Cocycle deformation of operator algebras

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Cocycle deformation

Assume Γ is a discrete group, $\mathcal{A} = \oplus_{s\in\Gamma}\mathcal{A}_s$ is a Γ-graded algebra, and Ω is a $\mathbb{C}^*$-valued 2-cocycle on Γ, so

$$\Omega(s,t)\Omega(st,u) = \Omega(t,u)\Omega(s,tu).$$

Then we can define a new product $\star$ on $\mathcal{A}$ by

$$a_s \star a_t = \Omega(s,t)^{-1} a_s a_t.$$

We want to generalize this construction to the analytic setting, replacing $\mathcal{A}$ by a C*-algebra and Γ by an arbitrary locally compact (quantum) group.

In which generality should we work?

If A is a C*-algebra and Γ is a locally compact group, an analogue of Γ-grading is a coaction of Γ on A.

If G is a of locally compact group, then an action of G on a C*-algebra can be thought of as a grading on A by the dual of G.

It is natural to try to cover at least these two cases.

For abelian group actions are in bijection with coactions of the dual.

The deformation for actions/coactions of $\mathbb{R}^d$ was defined by Rieffel, the particular case of actions of $\mathbb{T}^d$/coactions of $\mathbb{Z}^d$ was defined by Connes and Landi, and both constructions have been since studied by many authors.

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For finite group actions the construction of a new product $\star$ is also well-known.

More generally, assume $(\mathcal{H}, \hat{\Delta})$ is a Hopf algebra, $\mathcal{A}$ is an algebra and

$$\mathcal{H} \otimes \mathcal{A} \mapsto \mathcal{A}, \quad x \otimes a \mapsto x \rhd a,$$

is an action of $\mathcal{H}$ making $\mathcal{A}$ a left $\mathcal{H}$-module algebra. An invertible element $\Omega \in \mathcal{H} \otimes \mathcal{H}$ is called a 2-cocycle if

$$(\Omega \otimes 1)(\hat{\Delta} \otimes \iota)(\Omega) = (1 \otimes \Omega)(\iota \otimes \hat{\Delta})(\Omega).$$

Then a new product on $\mathcal{A}$ can be defined by

$$a \star b = m(\Omega^{-1} \rhd (a \otimes b)),$$

where $m \colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is the original product.

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Denote by $\mathcal{A}_\Omega$ the algebra $\mathcal{A}$ with new product. The algebra $\mathcal{A}_\Omega$ is a module algebra over the new Hopf algebra $\mathcal{H}_\Omega$ such that $\mathcal{H}_\Omega = \mathcal{H}$ as algebras, while the new coproduct on $\mathcal{H}_\Omega$ is defined by

$$\hat{\Delta}_\Omega(x) = \Omega\hat{\Delta}(x)\Omega^{-1}.$$

If $\mathcal{H}$ is the group algebra of a finite group G, there may exist cocycles that cannot be induced from abelian subgroups and are such that the Hopf algebra $\mathcal{H}_\Omega$ is neither commutative nor cocommutative (the simplest example is $G = D_8 \times \mathbb{Z}/2\mathbb{Z}$.) Since we need $\mathcal{H}_\Omega$ to recover back $\mathcal{A}$ from $\mathcal{A}_\Omega$, we see that even if we are interested only in actions and coactions of finite groups, this is too little for a good theory.

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Setup

Assume G is a locally compact quantum group, so we are given a von Neumann algebra $L^\infty(G)$ together with a coassociative normal unital injective $*$-homomorphism $\Delta\colon L^\infty(G)\to L^\infty(G)\bar\otimes L^\infty(G)$ such that there exist left and right invariant n.s.f. weights.

A unitary dual 2-cocycle on G is a unitary $\Omega\in L^\infty(\hat G)\bar\otimes L^\infty(\hat G)$ such that

$$(\Omega\otimes 1)(\hat\Delta\otimes\iota)(\Omega)=(1\otimes\Omega)(\iota\otimes\hat\Delta)(\Omega).$$

Assume we have a continuous left action of G on a C*-algebra A, so we are given an injective $*$-homomorphism $\alpha\colon A\to M(C_0(G)\otimes A)$ such that $(\Delta\otimes\iota)\alpha=(\iota\otimes\alpha)\alpha$ and such that $(C_0(G)\otimes 1)\alpha(A)$ is dense in $C_0(G)\otimes A$. We then want to define a deformation A_Ω of A.

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Deformation of $C_0(G)$

Consider the Fourier algebra $A(G) \subset C_0(G)$.
Identifying $A(G)$ with $L^\infty(\hat{G})_*$, define a new product on $A(G)$ by

$$a \star b = (a \otimes b)(\hat{\Delta}(\cdot)\Omega^*).$$

In general, there is no natural involution on $(A(G), \star)$.

Consider the multiplicative unitary $\hat{W} \in B(L^2(G) \otimes L^2(G))$ of $\hat{G}$.
(If G is a group, then $(\hat{W}\xi)(s,t) = \xi(ts,t)$.)
Then the cocycle identity can be written as

$$(\hat{\Delta} \otimes \iota)(\hat{W}\Omega^*)\Omega^*_{12} = (\hat{W}\Omega^*)_{13}(\hat{W}\Omega^*)_{23}.$$

This shows that the formula $\pi_\Omega(a) = (a \otimes \iota)(\hat{W}\Omega^*)$ defines a representation of $(A(G), \star)$ on $L^2(G)$.

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Denote by K the algebra $K(L^2(G))$ of compact operators on $L^2(G)$.

Theorem (Enock, De Commer)

The norm closure of the algebra $\pi_\Omega(A(G)) \subset B(L^2(G))$ is a C^*-algebra $C^*_r(\hat{G};\Omega)$. Furthermore, $\hat{W}\Omega^* \in M(K \otimes C^*_r(\hat{G};\Omega))$.

The result is not difficult to prove for regular quantum groups, which covers most known examples of cocycles.

We consider the C*-algebras $C^*_r(\hat{G};\Omega)$ as the deformations $C_0(G)_\Omega$ of $C_0(G)$ with respect to the left action by translations of G on $C_0(G)$.

For compact groups, the C*-algebras $C^*_r(\hat{G};\Omega)$ were introduced by Landstad and Wassermann around 1980.

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Quantization maps

By a result of De Commer, the von Neumann algebra $L^\infty(\hat{G})$ with the new coproduct $\hat{\Delta}_\Omega = \Omega\hat{\Delta}(\cdot)\Omega^*$ defines the dual of a locally compact quantum group G_Ω. We have

$$(\hat{W}_\Omega\Omega)_{23}\hat{W}_{12}(\hat{W}_\Omega\Omega)^*_{23} = (\hat{W}\Omega^*)_{12}(\hat{W}_\Omega\Omega)_{13}.$$

(For group duals this is a known identity proving quasi-equivalence of the regular representation λ and of $\lambda^\Omega \otimes \lambda^{\bar{\Omega}}$ for a cocycle Ω.)

This allows us to define, for $\nu \in K^* = B(L^2(G))_*$, maps

$$T_\nu \colon C_0(G) \to C^*_r(\hat{G};\Omega), \quad T_\nu(x) = (\iota \otimes \nu)(\hat{W}_\Omega\Omega(x \otimes 1)(\hat{W}_\Omega\Omega)^*).$$

It can be shown that the images of T_ν span a dense subspace of $C^*_r(\hat{G};\Omega)$.

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Deformed algebra

For an arbitrary continuous action $\alpha\colon A \to M(C_0(G) \otimes A)$ we define

$$A_\Omega \subset M(C_r^*(\hat{G};\Omega) \otimes A)$$

as the C*-algebra generated by elements of the form

$$(T_\nu \otimes \iota)\alpha(a),$$

for all $\nu \in K^*$ and $a \in A$.

(For $A = C_0(G)$ and $\alpha = \Delta$ the ends meet: we have an isomorphism $C_0(G)_\Omega \cong C_r^*(\hat{G};\Omega)$ mapping $(T_\nu \otimes \iota)\Delta(a)$ into $T_\nu(a)$.)

Examples

1. Assume G is the dual of a discrete group Γ, so Ω is a $\mathbb{T}$-valued 2-cocycle on Γ. Then $C_r^*(\Gamma;\Omega)$ is generated by the operators

$$\lambda_s^\Omega = \lambda_s \overline{\Omega(s,\cdot)} \ \text{ on } \ \ell^2(\Gamma),$$

satisfying $\lambda_{st}^\Omega = \Omega(s,t)\lambda_s^\Omega\lambda_t^\Omega$. The maps T_ν are

$$T_\nu(\lambda_s) = \nu(\lambda_s^{\bar\Omega})\lambda_s^\Omega.$$

Given an action of G, that is, a coaction $\alpha\colon A \to C_r^*(\Gamma)\otimes A$ of Γ, we see that A_Ω is generated by elements

$$\lambda_s^\Omega \otimes a_s, \quad \text{where} \quad \alpha(a_s) = \lambda_s \otimes a_s.$$

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2. Assume we have a continuous left action γ of $\hat{G}^{\mathrm{op}}$ on a C*-algebra B. We can define a (reduced) twisted crossed product by

$$\hat{G}^{\mathrm{op}} \ltimes_{\gamma,\Omega} B = [(J\hat{J}C_r^*(\hat{G};\Omega)\hat{J}J \otimes 1)\alpha(B)],$$

where the brackets denote the norm closure.

Consider the C*-algebra $A = \hat{G}^{\mathrm{op}} \ltimes_{\gamma} B$ equipped with the dual action $\alpha = \hat{\gamma}$ of G. Then it can be shown that that there exists a canonical isomorphism

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3. Assume G is a locally compact quantum group. The left and right action of G by translations on itself define a left action of $G \times G^{\mathrm{op}}$ on $C_0(G)$. Consider the dual cocycle

$$\Omega \otimes (\hat{J} \otimes \hat{J})\Omega(\hat{J} \otimes \hat{J})$$

on $G \times G^{\mathrm{op}}$. Then it can be shown that the deformation of $C_0(G)$ with respect to the action of $G \times G^{\mathrm{op}}$ and the above cocycle is canonically isomorphic to $C_0(G_\Omega)$.

4. Consider $G = V \cong \mathbb{R}^{2n}$. Fix a Euclidean norm and an orthogonal complex structure J on V. Identify $\hat{V}$ with V using the pairing $e^{i\langle x,y\rangle}$, fix a deformation parameter $h > 0$ and consider the 2-cocycle

$$\Omega_h(x, y) = e^{-\frac{ih}{2}\langle x,Jy\rangle} \text{ on } \hat{V} = V.$$

Consider the normal state ν_h on $B(L^2(V))$ defined by the function

$$\xi_h(x) = \left(\frac{h}{2\pi}\right)^{n/2} e^{-\frac{h}{4}\|x\|^2}.$$

We then have

$$\nu_h(\lambda_x^{\Omega_h}) = \nu_h(\lambda_x^{\bar{\Omega}_h}) = e^{-\frac{h}{4}\|x\|^2}.$$

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$$a \star_h b = \frac{1}{(\pi h)^{2n}} \int_{V\times V} \alpha_x(a)\alpha_y(b) e^{-\frac{2i}{h}\langle x, Jy\rangle} dx\, dy$$

for smooth $a, b \in A_\infty \subset A$, where the integral is understood as an oscillatory integral. The norm is defined by first defining a norm on $C_0^\infty(V)_h$ and then identifying $(A_\infty, \star_h)$ with the subalgebra

$$\alpha(A_\infty) \subset M(C_0(V)_h \otimes A).$$

We have a canonical isomorphism $C_0(V)_h \cong C^*(V; \Omega_h)$, so A_h and A_{Ω_h} can be considered as subalgebras of $M(C^*(V;\Omega_h) \otimes A)$. Then

$$A_h = A_{\Omega_h}.$$

Furthermore, our favorite quantization map $(T_{\nu_h} \otimes \iota)\alpha \colon A \to A_{\Omega_h}$ coincides with the map $\Phi_h \colon A \to A_h$,

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studied by Waldmann and his collaborators.

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5. Consider the group G diffeomorphic to $\mathbb{R}^{2n+2}$, with the group law

$$(a, v, t)(a', v', t') = (a + a', e^{-a'} v + v', e^{-2a'} t + t' + \frac{1}{2} e^{-a'} \omega_0(v, v')),$$

where $\omega_0(v, v') = \sum_{i=1}^{n} (v_i v'_{i+d} - v_{i+d} v'_i)$ is the standard symplectic form on $\mathbb{R}^{2n}$. Bieliavsky and Gayral have shown that for every $\theta > 0$ there exists a unitary cocycle Ω_θ on the dual of G defined by

$$\Omega_\theta^* = \int_{G\times G} K_\theta(x, y) \lambda_x \otimes \lambda_y \, dx \, dy,$$

where

$$\begin{aligned} K_\theta(x, y) &= \frac{4}{(\pi\theta)^{2n+2}} A(x, y) \exp\left\{ \frac{2i}{\theta} S(x, y) \right\}, \\ A(x, x') &= \big(\cosh(a) \cosh(a') \cosh(a - a') \big)^n \\ &\qquad \times \big(\cosh(2a) \cosh(2a') \cosh(2a - 2a') \big)^{1/2}, \\ S(x, x') &= \sinh(2a) t' - \sinh(2a') t + \cosh(a) \cosh(a') \omega_0(v, v'). \end{aligned}$$

Deformed action

Return to the general case. Assume the deformed quantum group G_Ω is regular. Then the formula

$$\alpha_\Omega(x) = (W_\Omega)^*_{12}(1 \otimes x)(W_\Omega)_{12}$$

defines a continuous left action of G_Ω on $A_\Omega \subset M(C^*_r(\hat{G};\Omega) \otimes A)$.

The action α_Ω might be well-defined without regularity assumptions. For example, we always have an action of G_Ω on $C_0(G)_\Omega \cong C^*_r(\hat{G};\Omega)$.

Theorem

Assume that G_Ω *is regular, the map* $x \mapsto \Omega_{21}\hat\alpha(x)\Omega_{21}^*$ *defines a continuous action of* $(\hat G_\Omega)^{\rm op}$ *on* $G \ltimes_\alpha A$ *and* $W_\Omega \otimes 1 \in M(C_0(G_\Omega)\otimes(G\ltimes_\alpha A))$*. Then*

$$A_\Omega \subset M(G\ltimes_\alpha A),$$

and this inclusion, together with $C_0(\hat G_\Omega)\subset M(G\ltimes_\alpha A)$*, defines an isomorphism*

$$G\ltimes_\alpha A \cong G_\Omega \ltimes_{\alpha_\Omega} A_\Omega.$$

For Hopf algebras this isomorphism was observed by Majid and others in the 90s.

For abelian locally compact groups G and continuous cocycles on $\hat G$ it was first used by Kasprzak to define deformations of C*-algebras.

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Induced cocycles

Assume G_1 is a closed quantum subgroup of G, so we have a normal unital injective $*$-homomorphism $\hat{\pi}\colon L^\infty(\hat{G}_1) \to L^\infty(\hat{G})$ respecting comultiplications, and $\Omega_1 \in L^\infty(\hat{G}_1)\bar{\otimes}L^\infty(\hat{G}_1)$ is a unitary dual 2-cocycle on G_1. We can induce it to a dual cocycle Ω on G, namely, define $\Omega = (\hat{\pi} \otimes \hat{\pi})(\Omega_1)$.

Given a continuous action α of G on A, by restriction we get a continuous action α_1 of G_1 on A. We can then consider the deformations A_Ω with respect to α and A_{Ω_1} with respect to α_1.

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Regular cocycles

The right action by translations of G on $C_0(G)$ defines a continuous right action $\beta\colon C_0(G)_\Omega \to M(C_0(G)_\Omega \otimes C_0(G))$.

Definition

We say that Ω is regular if $C_0(G)_\Omega \rtimes_\beta G$ is isomorphic to the algebra of compact operators on a Hilbert space.

Easy cases of regularity:

a) $\hat{G}$ is a genuine group;
b) G is compact.

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Theorem

Assume Ω *is a regular cocycle. Then*

$$A_\Omega = [(T_\nu \otimes \iota)\alpha(A) \mid \nu \in K^*] \ \text{ and } \ K \otimes A_\Omega \cong \hat{G}^{\mathrm{op}} \ltimes_{\hat{\alpha},\Omega} G \ltimes_\alpha A,$$

and the last isomorphism maps $K \otimes 1$ *onto* $\hat{G}^{\mathrm{op}} \ltimes_{\hat{\Delta}^{\mathrm{op}},\Omega} C_0(\hat{G})$.

If G is dual to a locally compact group Γ, then by the Packer-Raeburn stabilization trick we get

$$K \otimes K \otimes A_\Omega \cong \Gamma \ltimes_{\mathrm{Ad}\,\rho^\Omega \otimes \hat{\alpha}} (K \otimes (\hat{\Gamma} \ltimes_\alpha A)).$$

This implies, for example, that if Γ satisfies the strong Baum-Connes conjecture and Ω is homotopic to the trivial cocycle, then A_Ω is KK-equivalent to A.

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