Cocycle deformation of operator algebras

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(Joint work with J. Bhowmick, A. Sangha and L. Tuset)

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Assume $\Gamma$ is a discrete group, $\mathcal{A} = \bigoplus_{s \in \Gamma} \mathcal{A}_s$ is a $\Gamma$-graded algebra, and $\Omega$ is a $\mathbb{C}^*$-valued 2-cocycle on $\Gamma$, so

$$\Omega(s, t)\Omega(st, u) = \Omega(t, u)\Omega(s, tu).$$

Then we can define a new product $\star$ on $\mathcal{A}$ by

$$a_s \star a_t = \Omega(s, t)^{-1} a_s a_t.$$

We want to generalize this construction to the analytic setting, replacing $\mathcal{A}$ by a $\mathbb{C}^*$-algebra and $\Gamma$ by an arbitrary locally compact (quantum) group.
In which generality should we work?

If \( A \) is a \( \text{C}^* \)-algebra and \( \Gamma \) is a locally compact group, an analogue of \( \Gamma \)-grading is a coaction of \( \Gamma \) on \( A \).

If \( G \) is a of locally compact group, then an action of \( G \) on a \( \text{C}^* \)-algebra can be thought of as a grading on \( A \) by the dual of \( G \).

It is natural to try to cover at least these two cases.

For abelian group actions are in bijection with coactions of the dual. The deformation for actions/coactions of \( \mathbb{R}^d \) was defined by Rieffel, the particular case of actions of \( \mathbb{T}^d \)/coactions of \( \mathbb{Z}^d \) was defined by Connes and Landi, and both constructions have been since studied by many authors.
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If $A$ is a $C^*$-algebra and $\Gamma$ is a locally compact group, an analogue of $\Gamma$-grading is a coaction of $\Gamma$ on $A$.

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For finite group actions the construction of a new product $\ast$ is also well-known. More generally, assume $(\mathcal{H}, \hat{\Delta})$ is a Hopf algebra, $\mathcal{A}$ is an algebra and

$$\mathcal{H} \otimes \mathcal{A} \mapsto \mathcal{A}, \quad x \otimes a \mapsto x \triangleright a,$$

is an action of $\mathcal{H}$ making $\mathcal{A}$ a left $\mathcal{H}$-module algebra. An invertible element $\Omega \in \mathcal{H} \otimes \mathcal{H}$ is called a 2-cocycle if

$$(\Omega \otimes 1)(\hat{\Delta} \otimes \iota)(\Omega) = (1 \otimes \Omega)(\iota \otimes \hat{\Delta})(\Omega).$$

Then a new product on $\mathcal{A}$ can be defined by

$$a \ast b = m(\Omega^{-1} \triangleright (a \otimes b)),$$

where $m: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is the original product.
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where $m: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ is the original product.
Denote by $A_\Omega$ the algebra $A$ with new product. The algebra $A_\Omega$ is a module algebra over the new Hopf algebra $H_\Omega$ such that $H_\Omega = H$ as algebras, while the new coproduct on $H_\Omega$ is defined by

$$\hat{\Delta}_\Omega(x) = \Omega \hat{\Delta}(x) \Omega^{-1}.$$ 

If $H$ is the group algebra of a finite group $G$, there may exist cocycles that cannot be induced from abelian subgroups and are such that the Hopf algebra $H_\Omega$ is neither commutative nor cocommutative (the simplest example is $G = D_8 \times \mathbb{Z}/2\mathbb{Z}$.) Since we need $H_\Omega$ to recover back $A$ from $A_\Omega$, we see that even if we are interested only in actions and coactions of finite groups, this is too little for a good theory.
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Assume $G$ is a locally compact quantum group, so we are given a von Neumann algebra $L^\infty(G)$ together with a coassociative normal unital injective $\ast$-homomorphism $\Delta: L^\infty(G) \to L^\infty(G)\bar{\otimes}L^\infty(G)$ such that there exist left and right invariant n.s.f. weights.

A unitary dual 2-cocycle on $G$ is a unitary $\Omega \in L^\infty(\hat{G})\bar{\otimes}L^\infty(\hat{G})$ such that

$$ (\Omega \otimes 1)(\hat{\Delta} \otimes \iota)(\Omega) = (1 \otimes \Omega)(\iota \otimes \hat{\Delta})(\Omega). $$

Assume we have a continuous left action of $G$ on a $C^*$-algebra $A$, so we are given an injective $\ast$-homomorphism $\alpha: A \to M(C_0(G) \otimes A)$ such that $(\Delta \otimes \iota)\alpha = (\iota \otimes \alpha)\alpha$ and such that $(C_0(G) \otimes 1)\alpha(A)$ is dense in $C_0(G) \otimes A$. We then want to define a deformation $A_\Omega$ of $A$.

Our approach is motivated by the work of Kasprzak for abelian locally compact groups $G$ and continuous cocycles.
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Assume we have a continuous left action of \( G \) on a C\(^*\)-algebra \( A \), so we are given an injective \(*\)-homomorphism \( \alpha : A \to M(C_0(G) \otimes A) \) such that \((\Delta \otimes \iota)\alpha = (\iota \otimes \alpha)\alpha\) and such that \((C_0(G) \otimes 1)\alpha(A)\) is dense in \( C_0(G) \otimes A \). We then want to define a deformation \( A_\Omega \) of \( A \).

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Our approach is motivated by the work of Kasprzak for abelian locally compact groups $G$ and continuous cocycles.
Consider the Fourier algebra $A(G) \subset C_0(G)$. Identifying $A(G)$ with $L^\infty(\hat{G})_*$, define a new product on $A(G)$ by

$$a \star b = (a \otimes b)(\hat{\Delta}(\cdot)\Omega^*).$$

In general, there is no natural involution on $(A(G), \star)$. Consider the multiplicative unitary $\hat{W} \in B(L^2(G) \otimes L^2(G))$ of $\hat{G}$. (If $G$ is a group, then $(\hat{W}\xi)(s, t) = \xi(ts, t).$) Then the cocycle identity can be written as

$$(\hat{\Delta} \otimes \iota)(\hat{W}\Omega^*)\Omega^*_{12} = (\hat{W}\Omega^*)_{13}(\hat{W}\Omega^*)_{23}.$$

This shows that the formula $\pi_\Omega(a) = (a \otimes \iota)(\hat{W}\Omega^*)$ defines a representation of $(A(G), \star)$ on $L^2(G)$. 
Deformation of $C_0(G)$

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Denote by $K$ the algebra $K(L^2(G))$ of compact operators on $L^2(G)$.

**Theorem (Enock, De Commer)**

The norm closure of the algebra $\pi_\Omega(A(G)) \subset B(L^2(G))$ is a $C^*$-algebra $C^*_r(\hat{G}; \Omega)$. Furthermore, $\hat{W}\Omega^* \in M(K \otimes C^*_r(\hat{G}; \Omega))$.

The result is not difficult to prove for regular quantum groups, which covers most known examples of cocycles.

We consider the $C^*$-algebras $C^*_r(\hat{G}; \Omega)$ as the deformations $C_0(G)\Omega$ of $C_0(G)$ with respect to the left action by translations of $G$ on $C_0(G)$.

For compact groups, the $C^*$-algebras $C^*_r(\hat{G}; \Omega)$ were introduced by Landstad and Wassermann around 1980.
Denote by $K$ the algebra $K(L^2(G))$ of compact operators on $L^2(G)$.

**Theorem (Enock, De Commer)**

The norm closure of the algebra $\pi_\Omega(A(G)) \subset B(L^2(G))$ is a $C^*$-algebra $C_r^*(\hat{G}; \Omega)$. Furthermore, $\hat{W}\Omega^* \in M(K \otimes C_r^*(\hat{G}; \Omega))$.

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For compact groups, the $C^*$-algebras $C^*_r(\hat{G}; \Omega)$ were introduced by Landstad and Wassermann around 1980.
By a result of De Commer, the von Neumann algebra $L^\infty(\hat{G})$ with the new coproduct $\hat{\Delta}_\Omega = \Omega \hat{\Delta}(\cdot) \Omega^*$ defines the dual of a locally compact quantum group $G_\Omega$. We have

$$(\hat{W}_\Omega \hat{W})_{23} \hat{W}_{12} (\hat{W}_\Omega \hat{W})_{23}^* = (\hat{W}_\Omega^*)_{12} (\hat{W}_\Omega \hat{W})_{13}.$$  

(For group duals this is a known identity proving quasi-equivalence of the regular representation $\lambda$ and of $\lambda^\Omega \otimes \lambda^\tilde{\Omega}$ for a cocycle $\Omega$.)

This allows us to define, for $\nu \in K^* = B(L^2(G))_*$, maps

$$T_\nu : C_0(G) \to C^*_r(\hat{G}; \Omega), \quad T_\nu(x) = (\iota \otimes \nu)(\hat{W}_\Omega \hat{W}(x \otimes 1)(\hat{W}_\Omega \hat{W})^*).$$

It can be shown that the images of $T_\nu$ span a dense subspace of $C^*_r(\hat{G}; \Omega)$. 

Quantization maps

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It can be shown that the images of $T_\nu$ span a dense subspace of $C^*_r(\hat{G}; \Omega)$. 
For an arbitrary continuous action $\alpha : A \to M(C_0(G) \otimes A)$ we define

$$A_\Omega \subset M(C_r^*(\hat{G}; \Omega) \otimes A)$$

as the C*-algebra generated by elements of the form

$$(T_\nu \otimes \iota)\alpha(a),$$

for all $\nu \in K^*$ and $a \in A$.

(For $A = C_0(G)$ and $\alpha = \Delta$ the ends meet: we have an isomorphism $C_0(G)_\Omega \cong C_r^*(\hat{G}; \Omega)$ mapping $(T_\nu \otimes \iota)\Delta(a)$ into $T_\nu(a)$.)
1. Assume $G$ is the dual of a discrete group $\Gamma$, so $\Omega$ is a $\mathbb{T}$-valued 2-cocycle on $\Gamma$. Then $C^*_r(\Gamma; \Omega)$ is generated by the operators $\lambda^\Omega_s = \lambda_s \overline{\Omega(s, \cdot)}$ on $\ell^2(\Gamma)$, satisfying $\lambda^\Omega_{st} = \Omega(s, t) \lambda^\Omega_s \lambda^\Omega_t$. The maps $T_\nu$ are

$$T_\nu(\lambda_s) = \nu(\lambda\overline{s}) \lambda^\Omega_s.$$

Given an action of $G$, that is, a coaction $\alpha: A \to C^*_r(\Gamma) \otimes A$ of $\Gamma$, we see that $A_\Omega$ is generated by elements $\lambda^\Omega_s \otimes a_s$, where $\alpha(a_s) = \lambda_s \otimes a_s$.

This is exactly how Connes-Landi define $\theta$-deformations for $\Gamma = \mathbb{Z}^n$, and, more recently, Yamashita defines deformations for arbitrary discrete $\Gamma$. 
Examples

1. Assume $G$ is the dual of a discrete group $\Gamma$, so $\Omega$ is a $\mathbb{T}$-valued 2-cocycle on $\Gamma$. Then $C^*_r(\Gamma; \Omega)$ is generated by the operators

$$\lambda^\Omega_s = \lambda_s \Omega(s, \cdot) \text{ on } \ell^2(\Gamma),$$

satisfying $\lambda^\Omega_{st} = \Omega(s, t) \lambda^\Omega_s \lambda^\Omega_t$. The maps $T_\nu$ are

$$T_\nu(\lambda_s) = \nu(\lambda^\Omega_s) \lambda^\Omega_s.$$

Given an action of $G$, that is, a coaction $\alpha: A \to C^*_r(\Gamma) \otimes A$ of $\Gamma$, we see that $A_\Omega$ is generated by elements

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This is exactly how Connes-Landi define $\theta$-deformations for $\Gamma = \mathbb{Z}^n$, and, more recently, Yamashita defines deformations for arbitrary discrete $\Gamma$. 
2. Assume we have a continuous left action $\gamma$ of $\hat{G}^{\text{op}}$ on a $C^*$-algebra $B$. We can define a (reduced) twisted crossed product by

$$\hat{G}^{\text{op}} \ltimes_{\gamma, \Omega} B = [(J\hat{J}C_r^*(\hat{G}; \Omega)\hat{J}J \otimes 1)\alpha(B)],$$

where the brackets denote the norm closure.

Consider the $C^*$-algebra $A = \hat{G}^{\text{op}} \ltimes_{\gamma} B$ equipped with the dual action $\alpha = \hat{\gamma}$ of $G$. Then it can be shown that there exists a canonical isomorphism

$$A_{\Omega} \cong \hat{G}^{\text{op}} \ltimes_{\gamma, \Omega} B.$$
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$$A_\Omega \cong \hat{G}^{\text{op}} \ltimes_{\gamma, \Omega} B.$$
3. Assume $G$ is a locally compact quantum group. The left and right action of $G$ by translations on itself define a left action of $G \times G^{\text{op}}$ on $C_0(G)$. Consider the dual cocycle

$$\Omega \otimes (\hat{J} \otimes \hat{J})\Omega(\hat{J} \otimes \hat{J})$$

on $G \times G^{\text{op}}$. Then it can be shown that the deformation of $C_0(G)$ with respect to the action of $G \times G^{\text{op}}$ and the above cocycle is canonically isomorphic to $C_0(G_\Omega)$. 
4. Consider $G = V \cong \mathbb{R}^{2n}$. Fix a Euclidean norm and an orthogonal complex structure $J$ on $V$. Identify $\hat{V}$ with $V$ using the pairing $e^{i\langle x, y \rangle}$, fix a deformation parameter $h > 0$ and consider the 2-cocycle

$$\Omega_h(x, y) = e^{-\frac{ih}{2}\langle x, Jy \rangle} \quad \text{on} \quad \hat{V} = V.$$ 

Consider the normal state $\nu_h$ on $B(L^2(V))$ defined by the function

$$\xi_h(x) = \left(\frac{h}{2\pi}\right)^{n/2} e^{-\frac{h}{4}\|x\|^2}.$$ 

We then have

$$\nu_h(\lambda_x^{\Omega h}) = \nu_h(\lambda_x^{\hat{\Omega} h}) = e^{-\frac{h}{4}\|x\|^2}.$$ 

It can be shown that the elements

$$(T_{\nu_h} \otimes \nu)\alpha(a),$$

for all $a \in A$, span a dense subspace $A_{\Omega h}$. 
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It can be shown that the elements

$$(T_{\nu_h} \otimes \iota) \alpha(a),$$

for all $a \in A$, span a dense subspace $A_{\Omega_h}$. 
On the other hand, Rieffel defines a deformation $A_h$ of $A$ by

$$a \star_h b = \frac{1}{(\pi h)^{2n}} \int_{V \times V} \alpha_x(a) \alpha_y(b) e^{-\frac{2i}{h} \langle x, Jy \rangle} \, dx \, dy$$

for smooth $a, b \in A_\infty \subset A$, where the integral is understood as an oscillatory integral. The norm is defined by first defining a norm on $C_0^\infty(V)_h$ and then identifying $(A_\infty, \star_h)$ with the subalgebra

$$\alpha(A_\infty) \subset M(C_0(V)_h \otimes A).$$

We have a canonical isomorphism $C_0(V)_h \cong C^*(V; \Omega_h)$, so $A_h$ and $A_{\Omega_h}$ can be considered as subalgebras of $M(C^*(V; \Omega_h) \otimes A)$. Then

$$A_h = A_{\Omega_h}.$$  

Furthermore, our favorite quantization map $(T_{\nu_h} \otimes \iota)\alpha : A \to A_{\Omega_h}$ coincides with the map $\Phi_h : A \to A_h$,

$$\Phi_h(a) = \frac{1}{(\pi h)^n} \int_V e^{-\frac{1}{h} \|x\|^2} \alpha_x(a) \, dx,$$

studied by Waldmann and his collaborators.
On the other hand, Rieffel defines a deformation $A_h$ of $A$ by

$$a \ast_h b = \frac{1}{(\pi \hbar)^{2n}} \int_{V \times V} \alpha_x(a) \alpha_y(b) e^{-\frac{2i}{\hbar} \langle x, Jy \rangle} \, dx \, dy$$

for smooth $a, b \in A_\infty \subset A$, where the integral is understood as an oscillatory integral. The norm is defined by first defining a norm on $C_0^\infty(V)_\hbar$ and then identifying $(A_\infty, \ast_h)$ with the subalgebra

$$\alpha(A_\infty) \subset M(C_0(V)_\hbar \otimes A).$$

We have a canonical isomorphism $C_0(V)_\hbar \cong C^*(V; \Omega_\hbar)$, so $A_h$ and $A_{\Omega_\hbar}$ can be considered as subalgebras of $M(C^*(V; \Omega_\hbar) \otimes A)$. Then

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5. Consider the group $G$ diffeomorphic to $\mathbb{R}^{2n+2}$, with the group law

$$(a, v, t)(a', v', t') = (a + a', e^{-a'} v + v', e^{-2a'} t + t' + \frac{1}{2} e^{-a'} \omega_0(v, v')),$$

where $\omega_0(v, v') = \sum_{i=1}^{n} (v_i v'_i - v_{i+d} v'_i)$ is the standard symplectic form on $\mathbb{R}^{2n}$. Bieliavsky and Gayral have shown that for every $\theta > 0$ there exists a unitary cocycle $\Omega_\theta$ on the dual of $G$ defined by

$$\Omega^*_\theta = \int_{G \times G} K_\theta(x, y) \lambda_x \otimes \lambda_y \, dx \, dy,$$

where

$$K_\theta(x, y) = \frac{4}{(\pi \theta)^{2n+2}} A(x, y) \exp \left\{ \frac{2i}{\theta} S(x, y) \right\},$$

$$A(x, x') = (\cosh(a) \cosh(a') \cosh(a - a'))^n$$

$$\times (\cosh(2a) \cosh(2a') \cosh(2a - 2a'))^{1/2},$$

$$S(x, x') = \sinh(2a) t' - \sinh(2a') t + \cosh(a) \cosh(a') \omega_0(v, v').$$
Return to the general case. Assume the deformed quantum group $G_{\Omega}$ is regular. Then the formula

$$\alpha_{\Omega}(x) = (W_{\Omega})^{*}_{12}(1 \otimes x)(W_{\Omega})_{12}$$

defines a continuous left action of $G_{\Omega}$ on $A_{\Omega} \subset M(C_r^*(\hat{G}; \Omega) \otimes A)$.

The action $\alpha_{\Omega}$ might be well-defined without regularity assumptions. For example, we always have an action of $G_{\Omega}$ on $C_0(G)_{\Omega} \approx C_r^*(\hat{G}; \Omega)$. 

Theorem

Assume that $G_\Omega$ is regular, the map $\mu \mapsto \Omega_{21} \hat{\alpha} (\mu) \Omega_{21}^*$ defines a continuous action of $(\hat{G}_\Omega)^{\text{op}}$ on $G \ltimes_\alpha A$ and $W_\Omega \otimes 1 \in M(C_0(G_\Omega) \otimes (G \ltimes_\alpha A))$. Then

$$A_\Omega \subset M(G \ltimes_\alpha A),$$

and this inclusion, together with $C_0(\hat{G}_\Omega) \subset M(G \ltimes_\alpha A)$, defines an isomorphism

$$G \ltimes_\alpha A \cong G_\Omega \ltimes_{\alpha_\Omega} A_\Omega.$$

For Hopf algebras this isomorphism was observed by Majid and others in the 90s.

For abelian locally compact groups $G$ and continuous cocycles on $\hat{G}$ it was first used by Kasprzak to define deformations of C*-algebras.
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For abelian locally compact groups $G$ and continuous cocycles on $\hat{G}$ it was first used by Kasprzak to define deformations of $C^*$-algebras.
Induced cocycles

Assume $G_1$ is a closed quantum subgroup of $G$, so we have a normal unital injective $*$-homomorphism $\hat{\pi} : \ell^\infty(\hat{G}_1) \to \ell^\infty(\hat{G})$ respecting comultiplications, and $\Omega_1 \in \ell^\infty(\hat{G}_1) \bar{\otimes} \ell^\infty(\hat{G}_1)$ is a unitary dual 2-cocycle on $G_1$. We can induce it to a dual cocycle $\Omega$ on $G$, namely, define $\Omega = (\hat{\pi} \otimes \hat{\pi})(\Omega_1)$.

Given a continuous action $\alpha$ of $G$ on $A$, by restriction we get a continuous action $\alpha_1$ of $G_1$ on $A$. We can then consider the deformations $A_\Omega$ with respect to $\alpha$ and $A_{\Omega_1}$ with respect to $\alpha_1$.

**Theorem**

*We have a canonical isomorphism $A_\Omega \cong A_{\Omega_1}$.***
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Assume $G_1$ is a closed quantum subgroup of $G$, so we have a normal unital injective $*$-homomorphism $\hat{\pi}: L^\infty(\hat{G}_1) \to L^\infty(\hat{G})$ respecting comultiplications, and $\Omega_1 \in L^\infty(\hat{G}_1) \hat{\otimes} L^\infty(\hat{G}_1)$ is a unitary dual 2-cocycle on $G_1$. We can induce it to a dual cocycle $\Omega$ on $G$, namely, define $\Omega = (\hat{\pi} \otimes \hat{\pi})(\Omega_1)$.

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**Theorem**

*We have a canonical isomorphism $A_\Omega \simeq A_{\Omega_1}$.***
Regular cocycles

The right action by translations of $G$ on $C_0(G)$ defines a continuous right action $\beta : C_0(G)_\Omega \to M(C_0(G)_\Omega \otimes C_0(G))$.

**Definition**

We say that $\Omega$ is regular if $C_0(G)_\Omega \rtimes_\beta G$ is isomorphic to the algebra of compact operators on a Hilbert space.

Easy cases of regularity:

a) $\hat{G}$ is a genuine group;
b) $G$ is compact.

If $G$ is a group, regularity is closely related to the question whether the action of $G$ on $C_0(G)_\Omega$ is proper in appropriate sense.
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Theorem

Assume \( \Omega \) is a regular cocycle. Then

\[
A_\Omega = [(T_\nu \otimes \iota)\alpha(A) \mid \nu \in K^*] \quad \text{and} \quad K \otimes A_\Omega \cong \hat{G}^{\text{op}} \ltimes \alpha, \Omega G \ltimes \alpha A,
\]

and the last isomorphism maps \( K \otimes 1 \) onto \( \hat{G}^{\text{op}} \ltimes \hat{\Delta}^{\text{op}, \Omega} C_0(\hat{G}) \).

If \( G \) is dual to a locally compact group \( \Gamma \), then by the Packer-Raeburn stabilization trick we get

\[
K \otimes K \otimes A_\Omega \cong \Gamma \ltimes \text{Ad} \rho^{\Omega \otimes \hat{\alpha}} (K \otimes (\hat{\Gamma} \ltimes \alpha A)).
\]

This implies, for example, that if \( \Gamma \) satisfies the strong Baum-Connes conjecture and \( \Omega \) is homotopic to the trivial cocycle, then \( A_\Omega \) is KK-equivalent to \( A \).
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