SENSITIVITY ANALYSIS IN A MARKET WITH MEMORY

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ABSTRACT. A general market model with memory is considered. The formulation is given in terms of stochastic functional differential equations, which allow for flexibility in the modeling of market memory and delays. We focus on the sensitivity analysis of the dependence of option prices on the memory. This implies a generalization of the concept of delta. Our techniques use Malliavin calculus and Fréchet derivation. When it comes to option prices, we consider both the risk-neutral and the benchmark approaches and we compute the delta in both cases. Some examples are provided.

1. INTRODUCTION

In this paper we are interested in the study of price sensitivities of financial claims ("greeks") in markets with memory. The fundamental case we study is the so-called "delta", which is the sensitivity to the knowledge of the asset price at time $t = 0$. The delta typically takes the form

$$(1.1) \qquad \Delta(\eta) := \frac{\partial}{\partial \eta} p(\eta)$$

where

$$(1.2) \qquad p(\eta) = E_{Q^\eta} \left[\frac{\Phi(^\eta S_T)}{^\eta N(T)} \right]$$

is the price of the claim (or option) $\Phi(^\eta S_T)$ with respect to the underlying asset process $^\eta S_t$, $0 \leqslant t \leqslant T$ at maturity T. Here Φ is a pay-off function, $^\eta N(t)$, $0 \leqslant t \leqslant T$, some numéraire, and Q^η a certain probability measure (e.g. risk neutral measure). We assume that $^\eta S_t$, $0 \leqslant t \leqslant T$ describes e.g. a commodity or stock price process on a market with memory, that is we require that $^\eta S_t$, $0 \leqslant t \leqslant T$ depends on some memory η modeled e.g. by a function. Hence, we may interpret the price sensitivity in (1.1) as a "functional derivative" of the price with respect to the "market history" η.

Prices of goods or commodities exhibit frequently behavioral aspects that may be hard to interpret or model accurately. In the literature we find several works vindicating the presence of memory in markets, meaning that, prices of financial instruments are affected in some way by their values in the past. This phenomenon is discussed, for instance, in [1] and [2], where a market model with delay is presented and a Black-Scholes formula for the European call-option is derived. The model presented provides enough flexibility for a better fit than the classical Black-Scholes model when assessed against real market observations. Other authors dealing with markets with memory are for instance [4], [5], [9], [12] and [15]. In [14] the authors propose a model whose dynamics take the past of the prices into account in order to clarify the presence of random cyclical fluctuations in the market. We also mention the work [24], where a stochastic delay equation is used to model stock prices $^{\eta}S_t$, $0 \leqslant t \leqslant T$. Here, the occurrence of delay in the model is explained by the influence of insider traders on stock prices who have access to information about certain events prior to the beginning of the trading period. The model we employ in the present paper captures all models with memory mentioned here above.

In this paper we aim at analyzing sensitivities of prices of financial claims both in the *risk neutral valuation* and the *benchmark approach*. The *benchmark approach* for pricing contracts and financial options has been vigorously studied by e.g. [21], [19], [3], [6]. It has the main advantage of not requiring the existence of an *equivalent martingale measure* (risk neutral probability) in order to price claims, but the existence of a *numéraire* portfolio (i.e. the *growth optimal portfolio*), that is, a portfolio process for which discounted price processes are martingales with respect to the physical measure. Therefore, there is no necessity to change measure.

In the next section, we introduce a general *stochastic functional differential equation* (SFDE), which will then model a general asset price dynamics involving delay, and memory in general. Moreover, we present important results on stochastic and Fréchet derivatives that will be crucial in the computation of sensitivity parameters. Indeed we focus on the delta as the parameter of sensitivity with respect to the initial condition. We stress that the initial condition is the memory η in (1.1) that, in this framework, is a whole random path. Hence we identify the need to extend the concept of delta. Our techniques deal with Fréchet derivatives and Malliavin derivatives in Hilbert spaces. This paper contributes to the analysis of prices of financial derivatives or insurance linked-derivatives when the price of the underlying depends on its past. The computation of the delta and its very concept for markets with memory are tackled for the first time in this paper. From the mathematical point of view we derive, in the context of SFDE's, an explicit formula that connects the Fréchet derivative of the solution with respect to initial condition with its Malliavin derivative.

It is in Section 3 where we focus on the computation of the derivative of expectations in a general set-up. This results will then be applied in Section 4 where we study option pricing in the risk neutral valuation and in the benchmark approach. The option prices depend on the past of the underlying and we compute the sensitivity to this memory. We stress that our techniques suit path dependent options. Some examples of models with

delay or *memory* are presented. An appendix summarizing the used results in Malliavin calculus is given with the aim of providing a self-contained reading.

2. Stochastic functional differential equations

In this section we present the general setup for stochastic functional differential equations (SFDE's) that we will adopt to model delays in market dynamics. Our framework is inspired by and generalizes [1], [2] and [14].

2.1. **The model.** We consider $W = \{W(t, \omega); \ \omega \in \Omega, t \in [0, T]\}$ an m-dimensional standard Brownian motion on the complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)$ where the filtration is the one generated by the increments of W containing all P-null sets and $\mathcal{F} = \mathcal{F}_T$.

We are interested in stochastic processes $x : [-r, T] \times \Omega \to \mathbb{R}^d$, $r \geqslant 0$, with finite second order moments and a.s. continuous sample paths. So, one can look at x as a random variable $x : \Omega \to \mathcal{C}([-r, T], \mathbb{R}^d)$ in $L^2(\Omega, \mathcal{C}([-r, T], \mathbb{R}^d))$. In fact, we can look at x as

$$x : \Omega \to \mathcal{C}([-r, T], \mathbb{R}^d) \hookrightarrow L^2([-r, T], \mathbb{R}^d) \hookrightarrow \mathbb{R}^d \times L^2([-r, T], \mathbb{R}^d).$$

So, from now on, we denote $M_2([-r, T], \mathbb{R}^d) := \mathbb{R}^d \times L^2([-r, T], \mathbb{R}^d)$ the so-called Delfour-Mitter space endowed with the norm

$$(2.1) \qquad \|(v, \varphi)\| = \left(|v|^2 + \|\varphi\|_2^2\right)^{1/2}, (v, \varphi) \in M_2([-r, T], \mathbb{R}^d),$$

where $\| \cdot \|_2$ stands for the L^2-norm and $| \cdot |$ for the Euclidean norm in $\mathbb{R}^d$. For short we denote $M_2 := M_2([-r, 0], \mathbb{R}^d)$.

The interest of using such space comes from two facts. On the one hand, the space M_2 endowed with such norm has a Hilbert structure which allows for a Fourier representation of its elements. On the other hand, as we will see later on, the point 0 plays an important role and therefore we need to distinguish between two processes in $L^2([-r, 0])$ that have different images at the point 0. Finally, it is also a natural space to use since it coincides with the space of continuous functions $\mathcal{C}([-r, 0], \mathbb{R}^d)$ completed with respect to the norm presented in (2.1), by taking just the natural injection $i(\varphi(\cdot)) = (\varphi(0), \varphi(\cdot))$ for a $\varphi \in \mathcal{C}([-r, 0], \mathbb{R}^d)$ and by closing it.

In general, given a Banach space E, we denote by $\mathcal{L}^2(\Omega, E)$ the space of all random variables with finite second order moments taking values in E, and we endow this space with the seminorm

$$\|x\|_{L^2(\Omega, E)} = \left(\int_\Omega \|x(\omega)\|_E^2 P(d\omega)\right)^{1/2}.$$

So, $\mathcal{L}^2(\Omega, E)$ is a Fréchet space. It is important to differentiate between the space $\mathcal{L}^2(\Omega, E)$ and the space $L^2(\Omega, E) := \mathcal{L}^2(\Omega, E)/ \sim$ where the equivalence class is given by

$$x, y \in \mathcal{L}^2(\Omega, E) \ x \sim y \iff \|x - y\|_{\mathcal{L}^2(\Omega, E)} = 0.$$

Moreover, we can restrict ourselves into the subspace of $\mathcal{L}^2(\Omega, E)$ of all $(\mathcal{F}_t)_{t \in [0,T]}$-adapted processes $x \in L^2(\Omega, E)$, which means that, for all $t \in [0, T]$ the random variable $x(\cdot)(t) \in$

$L^2(\Omega, \mathbb{R}^d)$ is $\mathcal{F}_t$-measurable. We will denote the restriction of all $(\mathcal{F}_t)_{t \in [0,T]}$-adapted pro-cesses by $\mathcal{L}^2_A(\Omega, E)$. Respectively, $L^2_A(\Omega, E)$ denotes the subspace of $L^2(\Omega, E)$ of elements that admit an $(\mathcal{F}_t)_{t \in [0,T]}$-adapted modification. In our case $E = M_2$.

To deal with memory and delay we use the concept of segment of x. So, given a process x, some delay gap $r > 0$, and a specified time $t \in [0,T]$, we will consider the segment of x in its past time interval $[t-r, t]$. We denote it by $x_t(\omega, \cdot) : [-r, 0] \to \mathbb{R}^d$ defined as: $x_t(\omega, s) := x(\omega, t+s)$ for all $s \in [-r, 0]$. So $x_t(\omega, \cdot)$ is the segment of the ω-trajectory of the process x, and contains all the information from the past down to time $t - r$. Indeed a segment x_t is a $\mathcal{F}_t$-measurable random variable with values in M_2, i.e. $x_t(\omega, \cdot) \in M_2$ given $\omega \in \Omega$.

The segment of x relative to time $t = 0$, i.e. x_0, carries information from before $t = 0$. It represents the initial knowledge about the process x. Let us consider a trivially measurable variable $\eta \in L^2(\Omega, M_2)$. To shorten notation we write $\eta \in M_2$.

Consider then, the stochastic functional differential equation (SFDE),

(2.2)
$$\begin{cases} dx(t) = f(t, x_t)dt + g(t, x_t)dW(t), & t \in [0,T] \\ x_0 = \eta \in M_2 \end{cases}$$

where

$$f : [0,T] \times M_2 \longrightarrow \mathbb{R}^d$$

$$(t, \varphi) \longrightarrow f(t, \varphi)$$

and

$$g : [0,T] \times M_2 \longrightarrow L(\mathbb{R}^m, \mathbb{R}^d)$$

$$(t, \varphi) \longrightarrow g(t, \varphi).$$

Here $g(t, \varphi)$ is a full rank matrix. Under suitable hypotheses on the functionals f and g, one obtains existence and uniqueness of strong solutions (in the sense of L^2) of the SFDE (2.2). The solution is a process $x \in L^2(\Omega, M_2([-r, T], \mathbb{R}^d))$ admitting an $(\mathcal{F}_t)_{t \in [0,T]}$-adapted modification, that is, $x \in L^2_A(\Omega, M_2([-r, T], \mathbb{R}^d))$.

By uniqueness in $L^2(\Omega, M_2([-r, T], \mathbb{R}^d))$ we mean the following: given two processes $x^1, x^2 \in L^2(\Omega, M_2([-r, T], \mathbb{R}^d))$, we say that they are $L^2(\Omega, M_2([-r, T], \mathbb{R}^d))$-unique, or unique in the sense of $L^2(\Omega, M_2([-r, T], \mathbb{R}^d))$, if

$$\|x_1 - x_2\|_{L^2(\Omega, M_2([-r,T],\mathbb{R}^d))} = 0$$

i.e.,

$$\left(\int_\Omega \left(|x_1(\omega)(0) - x_2(\omega)(0)|^2 + \int_{-r}^T |x_1(\omega)(t) - x_2(\omega)(t)|^2 dt \right) P(d\omega) \right)^{1/2} = 0.$$

In such a case, we will just say that the two processes are L^2-unique or unique in the L^2 sense.

The hypotheses we need to ensure existence and uniqueness of solutions of the SFDE (2.2) are here below.

Hypotheses (H):
   (i) (Local Lipschitzianity) The drift and diffusion functionals f and g are Lipschitz on bounded sets in the second variable uniformly w.r.t. the first, i.e., for each integer $n \geqslant 0$, there is a Lipschitz contant L_n independent of $t \in [0, T]$ such that,

$$|f(t, \varphi_1) - f(t, \varphi_2)|_{\mathbb{R}^d} + \|g(t, \varphi_1) - g(t, \varphi_2)\|_{L(\mathbb{R}^m, \mathbb{R}^d)} \leqslant L_n \|\varphi_1 - \varphi_2\|_{M_2}$$

   for all $t \in [0, T]$ and functions $\varphi_1, \varphi_2 \in M_2$ such that $\|\varphi_1\|_{M_2} \leqslant n$, $\|\varphi_2\|_{M_2} \leqslant n$.
   (ii) (Linear growths) There exists a constant $C > 0$ such that,

$$|f(t, \psi)|_{\mathbb{R}^d} + \|g(t, \psi)\|_{L(\mathbb{R}^m, \mathbb{R}^d)} \leqslant C \left(1 + \|\psi\|_{M_2} \right)$$

   for all $t \in [0, T]$ and $\psi \in M_2$.

Now, we are in a position to state the theorem accurately.

Theorem 2.1 (Existence and Uniqueness)**.** *Given Hypotheses* **(H)** *on the coefficients f and g, the SFDE (2.2) has a solution ${}^\eta x \in L^2_A(\Omega, M_2([-r, T], \mathbb{R}^d))$ for a given initial condition $\eta \in M_2$ and it is unique in the sense of L^2.*

The solution (or better its adapted representative) is a process ${}^\eta x : \Omega \times [-r, T] \to \mathbb{R}^d$ such that

   (1) ${}^\eta x(t) = \eta(t)$, $t \in [-r, 0]$.
   (2) ${}^\eta x(\omega) \in M_2([-r, T], \mathbb{R}^d)$ *P-a.s.*
   (3) *For every $t \in [0, T]$, ${}^\eta x(t) : \Omega \to \mathbb{R}^d$ is $\mathcal{F}_t$-measurable.*

Proof. The proof is based on a similar approach as in the classical deterministic case by using successive Picard approximations and can be found in [16], Theorem 2.1. □

Hence, it makes sense to write

$$
{}^\eta x(t) = \begin{cases} \eta(0) + \int_0^t f(u, {}^\eta x_u) du + \int_0^t g(u, {}^\eta x_u) dW(u), \ t \in [0, T] \\ \eta(t), \ t \in [-r, 0]. \end{cases}
$$

Observe that the above integrals are well-defined. In fact, the process $(\omega, t) \mapsto x_t(\omega) \in M_2$ is adapted since x is pathcontinuous and adapted, and the composition with the deterministic coefficients f and g is then adapted as well.

Note that ${}^\eta x$ represents the solution starting off at time 0 with initial condition $\eta \in \mathcal{M}_2$.

One could consider the same dynamics but starting off at a later time, let us say, $s \in [0, T]$, with initial condition $\eta \in M_2$. Namely, we could consider:

(2.3)
$$\begin{cases} dx(t) = f(t, x_t)dt + g(t, x_t)dW(t), & t \in [s, T] \\ x(t) = \eta(t - s), & t \in [s - r, s]. \end{cases}$$

Again, under **(H)** the SFDE (2.3) has the solution,

(2.4) $${}^{\eta}x^s(t) = \begin{cases} \eta(0) + \int_s^t f(u, {}^{\eta}x_u^s)du + \int_s^t g(u, {}^{\eta}x_u^s)dW(u), & t \in [s, T] \\ \eta(t - s), & t \in [s - r, s] \end{cases}$$

The right hand superindex here, ${}^{\eta}x^s$, denotes the starting point. We will omit the superindex when starting at 0, ${}^{\eta}x^0 = {}^{\eta}x$. The interest of defining the solution starting at a later time comes from the semigroup property of the flow of the solution which we will present in the next subsection.

2.2. **Differentiability of the solution and properties.** Since we aim at studying the influence of the initial path η on the solution of (2.2) we need differentiability conditions on the coefficients in order to ensure existence of an at least once differentiable stochastic flow for (2.2).

In general, suppose we have E and F Banach spaces and $U \subseteq E$ an open set. We write $L(E, F)$ for the space of linear bounded operators from E to F endowed with the topology generated by the norms on each space. Then a functional $f : U \to F$ is said to be of class $\mathcal{C}^1$ if $Df : U \to L(E, F)$ is continuous on bounded sets in U. The derivative D is taken in the Fréchet sense. In the sequel, we will just focus on the Hilbert space $E = M_2$.

Now, following [17] we give the definition of stochastic flow.

Definition 2.2. *Denote by* $S([0, T]) := \{s, t \in [0, T] : 0 \leqslant s < t < T\}$. *Let* E *be a Banach space. A stochastic* $\mathcal{C}^1$-*semiflow on* E *is a random field* $X : S([0, T]) \times E \times \Omega \to E$ *satisfying the following properties:*

(i) X *is* $(\mathcal{B}(S([0, T])) \otimes \mathcal{B}(E) \otimes \mathcal{F}, \mathcal{B}(E))$-*measurable.*
(ii) *For each* $\omega \in \Omega$, *the map*

$$X(\cdot, \cdot, \cdot, \omega) : S([0, T]) \times E \longrightarrow E$$

$$(s, t, \eta) \longrightarrow X(s, t, \eta, \omega)$$

   is continuous.
(iii) *For fixed* $(s, t, \omega) \in S([0, T]) \times \Omega$ *the map*

$$X(s, t, \cdot, \omega) : E \longrightarrow E$$

$$\eta \longrightarrow X(s, t, \eta, \omega)$$

   is $\mathcal{C}^1$.

(iv) *If $0 \leqslant s \leqslant u \leqslant t$, $\omega \in \Omega$ and $x \in E$, then*

$$X(s, t, \eta, \omega) = X(u, t, X(s, u, \eta, \omega), \omega).$$

(v) *For all $(t, \eta, \omega) \in [0, T] \times E \times \Omega$, one has $X(t, t, \eta, \omega) = \eta$.*

In our setup, we consider the definition given above in the space $E = M_2$. Let us define $X(s, t, \eta, \omega) := {}^{\eta}x_t^s(\omega) = ({}^{\eta}x^s(t)(\omega), {}^{\eta}x_t^s(\omega)) \in M_2$ for $\omega \in \Omega$, $s \leqslant t$, where ${}^{\eta}x^s$ is the solution of the SFDE (2.3) with initial condition η. Observe here, that we make an abuse of notation when we write $x_t^s(\omega) \in M_2$, we already mean that $x_t^s(\omega)$ is of the form $(x^s(t)(\omega), x_t^s(\omega)) \in M_2$ where $x^s(t)(\omega) \in \mathbb{R}^d$ and $x_t^s(\omega) = \mathbf{1}_{[-r,0)}x_t^s(\omega) \in L^2([-r, 0], \mathbb{R}^d)$. We can see that X is indeed a Fréchet differentiable stochastic flow associated to (2.3) under the following conditions:

Hypotheses (D):

(i) The functional $f : [0, T] \times M_2 \to \mathbb{R}^d$ is jointly continuous. For each $t \in [0, T]$, the map

$$f(t, \cdot) : M_2 \longrightarrow \mathbb{R}^d$$

$$\varphi \longrightarrow f(t, \varphi)$$

is Lipschitz on bounded sets in M_2 uniformly with respect to $t \in [0, T]$. For each $t \in [0, T]$ the map

$$f(t, \cdot) : M_2 \longrightarrow \mathbb{R}^d$$

$$\varphi \longrightarrow f(t, \varphi)$$

is $\mathcal{C}^1$ uniformly with respect to $t \in [0, T]$.

(ii) For each $t \in [0, T]$, the functional $g(t, \cdot) : M_2 \to L(\mathbb{R}^m, \mathbb{R}^d)$ is $\mathcal{C}^1$, with Fréchet derivative $Dg(t, \cdot)$ globally bounded. For each $\varphi \in M_2$, the map

$$g(\cdot, \varphi) : [0, T] \longrightarrow L(\mathbb{R}^m, \mathbb{R}^d)$$

$$t \longrightarrow g(t, \varphi_t)$$

is square-integrable and locally of bounded variation. For each $t \in [0, T]$ and $\omega \in \Omega$ the functional

$$g(t, \cdot) : L^2([-r, T], \mathbb{R}^d) \longrightarrow L^2([0, T], L(\mathbb{R}^m, \mathbb{R}^d))$$

$$\varphi \longrightarrow g(t, \varphi_t)$$

is $\mathcal{C}^1$ and globally bounded.

We have the following result due to [17], Theorem 3.1.

Theorem 2.3. *Suppose that Hypotheses* **(D)** *are fulfilled and moreover that there exists a constant $C := C(T) > 0$ and $\gamma := \gamma(T) \in [0,1)$ such that*

$$(2.5) \qquad\qquad |f(t,\varphi)| \leqslant C \left(1 + \|\varphi\|_{M_2}^\gamma\right)$$

for all $t \in [0,T]$ and $\varphi \in M_2$. Then the following is true:

(i) *For each $\omega \in \Omega$, the map*

$$X(\cdot,\cdot,\cdot,\omega) : S([0,T]) \times M_2 \longrightarrow M_2$$

$$(s,t,\varphi) \longrightarrow X(s,t,\varphi,\omega)$$

is continuous and for fixed $(s,t,\omega) \in S([0,T]) \times \Omega$, the map

$$X(s,t,\cdot,\omega) : M_2 \longrightarrow M_2$$

$$\varphi \longrightarrow X(s,t,\varphi,\omega)$$

is $\mathcal{C}^1$.

(ii) *For each $\omega \in \Omega$ and $(s,t) \in S([0,T])$ with $t \geqslant s + r$ the map $X(s,t,\cdot,\omega) : M_2 \to M_2$ carries bounded sets into relatively compact sets. In particular, each Fréchet derivative $DX(s,t,\varphi,\omega) : M_2 \to M_2$ with respect to $\varphi \in M_2$, is a compact linear map for $t \geqslant s+r$, $\omega \in \Omega$.*

(iii) *The maps*

$$(s,t,\varphi,\omega) \mapsto X(s,t,\varphi,\omega) \in M_2$$
$$(s,t,\varphi,\omega) \mapsto DX(s,t,\varphi,\omega) \in L(M_2,M_2)$$
$$(s,t,\varphi,\omega) \mapsto \|DX(s,t,\varphi,\omega)\|_{L(M_2,M_2)} \in \mathbb{R}^+$$

are $(\mathcal{B}(S(T)) \otimes \mathcal{B}(M_2) \otimes \mathcal{F}, \mathcal{B}(M_2))$-measurable, $(\mathcal{B}(S(T)) \otimes \mathcal{B}(M_2) \otimes \mathcal{F}, \mathcal{B}_s(L(M_2,M_2)))$-measurable, and $(\mathcal{B}(S(T)) \otimes \mathcal{B}(M_2) \otimes \mathcal{F}, \mathcal{B}(\mathbb{R}^+))$-measurable, respectively.

Remark 2.4. *Note that condition (2.5) is sufficient. One may exchange it by other technical assumptions. We refer to [17] for a list of other sufficient conditions.*

Henceforward, we assume that the hypotheses from Theorem 2.3 are fulfilled. In such a case we know that the SFDE (2.2) is well-posed, it has a L^2-unique solution, and it admits a global Fréchet differentiable stochastic flow. See [17].

We will, from now on, use the following notation $X_t^s(\eta,\omega) := X(s,t,\eta,\omega) = {}^\eta x_t^s(\omega)$. By Theorem 2.3 and item (iv) in Definition 2.2 we have that equation (2.2) has the following property: For each $s,t \in [0,T]$, $s \leqslant t$,

$$X_t^0 = X_t^s \circ X_s^0.$$

Observe that this property defines a family of operators $\{X_t^s\}_{0 \leqslant s \leqslant t \leqslant T}$ that conforms to a semigroup. These operators will be used extensively in the sequel.

Finally, recall that we also have Malliavin differentiability of $^\eta x(t) \in L^2(\Omega, \mathbb{R}^d)$ and $^\eta x_t$, see [25]. The latter $x : \Omega \to M_2([-r,0], \mathbb{R}^d)$ is a random variable taking values in a Hilbert space. We have summarized the used elements of Malliavin calculus for Hilbert space valued random variables in the Appendix. We will denote by $\mathcal{D}_s$, $0 \leqslant s \leqslant T$, differentiation in the Malliavin sense.

In relation to (2.3) we also define the following family of operators. For any $u \in [-r,0]$, define $\rho_u : M_2 \to \mathbb{R}^d$ as the evaluation at u, that is, $\rho_u((v,\varphi)) := v\mathbf{1}_{\{0\}}(u) + \varphi(u)\mathbf{1}_{[-r,0)}(u)$ for any $(v, \varphi) \in M_2$. We observe here that the random variable $^\eta x(t)$ is an evaluation at 0 of the process $X_t^s = {}^\eta x_t^s$. Indeed, for $u \in [-r, 0]$,

$$\rho_u \circ X_t^s(\eta, \omega) = \rho_u({}^\eta x_t^s)(\omega) = {}^\eta x_t^s(u)(\omega) = {}^\eta x^s(t+u)(\omega).$$

Next result details an important relationship between the Malliavin derivative of the solution of (2.2) at s and the Fréchet derivative of (2.3) with respect to the initial path. We would like to highlight here that we wish to compare two objects with different natures.

On the one hand, the Malliavin derivative $\mathcal{D}_s {}^\eta x_t$ is a process which takes values in the space $L^2(\Omega, M_2)$, i.e. it is an equivalent class. On the other side, we are considering solutions to the SFDE (2.3) in a pathwise sense and then computing the Fréchet derivative $DX_t^s(\eta, \omega)$, which is an object in $L(M_2, M_2)$ for each $\omega \in \Omega$. In order, to compare the two we specify that we consider the representative of $\mathcal{D}_s X_t^0(\eta, \cdot)$ that is adapted, which we denote by $(\mathcal{D}_s X_t^0(\eta, \cdot))(\omega)$, $\omega \in \Omega^*$ where Ω^*, $P(\Omega^*) = 1$, is the set for which $\mathcal{D}_s X_t^0(\eta, \cdot)$ is adapted. Then for such representative and $\omega \in \Omega^*$ we compute the Fréchet derivative of the stochastic flow $X_t^0(\eta, \omega)$. Finally, we compare the two. This relation plays a crucial role in the study of the sensitivity of (2.2) to the initial path condition.

Theorem 2.5. *In the hypotheses of Theorem 2.3. For any $t \in [0,T]$ and $\eta \in M_2$, $X_t^0(\eta, \cdot)$ is Malliavin differentiable and for $s, t \in [0, T]$: $s \leqslant t$ and $\omega \in \Omega$, $X_t^s(\eta, \omega)$ is Fréchet differentiable with respect to $\eta \in M_2$. Moreover, we have the following relationship between random variables:*

$$(2.6) \qquad DX_t^s\left(X_s^0(\eta, \omega), \omega\right) = \left(\mathcal{D}_s X_t^0(\eta, \cdot)\right)(\omega) \; g_R^{-1}(s, {}^\eta x_s(\omega)) \; \rho_0, \quad P - a.s.$$

Here, fixed $s, t \in [0, T]$, $DX_s^t(X_s^0(\eta, \omega), \omega)$ stands for the Fréchet derivative of the flow $X_t^s(\cdot, \omega)$ given $\omega \in \Omega$, then evaluated at the point $X_s^0(\eta, \omega)$. Finally, g_R^{-1} denotes the right-inverse of the deterministic $(d \times m)$-matrix g.

Proof. For $t \in [0, T]$ the Malliavin and Fréchet differentiability of $^\eta x(t)$ have already been discussed in the previous section and can be found, respectively, in [25] and [16]. Hereafter, for simplicity in notation we omit the dependence in ω when confusion does not arise.

First of all, we make the following observation: for any $u \in [-r, 0]$

$$D\rho_u(X_t^s) = \rho_u \circ DX_t^s \text{ for any } u \in [-r, 0].$$

This implies that the segment process of the element $D\,{}^{\tilde\eta} x^s(t)$ is the same as the Fréchet derivative of the segment process $^{\tilde\eta} x_t^s$ for a given $\tilde\eta \in M_2$. The same holds for the Malliavin

derivative
$$\mathcal{D}_s \, \rho_u(X_t^0) = \rho_u \circ \mathcal{D}_s X_t^0.$$
but in this case one uses the chain rule in the framework of random variables taking values on Hilbert spaces, we refer to the appendix for further details. Such observation allows us to prove identity (2.6) in the finite dimensional case for arbitrary evaluations ρ_u, $u \in [-r, 0]$. In fact, it suffices to do it just at the point 0 thanks to the semigroup property of the flow of the solution.

Then we start by computing the Malliavin derivative $(\mathcal{D}_s \, {}^\eta x(t)) \, (\cdot)$ of the random variable $\omega \mapsto {}^\eta x(t)(\omega)$ at the point $s \in [0, t]$. So,

$$\mathcal{D}_s {}^\eta x(t) = \int_s^t \mathcal{D}_s[f(u, {}^\eta x_u)] du + g(s, {}^\eta x_s) + \int_s^t \mathcal{D}_s[g(u, {}^\eta x_u)] dW(u),$$

hence,

(2.7)
$$\mathcal{D}_s {}^\eta x(t) = \int_s^t D[f(u, {}^\eta x_u)] \circ \mathcal{D}_s({}^\eta x_u) du + g(s, {}^\eta x_s)$$
$$+ \int_s^t D[g(u, {}^\eta x_u)] \circ \mathcal{D}_s({}^\eta x_u) dW(u).$$

In the previous expression we used the chain rule for the Malliavin derivative of random variables taking values in a Banach space, see [22], Proposition 3.8. We include also an ad-hoc version of the chain rule in the appendix.

Thanks to Theorem 2.3, we know that the solution process ${}^\eta x$ admits a stochastic differentiable flow which we denote by $X_t^s(\eta, \omega) = {}^\eta x_t^s(\omega)$, $s \leqslant t$, $\omega \in \Omega$ and its evaluation at zero is namely $\rho_0(X_t^s(\eta, \omega)) = {}^\eta x^s(t, \omega)$. For any $0 \leqslant s \leqslant t \leqslant T$ we look at representative of the solution in a pathwise sense, as an operator with input $\eta \in M_2$ and output ${}^\eta x^s(t)(\omega)$ in $\mathbb{R}^d$. Hence, from (2.4) we have

$$\dot{x}^s(t) = \rho_0(\cdot) + \int_s^t f(u, \cdot) \circ X_u^s(\cdot) du + \int_s^t g(u, \cdot) \circ X_u^s(\cdot) dW(u)$$

where here the dot stands for the function η. Then, we compute the Fréchet derivative of the above operator at a generic point $\tilde{\eta} \in M_2$. To do so, we need to compute the derivative of the stochastic integral. It is not immediate that one may do so by exchanging integral with derivation. In order to justify that this can be done, one can refer to the work done by E. Fournié et al. in for instance [10] or [11] where the same approach is used for the computation of sensitivities. Thus

$$D \, {}^{\tilde{\eta}} x^s(t) = D\rho_0(\tilde{\eta}) + \int_s^t D[f(u, \cdot) \circ X_u^s(\cdot)](\tilde{\eta}) du + \int_s^t D[g(u, \cdot) \circ X_u^s(\cdot)](\tilde{\eta}) dW(u).$$

We recall that ρ_0 is a linear operator and its Fréchet derivative is itself, so $D\rho_0(\tilde{\eta}(\omega)) = \rho_0 \in L(M_2, \mathbb{R}^d)$. For the other two terms we apply the chain rule. First,

$$D[f(u, \cdot) \circ X_u^s(\cdot)](\tilde{\eta}) = Df(u, X_u^s(\tilde{\eta})) \circ DX_u^s(\tilde{\eta}),$$

where $Df(u, X_u^s(\tilde{\eta})) \in L(M_2, \mathbb{R}^d)$ and $DX_u^s(\tilde{\eta}) \in L(M_2, M_2)$. Finally, for g we have

$$Dg(u, X_u^s(\tilde{\eta})) \circ DX_u^s(\tilde{\eta}) \in L(M_2, L(\mathbb{R}^m, \mathbb{R}^d))$$

where $Dg(u, X_u^s(\tilde{\eta})) \in L(M_2, L(\mathbb{R}^m, \mathbb{R}^d))$. Thus, in a summary

(2.8)
$$D^{\tilde{\eta}} x^s(t) = \rho_0(\cdot) + \int_s^t Df(u, X_u^s(\tilde{\eta})) \circ DX_u^s(\tilde{\eta}) du$$

$$+ \int_s^t Dg(u, X_u^s(\tilde{\eta})) \circ DX_u^s(\tilde{\eta}) dW(u)$$

where both left-hand side and right-hand side are operators in $L(M_2, \mathbb{R}^d)$.

In particular, let us consider $\tilde{\eta} = X_s^0(\eta, \omega) = {}^\eta x_s^0(\omega) = {}^\eta x_s(\omega) \in M_2$ for $\omega \in \Omega^*$ where we recall that Ω^* is the full-measure set for which $\mathcal{D}_s X_t^0(\eta, \cdot)$ is adapted. We use the semigroup property of the flow $X_u^s \circ X_s^0 = X_u^0$ and we obtain:

(2.9)
$$D^{\tilde{\eta}} x^s(t) = \rho_0(\cdot) + \int_s^t D[f(u, {}^\eta x_u)] \circ DX_u^0(\eta) du$$

$$+ \int_s^t D[g(u, {}^\eta x_u)] \circ DX_u^0(\eta) dW(u).$$

At this point, we see that there is a similarity between equation (2.9) and equation (2.7). However, we note that equation (2.9) for ω fixed represents an operator in $L(M_2, \mathbb{R}^d)$ while in equation (2.7), after choosing the adapted representative $\omega \in \Omega^*$, $P(\Omega^*) = 1$, we have an operator in $L(\mathbb{R}^m, \mathbb{R}^d)$. Next step is then to transport equation (2.7) to (2.9). We may do it by means of an operator $\tau_s \in L(M_2, \mathbb{R}^m)$ such that, $\omega \in \Omega^*$,

$$g(s, {}^\eta x_s(\omega)) \circ \tau_s(\cdot) = \rho_0(\cdot).$$

We recall that $g(s, {}^\eta x_s(\omega)) \in L(\mathbb{R}^m, \mathbb{R}^d)$. So, we define:

$$\tau_s : M_2 \longrightarrow \mathbb{R}^m$$

$$\varphi \longrightarrow \tau_s(\varphi) = g_R^{-1}(s, {}^\eta x_s(\omega))\varphi(0)$$

where g_R^{-1} denotes the right-inverse of the $d \times m$ matrix $g(s, {}^\eta x_s(\omega))$, which is $m \times d$-dimensional, applied to $\varphi(0) \in \mathbb{R}^d$. Observe that the operator τ_s depends on the solution $x_s(\omega)$. In fact, in this case we have

$$\varphi \in M_2 \xrightarrow{\tau_s} g_R^{-1}(s, {}^\eta x_s(\omega))\varphi(0) \in \mathbb{R}^m \xrightarrow{\circ g(s, {}^\eta x_s(\omega))} g g_R^{-1}\varphi(0) = \varphi(0) \in \mathbb{R}^d.$$

Hence, $g(s, {}^\eta x_s(\omega)) \circ \tau_s(\cdot) = \rho_0(\cdot)$. Moreover for $\varphi \in M_2$

$$\left(\mathcal{D}_s\, {}^\eta x(t)\right)(\omega) \circ \tau_s(\varphi) = \int_s^t D[f(u, {}^\eta x_u(\omega))] \circ \left(\mathcal{D}_s({}^\eta x_u)\right)(\omega) \circ \tau_s(\varphi) du + \varphi(0)$$

$$+ \int_s^t D[g(u, {}^\eta x_u(\omega))] \circ \left(\mathcal{D}_s({}^\eta x_u)\right)(\omega) \circ \tau_s(\varphi) dW(u).$$

By uniqueness of the solutions (2.3) we obtain the desired formula. Indeed, denote $\mathcal{C} := \mathcal{C}([-r,0], \mathbb{R}^d)$. For all $\varphi \in M_2$ we will show that,

$$\left(D^{\,\tilde{\eta}} x^s(t)(\varphi)\right)_t \overset{L^2(\Omega, \mathcal{C})}{=} \left(\mathcal{D}_s{}^{\eta} x(t) \circ \tau_s(\varphi)\right)_t.$$

The argument relies on the fact that the *sup*-norm is weaker than the one in M_2. Again, for the sake of simplicity we skip the dependence on $\omega \in \Omega$. On the one hand,

$$\left(D^{\,\tilde{\eta}} x^s(t)(\varphi)\right)_t (\cdot) = \varphi(0) + \int_s^{t+\cdot} Df(u, {}^{\eta} x_u) \circ DX_u^s(\tilde{\eta})(\varphi) du$$

$$+ \int_s^{t+\cdot} Dg(u, {}^{\eta} x_u) \circ DX_u^s(\tilde{\eta})(\varphi) dW(u).$$

On the other hand,

$$\left(\mathcal{D}_s{}^{\eta} x(t) \circ \tau_s(\varphi)\right)_t (\cdot) = \varphi(0) + \int_s^{t+\cdot} D[f(u, {}^{\eta} x_u)] \circ \mathcal{D}_s({}^{\eta} x_u) \circ \tau_s(\varphi) du$$

$$+ \int_s^{t+\cdot} D[g(u, {}^{\eta} x_u)] \circ \mathcal{D}_s({}^{\eta} x_u) \circ \tau_s(\varphi) dW(u).$$

Therefore,

$$\left\| \left(D^{\,\tilde{\eta}} x^s(t)(\varphi)\right)_t - \left(\mathcal{D}_s{}^{\eta} x(t) \circ \tau_s(\varphi)\right)_t \right\|_{L^2(\Omega, \mathcal{C})}^2$$

$$= E\left[\sup_{\substack{t' \in [-r,0] \\ t+t' \geqslant s}} \left| \int_s^{t+t'} D[f(u, {}^{\eta} x_u)] \circ \left(DX_u^s(\tilde{\eta})(\varphi) - \mathcal{D}_s({}^{\eta} x_u) \circ \tau_s(\varphi)\right) du \right|^2 \right]$$

$$+ E\left[\sup_{\substack{t' \in [-r,0] \\ t+t' \geqslant s}} \left| \int_s^{t+t'} D[g(u, {}^{\eta} x_u)] \circ \left(DX_u^s(\tilde{\eta})(\varphi) - \mathcal{D}_s({}^{\eta} x_u) \circ \tau_s(\varphi)\right) dW(u) \right|^2 \right].$$

Then, Hölder's inequality, martingale inequality, the Itô isometry, and the continuity of Dh and Dg together with the fact that $[-r,0]$ is compact yield,

$$\left\| \left(D^{\,\tilde{\eta}} x^s(t)(\varphi)\right)_t - \left(\mathcal{D}_s{}^{\eta} x(t) \circ \tau_s(\varphi)\right)_t \right\|_{L^2(\Omega, \mathcal{C})}^2$$

$$\leqslant 2C_f^2 t \int_s^t \left\| DX_u^s(\tilde{\eta})(\varphi) - \mathcal{D}_s({}^{\eta} x_u) \circ \tau_s(\varphi) \right\|_{L^2(\Omega, \mathcal{C})}^2 du$$

$$+ 2MC_g^2 \int_s^t \left\| DX_u^s(\tilde{\eta})(\varphi) - \mathcal{D}_s({}^{\eta} x_u) \circ \tau_s(\varphi) \right\|_{L^2(\Omega, \mathcal{C})}^2 du$$

where C_f and C_g stand for the constants from the boundedness of Df and Dg respectively and M the constant coming from the martingale inequality. Gronwall's lemma gives the desired L^2-uniqueness. $\qquad\square$

Corollary 2.6. *In particular, under the conditions of Theorem 2.5, we also have the following relationship between the Malliavin derivative of ${}^{\eta(\omega)} x_t^s(\omega)$ and the Fréchet and*

with respect to $\eta \in M_2$,

$$(2.10) \qquad DX_t^0(\eta, \omega) = \left(\mathcal{D}_s X_t^0(\eta, \cdot)\right)(\omega) g_R^{-1}(s,^\eta x_s(\omega)) \rho_0 \circ DX_s^0(\eta, \omega), \quad P - a.s.$$

Proof. This is an immediate consequence of the semigroup property of the stochastic flow X, that is:

$$X_t^s \circ X_s^0 = X_t^0.$$

We compute the Fréchet derivative in both sides of the equation above at the point $\eta \in M_2$, $D[X_t^s \circ X_s^0](\eta) = DX_t^0(\eta)$ and use the chain rule

$$DX_t^s(X_s^0(\eta)) \circ DX_s^0(\eta) = DX_t^0(\eta).$$

The result follows by Theorem 2.5. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

3. Sensitivity analysis to the initial path condition

Having option pricing at focus, we present the necessary mathematical tools and theoretical formulae to study the sensitivity of the derivative prices to the initial condition. Specifically, we aim at giving expressions for the so-called delta. Note that we consider underlying price dynamics with memory, hence the initial condition is actually a whole process. Then we suggest a new definition for the parameter delta by extending classical concepts.

Before entering the specifics of the financial pricing frameworks, we detail the mathematical approach. Let us consider a function $\Phi : M_2 \to \mathbb{R}^+$ such that $\Phi(X_T^0(\eta)) \in L^2(\Omega)$, a fixed positive time $T < \infty$, and the functional

$$(3.1) \qquad\qquad\qquad p(\eta) = E\left[\Phi(^\eta x(T),\ ^\eta x_T)\right], \ \eta \in M_2$$

where $^\eta x$ is the solution of the SFDE (2.2) with η as initial condition. Recall that the stochastic flow of the solution is denoted by $X_T^0(\eta, \omega) = (^\eta x(T)(\omega),\ ^\eta x_T(\omega))$. Functionals of this type appear in pricing formulae of financial derivatives

$$\Phi(^\eta x(T),\ ^\eta x_T) : \Omega \longrightarrow \mathbb{R}^+$$

$$\omega \longrightarrow \Phi(X_T^0(\eta, \omega)).$$

The sensitivity of prices to the initial condition of the underlying is then to be reconducted to the study of variations of $p(\eta)$ to perturbations of η. The Fréchet derivative of p in η is a linear operator $Dp(\eta) \in L(M_2, \mathbb{R})$ and it describes the fluctuations of $p(\eta)$ around η. Hence, it is natural to define the delta as

$$(3.2) \qquad\qquad\qquad \Delta(\eta) := Dp(\eta) : M_2 \to \mathbb{R}$$

If one would like to produce an index of the robustness of prices to their initial path condition, then one could apply several definitions of robustness. For example, by taking

directional derivatives one would get

$$(3.3) \qquad \Delta_h := \lim_{\varepsilon \to 0} \frac{p(\eta + \varepsilon h) - p(\eta)}{\varepsilon} = \left. \frac{d}{d\varepsilon} p(\eta + \varepsilon h) \right|_{\varepsilon = 0}, \quad h \in M_2$$

and this represents the rate of change near η along the direction $h \in M_2$. Observe that the existence of $Dp(\eta)$ implies that the limit in (3.3) exists and it is finite.

Having an expression for $Dp(\eta)$ one could also take evaluations at functions $h \in M_2$ such that $\|h\| = 1$, i.e.

$$\Delta(h) := Dp(\eta)(h) \in \mathbb{R}$$

and compare $\Delta(h_1) \sim \Delta(h_2)$, $h_1, h_2 \in M_2$, $\|h_1\| = \|h_2\| = 1$.

Also one can simply use the operator norm as sensitivity parameter Δ:

$$(3.4) \qquad \Delta := |||Dp(\eta)||| := \sup_{\substack{\psi \in M_2 \\ \psi \neq 0}} \frac{|Dp(\eta)(\psi)|}{\|\psi\|_{M_2}}.$$

In this case the Δ in (3.4) gives a form of "worst case scenario" of all possible perturbations around η.

Our aim is then to give a formula for the evaluation of $Dp(\eta)$ and Δ. Our techniques are inspired by the Malliavin approach to the computation of the delta in a classical Brownian diffusion setup introduced by E. Fournié, J-M. Lasry, J. Lebuchoux, P-L Lions and N. Touzi in [10].

Next theorem gives an expression for $\Delta(\eta)$ which is independent of the Fréchet derivative of Φ for smooth payoff functions Φ. Thereafter, we will relax the smoothness assumption on Φ.

Theorem 3.1. *Let hypotheses from Theorem 2.3 be fulfilled and denote by $X_t^0(\eta, \omega)$, $\omega \in \Omega$, $t \in [0, T]$, $\eta \in M_2$, the flow associated to SFDE (2.2). Let $\Phi : M_2 \to [0, \infty)$ be a measurable function such that $\Phi(X_T^0(\eta)) \in L^2(\Omega)$. Consider the functional*

$$p(\eta) = E\left[\Phi(^\eta x(T), \, ^\eta x_T)\right].$$

Then for any bounded measurable function $a : [0, T] \to \mathbb{R}$ integrating to 1, we have that

$$(3.5) \qquad \Delta(\eta) = E\left[\Phi(X_T^0(\eta)) w^\Delta(\eta)\right]$$

where, for each η, $w^\Delta(\eta)$ is an element in $L^2(\Omega, L(M_2, \mathbb{R}))$ defined as

$$w^\Delta(\eta) := \int_0^T a(s) g_R^{-1}(s, \, ^\eta x_s) \rho_0 \circ DX_s^0(\eta) dW(s).$$

Moreover, we may define a delta-index as

$$\Delta := \sup_{\substack{\psi \in M_2 \\ \|\psi\|_{M_2} = 1}} |E\left[\Phi(^\eta x(T), \, ^\eta x_T) w^\Delta(\eta)(\psi)\right]|$$

Proof. Step 1: At first we consider the case $\Phi \in \mathcal{C}_b^1(M_2, \mathbb{R})$, i.e. Fréchet differentiable with continuous bounded derivative.

We will first show that the Fréchet derivative of the functional $p : M_2 \to \mathbb{R}$ indeed corresponds to the expectation of the pathwise Fréchet derivative of $\Phi(^\eta x(T)(\omega), \ ^\eta x_T(\omega))$. To do so, just observe the following, for $\eta, \psi \in M_2$

$$\left| p(\eta + \psi) - p(\eta) - E\left[D\Phi(^\eta x(T), \ ^\eta x_T) \right] \right|^2 =$$

$$= \left| E\left[\Phi(^{\eta+\psi}x(T), \ ^{\eta+\psi}x_T) - \Phi(^\eta x(T), \ ^\eta x_T) - D\Phi(^\eta x(T), \ ^\eta x_T) \right] \right|^2$$

$$\leqslant E\left[\left| \Phi(^{\eta+\psi}x(T), \ ^{\eta+\psi}x_T) - \Phi(^\eta x(T), \ ^\eta x_T) - D\Phi(^\eta x(T), \ ^\eta x_T) \right|^2 \right]$$

$$= \int_\Omega \left| \Phi(^{\eta+\psi}x(T)(\omega), \ ^{\eta+\psi}x_T(\omega)) - \Phi(^\eta x(T)(\omega), \ ^\eta x_T(\omega)) \right.$$

$$\left. - D\Phi(^\eta x(T)(\omega), \ ^\eta x_T(\omega)) \right|^2 P(d\omega).$$

Now since $\Phi \in C^1$ we know that for almost all $\omega \in \Omega$ we have

$$\left| \Phi(^{\eta+\psi}x(T)(\omega), \ ^{\eta+\psi}x_T(\omega)) - \Phi(^\eta x(T)(\omega), \ ^\eta x_T(\omega)) \right.$$

$$\left. - D\Phi(^\eta x(T)(\omega), \ ^\eta x_T(\omega)) \right|^2 \leqslant C\|\psi\|_{M_2}^2$$

for some constant $C > 0$. Then taking expectation at both sides it follows that

$$\left| p(\eta + \psi) - p(\eta) - E\left[D\Phi(^\eta x(T), \ ^\eta x_T) \right] \right|^2 \leqslant C\|\psi\|_{M_2}^2$$

and therefore

(3.6) $$Dp(\eta) = E\left[D\Phi(X_T^0(\eta)) \right].$$

Now we proceed to show (3.5) for the case of smooth $\Phi \in C^1(M_2, \mathbb{R})$. The chain rule gives

$$Dp(\eta) = E\left[\Phi'(^\eta x(T), ^\eta x_T) \circ DX_T^0(\eta) \right].$$

Here $\Phi'(X_T^0(\eta, \omega))$ denotes the Fréchet derivative of Φ at the point $X_T^0(\eta, \omega)$ which is an element in $L(M_2, \mathbb{R})$ and $DX_T^0(\eta, \omega) \in L(M_2, M_2)$. Now, we choose a bounded scalar function $a : \mathbb{R} \to \mathbb{R}$ that integrates to 1 and we use Corollary 2.6, then we obtain

$$DX_T^0(\eta, \omega) = \int_0^T a(s) DX_T^0(\eta, \omega) ds$$

(3.7) $$= \int_0^T \mathcal{D}_s X_t^0(\eta, \omega) g_R^{-1}(s, ^\eta x_s(\omega)) \rho_0 \circ DX_s^0(\eta, \omega) ds$$

Then plugging (3.7) inside (3.6), we have

$$Dp(\eta) = E\left[\Phi'(X_T^0(\eta)) \circ \int_0^T \mathcal{D}_s X_T^0(\eta) a(s) g_R^{-1}(s, {}^\eta x_s) \rho_0 \circ DX_s^0(\eta) ds\right]$$

$$= E\left[\int_0^T \Phi'(X_T^0(\eta)) \circ \mathcal{D}_s X_T^0(\eta) a(s) g_R^{-1}(s, {}^\eta x_s) \rho_0 \circ DX_s^0(\eta) ds\right]$$

$$= E\left[\int_0^T \mathcal{D}_s \Phi(X_T^0(\eta)) \circ a(s) g_R^{-1}(s, {}^\eta x_s) \rho_0 \circ DX_s^0(\eta) ds\right]$$

Next step is to use the duality formula for the Malliavin derivative. Observe though that now, $a(s) g_R^{-1}(s, {}^\eta x_s(\omega)) \rho_0 \circ DX_s^0(\eta, \omega) \in L(M_2, \mathbb{R}^m)$ then we write $\delta(a(\cdot) g_R^{-1}(\cdot, {}^\eta x.) \rho_0 \circ DX_\cdot^0(\eta)$ where $a(s) g_R^{-1}(s, {}^\eta x_s) \rho_0 \circ DX_s^0(\eta) \in L^2(\Omega, L(M_2, \mathbb{R}^m))$ and δ is the Skorokhod integral. Theory on Skorokhod integral of random variables taking values in a Banach space can be found in [13]. Nevertheless, note that when we apply a function $\psi \in M_2$ to the operator $a(s) g_R^{-1}(s, {}^\eta x_s) \rho_0 \circ DX_s^0(\eta)$, we obtain an element in $\mathbb{R}^m$ for which a classical duality formula can be used, see for instance [8], Theorem 3.14. Altogether, the Fréchet derivative $Dp(\eta) \in L(M_2, \mathbb{R})$ is given by

$$Dp(\eta) = E\left[\Phi(X_T^0(\eta)) \int_0^T a(s) g_R^{-1}(s, {}^\eta x_s) \rho_0 \circ DX_s^0(\eta) \delta W(s)\right]$$

$$= E\left[\Phi(X_T^0(\eta)) \delta\left(a(\cdot) g_R^{-1}(\cdot, {}^\eta x.) \rho_0 \circ DX_\cdot^0(\eta)\right)\right].$$

Observe also that the process $g_R^{-1}(s, {}^\eta x_s) \rho_0 \circ DX_s^0(\eta) \in L^2(\Omega, L(M_2, \mathbb{R}^m))$ is $\mathcal{F}_s$-measurable so the Skorokhod integral is actually an Itô integral.

Define $w^\Delta(\eta) := \int_0^T a(s) g_R^{-1}(s, {}^\eta x_s) \rho_0 \circ DX_s^0(\eta) \delta W(s)$, then

$$Dp(\eta) = E\left[\Phi(X_T^0(\eta)) w^\Delta(\eta)\right]$$

where $w^\Delta(\eta)$ is an element in $L(L^2(\Omega, M_2), L^2(\Omega, \mathbb{R}^+)) \hookrightarrow L^2(\Omega, M_2^*) \cong L^2(\Omega, M_2)$.

Take $\psi \in M_2$ and apply it to $Dp(\eta)$:

$$Dp(\eta)(\psi) = E\left[\Phi(X_T^0(\eta)) w^\Delta(\eta)\right](\psi) = E\left[\Phi(X_T^0(\eta)) w^\Delta(\eta)(\psi)\right]$$

$$= E\left[\int_0^T \left\{\mathcal{D}_s \Phi(X_T^0(\eta)) \cdot a(s) g_R^{-1}(s, {}^\eta x_s) \rho_0 \circ DX_s^0(\eta)\right\}(\psi) ds\right].$$

Now, we compute the integrand operator applied to $\psi : [-r, 0] \to \mathbb{R}^d$,

$$\left\{\mathcal{D}_s \Phi(X_T^0(\eta)) \cdot a(s) g_R^{-1}(s, {}^\eta x_s) \rho_0 \circ DX_s^0(\eta)\right\}(\psi) = \mathcal{D}_s \Phi(X_T^0(\eta)) \cdot a(s) g_R^{-1}(s, {}^\eta x_s) \rho_0 (DX_s^0(\eta)(\psi)).$$

Since $a(s) g_r^{-1}(s, {}^\eta x_s) \cdot \rho_0 \circ DX_s^0(\eta)(\psi)$ is an $\mathbb{R}^d$-valued random variable, we apply the finite-dimensional duality formula and get

$$(3.8) \qquad Dp(\eta)(\psi) = E\left[\Phi(X_T^0(\eta)) \int_0^T a(s) g_R^{-1}(s, {}^\eta x_s) \cdot \rho_0 \circ DX_s^0(\eta)(\psi) dW(s)\right]$$

and $w^\Delta(\eta)(\psi) = \int_0^T a(s) g_R^{-1}(s, {}^\eta x_s) \cdot \rho_0 \circ DX_s^0(\eta)(\psi) dW(s)$.

Step 2: Next, we consider that Φ is bounded and continuous (in particular $\Phi(X_T^0(\eta)) \in L^2(\Omega)$). Indeed, we can approximate Φ by a sequence of $\{\Phi\}_{n \geqslant 0} \subset C_b^1(M_2, \mathbb{R})$ such that $\Phi_n(\psi) \xrightarrow{n \to \infty} \Phi(\psi)$ for $\psi \in M_2$. Define

$$(3.9) \qquad \bar{\Delta}(\eta) := E[\Phi(X_T^0(\eta)) \int_0^T a(s) g_R^{-1}(s, {}^\eta x_s) \rho_0 \circ DX_s^0(\eta) dW(s)].$$

The objects $p(\eta)$ and $\bar{p}(\eta)$, $\eta \in M_2$ are well-defined since $\Phi(X_T^0(\eta)) \in L^2(\Omega)$ and using Cauchy-Schwarz inequality and Itô's isometry property we have that

$$|\bar{\Delta}(\eta)| \leqslant E[|\Phi(X_T^0(\eta))|^2]^{1/2} \left(\int_0^T E|a(s) g_R^{-1}(s, {}^\eta x_s) \rho_0 \circ DX_s^0(\eta)|^2 ds \right)^{1/2} < \infty$$

since a and g_R^{-1} are bounded and $E[|\rho_0 \circ DX_T^0(\eta)|^2] < \infty$ by Hypotheses **(D)**. Then we approximate $p_n(\eta) = E[\Phi_n(X_T^0(\eta))]$ and by the step 1 we have that $Dp_n(\eta) = E[\Phi_n(X_T^0(\eta)) w^\Delta(\eta)]$. Then $p_n(\eta) \to p(\eta)$ for all $\eta \in M_2$ and again using Cauchy-Schwarz inequality and Itô's isometry we have

$$|\Delta p_n(\eta) - \bar{\Delta}(\eta)| \leqslant E\left[|\Phi_n(X_T^0(\eta)) - \Phi(X_T^0(\eta))|^2 \right]^{1/2} E[|w^\Delta(\eta)|^2]^{1/2}.$$

Again $E[|w^\Delta(\eta)|^2]^{1/2} < \infty$ and since Φ_n and Φ are continuous and bounded we have

$$\sup_{\eta \in J} |Dp_n(\eta) - \bar{p}(\eta)| \xrightarrow{n \to \infty} 0$$

for all bounded closed subsets $J \subset M_2$. Thus, p defined is Fréchet differentiable with derivative $\Delta p(\eta) = \bar{\Delta}(\eta)$.

Step 3: Let us denote

$$\mathcal{G} := \{\Phi : M_2 \to [0, \infty), \text{ continuous and bounded}\}.$$

It is clear that $\mathcal{G}$ is a multiplicative class, i.e. $\psi_1, \psi_2 \in \mathcal{G}$ then $\psi_1 \psi_2 \in \mathcal{G}$.

Further, let $\mathcal{H}$ the class of functions $\Phi : M_2 \to [0, \infty)$, for which (3.5) holds. From step 2, $\mathcal{G} \subset \mathcal{H}$. Then, $\mathcal{H}$ is a monotone vector space on M_2, see e.g. [23, p.7] for definitions. Indeed, from dominated convergence we have monotonicity. In fact, if $\{\Phi\}_{n \geqslant 0} \subset \mathcal{H}$ such that $0 \leqslant \Phi_1 \leqslant \Phi_2 \leqslant \cdots \Phi_n \leqslant \cdots$ with $\lim_n \Phi_n = \Phi$ and Φ is bounded then $\Phi \in \mathcal{H}$. Furthermore, denote by $\sigma(\mathcal{G}) := \{f^{-1}(B), B \in \mathcal{B}(\mathbb{R}), f \in \mathcal{G}\}$ where $\mathcal{B}(\mathbb{R})$ denotes the Borel σ-algebra in $\mathbb{R}$. Then we are able to apply the monotone class theorem, see e.g. [23, Theorem 8] and conclude that $\mathcal{H}$ contains all bounded and $\sigma(\mathcal{G})$-measurable functions $\Phi : M_2 \to [0, \infty)$. Nevertheless, $\sigma(\mathcal{G})$ coincides with the Borel σ-algebra of M_2 since $\mathcal{G}$ contains all continuous bounded functions.

Step 4: The last step is to approximate any $\mathcal{B}(M_2)$-measurable function $\Phi : M_2 \to [0, \infty)$ such that $\Phi(X_T^0(\eta)) \in L^2(\Omega)$ by a sequence $\{\Phi\}_{n \geqslant 0}$ of bounded $\mathcal{B}(M_2)$-measurable functions. For instance,

$$\Phi_n(\psi) = \Phi(\psi) \mathbf{1}_{\{\Phi(\psi) \leqslant n\}}, \quad n \geqslant 0.$$

Then $\Phi_n \in \mathcal{H}$ for each $n \geqslant 0$. Define $\tilde{\Delta}(\eta) := E[\Phi(X_T^0(\eta))w^\Delta(\eta)]$. Then by Cauchy-Schwarz inequality and Itô's isometry again we obtain that

$$\sup_{\eta \in J} |Dp_{\Phi_n}(\eta) - \tilde{p}(\eta)| \leqslant C \sup_{\eta \in J} E[|\Phi_n(X_T^0(\eta)) - \Phi(X_T^0(\eta))|^2]^{1/2}$$

for some constant $C > 0$ and closed bounded subsets $J \subset M_2$. Finally, observe that

$$E[|\Phi_n(X_T^0(\eta)) - \Phi(X_T^0(\eta))|^2] \xrightarrow{n \to \infty} 0$$

thus proving the result. $\square$

We remark that the case in which Φ only depends on the initial value of the process (2.2) can be treated within the result above. In fact, we observe that for

(3.10) $p(\eta) = E\left[\Phi(^\eta x(T))\right] = E\left[\Phi(\rho_0(\ ^\eta x(T), ^\eta x_T))\right] = E\left[\tilde{\Phi}(X_T^0(\eta))\right]$

with

$$M_2 \xrightarrow{\rho_0} \mathbb{R}^d \xrightarrow{\Phi} \mathbb{R}$$
$$\overset{\tilde{\Phi}}{\overbrace{\phantom{M_2 \xrightarrow{\rho_0} \mathbb{R}^d \xrightarrow{\Phi} \mathbb{R}}}}$$

where we recall that ρ_0 is an evaluation at 0 that can also be seen as a projection onto $\mathbb{R}^d$ which we defined earlier as $\rho_0((v, \varphi)) = v$ for each $(v, \varphi) \in M_2$.

4. A MARKET MODEL WITH MEMORY AND THE DELTA

4.1. **Market model.** In the same filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)$ as before, we consider a market with a price process $S = \{S(t, \omega); \ t \in [0, T], \omega \in \Omega\}$ and a risk-less bond B with dynamics $dB(t) = B(t)\kappa(t)dt$ such that $B(0) = 1$ with $\kappa \in L^1([0, T], \mathbb{R}^+)$. One could consider several risky assets as well, but for the sake of simplicity of notation we restrict ourselves to the 1-dimensional case where also the Brownian motion is 1-dimensional.

For the price process we consider the SFDE with memory:

(4.1) $\begin{cases} \frac{dS(t)}{S(t)} = \mu(t, S_t)dt + \sigma(t, S_t)dW(t), & t \in [0, T] \\ S_0 = \eta \in M_2, & t \in [-r, 0] \end{cases}$

See (2.2) with $d = 1$ and $m = 1$. In equation (4.1) the functionals $\mu, \sigma : [0, T] \times M_2 \to \mathbb{R}$ are such that $S(\cdot)\mu(\cdot, S.)$ and $S(\cdot)\sigma(\cdot, S.)$ satisfy **(D)**. We still denote by $X_t^s(\eta, \omega) := {}^{\eta(\omega)}S_t(\omega)$ the stochastic flow associated to equation (4.1).

Note that if $r = 0$, no memory is included and we recover an SDE for which the Black-Scholes model is a particular case. Moreover, (4.1) with $r \geqslant 0$ includes the models with memory as presented by G. Stoica in [24], where he uses a model with delay by choosing μ and σ functions of $S(t - r)$. Also the cases studied in M. Arriojas, Y. Hu, S-E A. Mohammed and G. Pap, see [1] and [2], are covered by (4.1). Note that [2] presents a more general model than the one in [1] by taking $\mu(t, S_t) = \frac{1}{S(t)}f(t, S_t)$ for some functional f, and σ an evaluation at some point in the past. And again (4.1) generalizes the model in [5] where M-H Chang and R. K. Youree compute the price of a European option for a

model with distributed delays, that is, taking $\mu(t, S_t) = \int_{-r}^{0} S(t + u) d\nu_\mu(u)$ and $\sigma(t, S_t) = \int_{-r}^{0} S(t + u) d\nu_\sigma(u)$ for some measures ν_μ and ν_σ.

The model given in (4.1) under hypotheses (D) admits a unique strong solution

$$S(t) = \ ^\eta S(t) = \begin{cases} \eta(0) + \int_0^t S(u)\mu(u, S_u)du + \int_0^t S(u)\sigma(u, S_u)dW(u), & t \in [0, T] \\ \eta(t), & t \in [-r, 0]. \end{cases}$$

Recall that for all $t \in [0, T]$ the function $(t, \omega) \mapsto S_t(\omega, \cdot)$, $\omega \in \Omega$, is $\mathcal{F}_t$-measurable and μ and σ are jointly continuous deterministic functionals. Hence, the integrals are well-defined with $(\mathcal{F}_t)_{t\in[0,T]}$-adapted integrands and thus $S(t)$ is a semimartingale.

In this context, we can not say much about the distributions of $S(t)$ for a given $t \in [0, T]$, but, assuming that S_t is known at time $t \in [0, T]$, we know that the conditional distributions of $\frac{S(t)}{S(0)}|S_t$ satisfy

$$\frac{S(t)}{S(0)}\Big|S_t \sim \log N\left(\int_0^t \mu(u, S_u)du, \int_0^t \sigma^2(u, S_u)du\right).$$

On the other side, if we assume that the coefficients μ and σ are regular enough, namely that they admit an integral-type expression, with possibly delay, like

(4.2)
$$\begin{cases} \mu(t, S_t) = \mu(0, \eta) + \int_0^t \mu_1(u, S(u), S(u-r))du + \int_0^t \mu_2(u, S(u), S(u-r))dW_\mu(u), \\ \sigma(t, S_t) = \sigma(0, \eta) + \int_0^t \sigma_1(u, S(u), S(u-r))du + \int_0^t \sigma_2(u, S(u), S(u-r))dW_\sigma(u), \\ S(t) = \eta(t), \ t \in [-r, 0]. \end{cases}$$

where the random processes W_μ, W_σ may or may not be independent of W. Exploiting the integral representation (4.2) for the coefficients μ and σ we can solve equation (4.1) as follows: for $t \in [0, r]$ we have that $S(t - r) = \eta(t - r)$, which is known. So, the expressions in (4.2) for μ and σ have no longer delay, namely

$$\begin{cases} \mu(t, S_t) = \mu(0, \eta) + \int_0^t \mu_1(u, S(u), \eta(u-r))du + \int_0^t \mu_2(u, S(u), \eta(u-r))dW_\mu(u), \ t \in [0, r], \\ \sigma(t, S_t) = \sigma(0, \eta) + \int_0^t \sigma_1(u, S(u), \eta(u-r))du + \int_0^t \sigma_2(u, S(u), \eta(u-r))dW_\sigma(u), \ t \in [0, r], \\ S(t) = \eta(t), \ t \in [-r, 0]. \end{cases}$$

and therefore equation (4.1) becomes an ordinary SDE which can be solved for $t \in [0, r]$. Denote $S_1(t)$ the solution related to the interval $[0, r]$, then we can move to $t \in [r, 2r]$ using the same arguments, and so on. Thus the solution is the concatenation of all piece solutions

$$S(t) = \sum_{k=0}^{\infty} S_k(t)\mathbf{1}_{[(k-1)r, kr \wedge T]}(t).$$

Observe that price dynamics of type (4.1) are stochastic volatility models.

Given the model (4.1) we now proceed to obtain a pricing formula for price derivatives on the underlying price process S. First of all, we deal with the risk-neutral evaluation of such derivatives. In a second stage we will consider the benchmark approach.

4.2. **Risk-neutral pricing and the delta.** Let us consider the market model described above. The following version of Girsanov's theorem provides us with the existence of an equivalent martingale measure (or risk-neutral measure) Q, that is, a probability measure Q equivalent to P under which the process $\frac{S(t)}{B(t)}$ is a Q-martingale.

Theorem 4.1 (Girsanov). *Let $W^{Q^\eta}(t)$, $t \in [0,T]$ be an Itô process of the form*

$$dW^{Q^\eta}(t) = \theta_d(t, {}^\eta S_t)dt + dW(t),$$

where

(4.3)
$$\theta_d(t, {}^\eta S_t) := \frac{\mu(t, {}^\eta S_t) - \kappa(t)}{\sigma(t, {}^\eta S_t)},$$

also known as the market price of risk. Put

$$ {}^\eta M(t) := \exp\left\{ -\int_0^t \theta_d(u, {}^\eta S_u)dW(u) - \frac{1}{2}\int_0^t \theta_d^2(u, {}^\eta S_u)du \right\}.$$

Assume that ${}^\eta M(t)$, $t \in [0,T]$ is a martingale with respect to $(\mathcal{F}_t)_{t\in[0,T]}$ and P. Define the measure

$$dQ^\eta = {}^\eta M(T)dP.$$

Then Q^η is a probability measure on $\mathcal{F}_T$ and $W^{Q^\eta}(t)$ is a Brownian motion with respect to Q^η.

By applying Theorem 4.1 we obtain a unique risk-neutral measure Q^η. Observe that this market is complete. Moreover, notice that in the memory setting, the risk-neutral measure depends on the past values of S and hence on $\eta \in M_2$.

Let us consider now a (path dependent) option depending on $S_T \in L^2(\Omega, M_2)$. The payoff $\Phi(S_T)$, is given by $\Phi: M_2 \to \mathbb{R}^+$ such that $\Phi({}^\eta S_T) \in L^2(\Omega)$. An option like this one depends on the last portion of the price process, that is, on the values of $S(t)$ for every $t \in [T-r, T]$. Therefore, we will refer to such an option as a *European option with memory*.

At time $t = 0$, we can define the price operator p_{RN} of such an option under the risk neutral measure Q^η as a functional of the initial process $\eta \in M_2$. We denote by E_{Q^η} the expectation taken with respect to the measure Q^η. Then

(4.4)
$$\begin{aligned} p_{RN}: \quad M_2 &\longrightarrow \mathbb{R}^+ \\ \eta &\longmapsto p_{RN}(\eta) := \frac{1}{B(T)}E_{Q^\eta}[\Phi({}^\eta S_T)]. \end{aligned}$$

As presented in Section 3, the delta operator is then:

(4.5)
$$\begin{aligned} \Delta(\eta) := Dp_{RN}(\eta): \quad M_2 &\longrightarrow \mathbb{R} \\ \psi &\longmapsto Dp_{RN}(\eta)(\psi) := \frac{1}{B(T)}\Big(DE_{Q^\eta}[\Phi({}^\eta S_T)] \Big)(\psi). \end{aligned}$$

Observe that the risk-neutral measure depends on $\eta \in M_2$. In other words,

$$\Big(DE_{Q^\eta}[\Phi({}^\eta S_T)] \Big)(\psi) = \Big(DE[{}^\eta M(T)\Phi({}^\eta S_T)] \Big)(\psi).$$

By Theorem 3.1 we can compute the derivative with respect to η. In general, the product rule for Fréchet derivatives of operators taking values on Banach spaces does not necessarily hold but in our case it does since all operators have range in $\mathbb{R}$ and the usual product rule holds.

Recall that $\theta_d(t, {}^\eta S_t) = \frac{\mu(t, {}^\eta S_t) - \kappa(t)}{\sigma(t, {}^\eta S_t)}$ and that, by assumption $\mu(t, x(\omega))$ and $\sigma(t, x(\omega))$ are Fréchet differentiable with respect to $x(\omega) \in M_2$. Thus, ω-wise, by Theorem 3.1 and Lemma **??** we obtain

$$(4.6) \qquad Dp_{RN}(\eta)(\psi) = \frac{1}{B(T)} E_{Q^\eta}[\Phi({}^\eta S_T)\tilde{w}^\Delta(\eta)(\psi)],$$

where the weight $\tilde{w}^\Delta(\eta)$ is given by:

$$(4.7) \quad \tilde{w}^\Delta(\eta)(\psi) := D\log {}^\eta M(T)(\psi) + {}^\eta M(T)w^\Delta(\eta)(\psi)$$

$$= -\int_0^T \theta_d(t, {}^\eta S_t)D\theta_d(t, {}^\eta S_t)(\psi)dt - \int_0^T \theta_d(t, {}^\eta S_t)D\theta_d(t, {}^\eta S_t)(\psi)dW(t)$$

$$+ {}^\eta M(T)w^\Delta(\eta)(\psi)$$

$$= -\int_0^T \frac{\theta_d(t, {}^\eta S_t)}{\sigma(t, {}^\eta S_t)}[D\mu(t, {}^\eta S_t) - \theta_d(t, {}^\eta S_t)D\sigma(t, {}^\eta S_t)] \circ DX_t^0(\eta)dt$$

$$- \int_0^T \left[\frac{D\mu(t, {}^\eta S_t) - \theta_d(t, {}^\eta S_t)D\sigma(t, {}^\eta S_t)}{\sigma(t, {}^\eta S_t)}\right] \circ DX_t^0(\eta)dW(t).$$

$$+ {}^\eta M(T)w^\Delta(\eta)(\psi)$$

and, as before,

$$(4.8) \qquad w^\Delta(\eta)(\psi) = \int_0^T a(s)(\sigma(s, {}^\eta S_s)X_s^0(\eta))^{-1}\rho_0 \circ DX_s^0(\eta)(\psi)dW(s).$$

4.3. **Benchmark approach to pricing.** Hereafter, we retrieve the benchmark approach for the market model introduced before. We refer to [21], [19], [3] and [6] for an overview on the pricing of options in this approach. Here we summarize by saying that the foundation is in the existence of a suitable strictly positive portfolio defined through the following property: the market price processes expressed in units of this portfolio are P-martingales. For this reason this portfolio is generally known as P-*numéraire* portfolio. This property leads to martingale-type properties for processes linked to the market so that one can easily work under the real world measure. We provide here the so-called *real world pricing formula*, i.e. an alternative pricing formula written as the expectation under P of the option payoff expressed in units of the P-*numéraire*. Before proceeding we point out that the P-*numéraire* portfolio can be characterized as the *growth optimal portfolio*, see [20] or [21], i.e. as the solution to the optimization problem with log-utility.

Following this argument, let us consider a strategy $\pi = \{\pi(t) = (\pi_0(t), \pi(t))\}$, where $\pi_0(t)$ and $\pi(t)$ denote the portions of wealth invested in the bond B and in S respectively. Hence, $\pi_0(t) + \pi(t) = 1$, P-a.s. for all $t \in [0, T]$. Let V^π denote value process associated to the strategy π, with a fixed initial value at time $t = 0$, $V^\pi(0) = x$.

We derive now the growth optimal portfolio which we denote by $G(t, \omega), t \in [0, T], \omega \in \Omega$, by using the approach proposed in [21]. We wish to find a strictly positive and self-financing portfolio π^*, such that, the value associated to this portfolio is the one satisfying

$$V^{\pi^*}(T) = \sup_{\pi \in \mathcal{A}} E[\log V^{\pi}(T)]$$

where $\mathcal{A}$ denotes the set of all strictly positive and self-financing portfolios. So, $G := V^{\pi^*}$. The SFDE for the value process becomes,

(4.9)
$$\begin{cases} \frac{dV(t)}{V(t)} = [\kappa(t) + (\mu(t, S_t) - \kappa(t))\pi(t, S_t)]dt + \sigma(t, S_t)\pi(t, S_t)dW(t), \ t \in [0, T] \\ \frac{dS(t)}{S(t)} = \mu(t, S_t)dt + \sigma(t, S_t)dW(t), \quad t \in [0, T] \\ V(0) = x, \quad S_0 = \eta \in M_2 \ \ t \in [-r, 0] \end{cases}$$

which can be seen as a two-dimensional SFDE for $Y = (V, S)$. The fraction π depends also on the past of $S(t)$. Nevertheless, we stress that the M_2-valued random variable $S_t(\cdot)$ is known at time t. The process V is a semimartingale, with quadratic variation

$$[V, V](t) = \int_0^t V(u)^2 \sigma^2(u, S_u)\pi^2(u, S_u)du.$$

Applying Itô formula to $\log V(t)$ we get,

$$d \log V(t) = \left[\kappa(t) + (\mu(t, S_t) - \kappa(t)) \pi(t, S_t) - \frac{1}{2}\sigma^2(t, S_t)\pi^2(t, S_t) \right] dt + \sigma(t, S_t)\pi(t, S_t)dW(t).$$

By integrating and applying expectation, we get rid of the Brownian part. So we remain with a Lebesgue-type integral with integrand,

$$g_\pi(t, S_t) := \kappa(t) + (\mu(t, S_t) - \kappa(t)) \pi(t, S_t) - \frac{1}{2}\sigma^2(t, S_t)\pi^2(t, S_t).$$

For all $t \in [0, T]$ and given S_t, we have that the concave random variable g_π attains its maximum at

$$\pi^*(t, S_t) = \frac{\mu(t, S_t) - \kappa(t)}{\sigma^2(t, S_t)}.$$

Recall (4.3), then the SFDE for the growth optimal portfolio is obtained as the solution to,

(4.10)
$$\begin{cases} \frac{dG(t)}{G(t)} = [\kappa(t) + \theta_d^2(t, S_t)]dt + \theta_d(t, S_t)dW(t), \ t \in [0, T] \\ \frac{dS(t)}{S(t)} = \mu(t, S_t)dt + \sigma(t, S_t)dW(t), \quad t \in [0, T] \\ G(0) = 1, \ S_0 = \eta \in M_2, \quad t \in [-r, 0]. \end{cases}$$

Note that $G = {}^\eta G$ depends on η. For completeness, hereafter, we show that the benchmarked price $\frac{S}{G}$ is a P-martingale. First, we derive the expression for $d\frac{1}{G(t)}$ using Itô formula:

$$d\frac{1}{G(t)} = \frac{-1}{G(t)^2}dG(t) + \frac{1}{G(t)^3}d[G, G](t) = -\frac{1}{G(t)} \left[\kappa(t)dt + \theta_d(t, S_t)dW(t) \right].$$

Then, we compute $d\frac{S(t)}{G(t)}$ again using Itô formula and the expression for $d\frac{1}{G(t)}$:

$$d\frac{S(t)}{G(t)} = S(t)d\frac{1}{G(t)} + \frac{1}{G(t)}dS(t) + d\left[S,\frac{1}{G}\right](t) = \frac{S(t)}{G(t)}[\sigma(t,S_t) - \theta_d(t,S_t)]dW(t).$$

with initial condition $\frac{S(0)}{G(0)} = S(0) = \eta(0)$. We see that the *benchmarked* underlying security S is driftless, one can easily check that $\frac{S}{G}$ lies in $L^2([0,T]\times\Omega)$ which implies that the process $\frac{S(t)}{G(t)}$, $t \in [0,T]$ is a P-martingale.

Consider the European option with memory $\Phi(^\eta S_T) \in L^2(\Omega)$ and the value of its hedging portfolio $V(t,\omega), t \in [0,T], \omega \in \Omega$ with $V(T) = \Phi(S_T)$. As motivation for the concept of *fair portfolio* we review the following argument which shows that the benchmarked value process is driftless. Indeed, by the Itô formula, we have

$$d\frac{V(t)}{G(t)} = \left(\frac{\partial V}{\partial t}(t,S(t)) + \mu(t,S_t)S(t)\frac{\partial V}{\partial S}(t,S(t))\right.$$

$$+ \frac{1}{2}\sigma^2(t,S_t)S(t)^2\frac{\partial^2 V}{\partial S^2}(t,S(t)) - \kappa(t)V(t,S(t)) - S(t)\sigma(t,S_t)\theta_d(t,S_t)\frac{\partial V}{\partial S}(t,S(t))\left.\right)dt$$

$$+ \left(\frac{S(t)}{G(t)}\sigma(t,S_t)\frac{\partial V}{\partial S}(t,S(t)) - \frac{V(t)}{G(t)}\theta_d(t,S_t)\right)dW(t).$$

Observe that, $\mu(t,S_t) - \sigma(t,S_t)\theta_d(t,S_t) = \kappa(t)$, and the value of the hedging portfolio satisfies the Black-Scholes PDE. Hence the drift vanishes and

(4.11) $$d\frac{V(t)}{G(t)} = \left(\frac{S(t)}{G(t)}\sigma(t,S_t)\frac{\partial V}{\partial S}(t,S(t)) - \frac{V(t)}{G(t)}\theta_d(t,S_t)\right)dW(t).$$

Refer to [21] for the following concept.

Definition 4.2 (Fair portfolio)**.** *A portfolio value process $V(t), t \in [0,T]$, is* fair *if its benchmarked value process $\frac{V(t)}{G(t)}, t \in [0,T]$, is an $(\mathcal{F}_t)_{t\in[0,T]}$-martingale under the real world measure P.*

Under our model assumptions, the volatility term belongs to $L^2([0,T] \times \Omega)$ and $\frac{V(t)}{G(t)}$, $t \in [0,T]$ is then a P-martingale. So, from now on, all *hedging portfolios* are assumed to be *fair*.

The price $p_B(t)$, under the benchmark approach, of a European option with memory $\Phi(^\eta S_T)$ at time $t \in [0,T]$ is the value of the fair replicating portfolio

$$p(t) := {}^\eta V(t) = {}^\eta G(t)E\left[\frac{^\eta V(T)}{^\eta G(T)}|\mathcal{F}_t\right] = {}^\eta G(t)E\left[\frac{\Phi(^\eta S_T)}{^\eta G(T)}|\mathcal{F}_t\right].$$

There are several approaches on how to compute this price. One would be to use the Feynman-Kac formula in the delay setting studied by F. Yang and S-E. A. Mohammed in [25] under the assumption that $\Phi \in \mathcal{C}_2^2(M_2)$. We know, on the other hand, that the process S_t, $t \in [0,T]$ is Markovian, so we could use the following approach when the option only

depends on its past. Define for every $0 \leqslant t \leqslant T$,

$$u(t, T, \eta) := E\left[H({}^{\eta}S_T, {}^{\eta}G(T))\mid {}^{\eta}S_t\right]$$

for a real valued and positive function such that $E|H({}^{\eta}S_T, {}^{\eta}G(T))| < +\infty$ defined as $H(x, y) := \frac{\Phi(x)}{y}$. The Markovianity of ${}^{\eta}S_t$ allows us to write the price as,

$$\frac{p_B(t)}{{}^{\eta}G(t)} = u(t, T, {}^{\eta}S_t) = E\left[\frac{\Phi({}^{\eta}S_t S_T)}{{}^{\eta}G(T)}\right].$$

4.4. **The delta in the benchmark approach.** As before, we consider a European option $\Phi({}^{\eta}S_T) \in L^2(\Omega, \mathbb{R}^+)$. Note here that we need the assumption on Φ being Fréchet differentiable with bounded derivative to carry out our computations but that the final expression in independent of Φ' and can be relaxed as we did in Section 3. The initial time price of this option is given by,

$$
\begin{aligned}
\text{(4.12)} \qquad p_B : \quad & M_2 \longrightarrow \mathbb{R}^+ \\
& \eta \longmapsto p_B(\eta) = E\left[\frac{\Phi({}^{\eta}S_T)}{{}^{\eta}G(T)}\right].
\end{aligned}
$$

The Fréchet derivative at the point η is an operator,

$$Dp_B(\eta) : M_2 \longrightarrow \mathbb{R}^+$$

$$\psi \longrightarrow DE\left[\frac{\Phi({}^{\eta}S_T)}{{}^{\eta}G(T)}\right](\psi)$$

By virtue of Theorem 3.1 we may compute the delta by first computing the ω-wise Fréchet derivative of $\frac{\Phi({}^{\eta}S_T(\omega))}{{}^{\eta}G(T,\omega)}$ with the difference that now ${}^{\eta}G$ depends on η. Thus

$$
\begin{aligned}
\text{(4.13)} \qquad D\frac{\Phi({}^{\eta}S_T)}{{}^{\eta}G(T)} &= \frac{\Phi'({}^{\eta}S_T) \circ D^{\eta}S_T \, {}^{\eta}G(T) - \Phi({}^{\eta}S(T))D^{\eta}G(T)}{{}^{\eta}G(T)^2} \\
&= \underbrace{\frac{\Phi'({}^{\eta}S_T) \circ D^{\eta}S_T}{{}^{\eta}G(T)}}_{(I)} - \underbrace{\frac{\Phi({}^{\eta}S_T)D\log {}^{\eta}G(T)}{{}^{\eta}G(T)}}_{(II)}.
\end{aligned}
$$

For a more accurate expression we proceed similarly as in (4.7) to obtain (II) and for the term (I) we apply Theorem 2.5 and the method used in Section 4. Finally,

$$\text{(4.14)} \qquad Dp(\eta) = E\left[\Phi({}^{\eta}S_T)w^{\Delta}(\eta)\right],$$

where,

$$\text{(4.15)} \quad w^{\Delta}(\eta) = \delta\left(\frac{a(\cdot)\left(\sigma(\cdot, {}^{\eta}S_\cdot)X_\cdot^0(\eta)\right)^{-1}\rho_0 \circ DX_\cdot^0(\eta)}{{}^{\eta}G(T)}\right) - \frac{D\log {}^{\eta}G(T)}{{}^{\eta}G(T)} \in L(M_2, \mathbb{R}^+).$$

Here, the integral is a genuine Skorokhod integral. Once, again, if the option depends also on ${}^{\eta}S_T$, we apply the chain rule as mentioned in Section 4.

We use the classical Black-Scholes model (with no delay) to illustrate that this covers the case where the market has no delay, then we also use as example both the model proposed by U. Küchler and E. Platen in [14] and Arriojas, Hu, Mohammed and Pap in [1].

Example 4.3 (Black-Scholes model). *We choose the functionals f and g to be evaluations at zero followed by a multiplication by μ and σ respectively, and $B(t) = e^{\kappa t}$, so that we get,*

(4.16)
$$\begin{cases} dS(t) = \mu S(t)dt + \sigma S(t)dW(t), & t \in [0, T] \\ S(t) = \eta(t), & t \in [-r, 0]. \end{cases}$$

*Observe that the initial condition is just relevant at the point $t = 0$. Equation (4.16) satisfies trivially hypotheses **(D)** so we have existence of a stochastic flow $X_t^s(\eta, \omega) = {}^{\eta(\omega)}S_t^s(\omega)$. Then, the Fréchet derivative of X_s^0 at a point η given $\omega \in \Omega$ is an operator defined as follows,*

$$DX_s^0(\eta) : M_2 \longrightarrow M_2$$

$$\psi \longrightarrow DX_s^0(\eta)(\psi) := {}^\psi\alpha_s$$

where ${}^\psi\alpha_s$ is the segment of a process $\alpha \in L^2(\Omega, M_2([-r, T], \mathbb{R}))$ solution a linear SFDE with coefficients the Fréchet derivatives of f and g. Namely

$${}^\psi\alpha(s) = \begin{cases} \psi(0) + \mu \int_0^s Df(u, X_u^0(\eta))(\alpha_u)du + \sigma \int_0^s Dg(u, X_u^0(\eta))(\alpha_u)dW(u), s \in [0, T] \\ \psi(s), & s \in [-r, 0] \end{cases}$$

which in this case are evaluations at 0. So

$${}^\psi\alpha(s) = \begin{cases} \psi(0) + \mu \int_0^s \alpha(u)du + \sigma \int_0^s \alpha(u)dW(u), s \in [0, T] \\ \psi(s), & s \in [-r, 0]. \end{cases}$$

Hence,

$${}^\psi\alpha(s) = \psi(0)e^{(\mu - \frac{1}{2}\sigma^2)s + \sigma W(s)}.$$

Then, we apply formula (3.8) and choose, for simplicity, the scalar function a to be $a \equiv \frac{1}{T}$. So

$$Dp_{RN}(\eta)(\psi) = E\left[\Phi({}^\eta S_T)\frac{\psi(0)}{\eta(0)\sigma T}W(T)\right].$$

Finally, we can also consider

$$\Delta = |||Dp_{RN}(\eta)||| = \sup_{\substack{\psi \in M_2 \\ \|\psi\|=1}} |Dp_{RN}(\eta)(\psi)| = \left|E\left[\Phi({}^\eta S_T)\frac{W(T)}{\eta(0)\sigma T}\right]\right|,$$

modulus the measure under which the expectation is taken.

In reality, the price in the risk-neutral approach is computed under the risk-neutral measure and the numéraire used is the process $B(t)$, $t \in [0, T]$. If we wished to compute the Δ magnitude under the real world measure P, we should do it using $G(t)$, $t \in [0, T]$ as

numéraire. Note that G depends also on the initial condition. In this easy example we have, $G(t) = e^{(\kappa + \frac{\theta^2}{2})t + \theta W(t)}$ where $\theta = \frac{\mu - \kappa}{\sigma}$. So,

$$Dp_B(\eta)(\psi)E\left[\Phi(^\eta S_T)w^\Delta(\eta)(\psi)\right],$$

where

$$w^\Delta(\eta)(\psi) = \delta\left(\frac{\psi(0)}{T\eta(0)\sigma G(T)}\right) = \frac{1}{G(T)}\delta\left(\frac{\psi(0)}{T\eta(0)\sigma}\right) - \frac{\psi(0)}{T\eta(0)\sigma}\int_0^T \mathcal{D}_s\frac{1}{G(T)}ds.$$

Here we used the integration by parts formula for the Skorokhod integral, see for instance Theorem 3.15 in [8]. Moreover, $\mathcal{D}_s\frac{1}{G(t)} = \frac{1}{G(t)}\mathcal{D}_s\log\frac{1}{G(t)} = -\frac{1}{G(T)}\theta$ for all $s \in [0,T]$. So,

$$w^\Delta(\eta)(\psi) = \frac{1}{G(T)}\frac{\psi(0)}{T\eta(0)\sigma}\left(W(T) + \theta T\right).$$

Finally,

$$\Delta = \sup_{\substack{\psi \in M_2 \\ \|\psi\|=1}} |Dp_B(\eta)(\psi)| = \left|E\left[\frac{1}{G(T)}\frac{\Phi(^\eta S_T)}{T\eta(0)\sigma}\left(W(T) + \theta T\right)\right]\right| = \left|E_{Q^\eta}\left[\frac{\Phi(^\eta S_T)}{B(T)T\eta(0)\sigma}W^{Q^\eta}(T)\right]\right|.$$

As we can see, the expression is in concordance with the one usually used under the probability $dQ/dP = \frac{B(T)}{G(T)} = M(T)$.

Example 4.4 (Küchler-Platen model)**.** *We consider the market model with delay introduced by U. Küchler and E. Platen in [14]. The authors study the random cyclical fluctuations in commodity prices and propose an explanation and a model involving the presence of time delay. The model for the price process is explicitly written as*

$$S(t) = \alpha_1 \exp\left\{\alpha_2 Y(t) + \alpha_3 t\right\}, \quad t \in [0,T]$$

where α_1, α_2 and α_3 are parameters that can be estimated from market observations and explain economical aspects. Then the process $Y(t)$, $t \in [0,T]$ is governed by the following stochastic differential delay equation

(4.17)
$$dY(t) = \begin{cases} -\mu Y(t-r)dt + \sigma dW(t), & t \in [0,T] \\ \eta(t), & t \in [-r,0]. \end{cases}$$

Consequently, the price process is

(4.18)
$$S(t) = \alpha_1 \exp\left\{\alpha_2\left(\eta(0) - \mu\int_0^t Y(u-r)du + \sigma W(t)\right) + \alpha_3 t\right\}, \quad t \in [0,T].$$

Using Itô's formula we see that $S(t)$ satisfies

(4.19)
$$\frac{dS(t)}{S(t)} = \tilde{\mu}(t)dt + \tilde{\sigma}(t)dW(t)$$

where

$$\tilde{\mu}(t) := -C_1 Y(t-r) + C_2, \quad \tilde{\sigma}(t) = C_3$$

and

$$C_1 := \alpha_1\alpha_2\mu, \quad C_2 := \alpha_1\alpha_3 + \frac{1}{2}\alpha_2^2\sigma^2, \quad C_3 := \alpha_1\alpha_2\sigma.$$

Equation (4.19) depends on some external process $Y(t)$ in a non-Markovian manner, so S depends indeed on $\{\eta(t), \ t \in [-r,0]\}$.

Let then $\Phi(^\eta S_T)$ be a European option with memory. The price of such option in the benchmark setting is given by

$$p_B(\eta) = E\left[\frac{\Phi(^\eta S_T)}{^\eta G(T)}\right].$$

The delta operator is

(4.20) $$Dp_B(\eta) = E[\Phi(^\eta S_T)w^\Delta(\eta)],$$

with weight

(4.21) $$w^\Delta(\eta) = \frac{1}{C_3}\int_0^T \frac{a(s)\rho_0 \circ DX_s^0(\eta)}{^\eta G(T)X_s^0(\eta)}\delta W(s) - \frac{D^\eta G(T)}{^\eta G(T)},$$

where X here represents the flow of S. Then the Fréchet derivative of $\eta \mapsto \rho_0\left(^\eta X_s^0\right)$ is

$$D\rho_0\left(X_s^0(\eta)\right) = X_s^0(\eta)\alpha_2\left[\rho_0 - \mu\int_0^s D^\eta Y(u-r)du\right].$$

For a close expression, we assume that $t \in [0,r]$, then $Y(t-r) = \eta(t-r)$ and so $D^\eta Y(t-r) = \rho_{t-r}$. For an arbitrary $t \in [0,T]$ one may use an iteration argument for solving Y explicitly piecewise. Finally, an expression, under the benchmark approach, for $Dp_B(\eta)$ is

(4.22) $$Dp_B(\eta) = E\left[\Phi(^\eta S_T)w^\Delta(\eta)\right]$$

with

(4.23) $$w^\Delta(\eta) = \frac{\alpha_2}{C_3}\int_0^T \frac{a(s)}{^\eta G(T)}\left[\rho_0 - \mu\int_0^s \rho_{u-r}du\right]\delta W(s) - \frac{D^\eta G(T)}{^\eta G(T)}.$$

The derivative of $\theta_d(t,^\eta S_t)$ is given by

$$D\theta_d(t,^\eta S_t) = \frac{1}{C_3}\rho_{t-r} \circ DX_t^0(\eta).$$

So

$$\frac{D^\eta G(T)}{^\eta G(T)} = \frac{1}{C_3}\int_0^T \theta_d(t,^\eta S_t)\rho_{t-r} \circ DX_t^0(\eta)dt + \frac{1}{C_3}\int_0^T \rho_{t-r} \circ DX_t^0(\eta)dW(t).$$

Example 4.5 (Arriojas, Hu, Mohammed and Pap model)**.** *Consider now a slightly simplified version of the model proposed by authors in* [1] *with dynamics*

(4.24) $$\begin{cases} \frac{dS(t)}{S(t)} = \mu(t)S(t-r)dt + \sigma(t)dW(t), \ t \in [0,T] \\ S_0 = \eta \in C([-r,0],\mathbb{R}) \end{cases}$$

for befitting functions $\mu, \sigma : \mathbb{R} \to [0,\infty)$ such that $\sigma(t) > 0$ a.e. $t \in [0,T]$ and $\eta(0) > 0$ so that the model is feasible. The above equation is explicitly solvable as explained earlier

by stepwise construction. For simplicity, consider $t \in [0, r]$. Observe that the volatility term is free for delay, this is due to the fact that, an SFDE invoing discrete delay in the stochastic part does not admit a stochastic flow, see [16, p.144] or [17].

Then the price process is given by

$$(4.25) \qquad S(t) = \eta(0) \exp\left\{ \int_0^t \left(\mu(u)\eta(u-r) - \frac{1}{2}\sigma(u)^2 \right) du + \int_0^t \sigma(u)dW(u) \right\}.$$

As before, denoting by $X_t^0(\eta, \omega)$, $\omega \in \Omega$ the flow associated to equation (4.24). We have

$$DX_t^0(\eta) = X_t^0(\eta) \left(\frac{1}{\eta(0)}\rho_0 + \int_0^t \mu(u)\rho_{u-r}du \right).$$

Finally, the Δ-sensitivity under the risk-neutral measure is given by formula (4.6)

$$Dp_{RN}(\eta)(\psi) = \frac{1}{B(T)} E_{Q^\eta} \left[\Phi\left({}^\eta S_T \right) \tilde{w}^\Delta(\eta)(\psi) \right]$$

$$(4.26) \qquad \tilde{w}^\Delta(\eta)(\psi) := D \log {}^\eta M(T)(\psi) + {}^\eta M(T) w^\Delta(\eta)(\psi)$$

and here

$$(4.27) \qquad w^\Delta(\eta)(\psi) = \int_0^T \frac{a(s)}{\sigma(s)} \left(\frac{1}{\eta(0)}\psi(0) + \int_0^s \mu(u)\psi(u-r)du \right) dW(s).$$

APPENDIX A. THE MALLIAVIN DERIVATIVE

In this section we give further details on the Mallivin derivative $\mathcal{D}_s x_t$ of $x_t \in L^2(\Omega, M_2)$ as it appears in Section 2. For convenience we present two approaches which lead to the same concept. The authors in [22] develop a general Malliavin calculus for random variables taking values in a particular type of Banach spaces that include a wide class, also Hilbert spaces. The approach is similar to the one presented in [18]. Here, we exhibit the definition adapted to the case of a random variable taking values in a Hilbert space, namely M_2. Let $(\Omega, \mathcal{F}, P)$ be a probability space, let $\{W(h), h \in H\}$ be an *isonormal Gaussian process* with H a Hilbert space, which in our framework will be chosen to be $L^2([0, T])$. Then, consider the random variables of the type

$$F = f(W(h_1), \dots, W(h_n)) \otimes x, \text{ for } h_1, \dots, h_n \in H, \ x \in M_2, \ f \in \mathcal{C}_b^\infty(\mathbb{R}^n).$$

It follows that this class of random variables is dense in $L^2(\Omega; M_2)$. Then, we define the *Malliavin derivative* $\mathcal{D}F$ as the random variable $\mathcal{D}F : \Omega \to L(H, M_2)$ given by

$$\mathcal{D}F = \sum_{i=1}^n \partial_i f(W(h_1), \dots, W(h_n)) \otimes (h_i \otimes x)$$

Moreover, the operator $\mathcal{D}$ is closable from $L^2(\Omega, M_2)$ to $L^2(\Omega, L(H, M_2))$. It turns out that in fact, the image space $L(H, M_2)$ of $\mathcal{D}F$ is the space of Hilbert-Schmidt operators from H to M_2 commonly denoted by $S^2(H, M_2)$, i.e. for each $\omega \in \Omega$ the operator $\mathcal{D}F(\omega)$ is of Hilbert-Schimdt type and furthermore it is a known fact that such space is isometrically isomorphic to $H^* \otimes M_2 \cong H \otimes M_2$.

On the other hand, we may also give a more explicit representation of $\mathcal{D}_t F$ when $F \in L^2(\Omega, M_2)$. For our interest, $F = x_t \in L^2(\Omega, M_2)$ a random variable defined as $x_t : \Omega \to M_2$ such that $x_t(\omega, u) := x(\omega, t + u)$ for all $u \in [-r, 0]$. The space M_2 has a Hilbert structure given by the scalar product

$$\langle f, g \rangle = f(0)g(0) + \int_{-r}^{0} f(u)g(u)du, \ f, g \in M_2.$$

So, given $\omega \in \Omega$ we can expand the element $x_t(\omega, \cdot)$ by its Fourier series as follows

$$x_t(\omega, \cdot) \overset{M_2}{=} \sum_{k=0}^{\infty} \langle x_t(\omega, \cdot), e_k(\cdot) \rangle e_k(\cdot)$$

for a given Hilbertian basis $\{e_k\}_{k=0}^{\infty} \subset M_2$.

Since the above sum converges uniformly thanks to Parseval's identity, then the Malliavin derivative of x_t at a middle point $s \in [0, t]$ is

$$\mathcal{D}_s x_t(\omega, \cdot) = \sum_{k=0}^{\infty} \mathcal{D}_s \left(x_t(\omega, 0)e_k(0) + \int_{-r}^{0} x_t(\omega, u)e_k(u)du \right) e_k(\cdot)$$

$$= \sum_{k=0}^{\infty} \left(\mathcal{D}_s x(\omega, t)e_k(0) + \int_{-r}^{0} \mathcal{D}_s x(\omega, t + u)e_k(u)du \right) e_k(\cdot)$$

$$= \sum_{k=0}^{\infty} \langle \mathcal{D}_s x_t(\omega, \cdot), e_k(\cdot) \rangle e_k(\cdot).$$

Hence, $\mathcal{D}_s x_t(\omega, \cdot) = \{u \mapsto \mathcal{D}_s x(\omega, t + u)\} \in L^2(\Omega \times [-r, 0], \mathbb{R}^d) \cong L^2(\Omega, M_2)$ for every $u \in [-r, 0]$. As a process in $s \in [0, T]$, we can see $\mathcal{D}_s x_t(\omega, \cdot)$ as an element in $L^2([0, T] \times \Omega, M_2) \cong L^2(\Omega; H \otimes M_2)$ with $H = L^2([0, T])$ which is consistent with the approach introduced before.

Next, we give the chain rule for the Malliavin derivative used in Section 2, Theorem 2.5 for the case we are concerned with. See [22], Proposition 3.8 for the general result.

Proposition A.1 (Chain rule of the Malliavin derivative). *Let $f : M_2 \to \mathbb{R}^d$ be a Fréchet differentiable operator with continuous bounded derivative as the one in (2.2). Let $t \in [0, T]$ be a fixed time and $s \in [0, t]$. If $x_t \in L^2(\Omega, M_2)$ is Malliavin differentiable then $f(x_t) \in L^2(\Omega, \mathbb{R}^d)$ is also Malliavin differentiable with*

$$\mathcal{D}_s f(x_t) = Df(x_t)\mathcal{D}_s x_t.$$

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