifies $|\overline{\Phi}_n(s + t, y)| \leq N(1 + |y|)^m$. Then by lemma 2.7 [14] p144, $\lim_{n \rightarrow \infty} K_t^{\psi^n, s, y} = K_t^{\psi, s, y}$, $\forall t \in [0, T - s]$ if $\psi^n \rightarrow \psi$. We can choose step strategies ψ^n so that we need prove $K_t^{\psi, s, x}$ is a supermartingale for step strategies only. It suffices to prove that $\mathbb{E}_{s, x}^{\psi} [K_{t_2} / \mathcal{F}_{t_1}] \leq K_{t_1}$ (a.s.) for $t_2 \geq t_1$ if $u_t = u_{t_1} \in K$ on $[t_1, t_2)$ and $v_t = \beta[u_{t_1}] \in [0, n]$ for $t \in [t_1, t_2)$. We then have by the Lemma 5.19

$$\mathbb{E}_{s, y}^{\psi} [K_{t_2} / \mathbb{F}_{t_1}] \stackrel{\text{a.s.}}{=} \int_0^{t_1} f^{\psi_r}(s + r, Y_r^{\psi, s, y}) e^{-\varphi_r^{\psi, s, y}} dr \quad (76)$$

$$+ \mathbb{E}_{s, y}^{\psi} \left[\int_{t_1}^{t_2} f^{\psi_t}(s + t, Y_t) e^{-\varphi_t} dt + e^{-\varphi_{t_2}^{\psi, s, y}} \overline{\Phi}_n(s + t_2, Y_{t_2}) / \mathbb{F}_{t_1} \right] \quad (77)$$

$$= \int_0^{t_1} f^{\psi_r}(s + r, Y_r) e^{-\varphi_r^{\psi, s, y}} dr \quad (78)$$

$$+ e^{-\varphi_{t_1}^{\psi, s, y}} G_{s+t_1, s+t_2}^{\psi_{t_1}} \overline{\Phi}_n(s + t_2, Y_{t_1}^{\psi, s, y}). \quad (79)$$

So that it remains to prove the assertion (b) of the lemma.

(b) Our objective is to prove that $G_{s,t}\overline{\Phi}_n(t, y) \leq \overline{\Phi}_n(s, y) (= \sup_{\beta} \inf_u J_n^{u, v=\beta[u]}(s, y))$. We construct a sequence of subdivisions of $[t, T]$ whose diameter tends to zero:

$$t = s_0 = s_0^i < \dots < s_j^i < \dots < s_{n(i)}^i = T, \quad \max_j (s_{j+1}^i - s_j^i) \xrightarrow{i \rightarrow +\infty} 0.$$

By lemma 5.12, Proposition 5.8 and the assumptions (A1)-(A5), we get for $\psi_0 = (u_0, v_0) \in K \times [0, n]$

$$N(1 + |y|)^m \geq \overline{\Phi}_n(t, y) \geq G_{s_0^i, s_1^i} G_{s_1^i, s_2^i} \dots G_{s_{n(i)-1}^i, s_{n(i)}^i} g(y) \quad (80)$$

$$\geq G_{s_0^i, s_1^i}^{\psi_0} G_{s_1^i, s_2^i}^{\psi_0} \dots G_{s_{n(i)-1}^i, s_{n(i)}^i}^{\psi_0} g(y) \quad (81)$$

$$= \mathbb{E}_{s,y}^{\psi_0} \left[+ \int_0^{T-t} f^{\psi_0}(t+r, Y_r) e^{-\varphi r} dr + g(Y_{T-t}) e^{-\varphi T-t} \right] \quad (82)$$

$$\geq -N(1 + |y|)^m \quad (83)$$

where N does not depend on y . This implies that for every $u \in K$, $G_{s_0^i, s_1^i} G_{s_1^i, s_2^i} \dots G_{s_{n(i)-1}^i, s_{n(i)}^i} g(Y_{t-s}^{u, s, y})$ does not exceed $N(1 + |Y_{t-s}^{u, s, y}|)^m$, the latter expression having a finite mathematical expectation.

Then by $G_{s,t}^{\psi} G_{s_0^i, s_1^i} G_{s_1^i, s_2^i} \dots G_{s_{n(i)-1}^i, s_{n(i)}^i} g(y) = \mathbb{E}_{s,y}^{\psi} \left[\int_0^{t-s} f^{\psi}(s+r, Y_r) e^{-\varphi r} dr + w_0^i(Y_{t-s}) e^{-\varphi t-s} \right]$, and the application of Lebesgue's theorem and Theorem 5.13, we deduce

$$\lim_{i \rightarrow \infty} G_{s,t}^{\psi} w_0^i(y) = \mathbb{E}_{s,y}^{\psi} \left[\int_0^{t-s} f^{\psi}(s+r, Y_r) e^{-\varphi r} dr + w_0^i(Y_{t-s}) e^{-\varphi t-s} \right] \quad (84)$$

$$= G_{s,t}^{\psi} \overline{\Phi}_n(t, y). \quad (85)$$

Finally by lemma 5.12, we have

$$G_{s,t}^{\psi} \overline{\Phi}_n(t, y) = \lim_{i \rightarrow \infty} G_{s,t}^{\psi} G_{s_0^i, s_1^i} G_{s_1^i, s_2^i} \dots G_{s_{n(i)-1}^i, s_{n(i)}^i} g(y) \leq \overline{\Phi}_n(s, y).$$

■

Proof. (Lemma 5.16) First (b) is a consequence of (a). Indeed, by (a) with $r_t^u \equiv 0$, we have

$$\overline{\Phi}_n(s, y) = \inf_{u \in \mathcal{M}(s)} \mathbb{E}_{s,y}^u \left[\int_0^{\tau} [f^{ut} + n(g - \overline{\Phi}_n)_+](s+t, Y_t) e^{-\varphi t} dt + \overline{\Phi}_n(s+\tau, Y_{\tau}) e^{-\varphi \tau} \right]. \quad (86)$$

If $f_n^u := f^u + n(g - \overline{\Phi}_n)_+$, since $f_n^u \geq f^u$ and that $\overline{\Phi}_n \geq g_n$, we have for every $\tau = \tau^u$,

$$\overline{\Phi}_n(s, y) \geq \inf_{u \in \mathcal{M}(s)} \mathbb{E}_{s,y}^u \left[\int_0^{\tau} f^u(s+t, Y_t) e^{-\varphi t} dt + g_n(s+\tau, Y_{\tau}) e^{-\varphi \tau} \right] \quad (87)$$

$$\geq \sup_{\tau \in \mathcal{I}_{s,T}} \inf_{u \in \mathcal{M}(s)} \mathbb{E}_{s,y}^u \left[\int_0^{\tau} f^u(s+t, Y_t) e^{-\varphi t} dt + g_n(s+\tau, Y_{\tau}) e^{-\varphi \tau} \right]. \quad (88)$$

Furthermore if we define $\tau_0 \equiv \inf\{t \geq 0, \overline{\Phi}_n(s+t, Y_t^u) \leq g(s+t, Y_t^u)\}$, for $t \in [0, \tau_0^u]$ we have $f_n^u(s+t, Y_t^u) = f^u(s+t, Y_t^u)$, $\overline{\Phi}_n(s+t, Y_t^u) = g_n(s+t, Y_t^u)$ and in (86) we find

$$\overline{\Phi}_n(s, y) = \inf_{u \in \mathcal{M}(s)} \mathbb{E}_{s,y}^Y \left[\int_0^{\tau_0^u} f^u(s+t, Y_t) e^{-\varphi t} dt + g_n(s+\tau_0, Y_{\tau_0}^u) e^{-\varphi \tau_0} \right]. \quad (89)$$

Then combining (88) with (89), we deduce (b).

It remains to establish the assertion (a). In fact it is enough to prove it for $\tau^u = T - s$ and $r \equiv 0$. Let $\psi = (u, \bar{v}) \in \mathcal{B}_n$ we introduce

$$K_t = \bar{\Phi}_n(s+t, Y_t^{\psi, s, y}) e^{-\varphi_t^{\psi, s, y}} + \int_0^t f^{\psi_p}(s+p, Y_p^{\psi, s, y}) e^{-\varphi_p^{\psi, s, y}} dp,$$

and $\Phi_t = e^{\int_0^t \bar{v}_p dp}$. By the Lemma 5.14 the process K_t is a supermartingale. Thus applying the lemma of Appendix 2 in [14], the process $\rho_t \equiv K_t \phi_t - \int_0^t K_s d\phi_s$ is a supermartingale and

$$\bar{\Phi}_n(s, y) = \mathbb{E}\rho_0 \geq \mathbb{E}\rho_{T-s} \geq e^{n(T-s)} [\mathbb{E}K_{T-s} - \bar{\Phi}_n(s, y)] + \bar{\Phi}_n(s, y). \quad (90)$$

Using Fubini's theorem, we prove that

$$\mathbb{E}\rho_{T-s} = \mathbb{E}_{s, y}^u \left[g(T, Y_{T-s}) e^{-\varphi_{T-s}} + \int_0^{T-s} [f^{u_t} + \bar{v}_t (g - \bar{\Phi}_n)](s+t, Y_t) e^{-\varphi_t} dt \right].$$

Further we note the upper bound of the last expression with respect to \bar{v} is

$$\mathbb{E}_{s, y}^u \left[g(T, Y_{T-s}) e^{-\varphi_{T-s}} + \int_0^{T-s} [f^{u_t} + n(g - \bar{\Phi}_n)_+](s+t, Y_t) e^{-\varphi_t} dt \right]$$

Thus taking the upper bound and the lower bound with respect to respectively \bar{v} and u in (90), we find

$$\begin{aligned} \bar{\Phi}_n(s, x) &\geq \inf_{u \in \mathcal{M}(s)} \mathbb{E}_{s, y}^u \left[\int_0^{T-s} [f^{u_t} + n(g - \bar{\Phi}_n)_+](s+t, Y_t) e^{-\varphi_t} dt + g(T, Y_{T-s}) e^{-\varphi_{T-s}} \right] \\ &\geq e^{n(T-s)} \sup_{\bar{v}} \inf_{u \in \mathcal{M}(s)} (\mathbb{E}[K_{T-s}] - \bar{\Phi}_n(s, y)) + \bar{\Phi}_n(s, y). \end{aligned}$$

Finally the expression of K_{T-s} leads to $\sup_{\bar{v}} \inf_{u \in \mathcal{M}(s)} \mathbb{E}[K_{T-s}] - \bar{\Phi}_n(s, y) = 0$ so that the expected equality follows. ■

Proof. (Proposition 5.11) (i) follows from the Proposition 5.8.

(ii) Since $\mathcal{B}_n \subset \mathcal{B}_{n+1}$, the sequence $\bar{\Phi}_n(s, y)$ increases. Moreover by Corollary 5.17 $\bar{\Phi}_n \leq \Phi$ so that we define $\tilde{\Phi}(s, y) = \lim_{n \rightarrow \infty} \bar{\Phi}_n(s, y)$. By Corollary 5.17, $\tilde{\Phi}(s, y) \leq \Phi(s, y)$.

Recalling $\underline{\Phi}_n(s, y) \leq \bar{\Phi}_n(s, y)$ (see remark 5.10 (i)), let $\varepsilon > 0$ and $\alpha^\varepsilon \in \Delta_n(s)$ such that

$$\begin{aligned} \bar{\Phi}_n(s, y) \geq \underline{\Phi}_n(s, y) &\geq \sup_{v \in \mathcal{N}(s)} \mathbb{E}_{s, y}^{\alpha^\varepsilon, v} \left[\int_0^{T-s} [f^{\alpha_t^\varepsilon[v]} + v_t g](s+t, Y_t) e^{-\varphi_t} dt + g(T, Y_{T-s}) e^{-\varphi_{T-s}} \right] - \varepsilon \\ &\geq \mathbb{E}_{s, y}^{\alpha^\varepsilon, v} \left[\int_0^{T-s} [f^{\alpha_t^\varepsilon[v]} + v_t g](s+t, Y_t) e^{-\varphi_t} dt + g(T, Y_{T-s}) e^{-\varphi_{T-s}} \right] - \varepsilon, \forall v \in \mathcal{N}(s) \end{aligned} \quad (91)$$

We take for $\tau \in \mathcal{I}_{s, T}$, $v_t \equiv n\chi_{\tau \leq t}$. Then, using Fubini's theorem we get

$$\begin{aligned} \bar{\Phi}_n(s, y) &\geq \mathbb{E}_{s, y}^{\alpha^\varepsilon} \left[\int_\tau^{T-s} n e^{-n(t-\tau)} \left[\int_0^t f^{\alpha_p^\varepsilon}(s+p, Y_p) e^{-\varphi_p} dp + g(s+t, Y_t) e^{-\varphi_t} \right] dt \right] \\ &\quad + \mathbb{E}_{s, y}^{\alpha^\varepsilon} \left[e^{-n(T-s-\tau)} \left[\int_0^{T-s} f^{\alpha_t^\varepsilon}(s+t, Y_t) e^{-\varphi_t} dt + g(T, Y_{T-s}) e^{-\varphi_{T-s}} \right] \right] - \varepsilon. \quad (92) \end{aligned}$$

We introduce

$$\begin{aligned}\eta^{\alpha^\varepsilon, s, y}(t) &= \int_0^t f_p^{\alpha^\varepsilon}(s+p, Y_p^{\alpha^\varepsilon, s, y}) e^{-\varphi_p^{\alpha^\varepsilon, s, y}} dp + g(s+t, Y_t^{\alpha^\varepsilon, s, y}) e^{-\varphi_t^{\alpha^\varepsilon, s, y}}, t \leq T-s \\ \eta^{\alpha^\varepsilon, s, y}(t) &= \eta(T-s), t > T-s.\end{aligned}\quad (93)$$

Furthermore we introduce a random variable ξ which has an exponential distribution with a parameter equal to unity and which, in addition, does not depend on $\{\eta^{\alpha^\varepsilon, s, y}(t)\}$. We obtain $\overline{\Phi}_n(s, y) \geq \mathbb{E}_{s, y}^{\alpha^\varepsilon} \eta(\tau + \frac{1}{n}\xi) - \varepsilon$.

Therefore, by Lebesgue's theorem

$$\begin{aligned}\tilde{\Phi}(s, y) + \varepsilon &\geq \mathbb{E}_{s, y}^{\alpha^\varepsilon} \eta(\tau) \\ \tilde{\Phi}(s, y) + \varepsilon &\geq \sup_{\tau} \mathbb{E}_{s, y}^{\alpha^\varepsilon} \eta(\tau) \geq \inf_{\alpha} \sup_{\tau} \mathbb{E}_{s, y}^{\alpha} \eta(\tau)\end{aligned}$$

Letting ε tend to zero then $\tilde{\Phi}(s, y) \geq \Phi(s, y)$ and we conclude that $\tilde{\Phi}(s, y) = \Phi(s, y)$. The rest of the proof is a consequence of Dini's theorem and the property $|g_n(s, y)| \leq N(1 + |y|)^m$ with the same constant N for all n, s, y . ■

Proof. (Theorem 5.2)

Corollary 5.18, properties of supermartingales and $\Phi(s, y) = \rho_0$ imply

$$\begin{aligned}\Phi(s, y) = \mathbb{E}_{s, y}^{\psi}(\rho_0) \geq \mathbb{E}_{s, y}^u(\rho_\tau) &= \mathbb{E}_{s, y}^u \left[\int_0^\tau [f^{u_t}(s+t, Y_t) + \bar{v}_t \Phi(s+t, Y_t)] e^{-\varphi_t - \int_0^t \bar{v}_p dp} dt \right. \\ &\quad \left. + \Phi(s+\tau, Y_\tau) e^{-\varphi_\tau - \int_0^\tau \bar{v}_p dp} \right].\end{aligned}\quad (94)$$

which proves inequality (60).

Next let $\varepsilon > 0$ $\tau_{u, s, y}^\varepsilon = \inf\{t \geq 0 : \Phi(s+t, Y_t^{u, s, y}) \leq g(s+t, Y_t^{\alpha, s, y}) + \varepsilon\}$ and $\tau^u \leq \tau_{u, s, y}^\varepsilon$.

According to DPP (see Prop 5.8) for each n ,

$$\begin{aligned}\overline{\Phi}_n(s, y) &= \sup_{\beta \in \Delta_n(s)} \inf_{u \in \mathcal{M}(s)} E_{s, y}^{u, \beta} \left[\int_0^\tau (f^{u_t} + \beta[u]_t g)(s+t, Y_t) e^{-\varphi_t} dt \right. \\ &\quad \left. + \overline{\Phi}_n(s+\tau, Y_\tau) e^{-\varphi_\tau} \right] \\ &\leq \sup_{\beta \in \Delta(s)} \inf_{u \in \mathcal{M}(s)} E_{s, y}^{u, \beta} \left[\int_0^\tau (f^{u_t} + \beta[u]_t g)(s+t, Y_t) e^{-\varphi_t} dt \right. \\ &\quad \left. + \Phi(s+\tau, Y_\tau) e^{-\varphi_\tau} \right].\end{aligned}$$

Taking the limit as $n \rightarrow \infty$ and using the fact that $\overline{\Phi}_n \nearrow \Phi$ we have

$$\Phi(s, y) \leq \sup_{\beta \in \Delta(s)} \inf_{u \in \mathcal{M}(s)} E_{s, y}^{u, \beta} \left[\int_0^\tau (f^{u_t} + \beta[u]_t g)(s+t, Y_t) e^{-\varphi_t} dt + \Phi(s+\tau, Y_\tau) e^{-\varphi_\tau} \right]. \quad (95)$$

Now by (94) and the inequality $\Phi \geq g$ we get

$$\Phi(s, y) \geq E_{s, y}^{u, \beta} \left[\int_0^\tau (f^{u_t} + \beta[u]_t g)(s+t, Y_t) e^{-\varphi_t} dt + \Phi(s+\tau, Y_\tau) e^{-\varphi_\tau} \right]. \quad (96)$$

so that combining (95) and (96) we find

$$\Phi(s, y) = \sup_{\beta \in \Delta(s)} \inf_{u \in \mathcal{M}(s)} E_{s,y}^{u,\beta} \left[\int_0^\tau (f^{u_t} + \beta[u]_t g)(s+t, Y_t) e^{-\varphi t} dt + \Phi(s+\tau, Y_\tau) e^{-\varphi \tau} \right]. \quad (97)$$

Further, we take a sequence $u^i \in \mathcal{M}(s)$ and $\beta^i = \beta[u^i] \in \Delta(s)$ such that

$$\Phi(s, y) = \lim_{i \rightarrow \infty} E_{s,y}^{u^i} \left[\int_0^\tau (f^{u^i_t} + \beta^i[u^i]_t g)(s+t, Y_t) e^{-\varphi t} dt + \Phi(s+\tau, Y_\tau) e^{-\varphi \tau} - \int_0^\tau \beta^i[u^i]_p dp \right]. \quad (98)$$

From $g(s+t, Y_t^{u,s,y}) < \Phi(s+t, Y_t^{u,s,y}) - \varepsilon$ for $t < \tau_\varepsilon^{u,s,y}$, (94) and (98), we find

$$\varepsilon \lim_{i \rightarrow \infty} E_{s,y}^{u^i} \left[\int_0^\tau \beta^i[u^i]_t e^{-\varphi t} dt - \int_0^\tau \beta^i[u^i]_p dp \right] = 0. \quad (99)$$

By lemma 4.2 in [Krylov] p 153, (99) and (98) we deduce

$$\Phi(s, y) = \lim_{i \rightarrow \infty} E_{s,y}^{u^i} \left[\int_0^\tau f^{u^i_t}(s+t, Y_t) e^{-\varphi t} dt + \Phi(s+\tau, Y_\tau) e^{-\varphi \tau} \right]. \quad (100)$$

On the other hand by Corollary 5.18, the process

$$K_t^{u,s,y} = \Phi(s+t, Y_t^{u,s,y}) e^{-\varphi t} + E_{s,y}^{u^i} \left[\int_0^t f^{u^i_p}(s+p, Y_p) e^{-\varphi p} dp \right]$$

is a continuous supermartingale. Therefore according with lemma given in appendix 2 in [14], $K_t^{u,s,y} - \rho_t^{Y, \beta[u], s, y}$ is a supermartingale for each $\psi = (u, \beta[u])$. In particular, $E_{s,y}^u K_\tau \leq E_{s,y}^\psi \rho_\tau$ which together with (100) and (94) yields

$$\begin{aligned} \Phi(s, y) &= \lim_{i \rightarrow \infty} E_{s,y}^{u^i} K_\tau \leq \lim_{i \rightarrow \infty} E_{s,y}^{u^i, \beta^i[u^i]} \rho_\tau \\ &= \lim_{i \rightarrow \infty} E_{s,y}^{u^i} \left[\Phi(s+\tau, Y_\tau) e^{-\varphi \tau} - \int_0^\tau \beta^i[u^i]_p dp \right. \\ &\quad \left. + \int_0^\tau (f^{u^i_t} + \beta^i[u^i]_t \Phi)(s+t, Y_t) e^{-\varphi t} dt - \int_0^t \beta^i[u^i]_p dp \right]. \end{aligned}$$

Thus

$$\begin{aligned} \Phi(s, y) &\leq E_{s,y}^u \left[\int_0^\tau (f^{u_t} + \beta[u]_t \Phi)(s+t, Y_t) e^{-\varphi t} dt - \int_0^\tau \beta[u]_p dp \right. \\ &\quad \left. + \Phi(s+\tau, Y_\tau) e^{-\varphi \tau} - \int_0^\tau \beta[u]_p dp \right] \\ &\leq \Phi(s, y). \end{aligned}$$

The required equality is then proved. ■

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