Proceedings of Symposia in Pure Mathematics
Volume **00**, 1998

Two-primary Algebraic K-Theory of Spaces and Related Spaces of Symmetries of Manifolds

John Rognes

Abstract. We outline the link between automorphisms (symmetries) of manifolds and algebraic K-theory of spaces. Then we discuss recent two-primary calculations for the algebraic K-theory of a point, and obtain a two-primary description of the stable smooth h-cobordism spaces and pseudoisotopy spaces of discs in a range of degrees.

Introduction

This paper is based on the author's lecture at the Seattle algebraic K-theory conference in July 1997. It begins with an elementary introduction to the results of the theory relating spaces of symmetries of manifolds to algebraic K-theory. This expresses spaces of pseudoisotopies or h-cobordisms of manifolds in terms of F. Waldhausen's algebraic K-theory of spaces, alias the functor $X \mapsto A(X)$. For a deeper survey of these topics, see [**WW**].

By a theorem of B. Dundas, this algebraic K-theory of spaces can in principle be expressed in terms of algebraic K-theory of rings, together with topological cyclic homology of rings and spaces. The paper goes on to present some recent two-primary calculations of the topological cyclic homology of a point, of topological cyclic homology of the integers, and of algebraic K-theory of the integers, and assembles these to give an explicit two-primary description of the algebraic K-theory of a point in a range of degrees. This gives information about the spaces of pseudoisotopies and h-cobordisms of discs of high dimension, in this range of degrees.

Some ideas from geometric topology

Let us begin by asking some geometric questions. We will later point out how these relate to the algebraic K-theory of spaces.

Surgery. Surgery theory addresses the question: Which homotopy types contain manifolds ? That is, given a homotopy type of spaces, does there exist a manifold of that homotopy type ? Surgery theory reduces this question to relatively standard problems in algebraic topology. Now if the answer is yes then we

1991 *Mathematics Subject Classification.* 19Jxx, 55Q45, 57N37, 57N70, 57R50.

may also ask: How many manifolds there are in such a given homotopy type ? To make this question well-posed, we should really ask how many isomorphism classes of manifolds there are in this or that homotopy type, since otherwise the answer will not even be a set. Here isomorphism of manifolds can mean diffeomorphism if we are talking about smooth (C^∞-) manifolds, or homeomorphism if we are thinking of topological manifolds. Surgery theory also answers this refined question, in the sense that it is reduced to a question in algebraic topology.

Symmetries. But we may go on. Beyond asking how many isomorphism classes of manifolds there are in a homotopy type, we may ask in how many ways two abstractly isomorphic manifolds then in fact are isomorphic. That is, how many isomorphisms are there between them ? By fixing one choice of isomorphism we may assume that the two manifolds are one and the same, and in this case the question is: In how many ways is a manifold isomorphic to itself ? Such self-isomorphisms are precisely the automorphisms or symmetries of the manifold. So if we have some category of manifolds in mind, we may consider the set of self-maps of this manifold preserving the structures endowed upon it.

For example, considering Riemannian manifolds, smooth manifolds, topological manifolds or spaces, we have the following increasing chain of symmetries available:

$$\begin{aligned} \text{Isometries} &\subset \text{Diffeomorphisms} \\ &\subset \text{Homeomorphisms} \subset \text{Homotopy equivalences} \end{aligned}$$

These sets of symmetries can naturally be topologized, and thus become topological groups or monoids. We are fundamentally interested in understanding these *spaces of symmetries of manifolds.*

In the case of the sphere S^n, this amounts to the sequence of topological groups or monoids:

$$O(n+1) \to \mathrm{Diff}(S^n) \to \mathrm{Homeo}(S^n) \to G(S^n)$$

Here $O(n+1)$ denotes the orthogonal group of linear isometries of $\mathbb{R}^{n+1}$, or equivalently of S^n with the standard metric, while we write $G(X)$ for the grouplike monoid of self-homotopy equivalences of a space X. The first map above admits a left inverse (a retraction), essentially given by taking a diffeomorphism of S^n to the linear isomorphism induced by its derivative at, say, the north pole of S^n, orthogonalized by the Gram–Schmidt process. Hence $\mathrm{Diff}(S^n)$ splits off a factor $O(n+1)$, represented by the linear diffeomorphisms. We of course ask: What is the remaining factor ?

In general the topological types of these automorphism groups are infinite dimensional, i.e., large and complicated, so let us settle for studying their homotopy types.

QUESTION. *What is the homotopy type of these spaces of symmetries ?*

Here is an example of a theorem in this direction, proving a result called the Smale conjecture:

THEOREM (HATCHER [**H**]). *The natural map*

$$O(4) \xrightarrow{\simeq} \mathrm{Diff}(S^3)$$

is a homotopy equivalence.

The corresponding results for S^n with $n = 1, 2$ are much easier, while for $n \geq 4$ the homotopy type of the remaining factor is unknown.

h-cobordism, pseudoisotopy. Let us review some other kinds of symmetries that arise in geometric topology, and focus on the smooth category for definiteness.

DEFINITION. Let M, M' be (smooth) closed n-manifolds. A compact $(n+1)$-manifold W with boundary $\partial W \cong M \coprod M'$ is called a *cobordism* from M to M'. If the inclusions

$$M \xrightarrow{\simeq} W \xleftarrow{\simeq} M'$$

are homotopy equivalences, then W is called an *h-cobordism*.

Given an h-cobordism W as above, J.H.C. Whitehead showed how to associate an element $\tau(W, M) \in \mathrm{Wh}_1(\pi_1 M)$ to it, called its *Whitehead torsion*. Here $\mathrm{Wh}_1(\pi) = K_1(\mathbb{Z}\pi)/\{\pm\pi\}$ is the *Whitehead group* of the common fundamental group $\pi = \pi_1 M = \pi_1 W$. This is perhaps the first place where algebraic K-theory (through K_1-groups of group rings) enters into geometric topology. See [**Mi**] for a nice survey.

We recall that $\mathbb{Z}\pi$ is the group ring of π, $K_1(R) = GL(R)/E(R)$ is the Abelian group of infinite invertible matrices modulo infinite elementary matrices with coefficients in a ring R, and $\{\pm\pi\} \subset GL_1(\mathbb{Z}\pi) \subset GL(\mathbb{Z}\pi)$ naturally maps to $K_1(\mathbb{Z}\pi)$.

The main application of associating a Whitehead torsion element to an h-cobordism is to detect whether the h-cobordism can be trivialized or not. The trivial h-cobordism from M is the cylinder $M \times I$, with boundary $M \times \partial I = M \times 0 \coprod M \times 1$. We identify M with $M \times 0$ in the obvious way. Then an h-cobordism W is said to be *trivial* if it is isomorphic to this particularly trivial example. The s-cobordism theorem of Barden, Mazur and Stallings asserts that this is the case if and only if the associated Whitehead torsion element $\tau(W, M)$ is zero in $\mathrm{Wh}_1(\pi_1 M)$. See e.g. [**K**] for a proof.

s-COBORDISM THEOREM (BARDEN, MAZUR, STALLINGS). *Suppose* $\dim(M) \geq 5$. *If* $\tau(W, M) = 0$ *in* $\mathrm{Wh}_1(\pi_1 M)$ *then there exists a diffeomorphism*

$$\alpha\colon (W, M) \xrightarrow{\cong} (M \times I, M \times 0),$$

and conversely if such a trivialization exists then $\tau(W, M) = 0$.

Furthermore any element in $\mathrm{Wh}_1(\pi_1 M)$ *can be realized as the Whitehead torsion of an h-cobordism.*

This theorem tells us precisely when such a trivialization α exists. However, when one exists, it will not be unique ! So given another trivialization $\beta\colon (W, M) \to (M \times I, M \times 0)$, we can compare the two, and obtain a diffeomorphism

$$\psi = \beta\alpha^{-1}\colon (M \times I, M \times 0) \xrightarrow{\cong} (M \times I, M \times 0).$$

Such a trivialization ψ of the trivial h-cobordism is called a *pseudoisotopy* of M, or equivalently a *concordance* of M. It is an element in the space

$$P(M) = \mathrm{Diff}(M \times I \ \mathrm{rel}\ \ M \times 0)$$

of diffeomorphisms of $M \times I$ that fix (a neighborhood of) the lower edge $M \times 0$, which we call the *pseudoisotopy space* of M. Again this is a topological group, equal to the space of symmetries of a trivial h-cobordism.

The name 'concordance' may refer to how ψ compares different trivializations, while 'pseudoisotopy' expresses that such maps ψ generalize isotopies from the identity of M. For an isotopy $t \mapsto \phi_t \in \mathrm{Diff}(M)$ with $\phi_0 = 1_M$ defines a pseudoisotopy ψ by the formula $\psi(x,t) = (\phi_t(x), t)$. Here $(x,t) \in M \times I$.

The pseudoisotopy space is closely linked to the diffeomorphism spaces we considered at the outset. When W is a manifold with boundary (such as $M \times I$) we write $\mathrm{Diff}(W)$ for the topological group of diffeomorphisms fixing the boundary ∂W.

THEOREM (CERF [**C**]). *There is a fiber sequence*

$$\mathrm{Diff}(M \times I) \to P(M) \xrightarrow{r_1} \mathrm{Diff}(M)$$

onto the path components in $\mathrm{Diff}(M)$ *that are in the image of* r_1. *Here* r_1 *restricts a pseudoisotopy* ψ *acting on* $M \times I$ *to the upper edge* $M \times 1 \cong M$.

In a related vein, we can embed any h-cobordism W into $M \times I$, as a codimension 0 submanifold. (To see this, suppose W is an h-cobordism from M to M'. Realize $-\tau(W, M) \in \mathrm{Wh}_1(\pi_1 M)$ as the Whitehead torsion of an h-cobordism W' from M'. Then $W \cup_{M'} W'$ has zero Whitehead torsion, hence is isomorphic to $M \times I$, and contains W as a codimension 0 submanifold.) Restricting attention to h-cobordisms from M that arise as such codimension 0 submanifolds of $M \times I$, we can topologize the set of such, and form the *space of h-cobordisms* from M, denoted $H(M)$.

PROPOSITION [**W3**].
(a) $\pi_0 H(M) \cong \mathrm{Wh}_1(\pi_1 M)$. *Hence the isotopy classes of h-cobordisms from M are in bijection, via their Whitehead torsion, with the elements of the Whitehead group.*
(b) $\Omega H(M) \simeq P(M)$. *Hence the homotopy groups of* $H(M)$ *and* $P(M)$ *agree up to a shift by one degree.*

So the spaces $H(M)$, $P(M)$ and $\mathrm{Diff}(M)$ are closely related and have direct geometric interest.

Stabilization. As in homotopy theory these constructions can be stabilized, by increasing the dimension of the manifolds in question by multiplying them with cubes I^ℓ for $\ell \geq 0$. (This replaces a closed manifold M with a manifold with boundary, or even with corners, and so it becomes necessary to extend the preceding discussion to cover such cases too. In general a pseudoisotopy of M is required to fix $\partial M \times I$, in addition to $M \times 0$. This forces the stabilization map $P(M) \to P(M \times I)$ to involve some 'bending around collars'; see [**Ig, §2**].)

The stabilization process turns out to simplify the homotopy types of these spaces. We define

$$\mathcal{P}(M) = \mathrm{colim}_\ell\, P(M \times I^\ell)$$
$$\mathcal{H}(M) = \mathrm{colim}_\ell\, H(M \times I^\ell)$$

as the *stable pseudoisotopy space* and the *stable h-cobordism space* of M, respectively.

The miracle is that these functors now only depend on the homotopy type of M, and in fact take values in infinite loop spaces.

Theorem (Waldhausen [**W4**]). *There exists a homotopy functor*

$$\mathrm{Wh}\colon \mathit{Spaces} \to \Omega^\infty\text{-}\mathit{Spaces}$$
$$X \longmapsto \mathrm{Wh}(X)$$

such that

$$\Omega\,\mathrm{Wh}(M) \simeq \mathcal{H}(M)$$
$$\Omega^2\,\mathrm{Wh}(M) \simeq \mathcal{P}(M)$$

when M is a manifold.

We call $\mathrm{Wh}(X)$ the *Whitehead space* of X. There is actually one Whitehead space for each category of manifolds, and when necessary we will indicate the category (Diff or Top) by a superscript.

Algebraic K-theory of spaces

Roughly speaking, the algebraic K-theory of a space X can be thought of as the algebraic K-theory of the 'ring up to homotopy'

$$Q(\Omega X_+) = \mathrm{colim}_n\, \Omega^n\Sigma^n(\Omega X_+).$$

Here the loop space ΩX is an H-group, and can be modeled by an actual simplicial group called the *Kan loop group*. Hence we may think of ΩX as a topological group. The subscript $+$ denotes addition of a disjoint base point, and the spherical group ring $Q(\Omega X_+)$ is analogous to the usual integral group ring $\mathbb{Z}\pi$ on a group π. In fact, there is a 'ring homomorphism up to homotopy' from the spherical group ring $Q(\Omega X_+)$ to the integral group ring $\mathbb{Z}\pi$ when $\pi = \pi_1 X$, induced by the map to path components $\Omega X \to \pi_0\Omega X = \pi_1 X$ and a *linearization map* $Q(\pi_+) \to \mathbb{Z}\pi$.

We are being vague here, because there are many compatible definitions of the algebraic K-theory space $A(X)$ of the space X, but they all require some technical preparations that we do not wish to go into. As with the K-theory of rings, where the algebraic K-groups $K_i(R)$ appear in a unified way as the homotopy groups of a space $K(R)$, we can and will focus on the space (or spectrum) $A(X)$ as a whole, and only consider the homotopy groups $\pi_i A(X)$ when we are unable to do better.

In the simplest case, when $X = *$ is a point, we can give a precise definition of $A(*)$ and the linearization map $L\colon A(*) \to K(\mathbb{Z})$ as follows:

$$A(*) = \Omega B\big(\coprod_{k\ge 0} \mathrm{colim}_n\, BG(\vee^k S^n)\big) \xrightarrow{L} K(\mathbb{Z}) = \Omega B\big(\coprod_{k\ge 0} BGL_k(\mathbb{Z})\big)$$

Here $\vee^k S^n$ is the one-point union (wedge) of k copies of S^n. $G(-)$ denotes the monoid of self-homotopy equivalences, as before, and B denotes the bar construction. We may stabilize self-homotopy equivalences by suspension, and thus pass to the direct limit over n. The disjoint union of the spaces $\mathrm{colim}_n\, BG(\vee^k S^n)$ over all $k \ge 0$ admits a monoid pairing induced by wedge sum, taking a self-homotopy equivalence of $\vee^k S^n$ and one of $\vee^\ell S^n$ to one of $\vee^{k+\ell} S^n$. Applying ΩB to this topological monoid amounts to group completion, and is (in essence) equivalent to using Quillen's plus-construction.

The reduced homology of $\vee^k S^n$ is a copy of $\mathbb{Z}^k$ in degree n, and a self-homotopy equivalence of this space determines a linear isomorphism of $\mathbb{Z}^k$, or equivalently an element in $GL_k(\mathbb{Z})$. This construction is invariant under stabilization by suspensions, so induces maps $L\colon \operatorname{colim}_n BG(\vee^k S^n) \to BGL_k(\mathbb{Z})$, which relate wedge sum to Whitney (block) sum of matrices. Then the right hand space above is a model for the group completion of the nerve of the category of finitely generated free $\mathbb{Z}$-modules, i.e., for $K(\mathbb{Z})$, and passage to homology classes determines the linearization map $L\colon A(*) \to K(\mathbb{Z})$, as displayed.

In general, the association $X \mapsto A(X)$ is a homotopy functor:

$$A\colon \text{Spaces} \to \Omega^\infty\text{-Spaces}$$

Theorem (Waldhausen [**W3, W5**]).
(a) In the smooth category, there is a homotopy fiber sequence of infinite loop spaces, natural in X:

$$Q(X_+) \to A(X) \to \mathrm{Wh}^{\mathrm{Diff}}(X)\,.$$

There is a natural infinite loop splitting $A(X) \to Q(X_+)$, so $A(X) \simeq Q(X_+) \times \mathrm{Wh}^{\mathrm{Diff}}(X)$ as infinite loop spaces.

(b) In the topological category, there is a homotopy fiber sequence of infinite loop spaces, natural in X:

$$h(X, A(*)) \xrightarrow{\alpha} A(X) \to \mathrm{Wh}^{\mathrm{Top}}(X)\,.$$

Here $h(X, A(*)) = \Omega^\infty(A(*) \wedge X_+)$ is the value at the space X of the generalized homology theory associated to the spectrum $A(*)$. (The smash product $A(*) \wedge X_+$ is formed in the category of spectra.) The left map α is the assembly map in the algebraic K-theory of spaces.

So $A(*) \simeq QS^0 \times \mathrm{Wh}^{\mathrm{Diff}}(*)$ is central to both the smooth and the topological theory. In both categories $\mathrm{Wh}(M)$ was related to stable pseudoisotopy- and h-cobordism spaces in the section above. The following stability theorem tells us to what extent the stabilized theories agree with the unstable, geometrically relevant pseudoisotopy- and h-cobordism spaces.

Theorem (Igusa [**Ig**]). *Let M^n be a smooth n-manifold, and suppose $k \ll n/3$. (More precisely, suppose $n \geq \max\{2k+7, 3k+4\}$.) Then the stabilization map*

$$P(M) \to \mathcal{P}(M) \simeq \Omega^2\,\mathrm{Wh}(M)$$

is at least k-connected, in both the smooth and topological categories.

In particular, the map

$$P(D^n) \to \mathcal{P}(*) \simeq \Omega^2\,\mathrm{Wh}(*)$$

is roughly $n/3$-connected, and so $\pi_i P(D^n) \cong \pi_{i+2}\,\mathrm{Wh}^{\mathrm{Diff}}(*)$ for $i \ll n/3$ in the smooth category.

From here on we focus on the smooth category again.

Rational information

The linearization map $L\colon A(*) \to K(\mathbb{Z})$ is a rational equivalence, i.e., it induces an isomorphism of homotopy groups tensored with $\mathbb{Q}$. Combined with Borel's calculation of $K_i(\mathbb{Z}) \otimes \mathbb{Q}$, this gives a classical rational calculation of $A(*)$, and thus of $\mathrm{Wh}(*)$, $P(D^n)$ and $\mathrm{Diff}(D^n)$ in the stable range.

Theorem (Farrel and Hsiang [**FH**]). *Suppose $i \ll n/3$. Then*

$$\pi_i \mathrm{Diff}(D^n) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \textit{for } i \equiv 3 \bmod 4 \textit{ and } n \textit{ odd} \\ 0 & \textit{else.} \end{cases}$$

There are also reasonably explicit rational calculations for $A(X)$ for more general spaces X; see [**DHS**].

These results can be contrasted with the topological case, where the space $\mathrm{Homeo}(D^n)$ of homeomorphisms of D^n fixing the boundary is contractible by the Alexander trick: A contraction gradually reparametrizes a given homeomorphism over concentric discs of shrinking radius, while leaving the surrounding annulus fixed.

A more recent rational calculation involves the *cyclotomic trace map*

$$\mathrm{trc}_X \colon A(X) \to TC(X)$$

from the algebraic K-theory of spaces $A(X)$ to topological cyclic homology $TC(X)$; see [**BHM**].

Theorem (Bökstedt, Hsiang and Madsen). *The cyclotomic trace map*

$$\mathrm{trc}_* \colon A(*) \to TC(*)$$

is rationally injective.

As an application, these authors show that the K-theoretic assembly map

$$\alpha \colon K(\mathbb{Z}) \wedge B\Gamma_+ \to K(\mathbb{Z}\Gamma)$$

is rationally injective for groups Γ with $H_*(B\Gamma)$ finitely generated in each degree. This is the K-theoretic version of the Novikov conjecture for the group Γ.

Primary information

More recently, it has also become possible to access torsion information about the algebraic K-theory of spaces, and thus about the spaces of symmetries of manifolds. This is achieved by means of the following theorem of Dundas. We state it in its simplest interesting case:

Theorem (Dundas [**D**]). *The square*

$$\begin{array}{ccc} A(*) & \xrightarrow{L} & K(\mathbb{Z}) \\ \downarrow{\scriptstyle \mathrm{trc}_*} & & \downarrow{\scriptstyle \mathrm{trc}_{\mathbb{Z}}} \\ TC(*) & \xrightarrow{L} & TC(\mathbb{Z}) \end{array}$$

is homotopy Cartesian (after p-adic completion at any prime p). Hence $A(*)$ is homotopy equivalent to the homotopy fiber product of $TC(*)$ and $K(\mathbb{Z})$ over $TC(\mathbb{Z})$.

We pause to explain the diagram. Both algebraic K-theory and topological cyclic homology are spectrum-valued functors defined on a category of (strictly associative) ring spectra F, and the cyclotomic trace map $\mathrm{trc}_F \colon K(F) \to TC(F)$ is a natural transformation. This class of ring spectra contains the spherical group rings with underlying space $Q(\Omega X_+)$ for connected spaces X, as well as ordinary rings. Furthermore, the linearization map $L \colon Q(\Omega X_+) \to \mathbb{Z}\pi_1(X)$ is a morphism in this category of ring spectra. We write $\mathrm{trc}_X \colon A(X) \to TC(X)$ for the cyclotomic trace map in the case of the spherical group ring $Q(\Omega X_+)$. Then naturality of the cyclotomic trace map with respect to the linearization map asserts that there is a commutative square of spectra with $\pi = \pi_1 X$:

$$\begin{array}{ccc} A(X) & \xrightarrow{L} & K(\mathbb{Z}\pi) \\ \downarrow{\scriptstyle \mathrm{trc}_X} & & \downarrow{\scriptstyle \mathrm{trc}_{\mathbb{Z}\pi}} \\ TC(X) & \xrightarrow{L} & TC(\mathbb{Z}\pi) \end{array}$$

Dundas' theorem also tells us that this square is homotopy Cartesian. We recover the statement above in the case when $X = *$ is a point.

The history of this result begins with Goodwillie's theorem [**G1**] that relative K-theory is rationally equivalent to relative negative cyclic homology HC^- for nilpotent extensions of (simplicial) rings. This can be expressed as a rationally homotopy Cartesian square similar to the ones above. Replacing negative cyclic homology with topological cyclic homology, McCarthy [**McC**] proved that relative K-theory is also p-adically equivalent to relative topological cyclic homology TC for nilpotent extensions of (simplicial) rings. Goodwillie conjectured in his 1990 ICM talk [**G2**] that the same result should hold for maps of arbitrary (strictly associative) ring spectra that induce nilpotent extensions on π_0. This is what was proven by Dundas, and the versions of the theorem stated above amount to the special case of the linearization map $L \colon Q(\Omega X_+) \to \mathbb{Z}\pi_1 X$ of ring spectra. That map induces an isomorphism of rings on π_0, which certainly is a (trivial) nilpotent extension.

Vista

We now wish to use Dundas' theorem [**D**] to compute $A(*)$ completed at the prime 2. To do this, we first use the homotopy-theoretic description of $TC(*)$ from [**BHM**] to give a calculation of $\pi_* TC(*)$ in a range of degrees ($* \le 21$). Then we recall the calculation of $\pi_* TC(\mathbb{Z})$ at 2 from [**R5**], which in non-negative degrees agrees with the K-theory $K_*(\hat{\mathbb{Z}}_2)$ of the 2-adic integers. Next we review the 2-primary calculation of $K(\mathbb{Z})$ from [**RW**], which uses Voevodsky's proof of the Milnor conjecture [**V**]. Then $A(*)$ is in principle determined as the homotopy pullback in the square of Dundas' theorem. In practice this also involves determining the homotopical behavior of the maps $\mathrm{trc}_{\mathbb{Z}} \colon K(\mathbb{Z}) \to TC(\mathbb{Z})$ and $L \colon TC(*) \to TC(\mathbb{Z})$. The former map was described in [**R5**], and we have more recently used homotopy theoretic techniques to study the linearization map L in a range of degrees ($* \le 15$). As a conclusion, we are able to compute $\pi_* A(*)$, and thus $\pi_* \mathrm{Wh}(*)$ completed at 2, for $* \le 14$. This range of degrees is sufficient to allow the detection of certain v_1^4-periodic phenomena related to Bott periodicity and K-local spectra. In particular we can make statements about the 2-adic connectivity of the Hatcher–Waldhausen map [**W3, R1**]

$$hw \colon G/O \to \Omega\,\mathrm{Wh}(*)$$

in the smooth category. The aim for the remainder of the paper is to outline these homotopy-theoretic calculations.

Topological cyclic homology of a point: $TC(*)$

Let $C_q \subset S^1$ be the cyclic subgroup of order q. The *topological Hochschild homology* of a point $THH(*) = T(*)$ is an S^1-spectrum, and is C_q-equivariantly homotopy equivalent to the S^1-equivariant sphere spectrum QS^0 for each q:

$$THH(*) \simeq_{C_q} QS^0$$

Fixing a prime p and restricting attention to cyclic groups of order powers of p, there are *restriction* and *Frobenius* maps

$$R\colon THH(*)^{C_{p^n}} \to THH(*)^{C_{p^{n-1}}}$$
$$F\colon THH(*)^{C_{p^n}} \to THH(*)^{C_{p^{n-1}}}$$

for all $n \geq 1$. By definition the p-primary *topological cyclic homology* of a point is a homotopy limit

$$TC(*, p) = \mathrm{holim}_{R,F}\, THH(*)^{C_{p^n}} \simeq \mathrm{holim}_{R,F}\, Q(S^0)^{C_{p^n}}$$

over a suitable category generated by these restriction and Frobenius maps; see [**BHM, HM**] for more on these constructions. Hereafter we implicitly complete everything at p, and simply write $TC(*)$ for $TC(*, p)$.

The defining limit for $TC(*)$ can be analyzed in terms of the Segal–tom Dieck splitting [**S1, tD**]

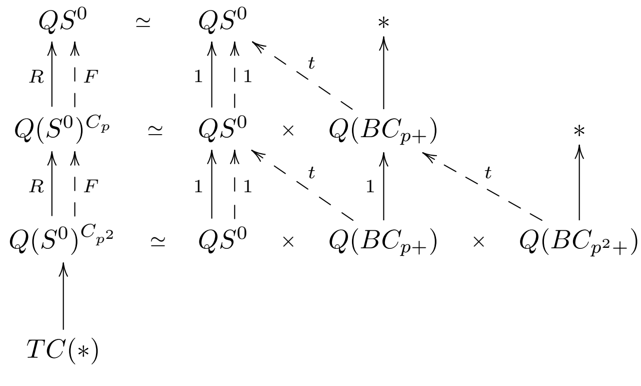
$$Q(S^0)^{C_{p^n}} \simeq \prod_{i=0}^{n} Q(BC_{p^i+}).$$

With respect to this factorization, the restriction map R is the identity on the ith factor for $0 \leq i < n$, and is trivial on the last (nth) factor. The Frobenius map F is the identity on the initial (0th) factor, and maps the ith factor to the $(i-1)$st factor by the Becker–Gottlieb transfer map [**BG**]

$$t\colon Q(BC_{p^i+}) \to Q(BC_{p^{i-1}+})$$

of the p-fold covering $BC_{p^{i-1}} \to BC_{p^i}$, for $0 < i \leq n$. So $R(x_0, x_1, \ldots, x_n) = (x_0, x_1, \ldots, x_{n-1})$, while $F(x_0, x_1, \ldots, x_n) = (x_0 + t(x_1), t(x_2), \ldots, t(x_n))$. See [**BHM, 5.18**].

The following diagram displays the first few relevant maps; the R-maps are solid and the F-maps are dashed.



The analysis gives the following calculation of $TC(*)$, which is a special case of a more general calculation of $TC(X)$ for any space X:

Theorem (Bökstedt, Hsiang and Madsen). *There is a homotopy Cartesian square*

$$\begin{array}{ccc} TC(*) & \xrightarrow{\alpha} & Q(\Sigma\mathbb{C}P^\infty_+) \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle \mathrm{trf}_{S^1}} \\ QS^0 & \xrightarrow{0} & QS^0. \end{array}$$

The composite map $A(*) \to TC(*) \xrightarrow{\beta} QS^0$ is the splitting map in the factorization $A(*) \simeq QS^0 \times \mathrm{Wh}(*)$.

More precisely, there exists a strictly commutative homotopy Cartesian square which is homotopy equivalent to the square displayed in the theorem. The S^1-equivariant transfer map trf_{S^1} is obtained from the 'dimension-shifting transfer' of [**LMS, p. 100**], which is a map of S^1-equivariant spectra

$$\tau\colon \Sigma^\infty_{S^1}(\mathbb{C}P^\infty_+) \to \Sigma^{-1}\Sigma^\infty_{S^1}(ES^1_+),$$

by passing to underlying non-equivariant spectra, delooping once, and mapping ES^1_+ to S^0.

A virtual Thom spectrum. Let γ^1 be the tautological complex line bundle over $\mathbb{C}P^\infty$. Working with spectra we can form the *virtual Thom spectrum* $\mathbb{C}P^\infty_{-1} = (\mathbb{C}P^\infty)^{-\gamma^1}$ of the formal negative of this line bundle. This spectrum has one cell in each even degree ≥ -2, corresponding to complex dimensions ≥ -1. Its connective cover is the suspension spectrum on $\mathbb{C}P^\infty_+$, and the attaching map of $\mathbb{C}P^\infty_+$ onto the (-2)-cell is the S^1-equivariant transfer map, up to a degree shift [**Ra**]. Hence there is a fiber sequence of underlying spaces:

$$\Omega^\infty(\Sigma\mathbb{C}P^\infty_{-1}) \to Q(\Sigma\mathbb{C}P^\infty_+) \xrightarrow{\mathrm{trf}_{S^1}} QS^0$$

We note that by [**BHM, 5.15**] there is a homotopy equivalence

$$Q(\Sigma\mathbb{C}P^\infty_+) \simeq \mathrm{holim}_n\, Q(BC_{p^n+})$$

(implicitly completed at p), where the homotopy limit is formed over the Becker–Gottlieb transfer maps t. The S^1-equivariant transfer trf_{S^1} is the map from this homotopy limit to the $n = 0$ term, which is QS^0.

Taking vertical homotopy fibers in the theorem above, we obtain:

Corollary. *There is a split fiber sequence of infinite loop spaces*

$$\Omega^\infty(\Sigma\mathbb{C}P^\infty_{-1}) \to TC(*) \xrightarrow{\beta} QS^0.$$

The splitting is given by the unit map $QS^0 \to TC(*)$.

Stable homotopy of $\mathbb{C}P^\infty_{-1}$. To compute the spectrum homotopy of $\mathbb{C}P^\infty_{-1}$ in a range, we use the Atiyah–Hirzebruch spectral sequence for stable homotopy theory:

$$E^2_{p,q} = H_p(\mathbb{C}P^\infty_{-1}; \pi^S_q) \Longrightarrow \pi_{p+q}\Omega^\infty(\mathbb{C}P^\infty_{-1})$$

Here classes $x_{2n} \in H_{2n}(\mathbb{C}P^\infty_{-1}) \cong \mathbb{Z}$ for $n \geq -1$ additively generate the entire homology. Hence the E^2-term of this spectral sequence has a copy of the stable homotopy groups of spheres (the stable stems) $\pi^S_* = \pi_* QS^0$, in each even column starting in filtration degree -2. Based on work by Mosher [**Mo**] and Mukai [**Mu1,**

Mu2, Mu3], the author as made such calculations in the range of total degrees ≤ 20, where there are approximately 100 nonzero differentials.

In the following theorem, the E^∞-representatives for permanent cycles are given on the form αx_{2n}, with $\alpha \in \pi_*^S$ given in Toda's notation [**T**]. Thus η, ν and σ are the Hopf maps, while μ, ζ and ρ are in the image of J-summand. We write $A \rtimes B$ for an extension of B by A as Abelian groups.

Theorem (Mosher, Mukai, Rognes). *The homotopy groups $\pi_n \Omega^\infty(\mathbb{C}P^\infty_{-1})$ are known for $n \leq 20$, and begin:*

n	$\pi_n \Omega^\infty(\mathbb{C}P^\infty_{-1})$	E^∞-*rep.*
-2	$\mathbb{Z}$	x_{-2}
-1	0	
0	$\mathbb{Z}$	$2x_0$
1	0	
2	$\mathbb{Z}$	$4x_2$
3	$\mathbb{Z}/8$	νx_0
4	$\mathbb{Z}$	$2x_4$
5	$\mathbb{Z}/2$	σx_{-2}
6	$\mathbb{Z}/2 \oplus \mathbb{Z}$	$\nu^2 x_0, 16x_6$
7	$\mathbb{Z}/2 \rtimes \mathbb{Z}/8$	$\mu x_{-2}, 2\sigma x_0$
8	$\mathbb{Z}/2 \oplus \mathbb{Z}$	$\nu^2 x_2, 8x_8$
9	$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/8$	$\eta^2\sigma x_0, \eta\bar{\nu} x_0, \sigma x_2$
10	$\mathbb{Z}$	$32x_{10}$
11	$\mathbb{Z}/8 \oplus \mathbb{Z}/4$	$\zeta x_0, 2\sigma x_4$
12	$\mathbb{Z}$	$16x_{12}$
13	$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \rtimes \mathbb{Z}/2$	$\rho x_{-2}, \zeta x_2, \eta^2\sigma x_4$

The extension in degree 7 is cyclic (yielding a copy of $\mathbb{Z}/16$), while the extension in degree 13 is unresolved.

Topological cyclic homology of the integers: $TC(\mathbb{Z})$

For p odd, $TC(\mathbb{Z}, p) = TC(\mathbb{Z})^\wedge_p$ was computed by Bökstedt and Madsen. For $p = 2$ the author's calculation of $TC(\mathbb{Z})^\wedge_2$ will appear in [**R2, R3, R4 and R5**]. The following two theorems describe the conclusion, where we implicitly complete at 2.

Theorem (McCarthy [**McC**], Hesselholt and Madsen [**HM**]). *There is a homotopy fiber sequence of spectra*

$$K(\hat{\mathbb{Z}}_2) \to TC(\mathbb{Z}) \to K(\mathbb{Z}, -1).$$

Hence $K(\hat{\mathbb{Z}}_2)$ is homotopy equivalent to the connective cover of $TC(\mathbb{Z})$.

Theorem (Rognes [**R5**]). *There are two homotopy fiber sequences of infinite loop spaces*

$$\begin{array}{ccccc} B\,\mathrm{Im}J_{\mathbb{C}} & \longrightarrow & K^{\mathrm{red}}(\hat{\mathbb{Z}}_2) & \longrightarrow & K(\hat{\mathbb{Z}}_2) \\ & & \downarrow & & \downarrow{\scriptstyle \mathrm{red}} \\ & & BBU & & \mathrm{Im}J_{\mathbb{C}}. \end{array}$$

Here $\mathrm{Im}J_{\mathbb{C}} \simeq K(\mathbb{F}_3)$ *is the complex image of* J*-spectrum, and* red *is a Galois reduction map.*

The fiber sequences in question each consist of a map going across, followed by a map going down. Hence this result expresses $K(\hat{\mathbb{Z}}_2)$ after 2-adic completion as a three-stage extension of known infinite loop spaces, each of which agrees with its (Bousfield) K-localization above degree 1. The extensions are also known, and induce split short exact sequences on the level of homotopy groups. So $K(\hat{\mathbb{Z}}_2)$ agrees with its K-localization above degree 1, and its homotopy groups

$$K_*(\hat{\mathbb{Z}}_2) \cong \pi_*\,\mathrm{Im}J_{\mathbb{C}} \oplus \pi_* BBU \oplus \pi_* B\,\mathrm{Im}J_{\mathbb{C}}$$

are completely known.

Algebraic K-theory of the integers: $K(\mathbb{Z})$

The 2-torsion in $K_*(\mathbb{Z})$ is found in [**RW**] using Voevodsky's proof of the Milnor Conjecture [**V**], Suslin and Voevodsky's subsequent identification [**SV**] of Bloch's higher Chow groups [**Bl**] with étale cohomology groups, a mod 2 version of the Bloch–Lichtenbaum spectral sequence [**BL**] converging to algebraic K-theory, and the topological data from the above calculation of $K(\hat{\mathbb{Z}}_2)$ to control the differentials in that spectral sequence.

The outcome is that Bökstedt's model $JK(\mathbb{Z})$ for the algebraic K-theory of the integers, defined in [**Bö**] as the homotopy fiber of the composite

$$\mathbb{Z} \times BO \xrightarrow{\psi^3 - 1} BSpin \xrightarrow{c} BSU$$

gives the correct answer for $K(\mathbb{Z})$, after localization or completion at 2. So $K(\mathbb{Z}) \simeq JK(\mathbb{Z})$ at 2, and there are 2-adic fiber sequences:

$$BBO \longrightarrow K(\mathbb{Z}) \longrightarrow \mathrm{Im}J_{\mathbb{C}}$$

$$K(\mathbb{Z}) \longrightarrow \mathbb{Z} \times BO \xrightarrow{c\circ(\psi^3-1)} BSU$$

$$\mathrm{Im}J_{\mathbb{R}} \longrightarrow K(\mathbb{Z}) \longrightarrow BBSO$$

In the last sequence, $\mathrm{Im}J_{\mathbb{R}}$ denotes the connective real image of J-spectrum, which agrees with the K-localization of the sphere spectrum above degree 1.

So $K(\mathbb{Z})$ agrees with its K-localization above degree 1, and $K(\mathbb{Z})^\wedge_2$ is completely known.

A fiber sequence

Recall the splittings $A(*) \simeq QS^0 \times \mathrm{Wh}(*)$ and $TC(*) \simeq QS^0 \times \Omega^\infty(\Sigma\mathbb{C}P^\infty_{-1})$. The cyclotomic trace map trc_* respects the projection to the QS^0-factors given by the older trace map to $THH(*) \simeq QS^0$ from [**W2**]. Hence we can fiber off a factor QS^0 from Dundas' theorem, and obtain two homotopy Cartesian squares:

$$\begin{array}{ccccc} \mathrm{Wh}(*) & \longrightarrow & A(*) & \xrightarrow{L} & K(\mathbb{Z}) \\ \downarrow{\scriptstyle \widetilde{\mathrm{trc}}} & & \downarrow{\scriptstyle \mathrm{trc}_*} & & \downarrow{\scriptstyle \mathrm{trc}_{\mathbb{Z}}} \\ \Omega^\infty(\Sigma\mathbb{C}P^\infty_{-1}) & \longrightarrow & TC(*) & \xrightarrow{L} & TC(\mathbb{Z}) \end{array}$$

Comparing vertical homotopy fibers then leads to the following fiber sequence

$$(*) \qquad \Omega\,\mathrm{Wh}(*) \xrightarrow{\Omega\widetilde{\mathrm{trc}}} \Omega^\infty(\mathbb{C}P^\infty_{-1}) \xrightarrow{\ell} \mathrm{hofib}(\mathrm{trc}_{\mathbb{Z}}).$$

Here ℓ is induced by the linearization map L.

The fiber of the cyclotomic trace map

The calculation of $TC(\mathbb{Z})$ at 2 simultaneously gave a complete description of the cyclotomic trace map

$$\mathrm{trc}_{\mathbb{Z}} \colon K(\mathbb{Z}) \to K(\hat{\mathbb{Z}}_2) \to TC(\mathbb{Z})$$

on homotopy. In particular we proved:

THEOREM (ROGNES [**R5**]). *The natural map*

$$\hat{\mathbb{Z}}_2 \cong K_{4i+1}(\mathbb{Z})^\wedge_2/(torsion) \to K_{4i+1}(\hat{\mathbb{Z}}_2)^\wedge_2/(torsion) \cong \hat{\mathbb{Z}}_2$$

is an isomorphism for all $i \geq 1$.

We can use this to compute $\pi_*\,\mathrm{hofib}(\mathrm{trc}_{\mathbb{Z}})$. For concreteness we give names to the generators of the groups in the following table, but do not explain the notation in detail. It respects the module action of π_*^S on the homotopy of the spectrum $\mathrm{hofib}(\mathrm{trc}_{\mathbb{Z}})$, so $\sigma \cdot \partial^2(1) = \partial^2(\sigma)$, as an example.

PROPOSITION. *The homotopy groups* $\pi_n\,\mathrm{hofib}(\mathrm{trc}_{\mathbb{Z}})$ *are known for all* n, *and begin:*

n	$\pi_n\,\mathrm{hofib}(\mathrm{trc}_{\mathbb{Z}})$	*gen.*
-2	$\mathbb{Z}$	$\partial^2(1)$
-1	0	
0	$\mathbb{Z}$	$\partial(f_1)$
1	0	
2	$\mathbb{Z}$	$\partial(f_3)$
3	$\mathbb{Z}/8 \rtimes \mathbb{Z}/2$	$\partial^2(\kappa_5), \eta^3$
4	$\mathbb{Z}/2$	$\partial(t_5)$
5	$\mathbb{Z}/2$	$\partial^2(\sigma)$

n	$\pi_n\,\mathrm{hofib}(\mathrm{trc}_{\mathbb{Z}})$	*gen.*
6	$\mathbb{Z}$	$\partial(f_7)$
7	$\mathbb{Z}/16$	$\partial(\sigma f_1)$
8	0	
9	0	
10	$\mathbb{Z}$	$\partial(f_{11})$
11	$\mathbb{Z}/8 \rtimes \mathbb{Z}/2$	$\partial^2(\kappa_{13}), \eta^2\mu$
12	$\mathbb{Z}/2$	$\partial(t_{13})$
13	$\mathbb{Z}/2$	$\partial^2(\rho)$

The extensions in degrees $\equiv 3 \bmod 8$ *are cyclic (yielding copies of* $\mathbb{Z}/16$*).*

Homotopy of the smooth Whitehead space

We now sketch how to determine the map $\pi_*(\ell)$ in a range of degrees, and to use the fiber sequence $(*)$ to describe $\pi_*\Omega\,\mathrm{Wh}(*)$.

To get started, note that $\mathrm{Wh}(*)$ is 1-connected. For $\pi_1\,\mathrm{Wh}(X) = \mathrm{Wh}_1(\pi_1 X)$ for all spaces X, and $\mathrm{Wh}_1(0) = 0$. It follows that $\pi_*(\ell)$ is an isomorphism for $* \le 0$.

Next, $\ell\colon \Omega^\infty(\mathbb{C}P^\infty_{-1}) \to \mathrm{hofib}(\mathrm{trc}_{\mathbb{Z}})$ is a spectrum map, so $\pi_*(\ell)$ is a π_*^S-module homomorphism. Combined with the isomorphisms in degrees $* \le 0$, this allows us to determine $\pi_*(\ell)$ in several higher degrees. Finally we use secondary composition methods involving Toda brackets to determine $\pi_*(\ell)$ for $* \le 14$.

This gives us $\pi_*\Omega\,\mathrm{Wh}(*)$ for $* \le 13$. (There remains an extension question in degree 13.) This result thus gives us $\pi_* A(*)$ and $\pi_* P(D^k)$ for large k, in a similar range of degrees.

Theorem. *The homotopy groups $\pi_n\Omega\,\mathrm{Wh}(*)$ are known (modulo odd torsion) for $n \le 13$, and begin:*

n	$\pi_n\Omega\,\mathrm{Wh}(*)$	$\pi_n G/O$
$0, 1$	0	0
2	$\mathbb{Z}/2$	$\mathbb{Z}/2$
3	0	0
4	$\mathbb{Z}$	$\mathbb{Z}$
5	0	0
6	$\mathbb{Z}/2$	$\mathbb{Z}/2$
7	0	0
8	$\mathbb{Z}/2 \oplus \mathbb{Z}$	$\mathbb{Z}/2 \oplus \mathbb{Z}$
9	$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/8$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$
10	$\mathbb{Z}/2$	$\mathbb{Z}/2$
11	$\mathbb{Z}/4$	0
12	$\mathbb{Z}$	$\mathbb{Z}$
13	$\mathbb{Z}/2 \rtimes \mathbb{Z}/2$	0

The extension in degree 13 *is unresolved.*

We have included the homotopy groups of G/O, the classifying space for smooth surgery normal invariants, for comparison. This is also the homotopy fiber of the j-map $j\colon BSO \to BSG$. There is a fiber sequence $\mathrm{Cok}J_\otimes \to G/O \to BSO_\otimes$ of infinite loop spaces, which only splits on the space level [**Ma, V.4**].

Using manifold models for the algebraic K-theory of spaces, Waldhausen defined a map

$$hw\colon G/O \to \Omega\,\mathrm{Wh}(*)$$

in [**W3**], and proved that it is 2-connected. From the table above, we see that the map might at best be 8-connected, and might even induce a split injection on homotopy. If so, the homotopy groups of the remainder term would begin with a $\mathbb{Z}/8$ in degree 9, a $\mathbb{Z}/4$ in degree 11, and a group of order four in degree 13.

Towards assembling a space level description of the homotopy type of $\Omega\,\mathrm{Wh}(*)$, we offer the following result. Here $\bar{\alpha}$ collapses the (-2)- and 0-cells of $\mathbb{C}P^\infty_{-1}$ to a point, $\bar{R}$ is the Segal retracting map [**S2**] that extends the inclusion $\mathbb{C}P^\infty \simeq BU(1) \to BU$, and c is the complexification map.

Proposition. *The diagram*

$$\begin{array}{ccccccc}
\mathrm{Cok}J & & & & & & \\
\downarrow & & & & & & \\
G/O & \xrightarrow{hw} & \Omega\,\mathrm{Wh}(*) & \xrightarrow{\Omega\widetilde{\mathrm{trc}}} & \Omega^\infty(\mathbb{C}P^\infty_{-1}) & \xrightarrow{\bar{\alpha}} & Q(\mathbb{C}P^\infty) \\
\downarrow & & & & & & \downarrow{\scriptstyle \bar{R}} \\
BSO & & & \xrightarrow{\quad\quad c \quad\quad} & & & BU
\end{array}$$

commutes, up to a homotopy automorphism of BSO.

With some extra work we get the following theorem, which improves somewhat on Bökstedt's theorem [**Bö**] that $hw\colon G/O \to \Omega\,\mathrm{Wh}(*)$ is a rational equivalence.

Theorem. *The Hatcher–Waldhausen map*

$$hw_*\colon \pi_* G/O \to \pi_*\Omega\,\mathrm{Wh}(*)$$

is at least 5*-connected, and induces* 2*-adic isomorphisms of homotopy groups modulo torsion.*

Corollary. *The first nontrivial k-invariant in* $\Omega\,\mathrm{Wh}(*)$ *is*

$$\beta\,\mathrm{Sq}^2 \in H^5(K(\mathbb{Z}/2,2);\mathbb{Z}).$$

Here β is the Bockstein map and Sq^2 is the Steenrod squaring operation.

As a concluding geometric interpretation of the connectivity of the Hatcher–Waldhausen map, we recall the rigid tubes map from [**W3**]. Let $\mathcal{T}(*)$ be the *stable tube space* of single smooth k-handles embedded in $D^n \times I$, attached to the base disc $D^n \times 0$, stabilized both with respect to the handle dimension k and codimension $n-k$. Likewise take as a model for BO the Grassmannian of k-dimensional subspaces of $\mathbb{R}^n$, stabilized both with respect to k and $n-k$. The *rigid tubes map*

$$BO \to \mathcal{T}(*)$$

takes a subspace $V^k \subset \mathbb{R}^n$ to a standardized smooth k-handle erected over the unit disc of V, attached to a thickening of the unit sphere of V.

The rigid tubes map has the same connectivity as the Hatcher–Waldhausen map. Hence our calculations show that, after 2-adic completion, the increased flexibility in the space of stable smooth tubes compared to the space of stable rigid tubes only affects the homotopy groups in degree 6 or higher, and possibly the first difference only appears in degree 9.

It remains an open problem to obtain a homotopy-theoretic understanding of the difference of these spaces of rigid or smooth tubes, or equivalently, of the fiber of the Hatcher–Waldhausen map.

References

[BG] J. C. Becker and D. H. Gottlieb, *The transfer map and fiber bundles*, Topology **14** (1975), 1–12.

[Bl] S. Bloch, *Algebraic cycles and higher K-theory*, Adv. Math. **61** (1986), 267–304.

[BL] S. Bloch and S. Lichtenbaum, *A spectral sequence for motivic cohomology*, Invent. Math. (to appear).

[Bö] M. Bökstedt, *The rational homotopy type of* $\Omega\,\mathrm{Wh}^{\mathrm{Diff}}(*)$, Algebraic Topology, Aarhus 1982 (I. Madsen and B. Oliver, eds.), LNM, vol. 1051, Springer, 1984, pp. 25–37.

[BHM] M. Bökstedt, W.-C. Hsiang and I. Madsen, *The cyclotomic trace and algebraic K-theory of spaces*, Invent. Math. **11** (1993), 465–540.

[BM] M. Bökstedt and I. Madsen, *Topological cyclic homology of the integers*, Asterisque **226** (1994), 57–143.

[Bou] A. K. Bousfield, *The localization of spectra with respect to homology*, Topology **18** (1979), 257–281.

[C] J. Cerf, *Topologie de certains espaces de plongements*, Bull. Soc. Math. France **89** (1961), 227–380.

[D] B. I. Dundas, *Relative K-theory and topological cyclic homology*, Acta. Math. **179** (1997), 223–242.

[DHS] W. Dwyer, W.-C. Hsiang and R. Staffeldt, *Pseudo-isotopy and invariant theory. I*, Topology **19** (1980), 367–385.

[FH] F. T. Farrell and W.-C. Hsiang, *On the rational homotopy groups of the diffeomorphism groups of discs, spheres and aspherical manifolds*, Algebraic and Geometric Topology, Proc. Symp. Pure Math., vol. 32, AMS, 1978, pp. 325–337.

[G1] T. Goodwillie, *Relative algebraic K-theory and cyclic homology*, Ann. of Math. **124** (1986), 347–402.

[G2] T. Goodwillie, *The differential calculus of homotopy functors*, Proc. of 1990 ICM in Kyoto, Japan, Math. Soc. of Japan, pp. 621–630.

[H] A. E. Hatcher, *A proof of the Smale conjecture,* $\mathrm{Diff}(S^3) \simeq O(4)$, Ann. of Math. **117** (1983), 553–607.

[HM] L. Hesselholt and I. Madsen, *On the K-theory of finite algebras over Witt vectors of perfect fields*, Topology **36** (1997), 29–101.

[Ig] K. Igusa, *The stability theorem for smooth pseudoisotopies*, K-Theory **2** (1988), 1–355.

[K] M. A. Kervaire, *Le theoreme de Barden–Mazur–Stallings*, Comment. Math. Helvet. **40** (1965), 31-42.

[LMS] L. G. Lewis, Jr., J. P. May and M. Steinberger, *Equivariant Stable Homotopy Theory*, Lecture Notes in Math., vol. 1213, Springer Verlag, 1986.

[Ma] J. P. May, with contributions by F. Quinn, N. Ray, and J. Tornehave, E_∞ *ring spaces and* E_∞ *ring spectra*, LNM, vol. 577, Springer, 1977.

[McC] R. McCarthy, *Relative algebraic K-theory and topological cyclic homology*, Acta. Math. **179** (1997), 197–222.

[Mi] J. W. Milnor, *Whitehead torsion*, Bull. AMS **72** (1966), 358–426.

[Mo] R. E. Mosher, *Some stable homotopy of complex projective space*, Topology **7** (1968), 179–193.

[Mu1] J. Mukai, *The* S^1*-transfer map and homotopy groups of suspended complex projective spaces*, Math. J. Okayama Univ. **24** (1982), 179–200.

[Mu2] J. Mukai, *On stable homotopy of the complex projective space*, Jap. J. Math. **19** (1993), 191–216.

[Mu3] J. Mukai, *The element* $\bar{\kappa}$ *is not in the image of the* S^1*-transfer*, Math. J. Okayama Univ. **36** (1994), 133–143.

[Ra] D. C. Ravenel, *The Segal conjecture for cyclic groups and its consequences*, Am. J. Math **106** (1984), 415–446.

[R1] J. Rognes, *The Hatcher-Waldhausen map is a spectrum map*, Math. Ann. **299** (1994), 529–549.

[R2] J. Rognes (after M. Bökstedt), *Trace maps from the algebraic K-theory of the integers*, J. Pure Appl. Algebra **125** (1998), 277-286.

[R3] J. Rognes, *The product on topological Hochschild homology of the integers with mod four coefficients*, J. Pure Appl. Algebra **134** (1999), 211–218.

[R4] J. Rognes, *Topological cyclic homology of the integers at two*, J. Pure Appl. Algebra **134** (1999), 219–286.

[R5] J. Rognes, *Algebraic K-theory of the two-adic integers*, J. Pure Appl. Algebra **134** (1999), 287–326.

[RW] J. Rognes and C. Weibel, *Two-primary algebraic K-theory of rings of integers in number fields*, J. Amer. Math. Soc. (to appear).

[S1] G. Segal, *Equivariant stable homotopy theory*, Proc. of ICM 1970 in Nice, France, vol. 2, 1971, pp. 59–63.

[S2] G. Segal, *The stable homotopy of complex projective space*, Quart. Jour. of Math. **24** (1973), 1–5.

[SV] A. A. Suslin and V. Voevodsky, *The Bloch–Kato conjecture and motivic cohomology with finite coefficients*, Preprint (www.math.uiuc.edu/K-theory) (1995).

[T] H. Toda, *Composition methods in homotopy groups of spheres*, Ann. of Math. Studies, vol. 49, Princeton University Press, 1962.

[tD] T. tom Dieck, *Transformations groups*, De Gruyter, 1987.

[V] V. Voevodsky, *The Milnor Conjecture*, Preprint (www.math.uiuc.edu/K-theory) (1996).

[W1] F. Waldhausen, *Algebraic K-theory of topological spaces. I*, Algebraic and Geometric Topology, Proc. Symp. Pure Math., vol. 22, AMS, 1978, pp. 35–60.

[W2] F. Waldhausen, *Algebraic K-theory of topological spaces. II*, Algebraic Topology, Proc. Symp., Aarhus 1978, LNM, vol. 763, Springer, 1979, pp. 356–394.

[W3] F. Waldhausen, *Algebraic K-theory of spaces, a manifold approach*, Current trends in algebraic topology, Part 1 (London, Ont., 1981), CMS Conf. Proc., vol. 2, AMS, 1982, pp. 141–184.

[W4] F. Waldhausen, *Algebraic K-theory of spaces*, Algebraic and geometric topology, Proc. Conf., New Brunswick/USA 1983, LNM, vol. 1126, Springer, 1985, pp. 318–419.

[W5] F. Waldhausen, *Algebraic K-theory of spaces, concordance, and stable homotopy theory*, Algebraic Topology and Algebraic K-Theory (W. Browder, ed.), Ann. of Math. Studies, vol. 113, Princeton University Press, 1987, pp. 392–417.

[WW] M. Weiss and B. Williams, *Automorphisms of manifolds*, Preprint (1997).

Dept. of Math., University of Oslo, Norway
E-mail address: `rognes@math.uio.no`