Brazilian Journal of Probability and Statistics
2014, Vol. 28, No. 2, 223–240
DOI: 10.1214/12-BJPS203

On free lunches in random walk markets with short-sale constraints and small transaction costs, and weak convergence to Gaussian continuous-time processes

Nils Chr. Framstad[a,b]
[a] *University of Oslo*
[b] *The Financial Supervisory Authority of Norway*

Abstract. This paper considers a sequence of discrete-time random walk markets with a safe and a single risky investment opportunity, and gives conditions for the existence of arbitrages or free lunches with vanishing risk, of the form of waiting to buy and selling the next period, with no shorting, and furthermore for weak convergence of the random walk to a Gaussian continuous-time stochastic process. The conditions are given in terms of the kernel representation with respect to ordinary Brownian motion and the discretisation chosen. Arbitrage and free lunch with vanishing risk examples are established where the continuous-time analogue is arbitrage-free under small transaction costs—including for the semimartingale modifications of fractional Brownian motion suggested in the seminal Rogers [*Math. Finance* **7** (1997) 95–105] article proving arbitrage in fBm models.

1 Introduction

If a continuous-time model for a financial market is discretised in time, will then the discretised version inherit its properties when it comes to free lunches, or absence of such? Asking the converse question: if a continuous-time model is the weak limit—"weak" because this topology gives neighbouring profits/loss process distributions for a given strategy—of a sequence of discrete-time models, will free lunch properties or no free lunch properties carry over the limit transition?

There is actually no guarantee that this will be the case. In Shiryaev's book (1999, Section VI.3), there are given stronger sufficient conditions for convergence to fair prices in terms of weak convergence of the *(driving noise, pricing kernel) pair*. This paper will show that if this joint convergence fails, then there is a wide range of problems where the arbitrage properties differ between the discretised prices and their weak limits, even when small transaction costs are introduced to the former.

This author's initial interest in the problem at hand, emerges from a work by Sottinen (2001), who establishes a sequence of discrete-time binary symmetric random walk (semimartingale) markets, which (a) converges weakly to a Black–Scholes market with prices being geometric fractional Brownian motion with

Key words and phrases. Stock price model, random walk, Gaussian processes, weak convergence.
Received June 2011; accepted June 2012.

Hurst parameter $H > 1/2$, and (b) admits an arbitrage obtained by waiting for the right moment to buy (if nonnegative drift) or short sell (if nonpositive drift) the stock, and unwinding the position the very next period; the "right moment" is of course when you might with probability one know that the stock market beats the money market even if tomorrow is a bad day (in which case you buy), or waiting for the conversely adverse stock market (in which case you short-sell). Now fractional Brownian motion is not a semimartingale, and as is well known since Rogers (1997) (for the positively autocorrelated parameter range), it will introduce arbitrages to canonical models where the ordinary Brownian motion does not. In view of this, there seems to have been a view that the result of Sottinen (2001) is due to specifics of the fBm, or at least its nonsemimartingale property, and this author admits to having fallen prey to this interpretation, which—as we shall see—is inaccurate.

This paper sets out to show that the phenomenon discovered by Sottinen (2001), is to be expected way more generally, including in the discretisation of arbitrage-free semimartingale price processes. As an example, we refer to Rogers (1997), who also proposes a parametrised semimartingale process whose moving average kernel converges to the fBm's—preserving the long memory which was the reason for suggesting fBm as a driving noise in the first place, but eliminating the short memory which caused the arbitrage. It turns out that when attempting to discretise in a manner akin to the construction of Sottinen (2001), the *long* memory will introduce arbitrages, and the arbitrage property is robust enough to withstand even the introduction of a small transaction cost. It is then essential that the discretised version has bounded downside (cf. the results of Guasoni (2006), Guasoni et al. (2008)). A different example, admitting free lunch with vanishing risk (FLVR) in the discretisation, is the Ornstein–Uhlenbeck process.

We shall on one hand give sufficient conditions for the existence of arbitrage or FLVR of the form (i) wait for a possible time to buy, and then (ii) sell next period. On the other, we give sufficient conditions for weak convergence of the discrete random walks to the Gaussian continuous-time counterparts. Because the results concerning arbitrages will require *bounded* innovations in the random walks, the weak convergence result (Theorem 2.2) will also be restricted to this case. Our main contributions compared to the previous literature (primarily Sottinen (2001)), are summarised as follows:

- We cover a fairly general class of Gaussian processes, and give examples to the existence of arbitrages/FLVRs of the above-mentioned form.
- Furthermore, we point out that the arbitrage for discretised fBm can emerge from the (originally desirable) long-run memory of the process, even if the short-run memory (which causes the arbitrage in the continuous-time model) is modified as to obtain the semimartingale property, for example, as suggested by Rogers (1997).

- We cover any negative drift term (a word which should be interpreted cautiously for nonsemimartingales) without shorting, as it turns out that the instantaneous growth from the noise term can tend to infinity.
- For the same reason the arbitrage may also admit sufficiently small *transaction costs*.
- We do not have to assume the discretised market to be binary (hence complete if arbitrage-free) with symmetric innovations. We will however assume bounded support, where the bound might depend on how fine the discretisation.
- Weak convergence of the driving noise is likewise shown in this more general setting.

2 The continuous-time and discrete-time market models

Our market has one "safe" asset, taken as numéraire and normalised to price $= 1$, and one "risky" asset $S^{(n)}$, which for each n is a discretisation of a continuously evolving stochastic process S. S will be constructed from a drift process A with time-derivative $a(t)$ and a driving noise Z, assumed to be a Gaussian moving average process with right-continuous sample paths and an adapted (hence upper limit of integration is t) kernel representation

$$\begin{aligned} Z(t) &= \int_{-\infty}^{t_0} K(t,s)\,\mathrm{d}W(s) + \int_{t_0}^{t} K(t,s)\,\mathrm{d}W(s) \\ &= J(t) + \int_{t_0}^{t} K(t,s)\,\mathrm{d}W(s), \end{aligned} \tag{2.1}$$

with respect to standard Brownian motion W, where K is a given function satisfying the following properties:

$$K \text{ is deterministic and piecewise-continuous,} \tag{2.2a}$$

$$K(t,s) = 0 \qquad \text{if } s \geq t \tag{2.2b}$$

and

$$\int_{-\infty}^{t} \big(K(t,s)\big)^2\,\mathrm{d}s < \infty \qquad \forall t. \tag{2.2c}$$

We assume that the agent enters the market at a given time $t_0 \geq 0$.

Notice that (2.2c) follows from the assumed Gaussian distribution, as the expression is in fact $\mathrm{E}[(Z(t))^2]$. Notice also that some representations involve K with a definition split in order to achieve square integrability, compensating the distant past. A frequently occurring representation form (cf., e.g., Cheridito (2004)) is $K(t,s) = \kappa(t-s) - \kappa(-s)$, and under the assumption that the kernel vanishes for $s \geq t$, then we have $\kappa(-s) = 0$ for $s \geq 0$. We shall later give particular attention to such a form of the type $K(t,s) = \kappa((t-s)_+)$ for $t > s > t_0$, where we not need to specify the definition below t_0.

We might choose to discretise W on the entire time line; however, J will merely enter as a drift term, and we can equally well discretise J directly. We shall choose to do the latter. Hence, we start by discretising the time scale (equidistantly) in intervals of length $1/n$, where for each n we define

$$s_i^{(n)} = t_0 + \frac{i - \lfloor nt_0 \rfloor}{n}, \tag{2.3}$$

where $\lfloor \cdot \rfloor$ is the floor function (rounding toward $-\infty$). Then we discretise W for $t > t_0$ by replacing its normalised increments $n^{1/2} \cdot [W(s_{i+1}^{(n)}) - W(s_i^{(n)})]$ by random variables $\xi_{i+1} = \xi_{i+1}^{(n)}$. Now discretise Z into

$$Z^{(n)}(t) = J\left(\frac{\lfloor nt \rfloor}{n}\right) + \sum_{i=\lfloor nt_0 \rfloor}^{\lfloor nt \rfloor - 1} K\left(\frac{\lfloor nt \rfloor}{n}, s_i^{(n)}\right) \cdot n^{-1/2} \xi_{i+1}^{(n)}. \tag{2.4}$$

For A we can take $A(t_0) = 0$, as we are interested in increments only; we therefore define A and its discretisation as

$$A(t) = \int_{t_0}^{t} a(s)\,\mathrm{d}s, \qquad A^{(n)}(t) = \frac{1}{n} \sum_{i=\lfloor nt_0 \rfloor}^{\lfloor nt \rfloor - 1} a\big(s_i^{(n)}\big) \tag{2.5}$$

and finally, S and its discretisation are assumed, respectively defined, to satisfy

$$S = G(A + Z), \qquad S^{(n)} = G\big(A^{(n)} + Z^{(n)}\big). \tag{2.6}$$

The canonical choice is G to be the exponential function, but we shall not need this specific property; for Proposition 3.2, we will however use convexity, and for Theorem 2.2 we shall need continuity. Except when K vanishes, the $S^{(n)}$ and the ξ_i sequence will generate the same filtration, so the first of the following assumptions is not very restrictive:

Assumption/notation 2.1. We assume formulae (2.1) through (2.6) to hold, and furthermore:

- The filtration will be generated by the $\{\xi_i\}$, so that the information at time t, is generated by $\{\xi_i\}_{i \le tn}$.
- By "step number j," we shall mean at time $s_j^{(n)}$. That means that the agent's first chance of trading, is not at step 0, but at prices noted at step $j_0 := \lfloor nt_0 \rfloor$. Should this lead to a singularity due to for example, $t_0 = 0$, $K(t, 0) = +\infty$, then we shall however eliminate this by assuming (without mention) that t_0 is > 0 and irrational.
- We shall use the term "j-measurable" to mean measurable at *step number* j, that is, at time $s_j^{(n)}$, and write $\mathrm{P}_j = \mathrm{P}_j^{(n)}$ for the probability measure conditional on the filtration generated up to this time/step and $\mathrm{E}_j = \mathrm{E}_j^{(n)}$ for the corresponding conditional expectation.

- The $\{\xi_i^{(n)}\}_{i,n}$ will be mutually independent and each $\xi_i^{(n)}$ bounded, and there exists some (common) constant $\nu > 0$ such that $\operatorname{ess\,inf} \xi_i^{(n)} < -\nu$ and $\operatorname{ess\,sup} \xi_i^{(n)} > \nu$.
- Since ξ_j is independent of the past, we shall suppress the dependence of law in terms like e.g., $\operatorname{ess\,sup}_{\xi_j}$ which will denote the supremum over the (P_j-)essential support of ξ_j.
- Two pieces of notation: K_1' shall denote the partial derivative with respect to the first variable. The symbol $\geqslant$ shall mean "no smaller than and not a.s. equal."
- a is assumed locally bounded, and G is assumed continuous and strictly increasing.

It should be remarked that it is unreasonable for an approximation to normalised standard Brownian motion that $\nu < 1$, but only in parts of Theorems 3.7 and 3.8 shall we actually need that 0 is interior in the support. Bounded support will however be essential for the arbitrage conditions, and the following result will be simplified by assuming a common bound:

Theorem 2.2 (Weak convergence). *Suppose* $\mathrm{E}[\xi_i] = 0$, $\mathrm{E}[\xi_i^2] = 1$ *and* $\operatorname{ess\,sup} |\xi_i| \le M < \infty$ *(all i, n). Then $Z^{(n)}$ converges weakly to Z, and for continuous G also $S^{(n)}$ to S, on the Skorohod space $D([t_0, T])$, every $T > t_0$.*

Proof. The drift and the already occurred part will represent no issue, and we can take $A = A^{(n)} = J = 0$ without loss of generality. Also, we can take G to be the identity, as weak limits commute with continuous functions G. Now convergence in finite-dimensional distributions follows like in Sottinen (2001, Theorem 1): by the CLT, the limit is Gaussian with zero mean; for the covariances, the independence of the ξ_i's yields, for $T \ge t \ge t_0$

$$\mathrm{E}\big[Z^{(n)}(T)Z^{(n)}(t)\big] = \sum_{i=\lfloor nt_0\rfloor}^{\lfloor nt\rfloor-1} K\Big(\frac{\lfloor nT\rfloor}{n}, s_i^{(n)}\Big)K\Big(\frac{\lfloor nt\rfloor}{n}, s_i^{(n)}\Big)\cdot \mathrm{E}\Big[\Big(\frac{\xi_{i+1}^{(n)}}{\sqrt{n}}\Big)^2\Big] \tag{2.7}$$

which is a Riemann sum converging to the desired value $\int_{t_0}^t K(T,s)K(t,s)\,\mathrm{d}s$.

It remains to prove tightness, which will follow by a set of sufficient conditions given in Whitt (2007, Lemma 3.11(ii.b)). For $T \ge t + h \ge t \ge t_0$, we have

$$\begin{aligned}
&\mathrm{E}_j\big[(Z^{(n)}(t+h) - Z^{(n)}(t))^2\big]\\
&\quad= \sum_{i=\lfloor nt_0\rfloor}^{\lfloor n(t+h)\rfloor-1}\Big(K\Big(\frac{\lfloor n(t+h)\rfloor}{n}, s_i^{(n)}\Big) - K\Big(\frac{\lfloor nt\rfloor}{n}, s_i^{(n)}\Big)\Big)^2 \mathrm{E}_j\Big[\Big(\frac{\xi_{i+1}^{(n)}}{\sqrt{n}}\Big)^2\Big]\\
&\quad\le \frac{M^2}{n}\sup_{t\in(t_0,T-h)}\sum_{i=\lfloor nt_0\rfloor}^{\lfloor n(t+h)\rfloor-1}\Big(K\Big(\frac{\lfloor n(t+h)\rfloor}{n}, s_i^{(n)}\Big) - K\Big(\frac{\lfloor nt\rfloor}{n}, s_i^{(n)}\Big)\Big)^2.
\end{aligned}$$

For each n, let t_n be an argsup. For any subsequence of t_n converging to a limit point $\bar{t}$, we have convergence as a Riemann sum:

$$\to M^2 \int_{t_0}^{\bar{t}+h} \big(K(\bar{t}+h,s) - K(\bar{t},s)\big)^2 \, \mathrm{d}s \tag{2.8}$$

which by square integrability tends to 0 as h does. □

For the discrete-time markets, we shall restrict ourselves to the following set of strategies:

Definition 2.3. Let $n < \infty$ be given. For any natural q, a "q-period strategy" ("single period strategy" if $q = 1$), consists of waiting until some stopping time $t_* = s_{j_*}^{(n)} \geq s_{j_0}^{(n)}$, buying a j_*-measurable number $u > 0$ of units, holding these until a stopping time $t^* = s_{j^*}^{(n)}$ where $j^* \in \{j_* + 1, \dots, j_* + q\}$ and then selling all u units.

The "net return" from this transaction is

$$R = R_{j_*, j^*} := u \cdot \big(S^{(n)}(t^*) - S^{(n)}(t_*)\big) - \big(\Lambda_* + \Lambda^*\big) \cdot \lambda, \tag{2.9}$$

where $\lambda\Lambda_*$ and $\lambda\Lambda^*$ are the respective transaction costs for buying and selling, allowed to depend on prices and units like in Assumption 2.4 below.

The reason for the "λ" parameter is that we will consider the properties for small transaction costs, and it will be convenient to scale by a number. The main results will be carried out under for fixed transaction costs and $u = 1$, and Proposition 3.2 will show that this is sufficiently general. For the time being, assume the more parsimonious form for Λ_*, Λ^*:

Assumption 2.4. $\Lambda_* = \Lambda_*(u, S^{(n)}(t_*))$ and $\Lambda^* = \Lambda^*(u, S^{(n)}(t_*), S^{(n)}(t^*))$ will be nonnegative functions, bounded in $(S^{(n)}(t_*), S^{(n)}(t^*))$, while λ will be a number ≥ 0.

Definition 2.5. We shall use the term "transaction cost λ" to imply that $u = 1$ and $\Lambda_* + \Lambda^* = 1$ (identically), and we shall refer to "the simple model (2.10)" the single period case of transaction cost λ where G is the identity.

This "simple model" will be the main focus. For this case, the net return on the event $\{j_* < \infty\}$ will be

$$\begin{aligned} &S^{(n)}\Big(t_0 + \frac{j_* + 1 - \lfloor nt_0 \rfloor}{n}\Big) - S^{(n)}\Big(t_0 + \frac{j_* - \lfloor nt_0 \rfloor}{n}\Big) - \lambda \\ &\quad = x_{j_*}^{(n)} + y_{j_*}^{(n)} + z_{j_*+1}^{(n)} - \lambda, \end{aligned} \tag{2.10a}$$

where we introduce the notation

$$x_j = x_j^{(n)} = \frac{1}{n}\{a(s_j^{(n)}) + J(s_{j+1}^{(n)}) - J(s_j^{(n)})\}, \tag{2.10b}$$

$$y_j = y_j^{(n)} = \frac{1}{\sqrt{n}} \sum_{i=j_0}^{j-1} [K(s_{j+1}^{(n)}, s_i^{(n)}) - K(s_j^{(n)}, s_i^{(n)})]\xi_{i+1}^{(n)}, \tag{2.10c}$$

$$z_{j+1} = z_{j+1}^{(n)} = \frac{1}{\sqrt{n}} K(s_{j+1}^{(n)}, s_j^{(n)})\xi_{j+1}^{(n)} \tag{2.10d}$$

adopting the convention that the empty sum, corresponding to $j = j_0$ $(= \lfloor nt_0 \rfloor)$, is zero. Notice that the $z_{j+1}^{(n)}$ term will represent the *innovation* from step j to $j+1$, and has subscript "$j+1$" since it is only $j+1$-measurable. The x_j and y_j, which correspond to the memory of the process as well as the drift a, are j-measurable. An arbitrage will occur in the market if the memory contribution dominates even in the worst-case innovation. The next section will make this more precise.

3 Free lunches: Sufficient conditions

Starting out with the definitions from the previous section, we now define arbitrage and free lunches with vanishing risk under our admissibility conditions. Informally, we have a FLVR if we can obtain an arbitrarily small downside to mean return ratio, and an arbitrage if one can have positive-mean return without downside. The arguably most natural, and also the strictest, concept of "downside" is the worst-case outcome, the essential supremum of the negative part, and we shall restrict ourselves to such a definition. The following definition appears notationally a bit cryptic, but will under q-period strategies coincide with the conventional definition of FLVR and arbitrage; informally, it says that downside should be arbitrarily small compared to expected return (and we note that the expectation term is finite, by boundedness of the ξ_i). Note however that this diverges on the outset—though not in substance, as we shall see below—from a commonplace assumption of fixed horizon. This fixed horizon is less natural here, where the strategies involve waiting first and only then is there a bound on the holding period.

Definition 3.1. Fix n, q both $< \infty$. Consider the condition

$$\frac{\operatorname{ess\,inf}^{(\mathrm{P}_{j_*})}[R_{j_*,j^*}|D_{j_*}]}{\mathrm{E}[R_{j_*,j^*}|D_j]} > -\delta \quad \text{and} \quad \mathrm{E}[R_{j_*,j^*}|D_{j_*}] \in (0,\infty). \tag{3.1}$$

- The market is said to admit *free lunch with vanishing risk* ("FLVR") *in q periods* if for every $\delta > 0$ there exist stopping times $j_* \geq j_0$ and $j^* \in \{j_*+1, \ldots, j_*+q\}$ and a j_*-measurable event D_{j_*} with $\mathrm{P}_{j_0}[D_{j_*}] > 0$, such that (3.1) holds.
- The FLVR is called an *arbitrage* if the FLVR definition holds also for $\delta = 0$.

- The simple model (2.10) will be said to admit FLVR, respectively, arbitrage, if the respective definition applies with $q = 1$ (i.e., $j^* = j_* + 1$).

Whenever necessary to distinguish the ex ante (at j_0) random variable which is either 0 (if D_j does not occur) or > 0 on one hand, from the D_j-conditional positive return on the other—colloquially speaking, the lottery ticket that yields $\geqslant 0$ from the actual positive lunch prize—we shall use terms like the "event" that the lunch "manifests itself."

Obviously, the lack of time bound makes no difference for an arbitrage; if there is an arbitrage according to this definition, then for some fixed Q, there is an arbitrage which is closed out within Q steps, that is, $j^* \leq Q$. Conversely, it does not matter that q is assumed deterministic; had we employed the same definition except with q being merely measurable and finite, we would have had an arbitrage for some deterministic q as well. The FLVR definition, on the other hand, might require an unbounded j_*. Informally, a FLVR is a sequence of lunches with uniformly positive mean, but where the risk tends to zero. This means that for any nonzero downside you choose as tolerance, then there is a fixed Q such that you have a lunch within your risk tolerance within Q steps. Letting downside tend to zero, then our setup allows Q to grow, as long as q obeys a fixed bound; compare this to the usual Black–Scholes setup, which rules out the strategy of trading and waiting for the unbounded stopping time until your position has made a given profit.

For the purpose of giving *sufficient* conditions for arbitrage/FLVR under *small* transaction cost—which is the main object of this section—the simple model (2.10), for which $q = 1$, turns out fairly close to general.

Proposition 3.2 (Free lunches in the simple model (2.10) vs. in the full model). *Fix $u > 0, n < \infty$. Assume that Λ^* of at most linear growth w.r.t. the last variable (the selling price). Then there is an arbitrage for sufficiently small λ, provided that so is the case in the simple model (2.10). If G is convex, then there is FLVR for sufficiently small λ, provided that so is the case in the simple model (2.10).*

Proof. The proof is less interesting, and is relegated to the Appendix. Notice that if $\xi_j K(s_{j+1}^{(n)}, s_j^{(n)})$ is upper bounded (for every j and n), then the at most linear growth condition will hold (since an arbitrage must be closed out in a bounded number of periods). □

Informally, an arbitrage in the simple model (2.10), occurs if at some bounded $j_* \in [j_0, \infty)$, given the information available then, the transaction costs plus the worst-case possible downside from the innovation $z_{j_*+1}^{(n)}$ will be more than fully compensated by the contribution from the dependence of the past (i.e., $x_{j_*+1}^{(n)} + y_{j_*+1}^{(n)}$); a FLVR occurs if it is "sufficiently more than compensated in mean" and "nearly fully compensated in worst-case." The following result is key for the arbitrage case.

Proposition 3.3 (Sufficient conditions for arbitrage in the simple model (2.10)). *Fix $n < \infty$. If for some natural $j \geq j_0$, we have*

$$\operatorname*{ess\,sup}_{\{\xi_i\}_{i=j_0+1,\ldots,j}} \{x_j + y_j\} + \operatorname*{ess\,inf}_{\xi_{j+1}} z_{j+1} \geq \bar{\lambda} \geq 0 \tag{3.2}$$

we have an arbitrage for all transaction costs $\lambda \in [0, \bar{\lambda})$ by choosing $j_* =$ this j. Furthermore, we have arbitrage for transaction cost $\bar{\lambda}$ if in addition there is a point probability that z_{j+1} attains its ess sup.

Proof. Suppose (3.2) holds for some $\bar{\lambda}$. Let D_j be the j-measurable event of attaining

$$x_j + y_j \geq \operatorname*{ess\,sup}_{\{\xi_i\}_{i=j_0+1,\ldots,j}} \{x_j + y_j\} - \varepsilon.$$

Then $\mathrm{P}_{j_0}[D_j] > 0$ for each $\varepsilon > 0$; let $\varepsilon \in (0, \bar{\lambda} - \lambda)$ if nonempty. Should the event D_j occur at step j, then (2.10a) is > 0 and so is the ess inf of (3.1), where then both numerator and denominator become positive. Arbitrage also for transaction cost $\bar{\lambda}$ holds if we have positive probability at $\varepsilon = 0$. □

So the discrete market will admit an arbitrage if there may occur a period so good that the contribution from this beneficial history, knocks out the innovation so much that the worst-case scenario is a profit. Evidently, this will not happen if K is a constant (i.e., ordinary Brownian motion), for then the history does not matter; on the other hand, if K is increasing in its first variable, then it is near-trivial to construct arbitrage examples by letting downside be bounded and the upside be unbounded. While one can certainly imagine a some modeler trying to use, for example, a shifted lognormal in order to model limited liability investments, a choice of a symmetric distribution for the ξ_i would arguably be more natural and innocuous-looking. But for a suitably wide range of models, a larger downside than upside will not even prevent an arbitrage, as we shall soon see. For the book-keeping of "good" and "bad" ξ_i outcomes, we shall denote their essential suprema/infima as follows:

$$M_i = M_i^{(n)} = \operatorname{ess\,sup} \xi_i^{(n)}, \tag{3.3a}$$

$$m_i = m_i^{(n)} = -\operatorname{ess\,inf} \xi_i^{(n)}. \tag{3.3b}$$

Standing at step j, then the worst that can happen in the next innovation, cf. (2.10c), is denoted β_j ("beta" for "bad"):

$$\beta_j = \beta_j^{(n)} = \begin{cases} -m_j & \text{if } K\big(s_{j+1}^{(n)}, s_j^{(n)}\big) \geq 0, \\ M_j & \text{otherwise.} \end{cases} \tag{3.4a}$$

Looking back in time, we define γ_{ij} ("gamma" for "good") to be the best possible history over $i = j_0, \dots, j$ (cf. (2.10b)):

$$\gamma_{ij} = \gamma_{ij}^{(n)} = \begin{cases} M_i & \text{if } K(s_{j+1}^{(n)}, s_i^{(n)}) \geq K(s_j^{(n)}, s_i^{(n)}), \\ -m_i & \text{otherwise.} \end{cases} \tag{3.4b}$$

We want to specify this in terms of time, not only steps. Suppose we are targeting an arbitrage within time T for the discretised model, choosing j_* to be the second-to-last step before time T, closing out the transaction before time T:

$$j_* + 1 = \lfloor nT - nt_0 + \lfloor nt_0 \rfloor \rfloor. \tag{3.5}$$

Then define $\Gamma(T, s) = \Gamma^{(n)}(T, s)$ as a left-continuous step function with values $\Gamma(T, s_i^{(n)}) = \gamma_{ij_*}$; extend it to be 0 for $s \in [s_{j_*}^{(n)}, T]$. Consider then $n^{-1/2} y_{j_*}$ and bear in mind that j_* depends on n chosen as (3.5). Then, provided that limits exist (again, K_1' denotes the derivative w.r.t. the first variable), we have

$$\lim_n (\operatorname{ess\,sup} y_{j_*} \sqrt{n}) \geq \int_{t_0}^{T} K_1'(T, s) \liminf_n \Gamma^{(n)}(T, s)\, \mathrm{d}s,$$

where the inequality follows from nonnegativity of the integrand and the Fatou lemma. So, given $\varepsilon > 0$, then for all large enough but finite n, there will be positive P_{j_0} measure of the event

$$D_{j_*} = \left\{ y_{j_*} \geq \left[\int_{t_0}^{T} K_1'(T, s)\Gamma^{(n)}(T, s)\, \mathrm{d}s - \varepsilon \right] n^{-1/2} \right\}. \tag{3.6}$$

This gives rise to the following result.

Theorem 3.4 (Sufficient conditions for arbitrage within time T in the simple model (2.10)). *Fix a $T > t_0$ and for each n, let j_* be given by (3.5). Assume that at T we have J Hölder continuous with exponent $\alpha > 1/2$, and furthermore that $K(t, s)$ is differentiable in the first variable, at $t = T$, for each $s \in (t_0, T)$. Then if*

$$\int_{t_0}^{T} K_1'(T, s)\Gamma^{(n)}(T, s)\, \mathrm{d}s > \left| K(s_{j_*+1}^{(n)}, s_{j_*}^{(n)})\beta_j^{(n)} \right| + \bar{\varepsilon} \tag{3.7}$$

holds for all large enough n, some $\bar{\varepsilon} > 0$, then for any n large enough, there is an arbitrage with sufficiently small transaction costs, by waiting until step j_*.

Proof. Under the assumptions, we would on the event in (3.6) have a net return of at least

$$\left[\frac{a(s_{j_*}^{(n)}) + J(s_{j_*+1}^{(n)}) - J(s_{j_*}^{(n)})}{n^{1/2}} + \int_{t_0}^{T} K_1'(T, s)\Gamma^{(n)}(T, s)\, \mathrm{d}s \right.$$
$$\left. - \left| K(s_{j_*+1}^{(n)}, s_{j_*}^{(n)})\beta_j^{(n)} \right| - \varepsilon \right] n^{-1/2} - \lambda$$

and the Hölder regularity ensures that the first term inside the bracket (i.e., $x_j\sqrt{n}$), will vanish as n grows. Then by (3.7), the net return will for large enough n be a positive random variable, even with small transaction costs. □

Remark 3.5. First, observe that if $\lim_{s\nearrow T} K(T,s)=0$, then this would lead to arbitrages. Second, note that Theorem 3.4 is stated for fixed T, but it is sufficient to look for some T where it applies. For example, if ξ_i have symmetric support for each n, then we can replace (3.7) by

$$\sup\left\{\int_{t_0}^{T} |K_1'(T,s)|\,\mathrm{d}s - \left|K\left(T, T-\frac{1}{n}\right)\right|\right\} > 0, \tag{3.8}$$

where the sup is taken over those $T > t_0$ for which $n(T-t_0)$ is integer. Now one can look for arbitrages by letting T grow.

Theorem 3.4 also applies to semimartingales. The corollary is stated only for the natural choice of symmetric innovations.

Corollary 3.6. *There are infinite-variation semimartingales Z, equalling weak limits of their discretisations $Z^{(n)}$ formed by i.i.d. bounded symmetric ξ_i, for which Theorem 3.4 applies.*

Proof. Put $t_0=0$ for simplicity. From Cheridito (2004, Theorem 3.9), it is sufficient for the semimartingale property that $K(t,s)=\kappa(t-s)$ on $t>s>0$ with κ being continuous and piecewise differentiable with $\kappa' \in \mathrm{L}^2((0,\infty))$, and under these conditions, total variation is infinite on compacts iff $\kappa(0^+)\neq 0$. Choose a $\kappa \geq 0$ with a global maximum at T, with $\kappa(T) > 2\kappa(0^+) > 0$; then it satisfies the hypothesis of Theorem 3.4, and we only need $\kappa(\vartheta)$ to be smooth and κ' to tend sufficiently fast to 0 as to be square integrable. □

The form where the dependence on (t,s) only appear through the difference, will cover many cases and simplify calculations. We introduce the conditions:

For each n, we have i-independent $m_i = m = m^{(n)}$ and $M_i = M = M^{(n)}$, (3.9a)

$$K(t,s) = \kappa\big((t-s)_+\big) \qquad \text{for } s \geq t_0, \text{ with } \kappa \text{ not constant on } (0,\infty) \tag{3.9b}$$

—the nonconstantness ruling out the ordinary Brownian motion. As seen above, this form covers a wide class of even semimartingales. We can then write y and z as

$$y_j = \frac{1}{\sqrt{n}} \sum_{i=j_0}^{j-1}\left[\kappa\left(\frac{j+1-i}{n}\right) - \kappa\left(\frac{j-i}{n}\right)\right]\xi_{i+1}^{(n)}, \tag{3.10a}$$

$$z_{j+1} = \frac{1}{\sqrt{n}}\kappa\left(\frac{1}{n}\right)\xi_{j+1}^{(n)}. \tag{3.10b}$$

Now consider the good outcomes γ_{ij}; if κ is monotone or $m = M$, then the series in (3.10a) will telescope. A nonmonotone κ only has more variation, which increases the sum, so the ess sup of y_j will therefore be at least

$$\left|\left(\kappa\left(\frac{j-j_0+1}{n}\right)-\kappa\left(\frac{1}{n}\right)\right)/\sqrt{n}\right| \cdot \begin{cases} M^{(n)} & \text{if } \kappa\left(\frac{j-j_0+1}{n}\right)\geq\kappa\left(\frac{1}{n}\right), \\ m^{(n)} & \text{otherwise} \end{cases} \tag{3.11}$$

(if we want to utilise the variation of κ, we could write in terms as a sum of $|\kappa'|$-terms, tending to the constant times the total variation of κ over the interval $(1/n, (j-j_0+1)/n)$). We have the following theorem.

Theorem 3.7 (Sufficient conditions for arbitrage in the simple model under the form (3.9)). *Suppose that J is Hölder continuous with index $\alpha > 1/2$ and furthermore that for some subsequence n_ℓ, we have $m^{(n_\ell)}$, and $M^{(n_\ell)}$ bounded. Then, each of the following conditions implies arbitrage for all large enough ℓ—and furthermore, for each of those n_ℓ, the arbitrage admits small enough transaction costs*:

(a) $\liminf_\ell |\kappa(1/n_\ell)| = 0$, *or κ changes sign.*
(b) *The total variation of κ over* $(0,\infty)$ (*i.e.*, $\int_0^\infty |\kappa'(\vartheta)|\,\mathrm{d}\vartheta$ *if κ' exists*), *is > than*

$$\liminf_\ell\left[|\kappa(1/n_\ell)|\cdot\frac{\max\{M^{(n_\ell)},m^{(n_\ell)}\}}{\min\{M^{(n_\ell)},m^{(n_\ell)}\}}\right].$$

Proof. In all cases, Hölder regularity ensures that $x_j\sqrt{n}$ will tend to 0 as n grows. Then:

(a) By (3.9b), κ takes some nonzero value. Suppose first that $\kappa(1/n_\ell) \searrow 0$ while $\kappa(\vartheta) > 0$. Then choosing a sequence of j's so that $(j-j_0+1)/n$ approximates ϑ from the appropriate side (recall that (2.2a) assumes only piecewise continuity), we obtain

$$\sqrt{n}\cdot[z_{j+1}+\operatorname{ess\,sup} y_{j+1}] \geq M\cdot\left[\kappa\left(\frac{j-j_0+1}{n}\right)-\kappa\left(\frac{1}{n}\right)-\frac{m}{M}\kappa\left(\frac{1}{n}\right)\right] \\ \to M\kappa(\vartheta) > 0.$$

The negative-sign case likewise converges to $m\cdot|\kappa(\vartheta)|$. Now suppose that κ changes sign, and by the previous part of the proof, we can assume that $\kappa(1/n_\ell)$ is bounded away from 0. Suppose that $\kappa(1/n_\ell) > 0 > \kappa(\vartheta)$. Then choosing j as above, we obtain

$$\sqrt{n}\cdot[z_{j+1}+\operatorname{ess\,sup} y_{j+1}] \geq m\cdot\left[\kappa\left(\frac{1}{n}\right)-\kappa\left(\frac{j-j_0+1}{n}\right)-\kappa\left(\frac{1}{n}\right)\right] \\ = -m\kappa\left(\frac{j-j_0+1}{n}\right)$$

which is positive whenever n_ℓ is large enough. The case with reversed signs follows likewise.

(b) $\sqrt{n}\,\text{ess sup}\, y_j$ exceeds $\min\{m^{(n)}, M^{(n)}\} \times$ total variation on $(\frac{1}{n}, \frac{j-j_0+1}{n})$, while even in the worst case, $\sqrt{n}z_j \geq -|\kappa(1/n)| \cdot \max\{m^{(n)}, M^{(n)}\}$. □

The Hölder regularity condition on J in Theorems 3.4 and 3.7 admits ramifications, as we need only bound the downside—it can be replaced by the condition that for each n we have $x_j \geq 0$ for infinitely many j, which by symmetry of W occurs in at least half of the cases (unconditionally, i.e., "$\mathrm{P}_{-\infty}$"). Should $x_j\sqrt{n}$ blow up as n grows, then it would be expected that $J(T, T-1/n)$ oscillates around 0, and we would be able to extract a subsequence where it adds positively to the return. Similar considerations would improve upon the next Theorem 3.8 as well. Before we state that result, it should be noted that the total variation criterion can also be improved upon in the setup of Theorem 3.4, if the variation of the step function corresponding to grid size $1/n$, diverges as time grows. When $|\kappa(1/n)| \to \infty$, and κ is monotone (anything else improves total variation) and does not change sign (if it does, Theorem 3.4 part (a) applies), we can still have free lunches with vanishing risk.

Theorem 3.8 (Sufficient conditions for FLVR under the form (3.9)). *Suppose zero transaction cost and that* $M^{(n)} = m^{(n)}$ *and* $\inf_i \mathrm{E}[\xi_i^{(n)}]/m^{(n)} > -1$. *Furthermore, assume that* κ *does not change sign, and that* $\inf_\vartheta |\kappa(\vartheta)| = 0 < \liminf_n |\kappa(1/n)|$ (*possibly* $= +\infty$). *Then either of the following is sufficient for FLVR*:

(a) *For given* n: $x_j^{(n)} \geq 0$ *for infinitely many* j, *and additionally*, $\lim_{\vartheta\to\infty} \kappa(\vartheta) = 0$.

(b) J *is Hölder continuous with index* $> 1/2$, $\liminf_n \frac{\inf_i \mathrm{E}[\xi_i^{(n)}]}{m^{(n)}} > -1$, *and* n *is large enough. The FLVR is an arbitrage if the positivity of the* x_j *is uniform* (*w.r.t.* j).

Proof. We prove only the case of positive κ. Just like in the proof of Theorem 3.7 part (a), $\sqrt{n}(\text{ess sup}\, y_j + \text{ess inf}\, z_{j+1})$ will telescope to $-m^{(n)}\kappa(\frac{j-j_0+1}{n})$, which can be made arbitrarily close to 0; a P_{j_0}-positive event will be it falling within ε/n of $-m^{(n)}\kappa(\frac{j-j_0+1}{n})$. It will turn out that this takes care of the numerator of (3.1) in the FLVR definition. For the denominator, we need the expected return:

$$\begin{aligned}
&x_j + \text{ess sup}\, y_j + \mathrm{E}[z_{j+1}] \\
&\quad = x_j + \frac{1}{\sqrt{n}}\left[-m\kappa\left(\frac{j-j_0+1}{n}\right) + m\kappa\left(\frac{1}{n}\right) + \kappa\left(\frac{1}{n}\right)\mathrm{E}\xi_{j+1}\right] \qquad (3.12) \\
&\quad = x_j + \frac{m}{\sqrt{n}}\left[-\kappa\left(\frac{j-j_0+1}{n}\right) + \kappa\left(\frac{1}{n}\right)\left(1 + \frac{\mathrm{E}\xi_{j+1}}{m}\right)\right].
\end{aligned}$$

(a) Now passing through a subsequence with nonnegative x_j, then $\kappa(\frac{j-j_0+1}{n})$ will vanish and (3.12) will be positive from the assumption; this takes care of the denominator of (3.1). Assuming that the numerator is negative (otherwise there is arbitrage), the ratio (3.1) will on the P_{j_0}-positive event of the history falling within $\varepsilon/\sqrt{n}$ of its ess sup, exceed

$$\left(-m\kappa\left(\frac{j-j_0+1}{n}\right)-\varepsilon\right)\Big/\left(m\left[-\kappa\left(\frac{j-j_0+1}{n}\right)+\kappa\left(\frac{1}{n}\right)\left(1+\frac{\mathrm{E}\xi_{j+1}}{m}\right)\right]\right)$$

which tends to 0 as j and ε^{-1} grow.

(b) The ratio (3.1) becomes—with $\varepsilon = \varepsilon_n$ possibly n-dependent—

$$\left(\frac{m}{\sqrt{n}}\left[\frac{x_j\sqrt{n}}{m}-\kappa\left(\frac{j-j_0+1}{n}\right)-\frac{\varepsilon}{m\sqrt{n}}\right]\right)$$
$$\Big/\left(\frac{m}{\sqrt{n}}\left[\frac{x_j\sqrt{n}}{m}-\kappa\left(\frac{j-j_0+1}{n}\right)-\frac{\varepsilon}{m\sqrt{n}}+\kappa\left(\frac{1}{n}\right)\left(1+\frac{\mathrm{E}\xi_{j+1}}{m}\right)\right]\right).$$

First, cancel $m/\sqrt{n}$. Then, observe that as in the proof of Theorem 3.7 part (a), we can choose j depending on n as to approximate the appropriate ϑ, or possibly the appropriate sequence of ϑ's, so that $\kappa(\frac{j-j_0+1}{n})$ vanishes in the limit, along with—by assumption—everything involving x_j. Then for suitably small ε, then $\kappa(1/n)$ will make the denominator (and hence the expectation) positive, while the numerator can be chosen arbitrarily small. □

Remark 3.9. Notice that the statements of Theorem 3.7 and of Theorem 3.8 part (b), do not depend on what t_0 is, and what history the agent faces upon entry. It is certainly not obvious that it should be this way. The setup of Theorem 3.4 does not rule out a priori that there could be an arbitrage initially, to be exploited at a later stage if a positive event D_j occurs, but which with positive probability disappears for good. (In more formal terms, the ξ_{j+1} could be drawn so that not only would $\mathrm{P}_{j_0+1}[D_j] = 0$, but also $\mathrm{P}_{j_0+1}[\mathrm{P}_i[D_j] > 0] = 0$ for all $i > j_0$.) But under the applicability of Theorem 3.7 or Theorem 3.8 part (b), this will not be the case: the arbitrage, respectively, FLVR, will show up for large enough n regardless of whether the agent enters after a long "bad" period which hampers future prospects; for fine enough discretisation, there will always be a positive event where the arbitrage could materialise. This is not to say that the probability of this event is independent of history, nor that the choice of positive event is—only the binary question of existence. Under these results, regardless how disadvantageous the development has been, the strategy of calmly waiting for lunch time, will always yield positive expected value.

4 Some examples and nonexamples

We will in the following discuss a few cases. Throughout this section, assume common symmetric support $[-m, m]$.

(a) Brownian motion is not prone to arbitrages in the discretised version; we have $K(t,s) = 1_{t>s>0}$, so there is no contribution from history.

(b) The Ornstein–Uhlenbeck process (mean-reverting to 0) admits the representation $\kappa(\vartheta) = \kappa(0) \cdot e^{-vt}$ with $v > 0$. This κ satisfies Theorem 3.8, which will yield FLVR (but, easily, not arbitrage) if choosing the distribution of the ξ_i to comply with the assumptions. If there is positive drift (mean-reversion to a positive level μ), then the discretised version admits arbitrage. It should be noted though, that in the continuous-time model, a portfolio of $\eta(t)$ yields a wealth process dynamics of $\eta(t)\,\mathrm{d}Z(t) = \eta(t)[v \cdot (\mu - Z(t)) + \sigma\,\mathrm{d}W(t)]$, where there is an arbitrarily big upside for given volatility level, by waiting for Z to become negatively large. However, the continuous model remains arbitrage-free, regardless of μ.

(c) Sottinen (2001) considers fractional Brownian motion with Hurst parameter $H > 1/2$, using the representation

$$K(t,s) = \int_s^t (u/s)^{H-1/2}(u-s)^{H-3/2}\,\mathrm{d}u$$

(up to an irrelevant positive constant), so that $K(t^+, t) = 0$ and K_1' is positive. Then by Remark 3.5 we will have Theorem 3.4 applying, as $J(t) = \int_0^{t_0} \int_s^t (u/s)^{H-1/2}(u-s)^{H-3/2}\,\mathrm{d}u\,\mathrm{d}W(s)$ is differentiable at $t = T^-$ (just interchange order of integration).

Furthermore, by Remark 3.9, the arbitrage holds regardless of history. No matter how bad (and how long!) the initial period until entry is, there is still a positive event that a free lunch will actually manifest.

(d) Maybe a more common representation for fractional Brownian motion is, for any $H \neq 1/2$ and up to a constant,

$$K(t,s) = (t-s)^{H-1/2} - \big((-s)_+\big)^{H-1/2},$$

corresponding to $\kappa(\vartheta) = \vartheta^{H-1/2}$. Let us assume that the ξ_i have the same support. κ is monotonous, so then conditions (3.9) hold. Now the results are different for positively ($H > 1/2$) and negatively ($H < 1/2$) autocorrelated fBm:

- In the case $H > 1/2$, $\kappa(0^+) = 0$ and κ is increasing. Furthermore, J is Hölder continuous of order up to H. Theorem 3.4 applies, by Remark 3.5.
- In the case $H < 1/2$, $\kappa(0^+) = \infty$ while $\kappa(\infty) = 0$. Then for at least half of the cases, Theorem 3.8 part (a) applies.

(e) Rogers (1997) proposes a modification of fractional Brownian motion, in order to eliminate the arbitrage but preserve the long run memory properties which motivated the use of fBm in finance in the first place. Rogers gives a specific (monotone) example

$$\kappa(\vartheta) = k \cdot \big(\vartheta^2 + v\big)^{(H-1/2)/2}, \tag{4.1}$$

but suggests more generally to choose κ such that $\kappa(0) = 1$, $\kappa'(0) = 1$, and has the same $\sim \vartheta^{H-1/2}$ behaviour for large ϑ. This behaviour, tending to ∞ for $H > 1/2$ and 0 for $H < 1/2$, is sufficient to yield the same results as for example (d).

(f) For a mix between a fractional and an (uncorrelated) ordinary Brownian motion, there is a very peculiar result by Cheridito (2001): if the fBm part has $H > 3/4$, then it behaves just as drift (which, e.g., means that it does not enter in the Black–Scholes formula). For $H \leq 3/4$, there is still arbitrage as if there were no Brownian component. However, in our case, such a process works like example (e) when $H > 1/2$: mixing in Brownian motion at volatility σ, we get $\kappa(t-s)$ replaced by $\sqrt{\sigma^2 + \kappa(t-s)^2} = (\sigma^2 + (t-s)^{2H-1})^{1/2}$ which for $H > 1/2$ works analogous to (4.1), and will admit arbitrage in the discretisation. There is apparently nothing happening at the Cheridito threshold of $3/4$.

We end this section by pointing out that not only are the discrete markets possibly different than their weak limits when it comes to *existence* of free lunches; the arbitrages themselves might occur from different properties of K. In the canonical models, either of the properties $\kappa(0^+) = 0$ and $\kappa(+\infty) = +\infty$ will lead to arbitrages, and the latter will be due to the long-run memory. The long-run memory was arguably the reason why fractional Brownian motion $\kappa(\vartheta) = \vartheta^{H-1/2}$ (with $H > 1/2$) was suggested in the first place as driving noise for financial markets, and the Rogers proposal of example (e) above, leaves that long memory in the process. Let assume that $t_0 = 0$, and that, coincidentally, $x_{j_0} = x_{j_0+1} = 0$ in order to isolate short-term effects. Fix n for the moment. Then the first-step innovation is symmetric, and the next one cannot lead to arbitrage either, as $\kappa(2/n) - \kappa(1/n) < \kappa(1/n)$ by concavity. The minimum number of steps (after j_0) for the arbitrage for the fBm case, is the smallest integer $> 2^{1/(H-1/2)}$ (equality suffices if the ess sup has point mass), so that the absolute minimum is 5 steps, obtained for H above ≈ 0.931. Of course, as the partition refines and n grows, this happens in shorter *time*, thus approaching the continuous-time setup where the profit instantly increases from 0 (Framstad (2004)). On the other hand, even when mixed with ordinary Brownian motion, the Examples (e) and (f) yield arbitrages; boosting up the ordinary Brownian part, will merely require a longer beneficial period before the arbitrage manifests itself. Those arbitrages are due to the *long run* behaviour—namely, the fact that κ tends to $+\infty$ on the long run.

5 Concluding remarks

We have seen that discrete-time random walk markets may behave radically different from their weak limits, as the former may admit arbitrages or FLVRs which vanish in the limit. Furthermore, quite a few of our estimates may be sharpened and the results likely ramified. That is in the author's opinion not a main concern. Rather, it has turned out that a type of result once interpreted as another objectional property of the fBm's, is simply to be expected if one models moving average processes in such a careless way.

One could certainly try to remedy the problem by choosing wide supports with low probability of the arbitrages manifesting themselves. Arguably, a practitioner should be worried even at far less radical modeling issues than the binary *existence of* arbitrage, and a quick fix which merely covers up the most obvious undesirable property, might not be an adequate solution to the inherent problem.

Appendix: Proof of Proposition 3.2

Put $u = q = 1$. Denote the buying and selling prices in the simple model (2.10) by ζ_* and ζ^*, and in the full model by $S_* = G(\zeta_*)$ and $S^* = G(\zeta^*)$. Observe first that we may take ζ_* bounded by restricting D_j without avoiding the property $\mathrm{P}_{j_0}[D_j] > 0$, and we will do so in the following. Assume that the simple model (2.10) has arbitrage for transaction cost $c > 0$; then

$$\zeta^* \geqslant \zeta_* + c$$

which by applying G and rearranging, is equivalent to

$$\begin{aligned} &S^* - S_* - \lambda\big(\Lambda^*(1, S_*, S^*) + \Lambda_*(1, S_*)\big) \\ &\quad \geqslant G(\zeta_* + c) - G(\zeta_*) - \lambda\big(\Lambda^*\big(1, G(\zeta_*), G\big(\zeta^*\big)\big) + \Lambda_*\big(1, G(\zeta_*)\big)\big) \end{aligned}$$

so we have arbitrage if the right-hand side is nonnegative, so it suffices that

$$0 < \lambda \leq \frac{G(\zeta_* + c) - G(\zeta_*)}{\Lambda^*(1, G(\zeta_*), G(\zeta^*)) + \Lambda_*(1, G(\zeta_*))}. \tag{A.1}$$

As already remarked, we can assume ζ_* bounded, so it suffices to bound Λ^* for given ζ_*. By at most linear growth, $\Lambda^*(1, S_*, S^*) \leq \lambda_0(S_*) + \lambda_1(S_*) \cdot S^*$, where λ_0, λ_1 are locally bounded functions of S_*, the return is

$$\begin{aligned} Y &= S^* - S_* - \lambda\big(\Lambda^*(1, S_*, S^*) + \Lambda_*(1, S_*)\big) \\ &\geq S^*\big(1 - \lambda\lambda_1(S_*)\big) - S_*\big(1 - \lambda\lambda_1(S_*)\big) - \lambda\lambda_1(S_*)S_* - \lambda(\Lambda_* + \lambda_0). \end{aligned}$$

By boundedness of S_* we can take $\lambda\lambda_1(S_*) < 1$, in which case the return will be $\geqslant 0$ if

$$S^* - S_* \geqslant \lambda \frac{\lambda_0 + \lambda_1 S_* + \Lambda_*}{1 - \lambda\lambda_1(S_*)}$$

which at least equals $\lambda \cdot 2(\lambda_0 + \lambda_1 S_* + \Lambda_*) =: \lambda\tilde{\Lambda}$, where $\tilde{\Lambda}$ is a locally bounded function of S_* alone. Hence, we can consider the problem with $\tilde{\Lambda}$ in place of Λ_* and assuming $\Lambda^* = 0$, and then the right-hand side of (A.1) will not depend on the selling price ζ^*. We are done with the arbitrage part of the proposition.

For FLVR, it suffices to point out that Jensen's inequality improves both the upside and downside for convex G, compared to for linear ones.

Acknowledgments

This paper has benefited from discussions with in particular Albert Shiryaev and Gisle James Natvig, and (indeed, more than usual) from comments from the journal's referees. All errors are mine. The content does not reflect the views of my affiliations.

References

Cheridito, P. (2001). Mixed fractional Brownian motion. *Bernoulli* **7**, 913–934. MR1873835

Cheridito, P. (2004). Gaussian moving averages, semimartingales and option pricing. *Stochastic Process. Appl.* **109**, 47–68. MR2024843

Framstad, N. C. (2004). Arbitrage with the ordinary chain rule. *Finance Letters* **2**.

Guasoni, P. (2006). No arbitrage under transaction costs, with fractional Brownian motion and beyond. *Math. Finance* **16**, 569–582. MR2239592

Guasoni, P., Rásonyi, M. and Schachermayer, W. (2008). Consistent price systems and face-lifting pricing under transaction costs. *Ann. Appl. Probab.* **18**, 491–520. MR2398764

Rogers, L. C. G. (1997). Arbitrage with fractional Brownian motion. *Math. Finance* **7**, 95–105. MR1434408

Shiryaev, A. N. (1999). *Essentials of stochastic finance*. *Facts, Models, Theory*. *Advanced Series on Statistical Science & Applied Probability* **3**. River Edge, NJ: World Scientific Publishing. Translated from the Russian manuscript by N. Kruzhilin. MR1695318

Sottinen, T. (2001). Fractional Brownian motion, random walks and binary market models. *Finance Stoch.* **5**, 343–355. MR1849425

Whitt, W. (2007). Proofs of the martingale FCLT. *Probab. Surv.* **4**, 268–302. MR2368952

Department of Economics
University of Oslo
P.O. Box 1095 Blindern
NO-0317 Oslo
Norway
E-mail: ncf+research@econ.uio.no