

# Isomorphisms between Heisenberg-invariant varieties

by

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**THESIS**  
*for the degree of*  
***Master in Mathematics***

(*Master i matematikk*)



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*May 2014*

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## Introduction

The main aim of this thesis is to investigate isomorphisms between certain Heisenberg-invariant varieties or when two Heisenberg-invariant varieties are isomorphic.

Our approach is as follows. First we will decompose the ring of homogeneous polynomials of degree  $k$  into irreducible Heisenberg-representations, and isomorphism-classes of Heisenberg-representations are obtained from these. These isomorphism-classes are sets  $\{V_{\alpha_1, \alpha_2, \dots, \alpha_k} | \forall (\alpha_1, \alpha_2, \dots, \alpha_k) \in \mathbb{C}^k\}$ , where  $V_{\alpha_1, \alpha_2, \dots, \alpha_k}$  is an irreducible Heisenberg-representation. As  $V_{\alpha_1, \alpha_2, \dots, \alpha_k}$  is the set of all linear-combinations of certain polynomials, it has zero set  $X_{\alpha_1, \alpha_2, \dots, \alpha_k}$ . And it is Heisenberg-invariant since  $V_{\alpha_1, \alpha_2, \dots, \alpha_k}$  is an Heisenberg-representation.

Now we will show that a certain subgroup of the normalizer of the Heisenberg-group (or rather its representation in  $GL(\mathbb{C}^6)$ ) in  $GL(\mathbb{C}^6)$  induces isomorphisms between certain Heisenberg-invariant varieties. This subgroup defines a group-action on  $\mathbb{C}^k$ . We will see that  $X_{\alpha_1, \alpha_2, \dots, \alpha_k}$  and  $X_{\beta_1, \beta_2, \dots, \beta_k}$  are isomorphic if  $(\alpha_1, \alpha_2, \dots, \alpha_k)$  and  $(\beta_1, \beta_2, \dots, \beta_k)$  belong to the same orbit of this group-action.

Last I will thank my supervisor, Kristian Ranestad, who has been very helpful all along.

# 1 The Heisenberg-group

The Heisenberg-group is the abstract-group generated by the three elements of order six,  $\sigma, \gamma$  and  $\epsilon$ , which satisfy the following three relations  $\gamma\sigma = \epsilon\sigma\gamma, \epsilon\sigma = \sigma\epsilon$  and  $\epsilon\gamma = \gamma\epsilon$ . Thus the Heisenberg-group consists of the elements  $\epsilon^r\sigma^s\gamma^t$  under group-multiplication

**Definition** H6= $\{\epsilon^r\sigma^s\gamma^t | \epsilon^6 = \sigma^6 = \gamma^6 = 1, \epsilon\sigma = \sigma\epsilon\sigma\gamma = \epsilon\gamma\sigma\}$  where  $\epsilon$  act as a primitive sixth-root of unity.

And from this definition we get the following multiplication rule.

**Proposition 1.1**  $\epsilon^r\sigma^s\gamma^t * \epsilon^{r'}\sigma^{s'}\gamma^{t'} = \epsilon^{r+r'-ts'}\sigma^{s+s'}\gamma^{t+t'}$

**Proof**

Let  $r', s'$  and  $t'$  be fixed and  $r, s$  and  $t$  vary. We use induction on  $s + t$

If  $s + t = 0 \Rightarrow s = t = 0$  and obviously

$$\epsilon^r\epsilon^{r'} = \epsilon^{r+r'}$$

Suppose it holds for  $s + t = k$ . Then if  $s'' + t'' = k + 1 \Rightarrow s'' = s + 1$  or  $t'' = t + 1$

Case I)  $s'' = s + 1$ , then  $t'' = t$  and

$$\begin{aligned} & \epsilon^r\sigma^{s''}\gamma^{t''}\epsilon^{r'}\sigma^{s'}\gamma^{t'} \\ & \epsilon^r\sigma^{s+1}\gamma^t\epsilon^{r'}\sigma^{s'}\gamma^{t'} = \\ & \sigma\epsilon^r\sigma^s\gamma^t\epsilon^{r'}\sigma^{s'}\gamma^{t'} = \\ & \sigma\epsilon^{r+r'-ts'}\sigma^{s+s'}\gamma^{t+t'} = \\ & \epsilon^{r+r'-ts'}\sigma^{(s+1)+s'}\gamma^{t+t'} = \\ & \epsilon^{r+r'-t''s'}\sigma^{s''+s'}\gamma^{t''+t'} \end{aligned}$$

Case II)  $t'' = t + 1$ , then  $s'' = s$  and

$$\begin{aligned} & \epsilon^r\sigma^{s''}\gamma^{t''}\epsilon^{r'}\sigma^{s'}\gamma^{t'} = \\ & \epsilon^r\sigma^s\gamma^{t+1}\epsilon^{r'}\sigma^{s'}\gamma^{t'} = \\ & \epsilon^r(\sigma^s\gamma)\gamma^t * \epsilon^{r'}\sigma^{s'}\gamma^{t'} = \\ & (\epsilon^r(\epsilon^s\gamma\sigma^s)\gamma^t)\epsilon^{r'}\sigma^{s'}\gamma^{t'} = \\ & \epsilon^s\gamma(\epsilon^r\sigma^s\gamma^t)\epsilon^{r'}\sigma^{s'}\gamma^{t'} = \\ & \epsilon^s\gamma\epsilon^{r+r'-ts'}\sigma^{s+s'}\gamma^{t+t'} = \\ & \epsilon^s\epsilon^{r+r'-ts'}(\gamma\sigma^{s+s'})\gamma^{t+t'} = \\ & \epsilon^s\epsilon^{r+r'-ts'}(\epsilon^{-(s+s')}\sigma^{s+s'}\gamma)\gamma^{t+t'} = \\ & \epsilon^{-s'}\epsilon^{r+r'-ts'}\sigma^{s+s'}\gamma^{(t+1)+t'} = \\ & \epsilon^{-s'}\epsilon^{r+r'-(t''-1)s'}\sigma^{s''+s'}\gamma^{t''+t'} = \\ & \epsilon^{r+r'-t''s'}\sigma^{s''+s'}\gamma^{t''+t'} \end{aligned}$$

**Definition** Let  $\mu_6$  be the group of 6th roots of unity under multiplication ;  
 $\mu_6 = \{\epsilon^r | r = 1, 2, 3, 4, 5, 6\}$

From the additivity in the multiplication rule in proposition 1.1 we see that  $\phi(\epsilon^r \sigma^s \gamma^t) = (s, t)$  is a group-homomorphism, and it gives rise to the following isomorphism of the quotient  $H6/(\mu_6)$

**Proposition 1.2**  $H6/(\mu_6) \cong \mathbb{Z}_6 \times \mathbb{Z}_6$ . Since  $|m\mu_6| = 6$ , and  $\mathbb{Z}_6 \times \mathbb{Z}_6$  obviously has 36 elements, we get that  $H6$  indeed has 216 elements.

**Proof** Let  $\phi : H6 \rightarrow \mathbb{Z}_6 \times \mathbb{Z}_6$  be given by  $\phi(\epsilon^r \sigma^s \gamma^t) = (s, t)$ . And  $\phi$  is a homomorphism;

$$\phi(\epsilon^r \sigma^s \gamma^t \epsilon^{r'} \sigma^{s'} \gamma^{t'}) = \phi(\epsilon^{r+r'-ts'} \sigma^{s+s'} \gamma^{t+t'}) = (s+s', t+t') = \phi(\epsilon^r \sigma^s \gamma^t) \phi(\epsilon^{r'} \sigma^{s'} \gamma^{t'}).$$

$\ker \phi = \{\epsilon^r \sigma^s \gamma^t | \phi(\epsilon^r \sigma^s \gamma^t) = (0, 0)\}$ , so if  $\epsilon^r * \sigma^s * \gamma^t \in \ker \phi$ , it follows that  $s = t = 0$

$$\Rightarrow \ker \phi = \mu_6$$

## 2 An irreducible representation of H6

A representation of  $H6$  on  $\mathbb{C}^6$ ,  $\rho : H6 \rightarrow GL(\mathbb{C}^6)$ , where  $\rho$  is a group-homomorphism will be defined by

$$\begin{aligned} \rho(\sigma) &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ \rho(\gamma) &= \begin{pmatrix} \epsilon^0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \epsilon^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \epsilon^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \epsilon^5 \end{pmatrix} \\ \rho(\epsilon) &= \epsilon * I \end{aligned}$$

So any element in the Heisenberg-group of type 5 is of the form  $\epsilon^r \sigma^s \gamma^t$  and can be represented as one of the following matrices

$$\begin{aligned}
& \left( \begin{array}{cccccc} 0 & \epsilon^{r+t} & 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon^{r+2t} & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon^{r+3t} & 0 & 0 \\ 0 & 0 & 0 & 0 & \epsilon^{r+4t} & 0 \\ 0 & 0 & 0 & 0 & 0 & \epsilon^{r+5t} \\ \epsilon^r & 0 & 0 & 0 & 0 & 0 \end{array} \right) \\
& \left( \begin{array}{cccccc} 0 & 0 & 0 & \epsilon^{r+3t} & 0 & 0 \\ 0 & 0 & 0 & 0 & \epsilon^{r+4t} & 0 \\ 0 & 0 & 0 & 0 & 0 & \epsilon^{r+5t} \\ \epsilon^r & 0 & 0 & 0 & 0 & 0 \\ 0 & \epsilon^{r+t} & 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon^{r+2t} & 0 & 0 & 0 \end{array} \right) \\
& \left( \begin{array}{cccccc} 0 & 0 & 0 & 0 & \epsilon^{r+4t} & 0 \\ 0 & 0 & 0 & 0 & 0 & \epsilon^{r+5t} \\ \epsilon^r & 0 & 0 & 0 & 0 & 0 \\ 0 & \epsilon^{r+t} & 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon^{r+2t} & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon^{r+3t} & 0 & 0 \end{array} \right) \\
& \left( \begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & \epsilon^{r+5t} \\ 0 & 0 & 0 & 0 & \epsilon^{r+4t} & 0 \\ 0 & 0 & 0 & \epsilon^{r+3t} & 0 & 0 \\ 0 & 0 & \epsilon^{r+2t} & 0 & 0 & 0 \\ 0 & \epsilon^{r+t} & 0 & 0 & 0 & 0 \\ \epsilon^r & 0 & 0 & 0 & 0 & 0 \end{array} \right) \\
& \left( \begin{array}{ccccc} \epsilon^r & 0 & 0 & 0 & 0 \\ 0 & \epsilon^{r+t} & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon^{r+3t} & 0 \\ 0 & 0 & 0 & 0 & \epsilon^{r+4t} \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)
\end{aligned}$$

### 3 The decomposition of the Homogeneous polynomials of degree 1,2,3,4 and 5 into irreducible Heisenberg-representations

The ring of homogenous polynomials of degree  $k$  will be denoted  $S_k$  for  $k = 0, 1, 2, 3, 4, 5$ . So

$$S_k = \{x_0^{i_0} x_1^{i_1} x_2^{i_2} x_3^{i_3} x_4^{i_4} x_5^{i_5} | i_0 + i_1 + i_2 + i_3 + i_4 + i_5 = k\}$$

We will decompose  $S_1, S_2, S_3, S_4$  and  $S_5$  into irreducible representations induced by the representation  $\text{span}\{x0, x1, x2, x3, x4, x5\}$  defined by

$$\begin{aligned}
\sigma(x_i) &= x_{i+1} \quad \text{for } i = 0, 1, 2, 3, 4, 5 \\
\gamma(x_i) &= \epsilon^i x_i
\end{aligned}$$

So the following representations below of  $S_1, S_2, S_3, S_4$  and  $S_5$  below are induced by this by this action of  $\sigma$  and  $\gamma$  on  $\text{span}\{x_0, x_1, x_2, x_3, x_4, x_5\}$  in the following sense

$$\begin{aligned} & \sigma(\text{span}\{f0(x_0, x_1, x_2, x_3, x_4, x_5), f1(x_0, x_1, x_2, x_3, x_4, x_5), \dots, fk(x_0, x_1, x_2, x_3, x_4, x_5)\}) \\ &= \text{span}\{f1(\sigma(x_0), \sigma(x_1), \sigma(x_2), \sigma(x_3), \sigma(x_4), \sigma(x_5)), \dots, fk(\sigma(x_0), \sigma(x_1), \sigma(x_2), \sigma(x_3), \sigma(x_4), \sigma(x_5))\} \\ & \text{and} \\ & \gamma(\text{span}\{f0(x_0, x_1, x_2, x_3, x_4, x_5), f1(x_0, x_1, x_2, x_3, x_4, x_5), \dots, fk(x_0, x_1, x_2, x_3, x_4, x_5)\}) \\ &= \text{span}\{f1(\gamma(x_0), \sigma(x_1), \gamma(x_2), \gamma(x_3), \gamma(x_4), \gamma(x_5)), \dots, fk(\gamma(x_0), \gamma(x_1), \gamma(x_2), \gamma(x_3), \gamma(x_4), \gamma(x_5))\} \end{aligned}$$

Thus they are clearly Heisenberg-invariant. This induces a group action on  $\mathbb{C}[x_0, x_1, x_2, x_3, x_4, x_5]$ , and thus a representation of H6 on  $\mathbb{C}[x_0, x_1, x_2, x_3, x_4, x_5]$ . Below is listed the general forms(isomorphism-classes) that these representations have, and which is denoted to the right of each one of them. We do not need to show that the below forms are Heisenberg-representations, but rather that if two of the irreducible components of S1,S2,S3,S4 or S5 have the same form they are indeed isomorphic as Heisenberg-representations. So each form below is an isomorphism-class of the irreducible components of S1,S2,S3,S4 and S5.

subsectionOne-dimensional irreducible Heisenberg-representations

### 3.0.1 Isomorphism-class $A_i$

Let  $V = \text{span}\{v_1\}$  and

$$\begin{aligned} \sigma(v_1) &= v_1, \\ \gamma(v_1) &= \epsilon^i * v_1 \text{ for } i = 0, 1, 2, 3, 4, 5 \end{aligned}$$

For each  $i = 0, 1, 2, 3, 4, 5$  we get an irreducible Heisenberg-representations; so there are six-irreducible one-dimensional representations.

## 3.1 Two-dimensional irreducible Heisenberg-representations

### 3.1.1 Isomorphism-class $B_1$

**Proposition 3.1** Let  $V = \text{span}\{v_1, v_2\}$  and

$$\begin{aligned} \sigma(v_i) &= v_{(i+1) \pmod 2} \\ \gamma(v_1) &= \epsilon^0 * v_1 \\ \gamma(v_2) &= \epsilon^3 * v_2 \end{aligned}$$

Let  $V = \text{span}\{v_1, v_2\}$  and

$$\begin{aligned} \sigma(v_i) &= v_{(i+1) \pmod 2} \\ \gamma(v_1) &= \epsilon^1 * v_1 \\ \gamma(v_2) &= \epsilon^4 * v_2 \end{aligned}$$

### 3.1.2 Isomorphism-class $B_3$

Let  $V = \text{span}\{v_1, v_2\}$  and

$$\begin{aligned}\sigma(v_i) &= v_{(i+1) \pmod 2} \\ \gamma(v_1) &= \epsilon^2 * v_1 \\ \gamma(v_2) &= \epsilon^5 * v_2\end{aligned}$$

## 3.2 Three-dimensional irreducible Heisenberg-representations

### 3.2.1 Isomorphism-class $C_1$

Let  $V = \text{span}\{v_1, v_2, v_3\}$  and

$$\begin{aligned}\sigma(v_i) &= v_{i+1 \pmod 3} \\ \gamma(v_1) &= \epsilon^0 * v_1 \\ \gamma(v_2) &= \epsilon^2 * v_2 \\ \gamma(v_3) &= \epsilon^4 * v_3\end{aligned}$$

### 3.2.2 Isomorphism-class $C_2$

Let  $V = \text{span}\{v_1, v_2, v_3\}$  and

$$\begin{aligned}\sigma(v_i) &= v_{i+1 \pmod 3} \\ \gamma(v_1) &= \epsilon^1 * v_1 \\ \gamma(v_2) &= \epsilon^3 * v_2 \\ \gamma(v_3) &= \epsilon^5 * v_2\end{aligned}$$

### 3.2.3 Isomorphism-class $C_3$

Let  $V = \text{span}\{v_1, v_2, v_3\}$  and

$$\begin{aligned}\sigma(v_i) &= v_{i-1 \pmod 3} \\ \gamma(v_1) &= \epsilon^0 * v_1 \\ \gamma(v_2) &= \epsilon^2 * v_2 \\ \gamma(v_3) &= \epsilon^4 * v_2\end{aligned}$$

### 3.2.4 Isomorphism-class $C_4$

Let  $V = \text{span}\{v_1, v_2, v_3\}$  and

$$\begin{aligned}\sigma(v_i) &= v_{i-1 \pmod 3} \\ \gamma(v_1) &= \epsilon^1 * v_1 \\ \gamma(v_2) &= \epsilon^3 * v_2 \\ \gamma(v_3) &= \epsilon^5 * v_2\end{aligned}$$

### 3.2.5 Isomorphism-class $C_5$

Let  $V = \text{span}\{v_1, v_2, v_3\}$  and

$$\begin{aligned}\sigma(v_i) &= v_{i+2} \pmod{3} \\ \gamma(v_1) &= \epsilon^0 * v_1 \\ \gamma(v_2) &= \epsilon^2 * v_2 \\ \gamma(v_3) &= \epsilon^4 * v_2\end{aligned}$$

### 3.2.6 Isomorphism-class $C_6$

Let  $V = \text{span}\{v_1, v_2, v_3\}$  and

$$\begin{aligned}\sigma(v_i) &= v_{i+2} \pmod{3} \\ \gamma(v_1) &= \epsilon^1 * v_1 \\ \gamma(v_2) &= \epsilon^3 * v_2\end{aligned}$$

### 3.2.7 Isomorphism-class $C_7$

Let  $V = \text{span}\{v_1, v_2, v_3\}$  and

$$\begin{aligned}\sigma(v_i) &= v_{i-2} \pmod{3} \\ \gamma(v_1) &= \epsilon^0 * v_1 \\ \gamma(v_2) &= \epsilon^2 * v_2 \\ \gamma(v_3) &= \epsilon^4 * v_2\end{aligned}$$

### 3.2.8 Isomorphism-class $C_8$

Let  $V = \text{span}\{v_1, v_2, v_3\}$  and

$$\begin{aligned}\sigma(v_i) &= v_{i-2} \pmod{3} \\ \gamma(v_1) &= \epsilon^1 * v_1 \\ \gamma(v_2) &= \epsilon^3 * v_2 \\ \gamma(v_3) &= \epsilon^5 * v_3\end{aligned}$$

## 3.3 Six-dimensional irreducible Heisenberg-representations

### 3.3.1 Isomorphism-class $D_1$

Let  $V = \text{span}\{v_1, v_2, v_3, v_4, v_5, v_6\}$  and

$$\begin{aligned}\sigma(v_i) &= v_{i+1} \pmod{6} \\ \gamma(v_1) &= \epsilon^5 * v_1 \\ \gamma(v_2) &= \epsilon^4 * v_2 \\ \gamma(v_3) &= \epsilon^3 * v_3 \\ \gamma(v_4) &= \epsilon^2 * v_4 \\ \gamma(v_5) &= \epsilon^1 * v_5 \\ \gamma(v_6) &= \epsilon^0 * v_6\end{aligned}$$

### 3.3.2 Isomorphism-class $D_2$

Let  $V = \text{span}\{v_1, v_2, v_3, v_4, v_5, v_6\}$  and Now we will see that the direct sum of irreducible components belonging to the same isomorphism-class splits as irreducible-representations of the normalizer of the Heisenberg-group in  $GL(\mathbb{C}^6), N[H6]$ . For  $S2$  we will see that

$$\begin{aligned} & \text{span}\{x_0^2 + x_3^2, x_1^2 + x_4^2, x_2^2 + x_5^2\} \oplus \text{span}\{x_0^2 - x_3^2, x_1^2 - x_4^2, x_2^2 - x_5^2\} \oplus \\ & \text{span}\{x_0x_4 + x_3x_1, x_1x_5 + x_4x_2, x_2x_0 + x_5x_3\} \oplus \text{span}\{x_0 * x_4 - x_3 * x_1, x_1 * x_5 - x_4x_2, x_2x_0 - x_5x_3\} \end{aligned}$$

splits in two irreducible representations of  $N[H6]$ ,namely

$$\begin{aligned} & \text{span}\{x_0^2 + x_3^2, x_1^2 + x_4^2, x_2^2 + x_5^2, x_1 * x_5 + x_4 * x_2, x_2 * x_0 + x_5 * x_3, x_0 * x_4 + x_3 * x_1\} \\ & \text{and} \end{aligned}$$

$$\text{span}\{x_0^2 - x_3^2, x_1^2 - x_4^2, x_2^2 - x_5^2, x_1 * x_5 - x_4 * x_2, x_2 * x_0 - x_5 * x_3, x_0 * x_4 + x_3 * x_1\}.$$

$$\begin{aligned} \sigma(v_i) &= v_{i-1} \pmod{6} \\ \gamma(v_1) &= \epsilon^1 * v_1 \\ \gamma(v_2) &= \epsilon^2 * v_2 \\ \gamma(v_3) &= \epsilon^3 * v_3 \quad \gamma(v_4) = \epsilon^4 * v_4 \\ \gamma(v_5) &= \epsilon^5 * v_5 \\ \gamma(v_6) &= \epsilon^6 * v_6 \end{aligned}$$

### 3.3.3 Isomorphism-class $D_3$

Let  $V = \text{span}\{v_1, v_2, v_3, v_4, v_5, v_6\}$  and

$$\sigma(v_i) = v_{i+2} \pmod{6} \quad \gamma(v_1) = \epsilon^1 * v_1 \quad \gamma(v_2) = \epsilon^2 * v_2 \quad \gamma(v_3) = \epsilon^3 * v_3 \quad \gamma(v_4) = \epsilon^4 * v_4 \quad \gamma(v_5) = \epsilon^5 * v_5 \quad \gamma(v_6) = \epsilon^6 * v_6$$

### 3.3.4 Isomorphism-class $D_4$

$$\begin{aligned} \sigma(v_i) &= v_{i-2} \pmod{6} \\ \gamma(v_1) &= \epsilon^1 * v_1 \\ \gamma(v_2) &= \epsilon^2 * v_2 \\ \gamma(v_3) &= \epsilon^3 * v_3 \\ \gamma(v_4) &= \epsilon^4 * v_4 \\ \gamma(v_5) &= \epsilon^5 * v_5 \\ \gamma(v_6) &= \epsilon^6 * v_6 \end{aligned}$$

Now to prove that if two of these irreducible components belong to the same Class above ( $A_1, A_2, \dots, B, B_2, \dots, C_1, \dots, D3, D4$ ), they are isomorphic as Heisenberg-representations.This is easy;a group action on a finitie-dimensional vector-space is uniquely up to ismormorphism defined by the action of the generators of G on the basis of V and thus the corresponding representation of G is also uniquely defined up to isomorphism.

We do not need to show that the below forms are Heisenberg-representations,but rather that if two of the irreducible components of S1,S2,S3,S4 or S5 have the same form they are indeed isomorphic as Heisenberg-representations

### 3.4 S1

$$S1 = \\ span\{x_0, x_1, x_2, x_3, x_4, x_5\}$$

### 3.5 S2

S2 has 21 elements and has a decomposition into the following irreducible Heisenberg-representations, that are grouped into isomorphism-classes.

$$S2 = \\ \oplus span\{x_0^2 + x_3^2, x_1^2 + x_4^2, x_2^2 + x_5^2\} (C_1) \\ \oplus span\{x_0^2 - x_3^2, x_1^2 - x_4^2, x_2^2 - x_5^2\} (C_1) \\ \oplus span\{x_0 * x_4 + x_3 * x_1, x_1 * x_5 + x_4 * x_2, x_2 * x_0 + x_5 * x_3\} (C_1) \\ \oplus span\{x_0 * x_4 - x_3 * x_1, x_1 * x_5 - x_4 * x_2, x_2 * x_0 - x_5 * x_3\} (C_1) \\ \oplus span\{x_0 * x_1 + x_3 * x_4, x_1 * x_2 + x_4 * x_5, x_2 * x_3 + x_5 * x_0\} (C_2) \\ \oplus span\{x_0 * x_3, x_1 * x_4, x_2 * x_5\} (C_2) \\ \oplus span\{x_0 * x_1 - x_3 * x_4, x_1 * x_2 - x_4 * x_5, x_2 * x_3 - x_5 * x_0\} (C_2)$$

,

### 3.6 S3

$$\begin{aligned}
& \oplus \text{span}\{x_0^3 + x_2^3 + x_4^3, x_1^3 + x_3^3 + x_5^3\} (B_1) \\
& \oplus \text{span}\{x_0^2 * x_3 + x_2^2 * x_5 + x_4^2 * x_1, x_1^2 * x_4 + x_3^2 * x_0 + x_5^2 * x_2\}, (B_1) \\
& \oplus \text{span}\{x_0 * x_1 * x_2 + x_2 * x_3 * x_4 + x_4 * x_5 * x_0, x_1 * x_2 * x_3 + x_3 * x_4 * x_5 + x_5 * x_0 * x_1\} (B_1) \\
& \oplus \text{span}\{x_0 * x_2 * x_4, x_1 * x_3 * x_5\} (B_1) \\
& \oplus \text{span}\{x_0^3 + \epsilon^4 * x_2^3 + \epsilon^2 * x_4^3, x_1^3 + \epsilon^4 * x_3^3 + \epsilon^2 * x_5^3\} (B_1) \\
& \oplus \text{span}\{x_0^2 * x_3 + \epsilon^4 * x_2^2 * x_5 + \epsilon^2 * x_4^2 * x_1, x_1^2 * x_4 + \epsilon^4 * x_3^2 * x_0 + \epsilon^2 * x_5^2 * x_2\} (B_1) \\
& \oplus \text{span}\{x_0 * x_1 * x_2 + \epsilon^4 * x_2 * x_3 * x_4 + \epsilon^2 * x_4 * x_5 * x_0, x_1 * x_2 * x_3 + \epsilon^4 * x_3 * x_4 * x_5 + \epsilon^2 * x_5 * x_0 * x_1\} \\
& \oplus \text{span}\{x_0 * x_2 * x_4, x_1 * x_3 * x_5\} (B_1) \\
& \oplus \text{span}\{x_0^3 + \epsilon^2 * x_2^3 + \epsilon^4 * x_4^3, x_1^3 + \epsilon^2 * x_3^3 + \epsilon^4 * x_5^3\} (B_1) \\
& , \oplus \text{span}\{x_0^2 * x_3 + \epsilon^2 * x_2^2 * x_5 + \epsilon^4 * x_4^2 * x_1, x_1^2 * x_4 + \epsilon^2 * x_3^2 * x_0 + \epsilon^4 * x_5^2 * x_2\} (B_1) \\
& \oplus \text{span}\{x_0 * x_1 * x_2 + \epsilon^2 * x_2 * x_3 * x_4 + \epsilon^4 * x_4 * x_5 * x_0, x_1 * x_2 * x_3 + \epsilon^2 * x_3 * x_4 * x_5 + \epsilon^4 * x_5 * x_0 * x_1\} \\
& \oplus \text{span}\{x_0 * x_2 * x_4, x_1 * x_3 * x_5\} (B_1) (B_1) \\
& \oplus \text{span}\{x_0^2 * x_1 + x_2^2 * x_3 + x_4^2 * x_5, x_1^2 * x_2 + x_3^2 * x_4 + x_5^2 * x_0\} (B_2) \\
& \oplus \text{span}\{x_0^2 * x_4 + x_2^2 * x_0 + x_4^2 * x_2, x_1^2 * x_5 + x_3^2 * x_1 + x_5^2 * x_3\} (B_2) \\
& \oplus \text{span}\{x_0 * x_1 * x_3 + x_2 * x_3 * x_5 + x_4 * x_5 * x_1, x_1 * x_2 * x_4 + x_3 * x_4 * x_0 + x_5 * x_0 * x_2\} (B_2) \\
& \oplus \text{span}\{x_0^2 * x_1 + \epsilon^4 * x_2^2 * x_3 + \epsilon^2 * x_4^2 * x_5, x_1^2 * x_2 + \epsilon^4 * x_3^2 * x_4 + \epsilon^2 * x_5^2 * x_0\} (B_2) \\
& \oplus \text{span}\{x_0^2 * x_4 + \epsilon^4 * x_2^2 * x_0 + \epsilon^2 * x_4^2 * x_2, x_1^2 * x_5 + \epsilon^4 * x_3^2 * x_1 + \epsilon^2 * x_5^2 * x_3\} (B_2) \\
& \oplus \text{span}\{x_0 * x_1 * x_3 + \epsilon^4 * x_2 * x_3 * x_5 + \epsilon^2 * x_4 * x_5 * x_1, x_1 * x_2 * x_4 + \epsilon^4 * x_3 * x_4 * x_0 + \epsilon^2 * x_5 * x_0 * x_2\} \\
& \oplus \text{span}\{x_0^2 * x_1 + \epsilon^2 * x_2^2 * x_3 + \epsilon^4 * x_4^2 * x_5, x_1^2 * x_2 + \epsilon^2 * x_3^2 * x_4 + \epsilon^4 * x_5^2 * x_0\} (B_2) \\
& \oplus \text{span}\{x_0^2 * x_4 + \epsilon^2 * x_2^2 * x_0 + \epsilon^4 * x_4^2 * x_2, x_1^2 * x_5 + \epsilon^2 * x_3^2 * x_1 + \epsilon^4 * x_5^2 * x_3\} (B_2) \\
& \oplus \text{span}\{x_0 * x_1 * x_3 + \epsilon^2 * x_2 * x_3 * x_5 + \epsilon^4 * x_4 * x_5 * x_1, x_1 * x_2 * x_4 + \epsilon^2 * x_3 * x_4 * x_0 + \epsilon^4 * x_5 * x_0 * x_2\} \\
& \oplus \text{span}\{x_0 * x_1 * x_4 + x_2 * x_3 * x_0 + x_4 * x_5 * x_2, x_1 * x_2 * x_5 + x_3 * x_4 * x_1 + x_5 * x_0 * x_3\} (B_3) \\
& \oplus \text{span}\{x_0^2 * x_2 + x_2^2 * x_4 + x_4^2 * x_0, x_1^2 * x_3 + x_3^2 * x_5 + x_5^2 * x_1\} (B_3) \\
& \oplus \text{span}\{x_1^2 * x_0 + x_2^2 * x_2 + x_5^2 * x_4, x_2^2 * x_1 + x_4^2 * x_3 + x_0^2 * x_5\} (B_3) \\
& \oplus \text{span}\{x_0 * x_1 * x_4 + \epsilon^4 * x_2 * x_3 * x_0 + \epsilon^2 * x_4 * x_5 * x_2, x_1 * x_2 * x_5 + \epsilon^4 * x_3 * x_4 * x_1 + \epsilon^2 * x_5 * x_0 * x_3\} \\
& \oplus \text{span}\{x_0^2 * x_2 + \epsilon^4 * x_2^2 * x_4 + \epsilon^2 * x_4^2 * x_0, x_1^2 * x_3 + \epsilon^4 * x_3^2 * x_5 + \epsilon^2 * x_5^2 * x_1\} (B_3) \\
& \oplus \text{span}\{x_1^2 * x_0 + \epsilon^4 * x_3^2 * x_2 + \epsilon^2 * x_5^2 * x_4, x_2^2 * x_1 + \epsilon^4 * x_4^2 * x_3 + \epsilon^2 * x_0^2 * x_5\} (B_3) \\
& \oplus \text{span}\{x_0 * x_1 * x_4 + \epsilon^2 * x_2 * x_3 * x_0 + \epsilon^4 * x_4 * x_5 * x_2, x_1 * x_2 * x_5 + \epsilon^2 * x_3 * x_4 * x_1 + \epsilon^4 * x_5 * x_0 * x_3\} \\
& \oplus \text{span}\{x_0^2 * x_2 + \epsilon^2 * x_2^2 * x_4 + \epsilon^4 * x_4^2 * x_0, x_1^2 * x_3 + \epsilon^2 * x_3^2 * x_5 + \epsilon^4 * x_5^2 * x_1\} (B_3) \\
& \oplus \text{span}\{x_1^2 * x_0 + \epsilon^2 * x_3^2 * x_2 + \epsilon^4 * x_5^2 * x_4, x_2^2 * x_1 + \epsilon^2 * x_4^2 * x_3 + \epsilon^4 * x_0^2 * x_5\} (B_3)
\end{aligned}$$

$S4 =$

$$\begin{aligned}
& \text{span}\{x_0^4 + x_3^4, x_1^4 + x_4^4, x_2^4 + x_5^4\}, \\
& \oplus \text{span}\{x_0x_1x_2x_3 + x_3x_4x_5x_0, x_1x_2x_3x_4 + x_4x_5x_0x_1, x_2x_3x_4x_5 + x_5x_0x_1x_2\} (C_1) \\
& \oplus \text{span}\{x_0x_1x_2x_5 + x_3x_4x_5x_2, x_1x_2x_3x_0 + x_4x_5x_0x_3, x_2x_3x_4x_1 + x_5x_0x_1x_4\} (C_1) \\
& \oplus \text{span}\{x_0^2x_2x_4 + x_3^2x_5x_1, x_1^2x_3x_5 + x_4^2x_0x_2, x_2^2x_4x_0 + x_5^2x_1x_3\} (C_1) \\
& \oplus \text{span}\{x_0^2x_1x_3 + x_3^2x_4x_0, x_1^2x_2x_4 + x_4^2x_5x_1, x_2^2x_3x_5 + x_5^2x_0x_2\} (C_1) \\
& \oplus \text{span}\{x_0^2x_2^2 + x_3^2x_5^2, x_1^2x_3^2 + x_4^2x_0^2, x_2^2x_4^2 + x_5^2x_1^2\} (C_1) \\
& \oplus \text{span}\{x_0^2x_1^2 + x_3^2x_4^2, x_1^2x_2^2 + x_4^2x_5^2, x_2^2x_3^2 + x_5^2x_0^2\} (C_1) \\
& \oplus \text{span}\{x_0^3x_2 + x_3^3x_5, x_1^3x_3 + x_4^3x_0, x_2^3x_4 + x_5^3x_1\} (C_1) \\
& \oplus \text{span}\{x_0^3x_4 + x_3^3x_1, x_1^3x_5 + x_4^3x_2, x_2^3x_0 + x_5^3x_3\} (C_1) \\
& \oplus \text{span}\{x_0^2x_3x_5 + x_3^2x_0x_2, x_1^2x_4x_0 + x_4^2x_1x_3, x_2^2x_5x_1 + x_5^2x_2x_4\} (C_1) \\
& \oplus \text{span}\{x_0^2x_1x_5 + x_3^2x_4x_2, x_1^2x_2x_0 + x_4^2x_5x_3, x_2^2x_3x_1 + x_5^2x_0x_4\} (C_1) \\
& \oplus \text{span}\{x_0^2x_3^2, x_1^2x_4^2, x_2^2x_5^2\} (C_1) \\
& \oplus \text{span}\{x_0^4 + \epsilon^3x_3^4, x_1^4 + \epsilon^3x_4^4, x_2^4 + \epsilon^3x_5^4\} (C_1) \\
& \oplus \text{span}\{x_0x_1x_2x_3 + x_3x_4x_5x_0, x_1x_2x_3x_4 + x_4x_5x_0x_1, x_2x_3x_4x_5 + x_5x_0x_1x_2\} (C_1) \\
& \oplus \text{span}\{x_0x_1x_2x_5 + \epsilon^3x_3x_4x_5x_2, x_1x_2x_3x_0 + \epsilon^3x_4x_5x_0x_3, x_2x_3x_4x_1 + \epsilon^3x_5x_0x_1x_4\} (C_1) \\
& \oplus \text{span}\{x_0^2x_2x_4 + \epsilon^3x_3x_5x_1, x_1^2x_3x_5 + \epsilon^3x_4x_0x_2, x_2^2x_4x_0 + \epsilon^3x_5x_1x_3\} (C_1) \\
& \oplus \text{span}\{x_0^2x_1x_3 + \epsilon^3x_3x_4x_0, x_1^2x_2x_4 + \epsilon^3x_4x_5x_1, x_2^2x_3x_5 + \epsilon^3x_5x_0x_2\} (C_1) \\
& \oplus \text{span}\{x_0^2x_2^2 + \epsilon^3x_3^2x_5^2, x_1^2x_3^2 + \epsilon^3x_4^2x_0^2, x_2^2x_4^2 + \epsilon^3x_5^2x_1^2\} (C_1) \\
& \oplus \text{span}\{x_0^2x_1^2 + \epsilon^3x_3^2x_4^2, x_1^2x_2^2 + \epsilon^3x_4^2x_5^2, x_2^2x_3^2 + \epsilon^3x_5^2x_0^2\} (C_1) \\
& \oplus \text{span}\{x_0^3x_2 + \epsilon^3x_3^3x_5, x_1^3x_3 + \epsilon^3x_4^3x_0, x_2^3x_4 + \epsilon^3x_5^3x_1\} (C_1) \\
& \oplus \text{span}\{x_0^3x_4 + \epsilon^3x_3^3x_1, x_1^3x_5 + \epsilon^3x_4^3x_2, x_2^3x_0 + \epsilon^3x_5^3x_3\} (C_1) \\
& \oplus \text{span}\{x_0^2x_3x_5 + \epsilon^3x_3^2x_0x_2, x_1^2x_4x_0 + \epsilon^3x_4^2x_1x_3, x_2^2x_5x_1 + \epsilon^3x_5^2x_2x_4\} (C_1) \\
& \oplus \text{span}\{x_0^2x_1x_5 + \epsilon^3x_3^2x_4x_2, x_1^2x_2x_0 + \epsilon^3x_4^2x_5x_3, x_2^2x_3x_1 + x_5^2x_0x_4\} (C_1) \\
& \oplus \\
& \oplus \text{span}\{x_0x_1x_2x_4 + x_3x_4x_5x_1, x_1x_2x_3x_5 + x_4x_5x_0x_2, x_2x_3x_4x_0 + x_5x_0x_1x_3\} (C_2) \\
& \oplus \text{span}\{x_0^2x_1x_2 + x_3^2x_4x_5, x_1^2x_2x_3 + x_4^2x_5x_0, x_3^2x_4x_0 + x_0^2x_1x_3\} (C_2) \\
& \oplus \text{span}\{x_0^3x_1 + x_3^3x_4, x_1^3x_2 + x_4^3x_5, x_2^3x_3 + x_5^3x_0\} (C_2) \\
& \oplus \text{span}\{x_0^3x_3 + x_3^3x_0, x_1^3x_4 + x_4^3x_1, x_2^3x_5 + x_5^3x_2\} (C_2) \\
& \oplus \text{span}\{x_0^3x_5 + x_3^3x_2, x_1^3x_0 + x_4^3x_3, x_2^3x_1 + x_5^3x_4\} (C_2) \\
& \oplus \text{span}\{x_0^2x_3x_4 + x_3^2x_0x_1, x_1^2x_4x_5 + x_4^2x_1x_2, x_2^2x_5x_0 + x_5^2x_2x_3\} (C_2) \\
& \oplus \text{span}\{x_0^2x_4x_5 + x_3^2x_1x_2, x_1^2x_5x_0 + x_4^2x_2x_3, x_2^2x_0x_1 + x_5^2x_3x_4\} (C_2) \\
& \oplus \text{span}\{x_0^2x_2x_3 + x_3^2x_5x_0, x_1^2x_3x_4 + x_4^2x_0x_1, x_2^2x_4x_5 + x_5^2x_1x_2\} (C_2) \\
& \oplus \text{span}\{x_0^2x_1x_4 + x_3^2x_4x_1, x_1^2x_2x_5 + x_4^2x_5x_2, x_2^2x_3x_0 + x_5^2x_0x_3\} (C_2) \\
& \oplus \text{span}\{x_0x_1x_2x_4 + \epsilon^3x_3x_4x_5x_1\} \\
& \oplus \text{span}\{x_0^2x_1x_2 + \epsilon^3x_3^2x_4x_5, x_1^2x_2x_3 + \epsilon^3x_4^2x_5x_0, x_3^2x_4x_0 + \epsilon^3x_0^2x_1x_3\} (C_2) \\
& \oplus \text{span}\{x_0^3x_1 + \epsilon^3x_3^3x_4, x_1^3x_2 + \epsilon^3x_4^3x_5, x_2^3x_3 + \epsilon^3x_5^3x_0\} (C_2) \\
& \oplus \text{span}\{x_0^3x_3 + \epsilon^3x_3^3x_0, x_1^3x_4 + \epsilon^3x_4^3x_1, x_2^3x_5 + \epsilon^3x_5^3x_2\} (C_2) \\
& \oplus \text{span}\{x_0^3x_5 + \epsilon^3x_3^3x_2, x_1^3x_0 + \epsilon^3x_4^3x_3, x_2^3x_1 + \epsilon^3x_5^3x_4\} (C_2) \\
& \oplus \text{span}\{x_0^2x_3x_4 + \epsilon^3x_3^2x_0x_1, x_1^2x_4x_5 + \epsilon^3x_4^2x_1x_2, x_2^2x_5x_0 + \epsilon^3x_5^2x_2x_3\} (C_2) \\
& \oplus \text{span}\{x_0^2x_4x_5 + \epsilon^3x_3^2x_1x_2, x_1^2x_5x_0 + \epsilon^3x_4^2x_2x_3, x_2^2x_0x_1 + \epsilon^3x_5^2x_3x_4\} (C_2) \\
& \oplus \text{span}\{x_0^2x_2x_3 + \epsilon^3x_3^2x_5x_0, x_1^2x_3x_4 + \epsilon^3x_4^2x_0x_1, x_2^2x_4x_5 + \epsilon^3x_5^2x_1x_2\} (C_2) \\
& \oplus \text{span}\{x_0^2x_1x_4 + \epsilon^3x_3^2x_4x_1, x_1^2x_2x_5 + \epsilon^3x_4^2x_5x_2, x_2^2x_3x_0 + \epsilon^3x_5^2x_1x_2\} (C_2) \\
& \oplus \text{span}\{x_0^2x_1x_4 + \epsilon^3x_3^2x_4x_1, x_1^2x_2x_5 + \epsilon^3x_4^2x_5x_2, x_2^2x_3x_0 + \epsilon^3x_5^2x_0x_3\} (C_2)
\end{aligned}$$

### 3.7 S5

$S5 =$

$$\begin{aligned}
& \text{span}\{x_0^5, x_1^5, x_2^5, x_3^5, x_4^5, x_5^5\} \ (D_1) \\
& \oplus \text{span}\{x_0x_1x_2x_3x_4, x_1x_2x_3x_4x_5, x_2x_3x_4x_5x_0, x_3x_4x_5x_0x_1, x_4x_5x_0x_1x_2, x_5x_0x_1x_2x_3\} \ (D_1) \\
& \oplus \text{span}\{x_0x_1x_2x_3x_5, x_1x_2x_3x_4x_0, x_2x_3x_4x_5x_1, x_3x_4x_5x_0x_2, x_4x_5x_0x_1x_3, x_5x_0x_1x_2x_4\} \ (D_1) \\
& \oplus \text{span}\{x_0^2x_1x_2x_3, x_1^2x_2x_3x_4, x_2^2x_3x_4x_5, x_3^2x_4x_5x_0, x_4^2x_5x_0x_1, x_5^2x_0x_1x_2\} \ (D_1) \\
& \oplus \text{span}\{x_0^2x_1x_2x_4, x_1^2x_2x_3x_5, x_2^2x_3x_4x_0, x_3^2x_4x_5x_1, x_4^2x_5x_0x_2, x_5^2x_0x_1x_3\} \ (D_1) \\
& \oplus \text{span}\{x_0^2x_1x_2x_5, x_1^2x_2x_3x_0, x_2^2x_3x_4x_1, x_3^2x_4x_5x_2, x_4^2x_5x_0x_3, x_5^2x_0x_1x_4\} \ (D_1) \\
& \oplus \text{span}\{x_0^2x_2x_3x_4, x_1^2x_3x_4x_5, x_2^2x_4x_5x_0, x_3^2x_5x_0x_1, x_4^2x_0x_1x_2, x_5^2x_1x_2x_3\} \ (D_1) \\
& \oplus \text{span}\{x_0^2x_2x_4x_5, x_1^2x_3x_5x_0, x_2^2x_4x_0x_1, x_3^2x_5x_1x_2, x_4^2x_0x_2x_3, x_5^2x_1x_3x_4\} \ (D_1) \\
& \oplus \text{span}\{x_0^2x_1x_3x_4, x_1^2x_2x_4x_5, x_2^2x_3x_5x_0, x_3^2x_4x_0x_1, x_4^2x_5x_1x_2, x_5^2x_0x_2x_3\} \ (D_1) \\
& \oplus \text{span}\{x_0^2x_1x_3x_5, x_1^2x_2x_4x_0, x_2^2x_3x_5x_1, x_3^2x_4x_0x_2, x_4^2x_5x_1x_3, x_5^2x_0x_2x_4\} \ (D_1) \\
& \oplus \text{span}\{x_0^2x_3x_4x_5, x_1^2x_4x_5x_0, x_2^2x_5x_0x_1, x_3^2x_0x_1x_2, x_4^2x_1x_2x_3, x_5^2x_2x_3x_4\} \ (D_1) \\
& \oplus \text{span}\{x_0^3x_1x_2, x_1^3x_2x_3, x_2^3x_3x_4, x_3^3x_4x_5, x_4^3x_5x_0, x_5^3x_0x_1\} \ (D_1) \\
& \oplus \text{span}\{x_0^3x_1x_3, x_1^3x_2x_4, x_2^3x_3x_5, x_3^3x_4x_0, x_4^3x_5x_1, x_5^3x_0x_2\} \ (D_1) \\
& \oplus \text{span}\{x_0^3x_1x_4, x_1^3x_2x_5, x_2^3x_3x_0, x_3^3x_4x_1, x_4^3x_5x_2, x_5^3x_0x_3\} \ (D_1) \\
& \oplus \text{span}\{x_0^3x_1x_5, x_1^3x_2x_0, x_2^3x_3x_1, x_3^3x_4x_2, x_4^3x_5x_3, x_5^3x_0x_4\} \ (D_1) \\
& \oplus \text{span}\{x_0^3x_2x_3, x_1^3x_3x_4, x_2^3x_4x_5, x_3^3x_5x_0, x_4^3x_0x_1, x_5^3x_1x_2\} \ (D_1) \\
& \oplus \text{span}\{x_0^3x_2x_4, x_1^3x_3x_5, x_2^3x_4x_0, x_3^3x_5x_1, x_4^3x_0x_2, x_5^3x_1x_3\} \ (D_1) \\
& \oplus \text{span}\{x_0^3x_2x_5, x_1^3x_3x_0, x_2^3x_4x_1, x_3^3x_5x_2, x_4^3x_0x_3, x_5^3x_1x_4\} \ (D_1) \\
& \oplus \text{span}\{x_0^3x_3x_4, x_1^3x_4x_5, x_2^3x_5x_0, x_3^3x_0x_1, x_4^3x_1x_2, x_5^3x_2x_3\} \ (D_1) \\
& \oplus \text{span}\{x_0^3x_3x_5, x_1^3x_4x_0, x_2^3x_5x_1, x_3^3x_0x_2, x_4^3x_1x_3, x_5^3x_2x_4\} \ (D_1) \\
& \oplus \text{span}\{x_0^2x_1^2x_2, x_1^2x_2^2x_3, x_2^2x_3^2x_4, x_3^2x_4^2x_5, x_4^2x_5^2x_0, x_5^2x_0^2x_1\} \ (D_1) \\
& \oplus \text{span}\{x_0^2x_1^2x_3, x_1^2x_2^2x_4, x_2^2x_3^2x_5, x_3^2x_4^2x_0, x_4^2x_5^2x_1, x_5^2x_0^2x_2\} \ (D_1) \\
& \oplus \text{span}\{x_0^2x_1^2x_4, x_1^2x_2^2x_5, x_2^2x_3^2x_0, x_3^2x_4^2x_1, x_4^2x_5^2x_2, x_5^2x_0^2x_3\} \ (D_1) \\
& \oplus \text{span}\{x_0^2x_1^2x_5, x_1^2x_2^2x_0, x_2^2x_3^2x_1, x_3^2x_4^2x_2, x_4^2x_5^2x_3, x_5^2x_0^2x_4\} \ (D_1) \\
& \oplus \text{span}\{x_0^2x_2^2x_3, x_1^2x_3^2x_4, x_2^2x_4^2x_5, x_3^2x_5^2x_0, x_4^2x_0^2x_1, x_5^2x_1^2x_2\} \ (D_1) \\
& \oplus \text{span}\{x_0^2x_2^2x_4, x_1^2x_3^2x_5, x_2^2x_4^2x_0, x_3^2x_5^2x_1, x_4^2x_0^2x_2, x_5^2x_1^2x_3\} \ (D_1) \\
& \oplus \text{span}\{x_0^2x_2^2x_5, x_1^2x_3^2x_0, x_2^2x_4^2x_1, x_3^2x_5^2x_2, x_4^2x_0^2x_3, x_5^2x_1^2x_4\} \ (D_1) \\
& \oplus \text{span}\{x_0^2x_3^2x_4, x_1^2x_4^2x_5, x_2^2x_5^2x_0\} \\
& \oplus \text{span}\{x_0^2x_3^2x_5, x_1^2x_4^2x_0, x_2^2x_5^2x_1, x_3^2x_0^2x_2, x_4^2x_1^2x_3, x_5^2x_2^2x_4\} \ (D_1) \\
& \oplus \text{span}\{x_0^3x_4x_5, x_1^3x_5x_0, x_2^3x_0x_1, x_3^3x_1x_2, x_4^3x_2x_3, x_5^3x_3x_4\} \ (D_1) \\
& \oplus \text{span}\{x_0^4x_2, x_1^4x_3, x_2^4x_4, x_3^4x_5, x_4^4x_0, x_5^4x_1\} \ (D_1) \\
& \oplus \text{span}\{x_0^4x_3, x_1^4x_4, x_2^4x_5, x_3^4x_0, x_4^4x_1, x_5^4x_2\} \ (D_1) \\
& \oplus \text{span}\{x_0^4x_4, x_1^4x_5, x_2^4x_0, x_3^4x_1, x_4^4x_2, x_5^4x_3\} \ (D_1) \\
& \oplus \text{span}\{x_0^4x_5, x_1^4x_0, x_2^4x_1, x_3^4x_2, x_4^4x_3, x_5^4x_4\} \ (D_1) \\
& \oplus \text{span}\{x_0^2x_1^3, x_1^2x_2^3, x_2^2x_3^3, x_3^2x_4^3, x_4^2x_5^3, x_5^2x_0^3\} \ (D_1) \\
& \oplus \text{span}\{x_0^2x_2^3, x_1^2x_3^3, x_2^2x_4^3, x_3^2x_5^3, x_4^2x_0^3, x_5^2x_1^3\} \ (D_1) \\
& \oplus \text{span}\{x_0^2x_3^3, x_1^2x_4^3, x_2^2x_5^3, x_3^2x_0^3, x_4^2x_1^3, x_5^2x_2^3\} \ (D_1) \\
& \oplus \text{span}\{x_0^2x_4^3, x_1^2x_5^3, x_2^2x_0^3, x_3^2x_1^3, x_4^2x_2^3, x_5^2x_3^3\} \ (D_1) \\
& \oplus \text{span}\{x_0^2x_5^3, x_1^2x_0^3, x_2^2x_1^3, x_3^2x_2^3, x_4^2x_3^3, x_5^2x_4^3\} \ (D_1) \\
& \oplus \text{span}\{x_0^2x_4^2x_5, x_1^2x_5^2x_0, x_2^2x_0^2x_1, x_3^2x_1^2x_2, x_4^2x_2^2x_3, x_5^2x_3^2x_4\} \ (D_1)
\end{aligned}$$

The irreducible representations above of  $S1, S2, S3, S4$  and  $S5$  are clearly Heisenberg-invariant and equivalence-classes of the group action  $H6 \times \mathbb{C}[x_0, x_1, x_2, x_3, x_4, x_5] \rightarrow \mathbb{C}[x_0, x_1, x_2, x_3, x_4, x_5]$ , since they are irreducible. That they are Heisenberg-representations because all the actions  $H6 \times V \rightarrow V$  (where  $V$  is one of the irreducible components) are inherited from the group-action  $H6 \times \mathbb{C}[x_0, x_1, x_2, x_3, x_4, x_5]$ , and thus they are group-actions.

Now we will see that the direct sum of irreducible components belonging to the same isomorphism-class splits as irreducible-representations of the normalizer of the Heisenberg-group in  $GL(\mathbb{C}^6), N[H6]$ . For  $S2$  we will see that

$$\text{span}\{x_0^2 + x_3^2, x_1^2 + x_4^2, x_2^2 + x_5^2\} \oplus \text{span}\{x_0^2 - x_3^2, x_1^2 - x_4^2, x_2^2 - x_5^2\} \oplus \\ \text{span}\{x_0x_4 + x_3x_1, x_1x_5 + x_4x_2, x_2x_0 + x_5x_3\} \oplus \text{span}\{x_0 * x_4 - x_3 * x_1, x_1 * x_5 - x_4x_2, x_2x_0 - x_5x_3\}$$

splits in two irreducible representations of  $N[H6]$ , namely

$$\text{span}\{x_0^2 + x_3^2, x_1^2 + x_4^2, x_2^2 + x_5^2, x_1 * x_5 + x_4 * x_2, x_2 * x_0 + x_5 * x_3, x_0 * x_4 + x_3 * x_1\}$$

and

$$\text{span}\{x_0^2 - x_3^2, x_1^2 - x_4^2, x_2^2 - x_5^2, x_1 * x_5 - x_4 * x_2, x_2 * x_0 - x_5 * x_3, x_0 * x_4 + x_3 * x_1\}.$$

Thus we first have to introduce the normalizer of  $H6$  in  $GL(\mathbb{C}^6)$ . We will first find its abstract group-structure, and then calculate it directly (in  $GL(V)$ )- which is necessary in order to find the action of the normalizer on spaces such as  $\text{span}\{x_0^2 + x_3^2, x_1^2 + x_4^2, (x_2^2 + x_5^2), x_1 * x_5 + x_4 * x_2, x_2 * x_0 + x_5 * x_3, (x_0 * x_4 + x_3 * x_1)\}$ . Now the Normalizer is the set  $\{n | nH6n^{-1} = H6\}$  under matrix multiplication.  $\rho(H6)$  is generated by  $\rho(\sigma)$  and  $\rho(\gamma)$ , so if  $n$  is an element of the normalizer then  $n\rho(\sigma)n^{-1}$  and  $n\rho(\gamma)n^{-1}$  will also generate  $\rho(H6)$ . So the normalizer consists of all matrices  $n$  in  $GL(\mathbb{C}^6)$  that satisfy the equations (\*) and (\*\*). (Here  $\rho$  is the Heisenberg-representation  $\rho : H6 \rightarrow GL(\mathbb{C}^6)$  defined above.)

$$\begin{aligned} n\rho(\sigma)n^{-1} &= (\epsilon * I)^r \rho(\sigma)^s \rho(\gamma)^t \quad (*) \\ n\rho(\gamma)n^{-1} &= (\epsilon * I)^{r'} \rho(\sigma)^{s'} \rho(\gamma)^{t'} \quad (**) \\ \implies -1 &= \det(\rho(\sigma)) = \det((\epsilon * I)^r) \det(\rho(\sigma)^s) * \det(\rho(\gamma)^t) = 1(-1)^s(-1)^t = (-1)^{s+t} \\ \implies -1 &= \det(\rho(\gamma)) = \det((\epsilon * I)^{r'}) \det(\rho(\sigma)^{s'}) \det(\rho(\gamma)^{t'}) = 1(-1)^{s'}(-1)^{t'} = (-1)^{s'+t'} \\ \implies s+t &\equiv 1 \pmod{2} \quad \text{or} \quad s'+t' \equiv 1 \pmod{2} \end{aligned}$$

But for  $n$  to be in the normalizer we must be sure that  $n\rho(\sigma)n^{-1}$  and  $n\rho(\gamma)n^{-1}$  actually generate  $\rho(H6)$ ; so there will be conditions on  $(s, t, s', t')$

$$\begin{aligned} &(\epsilon * I)^{r'} \rho(\sigma)^{s'} \rho(\gamma)^{t'} (\epsilon * I)^r \rho(\sigma)^s \rho(\gamma)^t \\ &= (\epsilon * I)(\epsilon * I)^r \rho(\sigma)^s \rho(\gamma)^t (\epsilon * I)^{r'} \rho(\sigma)^{s'} \rho(\gamma)^{t'} \end{aligned}$$

that by proposition 1.1 translates into

$$\begin{aligned} &\epsilon^{r+r'-t's} \sigma^{s+s'} \gamma^{t+t'} \\ &\epsilon^{r+r'-ts'+1} \sigma^{s+s'} \gamma^{t+t'} \\ &\Rightarrow ts' - t's \equiv 1 \pmod{6} \end{aligned}$$

so  $s + t \equiv 1 \pmod{2}$ ,  $s' + t' \equiv 1 \pmod{2}$  and  $ts' - t's \equiv 1 \pmod{6}$ . Now the set of such quadruples  $(s, t, s', t')$  or the set  $\{\begin{pmatrix} s & t \\ s' & t' \end{pmatrix} | s + t \equiv 1 \pmod{2}, s' + t' \equiv 1 \pmod{2}, ts' - t's \equiv 1 \pmod{6}\}$  under matrix-multiplication will form a group due to the following multiplication law. If

$$\begin{aligned} A\sigma * A^{-1} &= \epsilon^{r_1} * \sigma^{s_1} * \gamma^{t_1} \\ A * \gamma * A^{-1} &= \epsilon^{r'_1} * \sigma^{s'_1} * \gamma^{s'_1} \\ ,B * \sigma * B^{-1} &= \epsilon^{r_2} * \sigma^{s_2} * \gamma^{t_2} \\ B * \gamma * B^{-1} &= \epsilon^{r'_2} * \sigma^{s'_2} * \gamma^{t'_2} \\ \text{and} \\ (AB) * \sigma * (AB)^{-1} &= \epsilon^r * \sigma^s * \gamma^t \\ (AB) * \gamma * (AB)^{-1} &= \epsilon^{r'} * \sigma^{s'} * \gamma^{t'} \end{aligned}$$

then

$$\begin{pmatrix} s_1 & t_1 \\ s'_1 & s'_1 \end{pmatrix} \begin{pmatrix} s_2 & t_2 \\ s'_2 & s'_2 \end{pmatrix} = \begin{pmatrix} s & t \\ s' & t' \end{pmatrix}.$$

Thus we have an homomorphism from the group  $N[H6]$  to the group  $\{\begin{pmatrix} s & t \\ s' & t' \end{pmatrix} | s + t \equiv 1 \pmod{2}, s' + t' \equiv 1 \pmod{2}, ts' - t's \equiv 1 \pmod{6}\}$  under matrix-multiplication defined by  $\phi(A) = \{\begin{pmatrix} s & t \\ s' & t' \end{pmatrix}\}$ , with kernel precisely  $H6$ . Thus we have the following proposition

**Proposition 3.2**  $N[H6]/H6 \cong \{\begin{pmatrix} s & t \\ s' & t' \end{pmatrix} | s + t \equiv 1 \pmod{2}, s' + t' \equiv 1 \pmod{2}, ts' - t's \equiv 1 \pmod{6}\}$

From this we get the following corollary.

**Corollary 3.3**  $N[H6]/H6 = \langle \begin{pmatrix} 2 & 1 \\ 5 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 5 \\ 1 & 0 \end{pmatrix} \rangle$

**Proof** One only need to check that  $\begin{pmatrix} 2 & 1 \\ 5 & 0 \end{pmatrix}$  and  $\begin{pmatrix} 2 & 5 \\ 1 & 0 \end{pmatrix}$  generates  $N[H6]/H6$ . This can just be done by computing all possible products of these matrices.

Next we will calculate the normalizer directly. We saw that  $N[H6]/H6 \cong \langle \begin{pmatrix} 2 & 1 \\ 5 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 5 \\ 1 & 0 \end{pmatrix} \rangle$ , so we will find the matrices in  $N[H6]/H6$  corresponding to  $\begin{pmatrix} 2 & 1 \\ 5 & 0 \end{pmatrix}$  and  $\begin{pmatrix} 2 & 1 \\ 5 & 0 \end{pmatrix}$

**Proposition 3.4**

$$M1 * \sigma * M1^{-1} = \epsilon^r * \sigma^2 * \gamma^1$$

$$M1 * \gamma * M1^{-1} = \epsilon^{r'} * \sigma^5 * \gamma^0$$

$$M2 * \sigma * M2^{-1} = \epsilon^r * \sigma^2 * \gamma^5$$

$$M2 * \gamma * M2^{-1} = \epsilon^{r'} * \sigma^1 * \gamma^0$$

then

$$M1 = \begin{pmatrix} 1 & 1 & \epsilon^4 & 1 & 1 & \epsilon^4 \\ 1 & \epsilon^1 & 1 & \epsilon^3 & \epsilon^4 & \epsilon^3 \\ 1 & \epsilon^2 & \epsilon^2 & 1 & \epsilon^2 & \epsilon^2 \\ 1 & \epsilon^3 & \epsilon^4 & \epsilon^3 & 1 & \epsilon^1 \\ 1 & \epsilon^4 & 1 & 1 & \epsilon^4 & 1 \\ 1 & \epsilon^5 & \epsilon^2 & \epsilon^3 & \epsilon^2 & \epsilon^5 \end{pmatrix}$$

and

$$M2 = \begin{pmatrix} 1 & 1 & \epsilon^2 & 1 & 1 & \epsilon^2 \\ 1 & \epsilon^5 & 1 & \epsilon^3 & \epsilon^2 & \epsilon^3 \\ 1 & \epsilon^4 & \epsilon^4 & 1 & \epsilon^4 & \epsilon^4 \\ 1 & \epsilon^3 & \epsilon^2 & \epsilon^3 & 1 & \epsilon^5 \\ 1 & \epsilon^2 & 1 & 1 & \epsilon^2 & 1 \\ 1 & \epsilon^1 & \epsilon^4 & \epsilon^3 & \epsilon^4 & \epsilon^1 \end{pmatrix}$$

Now we will decompose  $S_2$  into irreducible representations of  $N[H6]$ . As mentioned earlier these are direct sums of irreducible Heisenberg-representations that are isomorphic as Heisenberg-representations (form the same isomorphism-class) First we will now see that the direct sum

$$\begin{aligned} & \text{span}\{x_0^2 + x_3^2, x_1^2 + x_4^2, x_2^2 + x_5^2\} \\ & \oplus \text{span}\{x_0^2 - x_3^2, x_1^2 - x_4^2, x_2^2 - x_5^2\} \\ & \oplus \text{span}\{x_0 * x_4 + x_3 * x_1, x_1 * x_5 + x_4 * x_2, x_2 * x_0 + x_5 * x_3\} \\ & \oplus \text{span}\{x_0 * x_4 - x_3 * x_1, x_1 * x_5 - x_4 * x_2, x_2 * x_0 - x_5 * x_3\} \end{aligned}$$

splits in two irreducible representations of  $N[H6]$  namely

$$\text{span}\{x_0^2 + x_3^2, x_1^2 + x_4^2, (x_2^2 + x_5^2), x_1 x_5 + x_4 x_2, x_2 x_0 + x_5 x_3, (x_0 x_4 + x_3 x_1)\}$$

and

$$\text{span}\{x_0^2 - x_3^2, x_1^2 - x_4^2, (x_2^2 - x_5^2), x_1 * x_5 - x_4 * x_2, x_2 * x_0 - x_5 * x_3, (x_0 * x_4 + x_3 * x_1)\}$$

The action of  $N[H6]/H6$  on

$$\text{span}\{x_0^2 + x_3^2, x_1^2 + x_4^2, x_2^2 + x_5^2, x_1 x_5 + x_4 x_2, x_2 x_0 + x_5 x_3, x_0 x_4 + x_3 x_1\}$$

is defined by the following lemmas.

Below we let the matrices M1,M2,M1M2,M2M1 act on the variables  $x_0, x_1, x_2, x_3, x_4, x_5$ ; let  $v$  be the vector  $(x_0, x_1, x_2, x_3, x_4, x_5)^\top$ , then this map sends  $x_0 \rightarrow y_0, x_1 \rightarrow y_1, x_2 \rightarrow y_2, x_3 \rightarrow y_3, x_4 \rightarrow y_4$  and  $x_5 \rightarrow y_5$ . Then let the map induced by M1 f.ex be  $f(x_0, x_1, x_2, x_3, x_4, x_5) = f(y_0, y_1, y_2, y_3, y_4, y_5)$ .

**Lemma 3.5** *M1 sends*

$$\alpha_1 * (x_0^2 + x_3^2) + \alpha_2 * (x_1^2 + x_4^2) + \alpha_3 * (x_2^2 + x_5^2) + \alpha_4 * (x_1 * x_5 + x_4 * x_2) + \alpha_5 * (x_2 * x_0 + x_5 * x_3) + \alpha_6 * (x_0 * x_4 + x_3 * x_1)$$

to

$$\begin{aligned}
& + (2 * \alpha_1 + 2 * \alpha_2 + 2 * \alpha_3 + 2 * \alpha_4 + 2 * \alpha_5 + 2 * \alpha_6) * (x_0^2 + x_3^2) \\
& + (2 * \alpha_1 + 2 * \epsilon^2 * \alpha_2 + 2 * \epsilon^4 * \alpha_3 + 2 * \alpha_4 + 2 * \epsilon^2 * \alpha_5 \\
& + 2 * \epsilon^4 * \alpha_6) * (x_1^2 + x_4^2) \\
& + (2 * \epsilon^2 * \alpha_1 + 2 * \alpha_2 + 2 * \epsilon^4 * \alpha_3 + 2 * \epsilon^2 * \alpha_4 \\
& + 2 * \alpha_5 + 2 * \epsilon^4 * \alpha_6) * (x_2^2 + x_5^2) \\
& + (4 * \epsilon^4 * \alpha_1 + 4 * \epsilon^4 * \alpha_2 + 4 * \epsilon^4 * \alpha_3 + (-2) * \epsilon^4 * \alpha_4 \\
& + (-2) * \epsilon^4 * \alpha_5 + (-2) * \epsilon^4 * \alpha_6) * (x_1 * x_5 + x_4 * x_2) \\
& + (4 * \epsilon^4 * \alpha_1 + 4 * \alpha_2 + 4 * \epsilon^2 * \alpha_3 + (-2) * \epsilon^4 * \alpha_4 + (-2) * \alpha_5 \\
& + (-2) * \epsilon^2 * \alpha_6) * (x_2 * x_0 + x_5 * x_3) \\
& + (4 * \alpha_1 + 4 * \epsilon^4 * \alpha_2 + 4 * \epsilon^2 * \alpha_3 + (-2) * \alpha_4 + (-2) * \epsilon^4 * \alpha_5 \\
& + (-2) * \epsilon^2 * \alpha_6) * ((x_0 * x_4 + x_3 * x_1))
\end{aligned}$$

and

**Lemma 3.6** *M2 sends*  $\alpha_1 * (x_0^2 + x_3^2) + \alpha_2 * (x_1^2 + x_4^2) + \alpha_3 * (x_2^2 + x_5^2)$   
 $+ \alpha_4 * (x_1 * x_5 + x_4 * x_2) + \alpha_5 * (x_2 * x_0 + x_5 * x_3) + \alpha_6 * (x_0 * x_4 + x_3 * x_1)$  to

$$\begin{aligned}
& (2 * \alpha_1 + 2 * \alpha_2 + 2 * \alpha_3 + 2 * \alpha_4 + 2 * \alpha_5 + 2 * \alpha_6) * (x_0^2 + x_3^2) \\
& + (2 * \alpha_1 + 2 * \epsilon^4 * \alpha_2 + 2 * \epsilon^2 * \alpha_3 + 2 * \alpha_2 + 2 * \epsilon^4 * \alpha_5 + 2 * \epsilon^2 * \alpha_6) * (x_1^2 + x_4^2) \\
& + (2 * \epsilon^4 * \alpha_1 + 2 * \alpha_2 + 2 * \epsilon^2 * \alpha_3 + 2 * \epsilon^4 * \alpha_4 + 2 * \alpha_5 + 2 * \epsilon^2 * \alpha_6) * (x_2^2 + x_5^2) \\
& + (4 * \epsilon^2 * \alpha_1 + 4 * \epsilon^2 * \alpha_2 + 4 * \epsilon^2 * \alpha_3 + (-2) * \epsilon^2 * \alpha_4 + (-2) * \epsilon^2 * \alpha_5 + (-2) * \epsilon^2 * \alpha_6) * (x_1 * x_5 + x_4 * x_2) \\
& + (4 * \epsilon^2 * \alpha_1 + 4 * \alpha_2 + 4 * \epsilon^4 * \alpha_3 + (-2) * \epsilon^2 * \alpha_4 + (-2) * \alpha_5 + (-2) * \epsilon^4 * \alpha_6) * ((x_2 * x_0 + x_5 * x_3) \\
& + (4 * \alpha_1 + 4 * \epsilon^2 * \alpha_2 + 4 * \epsilon^4 * \alpha_3 + (-2) * \alpha_4 + (-2) * \epsilon^2 * \alpha_5 + (-2) * \epsilon^4 * \alpha_6) * ((x_0 * x_4 + x_3 * x_1)))
\end{aligned}$$

Now the representation  $\text{span}\{x_0x_1 - x_3x_4, x_1x_2 - x_4x_5, x_2x_3 - x_5x_0\}$  of  $N[H6]/H6$  is defined by the next two lemmas

**Lemma 3.7** *M1 sends*  $\alpha_1 * (x_0 * x_1 - x_3 * x_4) + \alpha_2 * (x_1 * x_2 - x_4 * x_5) + \alpha_3 * (x_2 * x_3 - x_5 * x_0)$  to

$$\begin{aligned}
& (2(\epsilon + 1)\alpha_1 + 2 * (\epsilon^2 + \epsilon^1)\alpha_2 + 2(\epsilon^2 + \epsilon^3)\alpha_3)(x_0x_1 - x_3x_4) \\
& + (2(1 + \epsilon^5)\alpha_1 + 2(\epsilon^2 + \epsilon^3)\alpha_2 + 2(1 + \epsilon^5)\alpha_3)(x_1x_2 - x_4x_5) \\
& + (2(1 + \epsilon^1)\alpha_1 + 2(1 + \epsilon^5)\alpha_2 + 2(\epsilon^4 + \epsilon^5)\alpha_3)(x_2x_3 - x_5x_0)
\end{aligned}$$

and

**Lemma 3.8** *M2 sends*  $\alpha_1 * (x_0 * x_1 - x_3 * x_4) + \alpha_2 * (x_1 * x_2 - x_4 * x_5) + \alpha_3 * (x_2 * x_3 - x_5 * x_0)$  to

$$\begin{aligned}
& (2 * (\epsilon^5 + 1)\alpha_1 + 2 * (\epsilon^4 + \epsilon^5)\alpha_2 + 2 * (\epsilon^4 + \epsilon^3)\alpha_3) * (x_0 * x_1 - x_3 * x_4) \\
& + (2 * (1 + \epsilon^1)\alpha_1 + 2 * (\epsilon^4 + \epsilon^3)\alpha_2 + 2 * (1 + \epsilon^1)\alpha_3) * (x_1 * x_2 - x_4 * x_5) \\
& + (2 * (1 + \epsilon^5)\alpha_1 + 2 * (1 + \epsilon^1)\alpha_2 + 2 * (\epsilon^2 + \epsilon^5)\alpha_3) * (x_2 * x_3 - x_5 * x_0)
\end{aligned}$$

And  $\text{span}\{x_0^2 - x_3^2, x_1^2 - x_4^2, x_2^2 - x_5^2, x_0x_4 - x_3x_1, x_1x_5 + x_4, x_2x_0 - x_5x_3, x_0x_3, x_1x_4, x_2x_5, x_0x_1 + x_3x_4, x_1x_2 + x_4x_5, x_2x_3 + x_5x_0\}$  becomes a representation of  $N[H6]/H6$  by

**Lemma 3.9** *M1 sends*  $\alpha_1(x_0^2 - x_3^2) + \alpha_2(x_1^2 - x_4^2) + \alpha_3(x_2^2 - x_5^2)$   
 $+ \alpha_4((x_0x_4 - x_3x_1)) + \alpha_5((x_1x_5 + x_4x_2)) + \alpha_6((x_2x_0 - x_5x_3))$   
 $+ \alpha_7x_0x_3 + \alpha_8x_1x_4 + \alpha_9x_2x_5 + \alpha_{10}(x_0x_1 + x_3x_4)$   
 $+ \alpha_{11}(x_1x_2 + x_4x_5) + \alpha_{12}(x_2x_3 + x_5x_0)$  *to*

$$\begin{aligned}
& (4 * \alpha_1 + 4 * \epsilon * \alpha_2 + 4 * \epsilon^2 * \alpha_3 - 2 * \epsilon^2 * \alpha_4 - 2 * \epsilon^3 * \alpha_5 \\
& - 2 * \epsilon^4 * \alpha_6) * (x_0 * x_1 + x_3 * x_4) \\
& + (4 * \epsilon^4 * \alpha_1 + 4 * \epsilon * \alpha_2 + 4 * \epsilon^4 * \alpha_3 - 2 * \epsilon^4 * \alpha_4 - 2 * \epsilon^1 * \alpha_5 \\
& - 2 * \epsilon^4 * \alpha_6) * (x_1 * x_2 + x_4 * x_5) \\
& + (4 * \epsilon^4 * \alpha_1 + 4 * \epsilon^3 * \alpha_2 + 4 * \epsilon^2 * \alpha_3 - 2 * \epsilon^2 * \alpha_4 \\
& - 2 * \epsilon^1 * \alpha_5 - 2 * \alpha_6) * (x_2 * x_3 + x_5 * x_0) \\
& + (4 * \alpha_1 + 4 * \epsilon^3 * \alpha_2 + 4 * \alpha_3 + 4 * \alpha_4 + 4 * \epsilon^3 * \alpha_5 + 4 * \alpha_6) * x_0 * x_3 \\
& + (4 * \alpha_1 + 4 * \epsilon^5 * \alpha_2 + 4 * \epsilon^4 * \alpha_3 + 4 * \epsilon^4 * \alpha_4 + 4 * \epsilon^3 * \alpha_5 \\
& + 4 * \epsilon^4 * \alpha_6) * x_1 * x_4 \\
& + (4 * \alpha_1 + 4 * \epsilon^3 * \alpha_2 + 4 * \epsilon^4 * \alpha_3 + 4 * \epsilon^4 * \alpha_4 \\
& + 4 * \epsilon^5 * \alpha_5 + 4 * \alpha_6) * x_2 * x_5 \\
& + (\alpha_7 + \alpha_8 + \alpha_9 + 2 * \alpha_{10} + 2 * \alpha_{11} + 2 * \alpha_{12}) * (x_0^2 - x_3^2) \\
& + (\epsilon^3 * \alpha_7 + \epsilon^5 * \alpha_8 + \epsilon^1 * \alpha_9 + 2 * \epsilon^1 * \alpha_{10} \\
& + (-2) * \alpha_{11} + 2 * \epsilon^5 * \alpha_{12}) * (x_1^2 - x_4^2) \\
& + (\epsilon^2 * \alpha_7 + 1 * \alpha_8 + \epsilon^4 * \alpha_9 + 2 * \epsilon^4 * \alpha_{10} \\
& + (-2) * \alpha_{11} + 2 * \alpha_{12}) * (x_2^2 - x_5^2) \\
& + (2 * \alpha_7 + 2 * \epsilon^4 * \alpha_8 + 2 * \epsilon^2 * \alpha_9 + (-2) * \epsilon^2 * \alpha_{10} \\
& + (-2) * \alpha_{11} + (-2) * \epsilon^4 * \alpha_{12}) * ((x_0 * x_4 - x_3 * x_1)) \\
& + (2 * \epsilon^1 * \alpha_7 + 2 * \epsilon^1 * \alpha_8 + 2 * \epsilon^1 * \alpha_9 + (-2) * \epsilon^1 * \alpha_{10} \\
& + (-2) * \epsilon^1 * \alpha_{11} + (-2) * \epsilon^1 * \alpha_{12}) * (x_1 * x_5 - x_4 * x_2) \\
& + (2 * \epsilon^4 * \alpha_7 + 2 * \alpha_8 + 2 * \epsilon^2 * \alpha_9 + (-2) * \epsilon^2 * \alpha_{10} \\
& + (-2) * \epsilon^4 * \alpha_{11} + (-2) * \alpha_{12}) * ((x_2 * x_0 - x_5 * x_3))
\end{aligned}$$

and

**Lemma 3.10** *M2 sends*

$$\begin{aligned}
& \alpha_1 * (x_0^2 - x_3^2) + \alpha_2 * (x_1^2 - x_4^2) + \alpha_3 * (x_2^2 - x_5^2) \\
& + \alpha_4 * ((x_0 * x_4 - x_3 * x_1)) + \alpha_5 * ((x_1 * x_5 - x_4 * x_2)) \\
& + \alpha_6 * ((x_2 * x_0 - x_5 * x_3)) + \alpha_7 * x_0 * x_3 + \alpha_8 * x_1 * x_4 \\
& + \alpha_9 * x_2 * x_5 + \alpha_{10} * (x_0 * x_1 + x_3 * x_4) \\
& + \alpha_{11} * (x_1 * x_2 + x_4 * x_5) + \alpha_{12} * (x_2 * x_3 + x_5 * x_0)
\end{aligned}$$

to

$$\begin{aligned}
& (4 * \alpha_1 + 4 * \epsilon^5 * \alpha_2 + 4 * \epsilon^4 * \alpha_3 - 2 * \epsilon^4 * \alpha_4 \\
& - 2 * \epsilon^3 * \alpha_5 - 2 * \epsilon^2 * \alpha_6) * (x_0 * x_1 + x_3 * x_4) \\
& + (4 * \epsilon^2 * \alpha_1 + 4 * \epsilon^5 * \alpha_2 + 4 * \epsilon^2 * \alpha_3 \\
& - 2 * \epsilon^2 * \alpha_4 - 2 * \epsilon^5 * \alpha_5 - 2 * \epsilon^2 * \alpha_6) * (x_1 * x_2 + x_4 * x_5) \\
& + (4 * \epsilon^2 * \alpha_1 + 4 * \epsilon^3 * \alpha_2 + 4 * \epsilon^4 * \alpha_3 - 2 * \epsilon^4 * \alpha_4 \\
& - 2 * \epsilon^5 * \alpha_5 - 2 * \alpha_6) * (x_2 * x_3 + x_5 * x_0) \\
& + (4 * \alpha_1 + 4 * \epsilon^3 * \alpha_2 + 4 * \alpha_3 + 4 * \alpha_4 + 4 * \epsilon^3 * \alpha_5 + 4 * \alpha_6) * x_0 * x_3 \\
& + (4 * \alpha_1 + 4 * \epsilon^1 * \alpha_2 + 4 * \epsilon^2 * \alpha_3 + 4 * \epsilon^2 * \alpha_4 + 4 * \epsilon^3 * \alpha_5 \\
& + 4 * \epsilon^2 * \alpha_6) * x_1 * x_4 \\
& + (4 * \alpha_1 + 4 * \epsilon^3 * \alpha_2 + 4 * \epsilon^2 * \alpha_3 + 4 * \epsilon^2 * \alpha_4 \\
& + 4 * \epsilon^1 * \alpha_5 + 4 * \alpha_6) * x_2 * x_5 \\
& + (\alpha_7 + \alpha_8 + \alpha_9 + 2 * \alpha_{10} + 2 * \alpha_{11} + 2 * \alpha_{12}) * (x_0^2 - x_3^2) \\
& + (\epsilon^3 * \alpha_7 + \epsilon^1 * \alpha_8 + \epsilon^5 * \alpha_9 + 2 * \epsilon^5 * \alpha_{10} \\
& + (-2) * \alpha_{11} + 2 * \epsilon^1 * \alpha_{12}) * (x_1^2 - x_4^2) \\
& + (\epsilon^4 * \alpha_7 + 1 * \alpha_8 + \epsilon^2 * \alpha_9 + 2 * \epsilon^2 * \alpha_{10} \\
& + (-2) * \alpha_{11} + 2 * \alpha_{12}) * (x_2^2 - x_5^2) \\
& + (2 * \alpha_7 + 2 * \epsilon^2 * \alpha_8 + 2 * \epsilon^4 * \alpha_9 + (-2) * \epsilon^4 * \alpha_{10} + \\
& + (-2) * \alpha_{11} + (-2) * \epsilon^2 * \alpha_{12}) * (x_0 * x_4 - x_3 * x_1) \\
& + (2 * \epsilon^5 * \alpha_7 + 2 * \epsilon^5 * \alpha_8 + 2 * \epsilon^5 * \alpha_9 + (-2) * \epsilon^5 * \alpha_{10} \\
& + (-2) * \epsilon^5 * \alpha_{11} + (-2) * \epsilon^5 * \alpha_{12}) * (x_1 * x_5 - x_4 * x_2) \\
& + (2 * \epsilon^2 * \alpha_7 + 2 * \alpha_8 + 2 * \epsilon^4 * \alpha_9 + (-2) * \epsilon^4 * \alpha_{10} \\
& + (-2) * \epsilon^2 * \alpha_{11} + (-2) * \alpha_{12}) * ((x_2 * x_0 - x_5 * x_3))
\end{aligned}$$

We will not prove all of the lemmas as they involve rather long and tedious calculations, but merely that the action of M2 is analogous to that of M1; everywhere we can just replace  $\epsilon^\alpha$  with  $\epsilon^{-\alpha}$ .

**Lemma 3.11** *If M1 sends  $\alpha_1(x_0x_1 - x_3x_4) + \alpha_2(x_1x_2 - x_4x_5) + \alpha_3(x_2x_3 - x_5x_0)$  to*

$$\begin{aligned}
& (2(\epsilon^1 + 1)\alpha_1 + 2(\epsilon^2 + \epsilon^1)\alpha_2 + 2(\epsilon^2 + \epsilon^3)\alpha_3)(x_0x_1 - x_3x_4) \\
& + (2(1 + \epsilon^5)\alpha_1 + 2(\epsilon^2 + \epsilon^3)\alpha_2 + 2(1 + \epsilon^5)\alpha_3)(x_1x_2 - x_4x_5) \\
& + (2(1\epsilon^1)\alpha_1 + 2(1 + \epsilon^5)\alpha_2 + 2(\epsilon^4 + \epsilon^5)\alpha_3)(x_2x_3 - x_5x_0).
\end{aligned}$$

*then M2 sends  $\alpha_1 * (x_0 * x_1 - x_3x_4) + \alpha_2(x_1x_2 - x_4x_5) + \alpha_3(x_2x_3 - x_5x_0)$  to*

$$\begin{aligned}
& (2(\epsilon^{-1} + 1)\alpha_1 + 2(\epsilon^{-2} + \epsilon^{-1})\alpha_2 + 2(\epsilon^{-2} + \epsilon^{-3})\alpha_3)(x_0x_1 - x_3x_4) \\
& + (2(1 + \epsilon^{-5})\alpha_1 + 2(\epsilon^{-2} + \epsilon^{-3})\alpha_2 + 2(1 + \epsilon^{-5})\alpha_3)(x_1x_2 - x_4x_5) \\
& + (2(1\epsilon^{-1})\alpha_1 + 2(1 + \epsilon^{-5})\alpha_2 + 2(\epsilon^{-4} + \epsilon^{-5})\alpha_3)(x_2x_3 - x_5x_0).
\end{aligned}$$

*And similar for the actions on*

$$span\{x_0^2 + x_3^2, x_1^2 + x_4^2, x_2^2 + x_5^2, x_1x_5 + x_4x_2, x_2x_0 + x_5x_3, x_0x_4 + x_3x_1\}$$

and

$$\text{span}\{x_0^2 - x_3^2, x_1^2 - x_4^2, x_2^2 - x_5^2, x_1x_5 - x_4x_2, x_2 * x_0 - x_5x_3, x_0x_4 + x_3x_1\}.$$

**Proof** Look at

$$\begin{aligned} M1(x_0x_1 + x_3x_4) &= (M1_{11}x_0 + M1_{12}x_1 + M1_{13}x_2 + M1_{14}x_3 + M1_{15}x_4 + M1_{16}x_5) \\ &\quad (M1_{21}x_0 + M1_{22}x_1 + M1_{23}x_2 + M1_{24}x_3 + M1_{25}x_4 + M1_{26}x_5) \\ &\quad +(M1_{41}x_0 + M1_{42}x_1 + M1_{43}x_2 + M1_{44}x_3 + M1_{45}x_4 + M1_{46}x_5) \\ &\quad (M1_{51}x_0 + M1_{52}x_1 + M1_{53}x_2 + M1_{54}x_3 + M1_{55}x_4 + M1_{56}x_5) \end{aligned}$$

The coefficient of  $x_i x_j$  will be  $M1_{1i}M1_{2j} + M1_{1j}M1_{2i} + M1_{4i}M1_{5j} + M1_{4j}M1_{5i}$   
(\*) And the coefficient of  $x_i x_j$  in the expression

$$\begin{aligned} M2(x_0x_1 + x_3x_4) &= (M2_{11}x_0 + M2_{12}x_1 + M2_{13}x_2 + M2_{14}x_3 + M2_{15}x_4 + M2_{16}x_5) \\ &\quad (M2_{21}x_0 + M2_{22}x_1 + M2_{23}x_2 + M2_{24}x_3 + M2_{25}x_4 + M2_{26}x_5) \\ &\quad +(M2_{41}x_0 + M2_{42}x_1 + M2_{43}x_2 + M2_{44}x_3 + M2_{45}x_4 + M2_{46}x_5) \\ &\quad (M2_{51}x_0 + M2_{52}x_1 + M2_{53}x_2 + M2_{54}x_3 + M2_{55}x_4 + M2_{56}x_5) \end{aligned}$$

will be  $M2_{1i}M2_{2j} + M2_{1j}M2_{2i} + M2_{4i}M2_{5j} + M2_{4j}M2_{5i}$ . (\*\*)  
the two irreducible Heisenberg-representations  $\text{span}\{x_0x_1 + x_3x_4, x_1x_2 + x_4x_5, x_2x_3 + x_5x_0\}$   
and  $\text{span}\{x_3x_0, x_4x_1, x_2x_5\}$ .

So  $X_{\alpha_1, \alpha_2}$  is a Heisenberg-invariant variety. We say that  $X_{\alpha_1, \alpha_2}$  and  $X_{\beta_1, \beta_2}$  are isomorphic as Heisenberg-invariant varieties if  $V_{\alpha_1, \alpha_2}$  and  $V_{\beta_1, \beta_2}$  are isomorphic as Heisenberg-representations. An interesting question is when  $X_{\alpha_1, \alpha_2}$  and  $X_{\beta_1, \beta_2}$  are isomorphic as Heisenberg-invariant varieties; the next four lemmas gives an exact relation between  $(\alpha_1, \alpha_2)$  and  $(\beta_1, \beta_2)$ .

The entries of

$$M1 = \begin{pmatrix} 1 & 1 & \epsilon^4 & 1 & 1 & \epsilon^4 \\ 1 & \epsilon^1 & 1 & \epsilon^3 & \epsilon^4 & \epsilon^3 \\ 1 & \epsilon^2 & \epsilon^2 & 1 & \epsilon^2 & \epsilon^2 \\ 1 & \epsilon^3 & \epsilon^4 & \epsilon^3 & 1 & \epsilon^1 \\ 1 & \epsilon^4 & 1 & 1 & \epsilon^4 & 1 \\ 1 & \epsilon^5 & \epsilon^2 & \epsilon^3 & \epsilon^2 & \epsilon^5 \end{pmatrix}$$

is of the form  $M1_{ij} = \epsilon^{uij}$  where  $uij$  is 0,1,2,3,4 or 5 and then the entries of

$$M2 = \begin{pmatrix} 1 & 1 & \epsilon^2 & 1 & 1 & \epsilon^2 \\ 1 & \epsilon^5 & 1 & \epsilon^3 & \epsilon^2 & \epsilon^3 \\ 1 & \epsilon^4 & \epsilon^4 & 1 & \epsilon^4 & \epsilon^4 \\ 1 & \epsilon^3 & \epsilon^2 & \epsilon^3 & 1 & \epsilon^5 \\ 1 & \epsilon^2 & 1 & 1 & \epsilon^2 & 1 \\ 1 & \epsilon^1 & \epsilon^4 & \epsilon^3 & \epsilon^4 & \epsilon^1 \end{pmatrix}$$

is of the form  $M2_{ij} = \epsilon^{uij}$  where  $uij$  is 0,1,2,3,4 or 5

Thus the expression in (\*) will be sums of the form  $\sum \epsilon^{\gamma_u} \epsilon^{\gamma_v}$ . Now  $M2_{ij}^{-1} = M1_{ij} = \epsilon^{-uij}$  and thus the expression in (\*\*) will be of the form  $\sum \epsilon^{-\gamma_u} \epsilon^{-\gamma_v}$ . Now the result follows.

$$V_{\alpha_1, \alpha_2} = \text{span}\{\alpha_1(x_0x_1 + x_3x_4) + \alpha_2(x_2x_5), \alpha_1(x_1x_2 + x_4x_5) + \alpha_2(x_3x_0)\alpha_1(x_2x_3 + x_5x_0) + \alpha_2(x_4x_1)\}$$

is an irreducible Heisenberg-representation obtained from the two irreducible Heisenberg-representations  $\text{span}\{x_0x_1 + x_3x_4, x_1x_2 + x_4x_5, x_2x_3 + x_5x_0\}$  and  $\text{span}\{x_3x_0, x_4x_1, x_2x_5\}$ . Let  $X_{\alpha_1, \alpha_2}$  be the variety

$$\begin{aligned} & \{(x_0, x_1, x_2, x_3, x_4, x_5) | f(x_0, x_1, x_2, x_3, x_4, x_5) = 0 \forall \\ & c_1(\alpha_1(x_0x_1 + x_3x_4) + \alpha_2(x_2x_5)) + c_2(\alpha_1(x_1x_2 + x_4x_5) + \alpha_2(x_3x_0)) \\ & \quad + c_3(\alpha_1(x_2x_3 + x_5x_0) + \alpha_2(x_4x_1)), \\ & \quad c_1, c_2, c_3 \in \mathbb{C}\}, \\ & \text{And let } X_{\alpha_1, \alpha_2, \alpha_3} = \\ & = \{(x_0, x_1, x_2, x_3, x_4, x_5) | f(x_0, x_1, x_2, x_3, x_4, x_5) = 0 \forall \\ & f = \alpha_1(x_0^2x_1 + x_2^2x_3 + x_4^2x_5) + \alpha_2(x_1x_2x_4 + x_3x_4x_0 + x_5x_0x_2) \\ & \quad + \alpha_3(x_1^2x_5 + x_3^2x_1 + x_5^2x_3)\} \end{aligned}$$

So  $X_{\alpha_1, \alpha_2}$  is a Heisenberg-invariant variety. We say that  $X_{\alpha_1, \alpha_2}$  and  $X_{\beta_1, \beta_2}$  are isomorphic as Heisenberg-invariant varieties if  $V_{\alpha_1, \alpha_2}$  and  $V_{\beta_1, \beta_2}$  are isomorphic as Heisenberg-representations. An interesting question is when  $X_{\alpha_1, \alpha_2}$  and  $X_{\beta_1, \beta_2}$  are isomorphic as Heisenberg-invariant varieties; the next four lemmas gives an exact relation between  $(\alpha_1, \alpha_2)$  and  $(\beta_1, \beta_2)$ .

But first some preliminary results about the matrices  $M1M2$  and  $M2M1$ .

Remember that

$$M1 = \begin{pmatrix} 1 & 1 & \epsilon^4 & 1 & 1 & \epsilon^4 \\ 1 & \epsilon^1 & 1 & \epsilon^3 & \epsilon^4 & \epsilon^3 \\ 1 & \epsilon^2 & \epsilon^2 & 1 & \epsilon^2 & \epsilon^2 \\ 1 & \epsilon^3 & \epsilon^4 & \epsilon^3 & 1 & \epsilon^1 \\ 1 & \epsilon^4 & 1 & 1 & \epsilon^4 & 1 \\ 1 & \epsilon^5 & \epsilon^2 & \epsilon^3 & \epsilon^2 & \epsilon^5 \end{pmatrix}$$

and

$$M2 = \begin{pmatrix} 1 & 1 & \epsilon^2 & 1 & 1 & \epsilon^2 \\ 1 & \epsilon^5 & 1 & \epsilon^3 & \epsilon^2 & \epsilon^3 \\ 1 & \epsilon^4 & \epsilon^4 & 1 & \epsilon^4 & \epsilon^4 \\ 1 & \epsilon^3 & \epsilon^2 & \epsilon^3 & 1 & \epsilon^5 \\ 1 & \epsilon^2 & 1 & 1 & \epsilon^2 & 1 \\ 1 & \epsilon^1 & \epsilon^4 & \epsilon^3 & \epsilon^4 & \epsilon^1 \end{pmatrix}.$$

Thus

$$M1M2 = \begin{pmatrix} 2(2 + \epsilon^4) & 0 & 2(1 + 2\epsilon^2) & 0 & 2(1 + 2\epsilon^2) & 0 \\ 0 & 2(2 + \epsilon^4) & 0 & (2 + \epsilon^4) & 0 & 2\epsilon^2(2\epsilon^2 + 1) \\ 2(1 + 2\epsilon^2) & 0 & 2(1 + 2\epsilon^2) & 0 & 2(2 + \epsilon^4) & 0 \\ 0 & 2(1 + 2\epsilon^2) & 0 & 2(2 + \epsilon^4) & 0 & 2 * (1 + 2\epsilon^2) \\ 2 * (2 + \epsilon^4) & 0 & 2\epsilon^2(2 * \epsilon^2 + 1) & 0 & 2(2 + \epsilon^4) & 0 \\ 0 & 2 * (2 + \epsilon^4) & 0 & 2(1 + 2\epsilon^2) & 0 & 2 * (1 + 2\epsilon^2) \end{pmatrix}$$

and

$$M2M1 = \begin{pmatrix} 2(2+\epsilon^2) & 0 & 2(1+2\epsilon^4) & 0 & 2(1+2\epsilon^4) & 0 \\ 0 & 2(2+\epsilon^2) & 0 & (2+\epsilon^2) & 0 & 2\epsilon^4(\epsilon^4+1) \\ 2(1+2\epsilon^4) & 0 & 2(1+2\epsilon^4) & 0 & 2(2+\epsilon^2) & 0 \\ 0 & 2(1+2\epsilon^4) & 0 & 2(2+\epsilon^2) & 0 & 2(1+2\epsilon^4) \\ 2(2+\epsilon^2) & 0 & 2\epsilon^4 * (2\epsilon^4+1) & 0 & 2(2+\epsilon^2) & 0 \\ 0 & 2 * (2+\epsilon^2) & 0 & 2(1+2\epsilon^4) & 0 & 2 * (1+2\epsilon^4) \end{pmatrix}$$

If

$$M1 * \sigma * M1^{-1} = \epsilon^r * \sigma^2 * \gamma^1$$

$$M1 * \gamma * M1^{-1} = \epsilon^{r'} * \sigma^5 * \gamma^0$$

and,

$$M2 * \sigma * M2^{-1} = \epsilon^r * \sigma^2 * \gamma^5$$

$$M2 * \gamma * M2^{-1} = \epsilon^{r'} * \sigma^1 * \gamma^0$$

$$M1M2 * \sigma * (M1M2)^{-1} = \epsilon^r * \sigma^s * \gamma^t$$

$$M1M2 * \gamma * (M1M2)^{-1} = \epsilon^{r'} * \sigma^{s'} * \gamma^{t'}$$

then

$$\begin{pmatrix} 2 & 1 \\ 5 & 0 \end{pmatrix} * \begin{pmatrix} 2 & 5 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} s & t \\ s' & t' \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 4 & 1 \end{pmatrix} \quad (1)$$

$$\text{So } M1M2 * \sigma * (M1M2)^{-1} = \epsilon^r * \sigma^5 * \gamma^4 \quad M1M2 * \gamma * (M1M2)^{-1} = \epsilon^{r'} * \sigma^4 * \gamma^1.$$

This follows from the multiplication law on page 15.

So it follows from the proposition that

$$\langle M1M2, M2M1 \rangle = \langle \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 4 \\ 4 & 1 \end{pmatrix} \rangle .$$

So we have the following result.

### Proposition 3.12

$$\langle M1M2, M2M1 \rangle = \langle \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 4 \\ 4 & 1 \end{pmatrix} \rangle \text{ is isomorphic to the quaternion group}$$

x	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	k	-1	i
k	k	j	-i	-1

$$i = \begin{pmatrix} 5 & 4 \\ 4 & 1 \end{pmatrix}, j = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}, k = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}, \quad (2)$$

And these matrices satisfy the multiplication table above.

**Lemma 3.13** *M1M2 sends the element*

$$\begin{aligned} & c_1 * (\alpha_1(x_0x_1 + x_3x_4) + \alpha_2(x_2x_5)) + c_2(\alpha_1(x_1x_2 + x_4x_5) + \alpha_2(x_3x_0)) \\ & + c_3 * (\alpha_1(x_2x_3 + x_5x_0) + \alpha_2(x_4x_1)) \\ & \text{to} \\ & c_1 * (\beta_1(x_0x_1 + x_3x_4) + \beta_2(x_2x_5)) + c_2(\beta_1(x_1x_2 + x_4x_5) + \beta_2(x_3x_0)) \\ & + c_3 * (\beta_1(x_2x_3 + x_5x_0) + \beta_2(x_4x_1)) \end{aligned}$$

where  $\beta_1 = \alpha_2\epsilon^4 + 2\alpha_1$  og  $\beta_2 = \alpha_2 + 2\epsilon^2 * \alpha_1$ .

**Lemma 3.14**

*M2M1 sends the element*

$$\begin{aligned} & c_1(\alpha_1(x_0x_1 + x_3x_4) + \alpha_2(x_2x_5)) + c_2(\alpha_1(x_1x_2 + x_4x_5) + \alpha_2(x_3x_0)) \\ & + c_3(\alpha_1(x_2x_3 + x_5x_0) + \alpha_2(x_4x_1)) \\ & \text{to} \\ & c_1(\beta_1(x_0x_1 + x_3x_4) + \beta_2(x_2x_5)) + c_2(\beta_1(x_1x_2 + x_4x_5) + \beta_2(x_3x_0)) \\ & + c_3 * (\beta_1(x_2x_3 + x_5x_0) + \beta_2(x_4x_1)) \mathbb{C} \end{aligned}$$

where  $\beta_1 = \alpha_2 * \epsilon^2 + 2 * \alpha_1$  og  $\beta_2 = \alpha_2 + 2 * \epsilon^4 * \alpha_1$ .

**Lemma 3.15**

*M1M2 sends*

$$\begin{aligned} & \alpha_1(x_0^2x_1 + x_2^2x_3 + x_4^2x_5) + \alpha_2(x_1x_2x_4 + x_3x_4x_0 + x_5x_0x_2) \\ & + \alpha_3(x_1^2x_5 + x_3^2x_1 + x_5^2x_3) \\ & \text{to} \\ & \beta_1 * (x_0^2x_1 + x_2^2x_3 + x_4^2x_5) + \beta_2(x_1x_2x_4 + x_3x_4x_0 + x_5x_0x_2) \\ & + \beta_3(x_1^2x_5 + x_3^2x_1 + x_5^2x_3) \end{aligned}$$

where  $\beta_1 = (-3)(1 + 2\epsilon^2)\alpha_1 + (-3)(\epsilon^2 + 2\epsilon^4)\alpha_2$ ,  $\beta_2 = (-3)(2 + \epsilon^4)\alpha_1 + (-3)(1 + 2\epsilon^4)\alpha_2$  and  $\beta_3 = -24(2 + \epsilon^4)(2 + \epsilon^4)$ .

**Lemma 3.16**

*M2M1 sends*

$$\begin{aligned} & \mathbb{C}\alpha_1(x_0^2x_1 + x_2^2x_3 + x_4^2x_5) + \alpha_2(x_1x_2x_4 + x_3x_4x_0 + x_5x_0x_2) \\ & + \alpha_3(x_1^2x_5 + x_3^2x_1 + x_5^2x_3) \\ & \text{to} \\ & \beta_1(x_0^2x_1 + x_2^2x_3 + x_4^2x_5) + \beta_2(x_1x_2x_4 + x_3x_4x_0 + x_5x_0x_2) \\ & + \beta_3(x_1^2x_5 + x_3^2x_1 + x_5^2x_3) \end{aligned}$$

where  $\beta_1 = (-3) * (1 + 2 * \epsilon^4) * \alpha_1 + (-3) * (\epsilon^4 + 2 * \epsilon^2) * \alpha_2$ ,  
 $\beta_2 = (-3) * (2 + \epsilon^4) * \alpha_1 + (-3) * (1 + 2 * \epsilon^2) * \alpha_2$  and  $\beta_3 = -24 * (2 + \epsilon^4) * (2 + \epsilon^2)$ .

So the map *M1M2* and *M2M1* above induces the following group-actions on  $\mathbb{C}^2$  and  $\mathbb{C}^3$ . The group-action on  $\mathbb{C}^2$  is defined by *M1M2* that induces the

following map from  $\mathbb{C}^2$  to  $\mathbb{C}^2$  given by  $(\alpha_1, \alpha_2) \rightarrow (\alpha_2\epsilon^4 + 2\alpha_1, \alpha_2 + 2\epsilon^2\alpha_1)$  and  $M2M1$  which induce the map  $(\alpha_1, \alpha_2) \rightarrow (\alpha_2\epsilon^2 + 2\alpha_1, \alpha_2 + 2\epsilon^4\alpha_1)$ , The group-action on  $\mathbb{C}^3$  is defined by  $M1M2$  that induces the map from  $\mathbb{C}^3$  to  $\mathbb{C}^3$  given by

$$\begin{aligned}
(\alpha_1, \alpha_2, \alpha_3) &\rightarrow \\
((-3)(1+2\epsilon^2)\alpha_1 + (-3)(\epsilon^2 + 2\epsilon^4)\alpha_2, \\
(-3)(2+\epsilon^4)\alpha_1 + (-3)(1+2*\epsilon^4)\alpha_2, -24(2+\epsilon^4)*2+\epsilon^4) \\
\text{and } M2M1 \text{ induces the map} \\
(\alpha_1, \alpha_2, \alpha_3) &\rightarrow \\
((-3)*(1+2\epsilon^4)*\alpha_1 + (-3)(\epsilon^4 + 2\epsilon^2)\alpha_2, \\
(-3)*(2+\epsilon^2)*\alpha_1 + (-3)*(1+2*\epsilon^2)*\alpha_2, -24*(2+\epsilon^2)*(2+\epsilon^2)).
\end{aligned}$$

This defines a group action of  $\langle M1M2, M2M1 \rangle$  or of the quaternion group on  $\mathbb{C}^2$  and  $\mathbb{C}^3$ ; and this group-action partitions  $\mathbb{C}^2$  into equivalence classes(orbits). So if  $(\alpha_1, \alpha_2)$  and  $(\beta_1, \beta_2)$  are of the same equivalence class(or orbit),then  $X_{\alpha_1, \alpha_2}$  and  $X_{\beta_1, \beta_2}$  are isomorphic as varieties(\*).  $\langle M1M2, M2M1 \rangle$  also defines a group-action on  $\mathbb{C}^3$ , and here  $X_{\alpha_1, \alpha_2, \alpha_3}$  and  $X_{\beta_1, \beta_2, \beta_3}$  are also isomorphic as varieties if  $(\alpha_1, \alpha_2, \alpha_3)$  ( $\beta_1, \beta_2, \beta_3$ ) belong to the same orbit of this action(\*\*).

The isomorphisms between the varieties above follows from the fact that  $\langle M1M2, M2M1 \rangle$  is a subgroup of  $N[H6]$ ,the normalizer of  $H6$ . So in a sense the quaternion-group is the key to the solution of the question when two  $X_{\alpha_1, \alpha_2}$  and  $X_{\beta_1, \beta_2}$  are isomorphic as Heisenberg-invariant varieties.

**Conjecture** The statements (\*) and (\*\*) above holds for irreducible Heisenberg-varieties that are obtained in an analogous manner as  $X_{\alpha_1, \alpha_2}$  and  $X_{\alpha_1, \alpha_2, \alpha_3}$

Lastly we will prove that the maps induced by  $M2M1$  can be obtained from the map induced by  $M1M2$  by replacing every  $\epsilon^u$  with  $\epsilon^{-u}$ .So it is given by

$$\begin{aligned}
(\alpha_1, \alpha_2) &\rightarrow \\
(\alpha_2\epsilon^{-4} + 2\alpha_1, \alpha_2 + 2\epsilon^{-2}\alpha_1) \\
\text{and} \\
(\alpha_1, \alpha_2, \alpha_3) &\rightarrow \\
((-3)(1+2\epsilon^{-2})\alpha_1 + (-3)(\epsilon^2 + 2\epsilon^{-4})\alpha_2, \\
(-3)(2+\epsilon^{-4})\alpha_1 + (-3)(1+2\epsilon^{-4})\alpha_2, -24(2+\epsilon^{-4})(2+\epsilon^{-4}))
\end{aligned}$$

. Then we will prove tha we can replace  $\epsilon^u$  with  $\epsilon^{-u}$ .

**Proof** Look at

$$\begin{aligned}
M1M2(x_0x_1 + x_3x_4) = \\
(M1M2_{11}x_0 + M1M2_{12}x_1 + M1M2_{13}x_2 + M1M2_{14}x_3 + M1M2_{15}x_4 + M1M2_{16}x_5) \\
(M1M2_{21}x_0 + M1M2_{22}x_1 + M1M2_{23}x_2 + M1M2_{24}x_3 + M1M2_{25}x_4 + M1M2_{26}x_5) + \\
(M1M2_{41}x_0 + M1M2_{42}x_1 + M1M2_{43}x_2 + M1M2_{44}x_3 + M1M2_{45}x_4 + M1M2_{46}x_5) \\
(M1M2_{51}x_0 + M1M2_{52}x_1 + M1M2_{53}x_2 + M1M2_{54}x_3 + M1M2_{55}x_4 + M1M2_{56}x_5)
\end{aligned}$$

The coefficient of  $x_i x_j$  will be

$$M1M2_{1i}M1M2_{2j} + M1M2_{1j}M1M2_{2i} + M1M2_{4i}M1M2_{5j} + M1M2_{4j}M1M2_{5i} \quad (*)$$

And the coefficient of  $x_i x_j$  in the expression

$$\begin{aligned} M2M1(x_0x_1 + x_3x_4) = & \\ & (M2M1_{11}x_0 + M2M1_{12}x_1 + M2M1_{13}x_2 + M2M1_{14}x_3 + M2M1_{15}x_4 + M2M1_{16}x_5) \\ & (M2M1_{21}x_0 + M2M1_{22}x_1 + M2M1_{23}x_2 + M2M1_{24}x_3 + M2M1_{25}x_4 + M2M1_{26}x_5) + \\ & (M2M1_{41}x_0 + M2M1_{42}x_1 + M2M1_{43}x_2 + M2M1_{44}x_3 + M2M1_{45}x_4 + M2M1_{46}x_5) \\ & (M2M1_{51}x_0 + M2M1_{52}x_1 + M2M1_{53}x_2 + M2M1_{54}x_3 + M2M1_{55}x_4 + M2M1_{56}x_5) \end{aligned}$$

The coefficient of  $x_i x_j$  will be

$$M2M1_{1i}M2M1_{2j} + M2M1_{1j}M2M1_{2i} + M2M1_{4i}M2M1_{5j} + M2M1_{4j}M2M1_{5i} \quad (*)$$

Since

$$M1M2_{ij} = \epsilon^{a_{i1}} * \epsilon^{b_{1j}} + \epsilon^{a_{i2}} * \epsilon^{b_{2j}} + \epsilon^{a_{i3}} * \epsilon^{b_{3j}} + \epsilon^{a_{i4}} * \epsilon^{b_{4j}} + \epsilon^{a_{i5}} * \epsilon^{b_{5j}} + \epsilon^{a_{i6}} * \epsilon^{b_{6j}}$$

, where  $a_{i1}, a_{i2} \dots a_{i6}, b_{1j}, \dots, b_{6j}$  are elements of the set  $\{0, 1, 2, 3, 4, 5\}$ , the expression in  $(*)$  will be of the form  $\sum \epsilon^{\gamma_u} \epsilon^{\gamma_v} \epsilon^{\gamma_w} \epsilon^{\gamma_x}$ . But since

$$M2M1_{ij} = \epsilon^{-a_{i1}} * \epsilon^{-b_{1j}} + \epsilon^{-a_{i2}} * \epsilon^{-b_{2j}} + \epsilon^{-a_{i3}} * \epsilon^{-b_{3j}} + \epsilon^{-a_{i4}} * \epsilon^{-b_{4j}} + \epsilon^{-a_{i5}} * \epsilon^{-b_{5j}} + \epsilon^{-a_{i6}} * \epsilon^{-b_{6j}},$$

the expression in  $(**)$  is of the form  $\sum \epsilon^{-\gamma_u} \epsilon^{-\gamma_v} \epsilon^{-\gamma_w} \epsilon^{-\gamma_x}$ . And similar for

$M1M2(x_2x_5), M1M2(x_1x_2+x_4x_5), M1M2(x_3x_0), M1M2(x_2x_3+x_5x_0)$  and  $M1M2(x_4x_1)$ .

So lemma 4.12 follows from 4.10 by replacing  $\epsilon^\alpha$  with  $\epsilon^{-\alpha}$  in the expressions for  $\beta_1$  and  $\beta_2$ . Same argument will also prove lemma 4.13

## References

- [1] Jacobsen, Robin Bjoernetun, Masteroppgave for graden matematikk, Matematisk Institutt, UIO 1999.
- [2] Fauskrud,Lars, Spesielle Cremona-Transformasjoner, Cand.Scient. Oppgave,Matematisk Institutt,UIO Authumn 1996
- [3] Hartshorne,Robin Algebraic Geometry,1977 Springer, Berlin