A Singer construction in motivic homotopy theory

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Contents

Introduction

The work pursued in this thesis concerns two fields related inside homotopy theory. On the one hand, it draws on the work initiated in [31] by Fabien Morel and Vladimir Voevodsky on the homotopy theory of schemes. This work opens the possibility of transfering methods from algebraic topology to the study of schemes and varieties. On the other hand, we will use methods developed for the study of classifying spaces of finite groups which have been explored in stable equivariant homotopy theory.

To be more specific, our interest lies in transfering some of the work on Segal's Burnside ring conjecture in algebraic topology to motivic homotopy theory. This conjecture, now a proven result by the work of Gunnar Carlsson [8] in the 1980s, concerns the stable cohomotopy of classifying spaces of groups. For a finite group G, there is a map

$$R(G) \longrightarrow KU^0(BG)$$

from its representation ring $R(G)$ to the K-theory of its classifying space BG. After completing $R(G)$ at its augmentation ideal and extending this map by continuity, Atiyah [3] proved that the resulting map is an isomorphism. Segal [36] worked on what would happen if one replaced K-theory with stable cohomotopy, thereby trying to compute $\pi^0_S(BG)$. His conjecture was that a replacement for $R(G)$ was $A(G)$, the Burnside ring of isomorphism classes of finite G-sets, and that the same process would yield an isomorphism after passing to its completion.

The conjecture was generalized, reformulated and proved in special cases along the way to a full proof. Our work relates to a construction appearing on the algebraic side of the considerations that concern the case where $G = C_p$, the cyclic group of order p. In the case where $G = \mathbb{Z}/2$ one has

$$B\mathbb{Z}/2 \simeq \mathbb{R}P^\infty.$$

In this case Lin used the Adams spectral sequence to verify Segal's conjecture. In his paper [24], he constructs an inverse system of spectra $\mathbb{R}P^n_k$ by using stunted projective spaces and James periodicity. Let $D(X)$ be the S-dual of a spectrum X. Then there is a relation

$$\pi^i(X) \cong \pi_{-i}(DX),$$

and it is possible to show that

$$D(\mathbb{R}P_0^n) \simeq \Sigma(\mathbb{R}P_{-n-1}^{-1}).$$

Theorem 1.2 in Lin's article treats $\mathbb{R}P_0^\infty$ by describing the groups

$$\lim_k \pi^i(\mathbb{R}P_0^k) \cong \lim_k \pi_{-i-1}(\mathbb{R}P_{-k-1}^{-1})$$

for different i. For the spectra $\mathbb{R}P_k^\infty$, the Adams spectral sequence permits a calculation of the groups $[S^i, \mathbb{R}P_k^\infty]$ (theorem 1.3 in [24]). The calculation of the E_2-term over $\mathcal{A}$, the mod 2 Steenrod algebra, boils down to calculating the system of groups

$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(\mathbb{R}P_k^\infty), \mathbb{Z}/2)$$

and in the end

$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{Z}/2[x, x^{-1}], \mathbb{Z}/2)$$

after passing to the limit. Remarkably, Lin, together with Adams, Davis and Mahowald ([25]) show that

$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{Z}/2[x, x^{-1}], \mathbb{Z}/2) \cong \mathrm{Ext}_{\mathcal{A}}^{s,t+1}(\mathbb{Z}/2, \mathbb{Z}/2).$$

This isomorphism can be understood on more conceptual grounds: It is a special case of the so-called Singer construction $R_+(-)$ which can be defined for modules over $\mathcal{A}$. It comes equipped with an $\mathcal{A}$-linear map

$$\epsilon : R_+(\mathbb{Z}/2) \longrightarrow \mathbb{Z}/2$$

such that the induced map

$$\epsilon^* : \mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathrm{Ext}_{\mathcal{A}}^{*,*}(R_+(\mathbb{Z}/2), \mathbb{Z}/2).$$

is an isomorphism. It is put to use in calculating the relevant Ext-groups in the case where G is elementary abelian in [1]. Carlsson's work shows that this step is the base on which the general conjecture rests.

In [8], Carlsson comments that Segal's conjecture in its original form is hard to generalize due to its focus on a map that only involves the zeroth stable cohomotopy group of BG. Therefore, he generalizes it to a form which has better chances of success and, based on previous calculations, he shows that one has an isomorphism of rings

$$\gamma : \widehat{\pi}_G^*(S^0)_I \longrightarrow \pi_G^*(EG_+)(\cong \pi_S^*(BG_+))$$

involving completed equivariant (stable) cohomotopy groups. Here I is the augmentation ideal of $A(G) \cong \pi_G^0(S^0)$ and EG is a contractible G-CW complex with G acting freely. Hence Carlsson's work implies Segal's original conjecture.

We follow considerations taken from the introduction of [18] and we will always assume that G is finite. There is an equivariant cofiber sequence

$$EG_+ \longrightarrow S^0 \longrightarrow \tilde{EG}$$

induced by the collapse map $EG_+ \to S^0$. Given a G-space X, there is a G-map

$$X \cong F(S^0, X) \longrightarrow F(EG_+, X).$$

Smashing with the above sequence and taking fixed points, we end up with the commutative diagram

$$\begin{array}{ccccc} (EG_+ \wedge X)^G & \longrightarrow & X^G & \longrightarrow & (\tilde{EG} \wedge X)^G \\ \downarrow \simeq & & \downarrow \Gamma & & \downarrow \hat{\Gamma} \\ (EG_+ \wedge F(EG_+, X))^G & \longrightarrow & (F(EG_+, X))^G & \longrightarrow & (\tilde{EG} \wedge F(EG_+, X))^G \end{array}$$

We borrow some notation from [26]: Let X be a genuine G-spectrum and let

$$X^{hG} := (F(EG_+, X))^G,$$
$$X_{hG} := (EG_+ \wedge X)/G \simeq (EG_+ \wedge X)^G$$

and

$$X^{tG} := (\tilde{EG} \wedge F(EG_+, X))^G$$

These are refered to as the homotopy fixed points, the homotopy orbits and the Tate construction of X respectively. Specializing to the case where $X = S$, the sphere spectrum in the category of genuine G-spectra, and $G = C_2$ we get a commutative diagram

$$\begin{array}{ccccc} S_{hC_2} & \longrightarrow & S^{C_2} & \longrightarrow & S \\ \downarrow \simeq & & \downarrow \Gamma & & \downarrow \hat{\Gamma} \\ S_{hC_2} & \longrightarrow & S^{hC_2} & \longrightarrow & S^{tC_2} \end{array}$$

In the introduction of [18] it is observed that this is equivalent to

$$\begin{array}{ccccc} \mathbb{R}P^\infty_+ & \longrightarrow & S^{C_2} & \longrightarrow & S \\ \downarrow \simeq & & \downarrow \Gamma & & \downarrow \hat{\Gamma} \\ \mathbb{R}P^\infty_+ & \longrightarrow & D(\mathbb{R}P^\infty_+) & \longrightarrow & \Sigma\mathbb{R}P^\infty_{-\infty} \end{array}$$

with

$$\mathbb{R}P^\infty_{-\infty} := \mathrm{holim}_k\, \mathbb{R}P^\infty_{-k}$$

and

$$\mathbb{R}P^\infty_{-k} := Th(-k\gamma^1 \downarrow \mathbb{R}P^\infty).$$

Here $k\gamma^1$ is the k-fold sum of γ^1, the tautological line bundle over $\mathbb{R}P^\infty$, and $Th(-k\gamma^1 \downarrow \mathbb{R}P^\infty)$ denotes the Thom spectrum of the virtual vector bundle $-k\gamma^1$. It is possible to rephrase Segal's conjecture as a homotopy limit problem where we want to show that Γ is an equivalence and this happens if and only if $\widehat{\Gamma}$ is too. For this, we may use an inverse limit of Adams spectral sequences just as Lin did in his work. It is here that Singer's construction comes to the aid: Define

$$H_c^*(\Sigma\mathbb{R}P_{-\infty}^\infty) := \operatorname{colim}_{-k} H^*(\Sigma\mathbb{R}P_{-k}^\infty),$$

something we will call the continous cohomology of $\Sigma\mathbb{R}P_{-\infty}^\infty$. In this notation, $H^*(-)$ means cohomology with coefficients in $\mathbb{Z}/2$. As a module over the Steenrod algebra, it is isomorphic to $R_+(\mathbb{Z}/2)$. Now, we have a map of spectra

$$S \longrightarrow \Sigma\mathbb{R}P_{-\infty}^\infty$$

realizing

$$\epsilon : R_+(\mathbb{Z}/2) \longrightarrow \mathbb{Z}/2$$

after evaluating cohomology. Because of the resulting Ext-isomorphism, this means the E_2-terms of the Adams spectral sequence converging to $\pi_*(S)\hat{_2}$ and the inverse limit over k of the Adams spectral sequences converging to $\pi_*(\Sigma\mathbb{R}P_{-k}^\infty)\hat{_2}$ associated to $\Sigma\mathbb{R}P_{-\infty}^\infty$ are isomorphic. Hence, the map

$$S \to \Sigma\mathbb{R}P_{-\infty}^\infty$$

inducing ϵ is an equivalence after completion at 2.

The world of motivic homotopy is a modern development related to classical algebraic topology. Since its birth in [31] it has been used for studying cohomology theories of algebraic varieties among which motivic cohomology has been the main focus. These are powerful techniques and the study of cohomology operations on motivic cohomology has played a part in the ideas that led to proofs of the Milnor and Bloch-Kato conjectures. To explore this theory further it would be interesting to know that constructions from the classical theory could be set up to work in this framework.

Although we do not prove a motivic version of Segal's conjecture, we show that there exists a construction entirely similar to the one introduced by Singer for modules over the motivic Steenrod algebra. This is theorem 3.3.6. Also, for the module playing the part of $\mathbb{Z}/2$, we check that the Singer construction can be realized as the continous motivic cohomology of an inverse tower of motivic spectra just as it was done in Lin's work. This is part 4.1, culminating in proposition 4.1.37. Finally, we give an application, proving that there is an equivalence resembling the one we saw above although the completion is different. This is theorem 2.0.2.

Chapter 1

The basic categories

Let F be a field. We will study the motivic homotopy theory of smooth, separated schemes of finite type over F and our notation for this category will be Sm/F. The basic reference for these ideas is [31] and additional details may also be found in [34]. A good introduction with a lot of motivation can also be found in [39].

Motivic homotopy theory comes about in a way that mimicks the definition of homotopy in the some relevant category of spaces. Namely, given such a category, one defines weak equivalences between them and invert them to form the associated homotopy category.

If we want to represent generalized cohomology theories, we define spectra. These are collections of pointed spaces $\{E_i\}$ and bonding maps

$$\Sigma E_i \to E_{i+1}$$

between them. For each spectrum we can define stable homotopy groups and given a map of spectra, we say that it is stable weak equivalence if the resulting map on all stable homotopy groups is an isomorphism. If such maps are inverted we end up with the stable homotopy category SH. By the Brown representability theorem, every cohomology theory $E^*(-)$ on spectra can be represented by an object in SH which is unique up to isomorphism.

The category of schemes is too rigid to study homotopy theory directly. For instance, it will be of interest to form topological constructions that might destroy the scheme structure. Hence we embed some category of schemes, e.g. Sm/F, into some larger category where these constructions can be performed harmlessly.

As discussed in [39], this is done in several steps. One first embeds the relevant schemes into presheaves $Pre(Sm/F)$ on Sm/F by sending a scheme to the presheaf it represents. This category has all small limits and colimits which was what we wanted. This is not enough however since we also want certain pushout squares to be preserved after the embedding. To solve this we pass to sheaves $Shv_{Nis}(Sm/F)$ which *do* respect these pushouts. The sheaf property can be defined with respect to any Grothendieck topology, but in motivic homotopy theory, the relevant one is the Nisnevich topology. This topology is subcanonical so the presheaf represented by a smooth scheme is in fact a sheaf.

At this point, we are ready to build in the notion of homotopy. We can embed $Shv_{Nis}(Sm/F)$ into the category of simplicial sheaves

$$\Delta^{op} Shv_{Nis}(Sm/F)$$

by giving a sheaf the constant (discrete) simplicial structure. The sheaf associated to Spec(F) plays the role of the basepoint in this category. There is also a pointed version of this category which we denote using

$$\Delta^{op} Shv_{Nis}(Sm/F)_{\bullet}.$$

The forgetful functor

$$\Delta^{op} Shv_{Nis}(Sm/F)_{\bullet} \to \Delta^{op} Shv_{Nis}(Sm/F)$$

comes with a left adjoint that takes X to $X_+ := X \amalg \operatorname{Spec}(F)$. It is here we can define a model structure which lets us speak of the notion of homotopy: In a model category we define a class of weak equivalences and each such category has an associated homotopy category where the images of the weak equivalences are formally inverted. This process is called localization, a method first used by Bousfield. For the precise definitions and main results, the reader may consult [19]. The category $\Delta^{op} Shv_{Nis}(Sm/F)$ can be given many model structures (objectwise/local, injective/projective,..) with pointed versions for $\Delta^{op} Shv_{Nis}(Sm/F)_{\bullet}$. We can freely choose between the ones that are Quillen equivalent since the resulting homotopy categories will be equivalent. Let us denote one such homotopy category using H_{Nis}.

In addition to the weak equivalences we have defined so far, we want the affine line $\mathbb{A}^1$ to play the part of the unit interval. Again, this is done by using localization. First, call a space Z $\mathbb{A}^1$-local if

$$\operatorname{Hom}_{H_{Nis}}(X, Z) \to \operatorname{Hom}_{H_{Nis}}(X \times \mathbb{A}^1, Z)$$

is an isomorphism for all $X \in Sm/F$. A map $P \to Q$ in H_{Nis} is an $\mathbb{A}^1$-weak equivalence if

$$\operatorname{Hom}_{H_{Nis}}(Q, Z) \to \operatorname{Hom}_{H_{Nis}}(P, Z)$$

if an isomorphism for all $\mathbb{A}^1$-local Z. If H_{Nis} is further localized with respect to $\mathbb{A}^1$-weak equivalences, we end up with the relevant category in which to study motivic homotopy theory.

Definition 1.0.1. *Let $H_\bullet(F)$ be the $\mathbb{A}^1$-localized category obtained from $\Delta^{op}Shv_{Nis}(Sm/F)_\bullet$. We will refer to objects in this category as pointed motivic spaces.*

In this way we force the maps

$$X \times \mathbb{A}^1 \to X$$

to be weak equivalences.

Let S_s^1 be the constant simplicial sheaf with value $\Delta^1/\partial\Delta^1$ and let S_t^1 be the simplicial sheaf represented by $\mathbb{G}_m$ with 1 as its basepoint. The category $\Delta^{op}Shv_{Nis}(Sm/F)_\bullet$ is symmetric monoidal with respect to the smash product and this property is inherited by $H_\bullet(F)$. We let

$$S_s^n := (S_s^1)^{\wedge n}$$

and

$$S_t^n := (S_t^1)^{\wedge n}$$

for $n \geq 0$. Additionally, we write

$$S^{p,q} := S_s^{p-q} \wedge S_t^q$$

for $p \geq q \geq 0$. Given $X \in H_\bullet(F)$, the smash products $X \wedge S_s^1$ and $X \wedge S_t^1$ will be denoted $\Sigma_s X$ and $\Sigma_t X$ respectively. Let $T := \mathbb{A}^1/(\mathbb{A}^1 - 0)$. This is refered to as the Tate object in the literature. There are isomorphisms $T \cong S_s^1 \wedge S_t^1 \cong \mathbb{P}^1$, with $\mathbb{P}^1$ pointed at infinity. Finally, the smash product of $X \wedge T$ will be denoted $\Sigma_T X$.

We will work with spectra and need to see how these are handled in the motivic world. A T-spectrum is a sequence of pointed motivic spaces $E_0, E_1, \ldots$ with bonding maps

$$\Sigma_T E_i \to E_{i+1}.$$

A map $E \to F$ between these are collections of maps $E_i \to F_i$ commuting with the bonding maps. We let $Spt(F)$ denote this category. For a spectrum we can define the presheaf

$$\pi_{p,q}^s(E)(X) := \lim_n \mathrm{Hom}_{H_\bullet(F)}(S^{p+2n,q+n} \wedge X_+, E_n); X \in Sm/F$$

and its associated sheaf $\pi_{p,q}^s(E)_{Nis}$. A stable weak equivalence of T-spectra is a map inducing an isomorphism of sheaves $\pi_{p,q}^s(E)_{Nis} \to \pi_{p,q}^s(F)_{Nis}$. We localize and define $SH(F)$ to be the category obtained after inverting the stable weak equivalences. The suspension functor Σ_T defined on $SH(F)$ becomes invertible and the objects in this category represent bigraded cohomology theories on Sm/F by defining

$$E^{p,q}(X) := \lim_n \mathrm{Hom}_{H_\bullet(F)}(S^{-p+2n,-q+n} \wedge X_+, E_n); X \in Sm/F.$$

Examples of such cohomology theories are motivic cohomology, algebraic K-theory and algebraic cobordism, all of which are mentioned in [39].

Chapter 2

Overview and the main argument

As alluded to in the introduction, the Singer construction is an algebraic construct providing us with relevant homological information about the Adams spectral sequence in some special cases. In turn, this makes it possible to use the spectral sequence to describe the stable homotopy groups of classifying spaces after some appropriate completion. In this chapter we review the argument and what we need to set this up motivically.

In chapter 3, the motivic Steenrod algebra A is reviewed and given a left module M over it, we check that $R_+(M)$ can be constructed as $\mathrm{colim}_n\, B(n) \otimes_{A(n-1)} M$, where $B(n)$ is an $A(n)$-$A(n-1)$-bimodule for finitely generated subalgebras of A and each morphism in the colimit system is an additive isomorphism as discussed in [1]. The main motivation will be the usage of $R_+(M)$ with $M = \mathbb{M}_p$ where $\mathbb{M}_p := H^{*,*}(\mathrm{Spec}(F); \mathbb{Z}/p)$ serves as the motivic stand-in for $\mathbb{Z}/p$. The identification of A relies on the characteristic of the base field over which all spaces and spectra are defined to be 0 as explained in [42]. In any case, there is an A-linear map

$$\epsilon : R_+(\mathbb{M}_p) \longrightarrow \mathbb{M}_p$$

inducing an isomorphism

$$\epsilon^* : \mathrm{Ext}_A^{*,(*,*)}(\mathbb{M}_p, \mathbb{M}_p) \cong \mathrm{Ext}_A^{*,(*,*)}(R_+(\mathbb{M}_p), \mathbb{M}_p).$$

This follows from proposition 3.3.4 and theorem 3.3.6.

Next, in chapter 4, we realize $R_+(\mathbb{M}_2)$ using an inverse tower of motivic spectra $\underline{L}_{-k}^{\infty}$ and the A-module $H_c^{*,*}(\underline{L}_{-\infty}^{\infty}) := \mathrm{colim}_k\, H^{*,*}(\underline{L}_{-k}^{\infty})$ for $\underline{L}_{-\infty}^{\infty} := \mathrm{holim}_{-k}\, \underline{L}_{-k}^{\infty}$. This is done in the exact same manner as we saw above with the only difference lying in the definition of the motivic Thom spectra due to the fact that we may not work with orthogonal complements in algebraic geometry. See 4.1.23 for the precise definition of $\underline{L}_{-k}^{\infty}$. This particular tower requires that we restrict our work to motivic cohomology with mod 2 coefficients if the algebraic identifications are to work out correctly. There is most likely a similar tower for the odd case but this eludes the author at the moment. The main result is the existence of the A-linear isomorphisms

$$R_+(\mathbb{M}_2) \cong \Sigma^{1,0} H_c^{*,*}(\underline{L}^{\infty}_{-\infty})$$

which is a consequence of proposition 4.1.37.

Finally, in part 4.2, convergence properties of the motivic Adams spectral sequence are reviewed and we check that all the building blocks going into the argument above are indeed working. This limits the generality of the base field F and forces $char(F)$ to be 0 as we are using theorem 1 in [21]. We look at the tower

$$\underline{L}^{\infty}_{-\infty} \longrightarrow \cdots \longrightarrow \underline{L}^{\infty}_{-1} \longrightarrow \underline{L}^{\infty}.$$

At each step there is a motivic Adams spectral sequence strongly converging to the homotopy groups of $\widehat{\underline{L}^{\infty}_{-k}}$, the $2,\eta$-completion of $\underline{L}^{\infty}_{-k}$ where $\eta \in \pi_{1,1}(S)$ is the algebraic Hopf map

$$\mathbb{A}^2 \setminus 0 \to \mathbb{P}^1$$

sending $(x, y) \mapsto [x : y]$. The E_2-terms are

$$\mathrm{Ext}_A^{*,(*,*)}(H^{*,*}(\underline{L}^{\infty}_{-k}), \mathbb{M}_2)$$

and we take the inverse limit of these spectral sequences to form a new spectral sequence converging strongly to $\widehat{\underline{L}^{\infty}_{-\infty}}$ with E_2-term

$$E_2^{s,(t,*)}(\underline{L}^{\infty}_{-\infty}) \cong \mathrm{Ext}_A^{s,(t,*)}(H_c^{*,*}(\underline{L}^{\infty}_{-\infty}), \mathbb{M}_2).$$

This is theorem 4.2.23. For this result to work properly we need to assume that $\mathbb{M}_2$ is a finite dimensional vector space over $\mathbb{Z}/2$ in each bidegree since we need certain inverse limit groups to vanish.

Given these provisos, we state our main result:

Theorem 2.0.2. *Assume that $char(F) = 0$, that $p = 2$ and that $\mathbb{M}_2$ is a finite dimensional vector space over $\mathbb{Z}/2$ in each bidegree. Then the inverse limit spectral sequence described above induces a $\pi_{*,*}(-)$-isomorphism*

$$S \to \Sigma^{1,0} \underline{L}^{\infty}_{-\infty}$$

after $2, \eta$-completion.

Proof. Since the inverse limit spectral sequence satisfies

$$E_2^{s,(t,*)}(\underline{L}^{\infty}_{-\infty}) \cong \mathrm{Ext}_A^{s,(t,*)}(\operatorname*{colim}_k H^{*,*}(\underline{L}^{\infty}_{-k}), \mathbb{M}_2)$$

and

$$R_+(\mathbb{M}_2) \cong \Sigma^{1,0} H_c^{*,*}(\underline{L}^{\infty}_{-\infty}),$$

the map

$$\epsilon^* : \mathrm{Ext}_A^{*,(*,*)}(\mathbb{M}_2, \mathbb{M}_2) \cong \mathrm{Ext}_A^{*,(*,*)}(R_+(\mathbb{M}_2), \mathbb{M}_2)$$

sets up an isomorphism between the E_2-term of this spectral sequence and the E_2-term of the motivic Adams spectral sequence converging to the homotopy groups of the $2, \eta$-completion of the motivic sphere spectrum. The reader may find a picture of the E_2-term in appendix A in [14]. In particular, we have

$$\mathrm{Hom}_A(\mathbb{M}_2, \mathbb{M}_2) \cong \mathrm{Hom}_A(R_+(\mathbb{M}_2), \mathbb{M}_2) = \mathrm{Ext}_A^{0,(0,0)}(H_c^{*,*}(\Sigma^{1,0}\underline{L}^{\infty}_{-\infty}), \mathbb{M}_2).$$

The identity morphism generates

$$\mathrm{Hom}_A^{0,0}(\mathbb{M}_2, \mathbb{M}_2) = \mathbb{Z}/2$$

and is an infinite cycle of the Adams spectral sequence of the completed sphere: The E_2-term is 0 when s or $t-s$ is negative and so any differential to or from this group is trivial. The same vanishing must then also hold for the inverse limit spectral sequence. From this we know that

$$\mathrm{Hom}_A^{0,0}(R_+(\mathbb{M}_2), \mathbb{M}_2) = \mathbb{Z}/2$$

is generated by $1 \in \mathbb{Z}/2$ which corresponds to ϵ. This will then be an infinite cycle.

This cycle will correspond to a class in $\pi_{0,0}(\widehat{(\Sigma^{1,0}\underline{L}^{\infty}_{-\infty})})$. Hence there is a map

$$f : S^{0,0} \to \widehat{\Sigma^{1,0}\underline{L}^{\infty}_{-\infty}}.$$

Now, define

$$f_k : S^{0,0} \to \widehat{\Sigma^{1,0}\underline{L}^{\infty}_{-k}}$$

to be the composition

$$S^{0,0} \to \widehat{\Sigma^{1,0}\underline{L}^{\infty}_{-\infty}} \to \widehat{\Sigma^{1,0}\underline{L}^{\infty}_{-k}}$$

where the first map is always f. The induced maps

$$f_k^* : H^{*,*}(\widehat{\Sigma^{1,0}\underline{L}^{\infty}_{-k}}) = H^{*,*}(\Sigma^{1,0}\underline{L}^{\infty}_{-k}) \to H^{*,*}(S^{0,0}) = \mathbb{M}_p$$

are compatible so there is an induced map

$$f^* : H^{*,*}(\Sigma^{1,0}\underline{L}^{\infty}_{-\infty}) \to \mathbb{M}_p$$

Given the correspondence between the two spectral sequences, this map is ϵ under the identification $H_c^{*,*}(\Sigma^{1,0}\underline{L}^{\infty}_{-\infty}) = R_+(\mathbb{M}_p)$. From the maps f_k we get induced maps of spectral sequences

$$E_r^{*,(*,*)}(S^{0,0}) \to E_r^{*,(*,*)}(\Sigma^{1,0}\underline{L}^{\infty}_{-k})$$

At the E_2-terms, these are

$$\mathrm{Ext}_A^{*,(*,*)}(\mathbb{M}_p, \mathbb{M}_p) \to \mathrm{Ext}_A^{*,(*,*)}(H^{*,*}(\Sigma^{1,0}\underline{L}^{\infty}_{-k}), \mathbb{M}_p),$$

and they converge to the homomorphism

$$\pi_{*,*}(\widehat{S^{0,0}}) \to \pi_{*,*}(\widehat{\Sigma^{1,0}\underline{L}^{\infty}_{-k}}).$$

Passing to the limit, there is an induced map of spectral sequences

$$E_r^{*,(*,*)}(S^{0,0}) \to E_r^{*,(*,*)}(\Sigma^{1,0}\underline{L}^{\infty}_{-\infty})$$

given at the E_2-term as

$$\mathrm{Ext}_A^{*,(*,*)}(\mathbb{M}_p, \mathbb{M}_p) \to \mathrm{Ext}_A^{*,(*,*)}(H_c^{*,*}(\Sigma^{1,0}\underline{L}^{\infty}_{-\infty}), \mathbb{M}_p).$$

It converges to the homomorphism

$$\pi_{*,*}(\widehat{S^{0,0}}) \to \pi_{*,*}(\widehat{\Sigma^{1,0}\underline{L}^{\infty}_{-\infty}}) \cong \lim_k \pi_{*,*}(\widehat{\Sigma^{1,0}\underline{L}^{\infty}_{-k}}).$$

Under the identification $H_c^{*,*}(Y) = R_+(\mathbb{M}_p)$, this map corresponds to ϵ so the map of E_2-terms is the familiar Ext-isomorphim. This implies that the map of the E_r-term is an isomorphism for all r which in turn implies the isomorphism of abutments

$$\pi_{*,*}(\widehat{S^{0,0}}) \to \pi_{*,*}(\widehat{\Sigma^{1,0}\underline{L}^{\infty}_{-\infty}})$$

since the spectral sequences are strongly convergent. □

At numerous places in the text we shall not restrict our work to the case $p = 2$ and this is due to the fact that many considerations work perfectly fine for odd p. At some point someone may construct an inverse tower of spectra for these cases too so the computations made may come in handy.

Chapter 3

The algebra

3.1 The motivic Steenrod algebra and its dual

From now on we let F be a field of characteristic 0. There are several reasons for this and we will comment on these matters in remark 3.1.7. The basic algebraic object with which we will work is $H^{*,*}(\mathrm{Spec}(F);\mathbb{Z}/p)$, the bigraded motivic cohomology ring of a point (with p a rational prime). An element h in $H^{a,b}(\mathrm{Spec}(F);\mathbb{Z}/p)$ is said to have degree a and weight b, or sometimes bidegree (a,b). The notation $\mathrm{bideg}(\mathsf{h}) = (a,b)$, $\deg(\mathsf{h}) = a$ and $\mathrm{wt}(\mathsf{h}) = b$ may also be used. Following notational practice from [14], we let

$$\mathbb{M}_p := H^{*,*}(\mathrm{Spec}(F);\mathbb{Z}/p).$$

Some facts on $\mathbb{M}_p$ will need to be recollected and for this we define other well-known algebraic objects:

Definition 3.1.1. *For a field F, let*

$$T(F^\times) := \mathbb{Z} \oplus F^\times \oplus (F^\times \otimes F^\times) \oplus \cdots$$

be the free, graded algebra on the group $F^\times$ of units of F. Define

$$K_*^M(F) := T(F^\times)/(x \otimes (1-x) : x \in F^\times, x \neq 0, 1)$$

This is a graded ring refered to as the Milnor K-theory of F.

Conventions have us denote the elements in $K_n^M(F)$ by $\{x_1, \ldots, x_n\}$. For more on Milnor K-theory, the reader should see chapter 4 and 7 in [17].

Definition 3.1.2. *For a field F, let*

$$\mu_p(F) := \mathrm{Spec}(F[x]/(x^p - 1))$$

be the p-th roots of unity in F.

Proposition 3.1.3. *Let F be a field that admits resolution of singularities. Then $\mathbb{M}_p^{0,1} \cong \mu_p(F)$ and $\mathbb{M}_p^{a,a} \cong K_a^M(F)/p$ for nonnegative integers a.*

Proof. The identification $H^{0,1}(\mathrm{Spec}(F); \mathbb{Z}/p) \cong \mu_p(F)$ can be found in [28], corollary 4.9. The second is a consequence of theorem 5.1 (in the same reference) which states that

$$H^{a,a}(\mathrm{Spec}(F); \mathbb{Z}) \cong K_a^M(F).$$

The long exact sequence in motivic cohomology associated to

$$0 \longrightarrow \mathbb{Z} \xrightarrow{*p} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/p \longrightarrow 0$$

and the fact that $H^{a+1,a}(\mathrm{Spec}(F); \mathbb{Z})$ vanishes (lemma 3.2 part 2 in [38]) settles the result. □

Let τ be a generator of $\mathbb{M}_p^{0,1} \cong \mu_p(F)$ and ρ be the class of $\{-1\}$ in $\mathbb{M}_p^{1,1} \cong F^\times/(F^\times)^p$. We remark that $\rho = 0$ when p is odd. The following result on $\mathbb{M}_p$ will be helpful to us:

Lemma 3.1.4. *Let F be our field of characteristic* 0. *Then*

$$\mathbb{M}_p^{a,b} = 0$$

when $a < 0$ and when $a > b$.

Proof. The first condition relies on the Bloch-Kato conjecture. It implies that $H^{a,b}(\mathrm{Spec}(F); \mathbb{Z}/p) \cong H^a_{\acute{e}t}(\mathrm{Spec}(F); \mu_p^{\otimes b})$, and this vanishes for negative degrees by construction. The references for this is theorem 6.17 in [43]. The second condition is theorem 3.6 in [28]. □

On the last result: One should read the comments in the introduction of [45]. Here Weibel comments that this result rests on three lemmas from an earlier version of Voevodskys paper, some of which are suspected to be false. He states that these problems are circumvented by using a result from his paper and gives references to the remaining lemmas.

Consider two bigraded modules M and N over $\mathbb{Z}/p$, both with bidegrees consisting of a degree and a weight. We define an isomorphism of bigraded $\mathbb{Z}/p$-modules

$$\mathrm{Tw}^{\deg} : M \otimes_{\mathbb{Z}/p} N \longrightarrow N \otimes_{\mathbb{Z}/p} M$$

by the association $\mathrm{Tw}^{\deg}(m \otimes n) = (-1)^{\deg(m)\deg(n)} n \otimes m$, the twist map that only takes degrees into account. The cup product makes $\mathbb{M}_p$ a bigraded algebra over $\mathbb{Z}/p$, commutative in that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{M}_p \otimes_{\mathbb{Z}/p} \mathbb{M}_p & \xrightarrow{\mathrm{Tw}^{\deg}} & \mathbb{M}_p \otimes_{\mathbb{Z}/p} \mathbb{M}_p \\ & {\scriptstyle \cup} \searrow \quad \swarrow {\scriptstyle \cup} & \\ & \mathbb{M}_p & \end{array}$$

In his paper [41], Voevodsky studies bistable operations on motivic cohomology with coefficients in $\mathbb{Z}/p$. The two ways of suspending in $H_\bullet(F)$ add complexity to the study of operations on $H^{*,*}(X;\mathbb{Z}/p)$ with $X \in H_\bullet(F)$. There are natural isomorphisms

$$\sigma_s : H^{*,*}(X;\mathbb{Z}/p) \longrightarrow H^{*+1,*}(\Sigma_s X;\mathbb{Z}/p)$$

and

$$\sigma_t : H^{*,*}(X;\mathbb{Z}/p) \longrightarrow H^{*+1,*+1}(\Sigma_t X;\mathbb{Z}/p),$$

sometimes refered to as the simplicial and algebraic suspension isomorphisms.

Definition 3.1.5. *A bistable operation of bidegree (u, v) on $H^{*,*}(-;\mathbb{Z}/p)$ is a collection of natural transformations*

$$\theta : H^{a,b}(-;\mathbb{Z}/p) \longrightarrow H^{a+u,b+v}(-;\mathbb{Z}/p)$$

(for all $(a, b) \in \mathbb{Z}\times\mathbb{Z}$) of functors on $H_\bullet(F)$ that commute with both σ_s and σ_t.

The set of all such operations will be denoted A^{bist} and this is a bigraded noncommutative algebra over $\mathbb{Z}/p$ with multiplication given by composition of operations.

Voevodsky constructs operations P^k and B^k (for integers $k \geq 0$) of bidegrees $(2k(p-1), k(p-1))$ and $(2k(p-1)+1, k(p-1))$ respectively. These are shown to satisfy $P^0 = 1$, $\beta B^k = 0$ and $\beta P^k = B^k$ where β is the Bockstein operation

$$\beta : H^{*,*}(X;\mathbb{Z}/p) \longrightarrow H^{*+1,*}(X;\mathbb{Z}/p)$$

induced from the short exact sequence

$$0 \longrightarrow \mathbb{Z}/p \longrightarrow \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p \longrightarrow 0.$$

Definition 3.1.6. *Let A be the subalgebra of A^{bist} generated by operations of the form $u \mapsto \mathsf{h}u$ ($\mathsf{h} \in \mathbb{M}_p$) in addition to β and the P^k's. A is called the motivic Steenrod algebra. The inclusion of bigraded $\mathbb{Z}/p$-algebras*

$$\mathbb{M}_p \rightarrowtail A$$

taking h to the map $u \mapsto \mathsf{h}u$ will be denoted η.

A complication for things to come is the fact that this inclusion is not central, so A is *not* an algebra over $\mathbb{M}_p$.

Remark 3.1.7. At the beginning of this section we assumed that the characteristic of our ground field F was 0 and lemma 3.1.4 was one reason why. The bounds on $\mathbb{M}_p$ are needed for calculations later on. Another important reason for this lies in the relationship between A and A^{bist}. It is expected that these coincide, but at the present, this can only be proved with our basic assumption. See section 3.4 of [42] for more on this.

Following conventions from algebraic topology, one writes

$$Sq^{2k+\epsilon} := \beta^{\epsilon} P^k$$

($\epsilon \in \{0, 1\}$) when $p = 2$. These operations satisfy analogues of the Cartan formula and Adem relations which we record for use in proving lemma 4.1.19 and proposition 4.1.37:

Proposition 3.1.8. *Let X and Y be motivic spaces, $x \in H^{*,*}(X; \mathbb{Z}/p)$, $y \in H^{*,*}(Y; \mathbb{Z}/p)$. Then we have*

$$P^k(x \times y) = \sum_{i=0}^{k} P^i(x) \times P^{k-i}(y),$$

when p is odd, and

$$Sq^{2i}(x \times y) = \sum_{a=0}^{i} Sq^{2a}(x) \times Sq^{2i-2a}(y) + \tau \sum_{b=0}^{i-1} Sq^{2b+1}(x) \times Sq^{2i-2b-1}(y),$$

$$\begin{aligned} Sq^{2i+1}(x \times y) = &\sum_{a=0}^{i} (Sq^{2a+1}(x) \times Sq^{2i-2a}(y) + Sq^{2a}(x) \times Sq^{2i-2a-1}(y)) \\ &+ \rho \sum_{b=0}^{i-1} Sq^{2b+1}(x) \times Sq^{2i-2b-1}(y) \end{aligned}$$

when $p = 2$.

Proof. Proposition 9.7 in [41]. □

Theorem 3.1.9. *Let $0 < a < 2b$. Then we have*

$$Sq^a Sq^b = \sum_{i=0}^{\lfloor a/2 \rfloor} \binom{b-1-i}{a-2i} Sq^{a+b-i} Sq^i$$

when a is odd and

$$Sq^a Sq^b = \sum_{i=0}^{\lfloor a/2 \rfloor} \tau^{\varepsilon_i} \binom{b-1-i}{a-2i} Sq^{a+b-i} Sq^i + \rho \sum_{i=1, i \equiv b(2)}^{\lfloor a/2 \rfloor} \binom{b-1-i}{a-2i} Sq^{a+b-i-1} Sq^i$$

when a is even. Here,

$$\varepsilon_i = \begin{cases} 1 & \text{if } b \text{ is even and } i \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This is theorem 10.2 in [41]. However, the statement given there is incorrect. Voevodsky follows Steenrod's work in [16] but the terms involving the factor ρ should read as above. □

Theorem 3.1.10. *Let p be an odd prime. If $0 < a < pb$, we have*

$$P^a P^b = \sum_{i=0}^{\lfloor a/p \rfloor} (-1)^{a+i} \binom{(p-1)(b-i)-1}{a-pi} P^{a+b-i} P^i$$

If $0 < a \leq pb$ then

$$P^a \beta P^b = \sum_{i=0}^{\lfloor a/p \rfloor} (-1)^{a+i} \binom{(p-1)(b-i)}{a-pi} \beta P^{a+b-i} P^i + \sum_{i=0}^{\lfloor (a-1)/p \rfloor} (-1)^{a+i-1} \binom{(p-1)(b-i)-1}{a-pi-1} P^{a+b-i} \beta P^i$$

Proof. This is theorem 10.3 in [41]. There, one finds one of the index boundaries stated incorrectly: The Adem relation for $P^a \beta P^b$ reads $0 \leq a \leq pb$ where it should be as above. □

As a consequence, similar to the situation for the classical Steenrod algebra, one may show that as a left $\mathbb{M}_p$-module, A is freely generated by the admissible monomials in these generators (this is lemma 11.1 and corollary 11.5 in [41]): For p odd, a monomial

$$P^I := \beta^{\epsilon_0} P^{k_1} \cdots P^{k_s} \beta^{\epsilon_s}$$

is said to be admissible if the sequence $I = (\epsilon_0, k_1, \ldots, k_s, \epsilon_s)$ of nonnegative integers satifies $k_i \geq pk_{i+1} + \epsilon_i$ for $1 \leq i < s$. Here, $\epsilon_i \in \{0, 1\}$. For $p = 2$ monomials of the form

$$Sq^I := Sq^{k_1} \cdots Sq^{k_s}$$

are said to be admissible if the sequence $I = (k_1, \ldots, k_s)$ of nonnegative integers satifies $k_i \geq 2k_{i+1}$ for $1 \leq i < s$. We will use the notation $A \cong \mathbb{M}_p\{P^I \mid I \text{ admissible}\}$ when p is odd and $A \cong \mathbb{M}_p\{Sq^I \mid I \text{ admissible}\}$ when $p = 2$ with I admissible according to the two different cases.

Definition 3.1.11. *Let M be a bigraded left $\mathbb{M}_p$-module. If M is generated by some set $\{b^I\}$ and only finitely many of these lie in any given bidegree then M is said to have finite type. Similarly, a bigraded right $\mathbb{M}_p$-module is said to have finite type if only finitely many of its right generators has any given bidegree.*

An inspection of the bidegrees of the admissible monomials given above reveals that A is a free left $\mathbb{M}_p$-module and of finite type. It is also bounded below by both axes in the degree-weight plane.

A can be given more structure, as expected: Voevodsky defines a morphism of rings (this is lemma 11.8 in [41])

$$\Delta : A \longrightarrow (A \otimes_l A)_r$$

defined as follows: Given $a \in A$, $\Delta(a)$ is equal to $\Sigma a' \otimes a'$, the unique element such that

$$a(xy) = \Sigma(-1)^{\deg(a')p} a'(x)a''(y)$$

for any $x \in \tilde{H}^{p,*}(X)$ and $y \in \tilde{H}^{*,*}(Y)$. Here $A \otimes_l A$ is taken to mean the *left module tensor product* and $(A \otimes_l A)_r$ is a submodule (over $\mathbb{Z}/p$) of $A \otimes_l A$ carrying a ring-structure induced from $A \otimes_{\mathbb{Z}/p} A$:

Definition 3.1.12. *Let M and N be bigraded, left $\mathbb{M}_p$-modules. Then $M \otimes_l N$ is defined as the coequalizer in the following diagram:*

$$\begin{array}{ccccc} \mathbb{M}_p \otimes_{\mathbb{Z}/p} M \otimes_{\mathbb{Z}/p} N & \xrightarrow{a_M \otimes N} & M \otimes_{\mathbb{Z}/p} N & \longrightarrow & M \otimes_l N \\ \downarrow{\scriptstyle \mathrm{Tw}^{\deg} \otimes N} & \nearrow{\scriptstyle M \otimes a_N} & & & \\ M \otimes_{\mathbb{Z}/p} \mathbb{M}_p \otimes_{\mathbb{Z}/p} N & & & & \end{array}$$

where the $a_?$-maps are given by the left action of $\mathbb{M}_p$ on M and N.

The existence and uniqueness of Δ is proved following the proof Milnor gives in part 3 of [29]. Where Milnor proves uniqueness using the vanishing of the cohomology groups of Eilenberg-Maclane spaces, Voevodsky uses a similar condition of the motivic cohomology of classifying spaces which is given in [41] lemma 11.4.

The topological steenrod algebra admits a multiplicative coproduct so one could be tempted to think that this applies to A as well. However, since $\mathbb{M}_p$ is not central in A, Δ is not left $\mathbb{M}_p$-linear, hence we must be careful when we want to check that Δ is multiplicative. In [41] part 11, Voevodsky observes that we can let $A \otimes_l A$ act on $A \otimes_{\mathbb{Z}/p} A$ by the formula

$$(a \otimes b)(c \otimes d) = (-1)^{\deg(b)\deg(c)} ac \otimes bd$$

where $a \otimes b \in A \otimes_l A$ and $c \otimes d \in A \otimes_{\mathbb{Z}/p} A$, and pass to the quotient $A \otimes_l A$. Then Δ lands in a special subset of $A \otimes_l A$:

Definition 3.1.13. *Let $f \in A \otimes_l A$ and $x, y \in A \otimes_{\mathbb{Z}/p} A$ with the property that x and y are identified in $A \otimes_l A$ under the projection*

$$A \otimes_{\mathbb{Z}/p} A \longrightarrow A \otimes_l A.$$

Then f is operator-like *if we have $fx = fy \in A \otimes_l A$. The collection of such elements is denoted $(A \otimes_l A)_r$.*

One observes that $(A \otimes_l A)_r$ is in fact a ring: For x, y as in the definition and f, g operator-like, we have $fx = fy$ and hence $gfx = gfy$ in $A \otimes_l A$.

For a bigraded right $\mathbb{M}_p$-module M and a bigraded left $\mathbb{M}_p$-module N, the notation $M \otimes_{\mathbb{M}_p} N$ will be reserved for the standard way of forming tensor products of graded modules over the bigraded $\mathbb{Z}/p$-algebra $\mathbb{M}_p$ by which we mean the coequalizer in the diagram

$$M \otimes_{\mathbb{Z}/p} \mathbb{M}_p \otimes_{\mathbb{Z}/p} N \underset{M \otimes a_N}{\overset{a_M \otimes N}{\rightrightarrows}} M \otimes_{\mathbb{Z}/p} N \longrightarrow M \otimes_{\mathbb{M}_p} N$$

Remark 3.1.14. The considerations on the different tensor products over $\mathbb{M}_p$ come about as $A \otimes_l A$ is not an $\mathbb{M}_p$-algebra since $\mathbb{M}_p$ is not in the center of A. We also remark that our notation is different from the one used by Voevodsky in [41] where he uses $A \otimes_{\mathbb{M}_p} A$ instead of our $A \otimes_l A$ and $A \otimes_{r,l} A$ instead of our $A \otimes_{\mathbb{M}_p} A$.

Definition 3.1.15. *For any left $\mathbb{M}_p$-module M we define*

$$M^\vee := \mathrm{Hom}_{\mathbb{M}_p}(M, \mathbb{M}_p)$$

(left $\mathbb{M}_p$-module homomorphisms).

We will refer to this process as $\mathbb{M}_p$-dualization and say that $M^\vee$ is the $\mathbb{M}_p$-dual of M. $M^\vee$ is a left module over $\mathbb{M}_p$ with its left action given by $(\mathsf{h}f)(m) := (-1)^{\deg(f)\deg(\mathsf{h})} f(\mathsf{h}m)$ ($\mathsf{h} \in \mathbb{M}_p$, $f \in \mathrm{Hom}_{\mathbb{M}_p}(M, \mathbb{M}_p)$ and $m \in M$). If in addition M is a bimodule over $\mathbb{M}_p$, $M^\vee$ is also a bimodule with a right action given by $(f\mathsf{h})(m) := (-1)^{\deg(m)\deg(\mathsf{h})} f(m\mathsf{h})$. These actions produce elements in $\mathrm{Hom}_{\mathbb{M}_p}(M, \mathbb{M}_p)$ since $\mathbb{M}_p$ is graded commutative: It is obvious that these maps respect addition of elements in M so we check that they are graded left $\mathbb{M}_p$-linear maps. For $\mathsf{h}' \in \mathbb{M}_p$ we have

$$\begin{aligned}(\mathsf{h}f)(\mathsf{h}'m) &= (-1)^{\deg(f)\deg(\mathsf{h})} f(\mathsf{h}\mathsf{h}'m) \\ &= (-1)^{\deg(\mathsf{h})\deg(\mathsf{h}')}(-1)^{\deg(f)\deg(\mathsf{h}')}\mathsf{h}'(-1)^{\deg(f)\deg(\mathsf{h})} f(\mathsf{h}m) \\ &= (-1)^{(\deg(\mathsf{h})+\deg(f))\deg(\mathsf{h}')}\mathsf{h}'(\mathsf{h}f)(m)\end{aligned}$$

and

$$\begin{aligned}(f\mathsf{h})(\mathsf{h}'m) &= (-1)^{\deg(\mathsf{h}'m)\deg(\mathsf{h})} f(\mathsf{h}'m\mathsf{h}) \\ &= (-1)^{(\deg(\mathsf{h}')+\deg(m))\deg(h)}(-1)^{\deg(f)\deg(\mathsf{h}')}\mathsf{h}' f(m\mathsf{h}) \\ &= (-1)^{(\deg(f)+\deg(\mathsf{h}))\deg(\mathsf{h}')}\mathsf{h}'(-1)^{\deg(m)\deg(\mathsf{h})} f(m\mathsf{h}) \\ &= (-1)^{(\deg(f)+\deg(\mathsf{h}))\deg(\mathsf{h}')}\mathsf{h}'(f\mathsf{h})(m)\end{aligned}$$

so these maps are indeed homomorphisms of graded left $\mathbb{M}_p$-modules.

Remark 3.1.16. Now, consider $A^\vee$. The motivic Steenrod algebra A is a free left $\mathbb{M}_p$-module on the admissible monomials P^I or Sq^I (where I is an admissible sequence). This implies that

$$A^\vee \cong \prod_I \mathbb{M}_p\{Sq^I\}^\vee \text{ or } A^\vee \cong \prod_I \mathbb{M}_p\{P^I\}^\vee$$

as a left module over $\mathbb{M}_p$. Because of our assumptions on F and the fact that A is of finite type and in a sense bounded below, the limitations on bidegrees will, as we will prove later, imply that

$$A^\vee \cong \bigoplus_I \mathbb{M}_p\{Sq^I\}^\vee \text{ or } A^\vee \cong \bigoplus_I \mathbb{M}_p\{P^I\}^\vee$$

and so $A^\vee$ has a basis consisting of the $\mathbb{M}_p$-dual basis of $\{Sq^I \mid I \text{ admissible}\}$ or $\{P^I \mid I \text{ admissible}\}$. Thus $A^\vee$ is a free left $\mathbb{M}_p$-module. From bidegree considerations, it also of finite type and bounded below. This also implies that $A \cong (A^\vee)^\vee$ as left $\mathbb{M}_p$-modules. We define $\{\vartheta(I) \mid I \text{ admissible}\}$ to be the $\mathbb{M}_p$-dual basis of $\{Sq^I \mid I \text{ admissible}\}$ when $p = 2$ and $\{P^I \mid I \text{ admissible}\}$ when p is odd.

There is a composition of maps

$$A^\vee \otimes_{\mathbb{Z}/p} A^\vee \twoheadrightarrow A^\vee \otimes_l A^\vee \xrightarrow{G} (A \otimes_l A)^\vee \longrightarrow ((A \otimes_l A)_r)^\vee$$

where the first map is the quotient map defining the coequalizer, the second map is defined by letting

$$G(\alpha \otimes \beta)(a \otimes b) = (-1)^{\deg(a)\deg(\beta)}\alpha(a)\beta(b)$$

($\alpha, \beta \in A^\vee$ and $a, b \in A$) and the third is the $\mathbb{M}_p$-dual of the inclusion. The composite will be denoted $\underline{G}$.

Definition 3.1.17. *Let*

$$\phi : A^\vee \otimes_{\mathbb{Z}/p} A^\vee \longrightarrow A^\vee$$

be defined as $\underline{G}$ *concatenated with the* $\mathbb{M}_p$*-dual of* Δ.

Proposition 3.1.18. *The map* ϕ *gives* $A^\vee$ *the structure of an associative algebra over* $\mathbb{Z}/p$*. It is graded commutative with respect to the degree.*

Proof. This is proposition 12.1 in [41]. □

To go further we need some helpful results:

Lemma 3.1.19. *Let* M *be a bigraded* $\mathbb{M}_p$*-bimodule and* N *a bigraded left* $\mathbb{M}_p$*-module. Then there is a morphism of bigraded left* $\mathbb{M}_p$*-modules*

$$\theta : M^\vee \otimes_{\mathbb{M}_p} N^\vee \longrightarrow (M \otimes_{\mathbb{M}_p} N)^\vee$$

defined by letting

$$\theta(f \otimes g)(m \otimes n) = (-1)^{\deg(g)\deg(m)} f(mg(n)),$$

where $f \in M^\vee$, $g \in N^\vee$, $m \in M$ *and* $n \in N$.

Proof. The morphism is a specialization of lemma 3.3 a) in [5]. □

Lemma 3.1.20. *Let* M *and* N *be as in the last lemma and assume that* N *is free on a set* $\{b^I\}$*. If the canonical morphisms*

$$M^\vee \otimes_{\mathbb{M}_p} \bigoplus_I \mathbb{M}_p\{b^I\}^\vee \longrightarrow M^\vee \otimes_{\mathbb{M}_p} \prod_I \mathbb{M}_p\{b^I\}^\vee$$

and

$$\bigoplus_I (M \otimes_{\mathbb{M}_p} \mathbb{M}_p\{b^I\})^\vee \longrightarrow \prod_I (M \otimes_{\mathbb{M}_p} \mathbb{M}_p\{b^I\})^\vee$$

(of left $\mathbb{M}_p$*-modules) are isomorphisms, then* θ *is an isomorphism.*

Proof. We argue with a commutative diagram:

$$\begin{array}{ccccc}
 & & M^\vee \otimes_{\mathbb{M}_p} N^\vee & \xrightarrow{\theta} & (M \otimes_{\mathbb{M}_p} N)^\vee \\
 & \swarrow^{\mathcal{A}} & \downarrow & & \downarrow^{\mathcal{E}} \\
M^\vee \otimes_{\mathbb{M}_p} \prod_I \mathbb{M}_p\{b^I\}^\vee & \longrightarrow & \prod_I (M^\vee \otimes_{\mathbb{M}_p} \mathbb{M}_p\{b^I\}^\vee) & \xrightarrow{\prod_I \theta} & \prod_I (M \otimes_{\mathbb{M}_p} \mathbb{M}_p\{b^I\})^\vee \\
\uparrow^{\mathcal{B}} & & \uparrow & & \uparrow^{\mathcal{D}} \\
M^\vee \otimes_{\mathbb{M}_p} \bigoplus_I \mathbb{M}_p\{b^I\}^\vee & \xrightarrow{\mathcal{C}} & \bigoplus_I (M^\vee \otimes_{\mathbb{M}_p} \mathbb{M}_p\{b^I\}^\vee) & \xrightarrow{\bigoplus_I \theta} & \bigoplus_I (M \otimes_{\mathbb{M}_p} \mathbb{M}_p\{b^I\})^\vee
\end{array}$$

If $N \cong \bigoplus_I \mathbb{M}_p\{b^I\}$ then $N^\vee \cong \prod_I \mathbb{M}_p\{b^I\}^\vee$ and

$$M^\vee \otimes_{\mathbb{M}_p} N^\vee \cong M^\vee \otimes_{\mathbb{M}_p} \prod_I \mathbb{M}_p\{b^I\}^\vee.$$

This isomorphism is named $\mathcal{A}$ in the diagram. We also have

$$M \otimes_{\mathbb{M}_p} N \cong M \otimes_{\mathbb{M}_p} \bigoplus_I \mathbb{M}_p\{b^I\} \cong \bigoplus_I M \otimes_{\mathbb{M}_p} \mathbb{M}_p\{b^I\}$$

so

$$(M \otimes_{\mathbb{M}_p} N)^\vee \cong \prod_I (M \otimes_{\mathbb{M}_p} \mathbb{M}_p\{b^I\})^\vee$$

which is isomorphism $\mathcal{E}$. That $\mathcal{C}$ is an isomorphism is general theory. Now θ is an isomorphism when N is free on one generator b since we have the following commutative diagram:

$$\begin{array}{ccccc} M^\vee \otimes_{\mathbb{M}_p} (\mathbb{M}_p\{b\})^\vee & \xrightarrow{\cong} & M^\vee \otimes_{\mathbb{M}_p} \mathbb{M}_p\{b\}^\vee & \xrightarrow{\cong} & \Sigma^{-\operatorname{bideg}(b)} M^\vee \\ & \searrow^{\theta} & & & \downarrow \cong \\ & & (M \otimes_{\mathbb{M}_p} \mathbb{M}_p\{b\})^\vee & \xleftarrow{\cong} & (\Sigma^{\operatorname{bideg}(b)} M)^\vee \end{array}$$

Thus the morphism $\bigoplus_I \theta$ must also be an isomorphism. If we can show that morphisms $\mathcal{B}$ and $\mathcal{D}$ are isomorphims, then θ is also an isomorphism by commutativity of the diagram. These are direct consequences of the assumptions in the hypothesis. □

Lemma 3.1.21. *The conditions of the last lemma are met when $M = N = A$ and hence there is an isomorphism*

$$\theta : A^\vee \otimes_{\mathbb{M}_p} A^\vee \longrightarrow (A \otimes_{\mathbb{M}_p} A)^\vee$$

of left modules over $\mathbb{M}_p$.

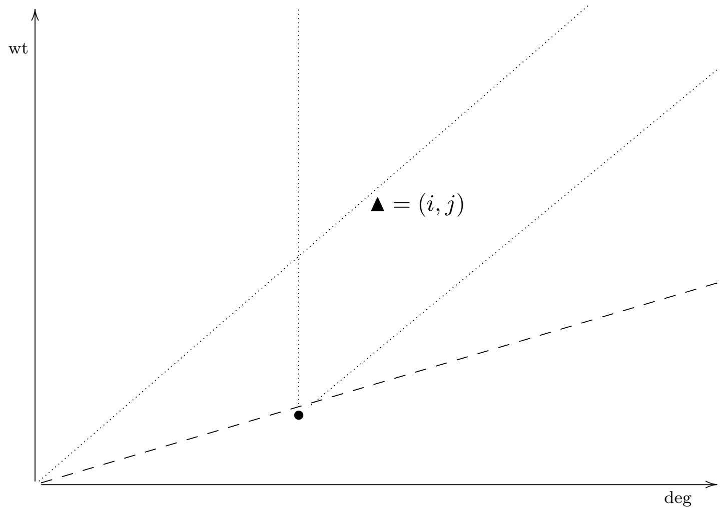
Proof. We check that there are isomorphisms

$$\bigoplus_I \mathbb{M}_p\{\vartheta(I)\} \cong \prod_I \mathbb{M}_p\{\vartheta(I)\}$$

and

$$\bigoplus_I (A \otimes_{\mathbb{M}_p} \mathbb{M}_p\{P^I\})^\vee \cong \prod_I (A \otimes_{\mathbb{M}_p} \mathbb{M}_p\{P^I\})^\vee$$

where $\{P^I\}$ is the basis of admissible monomials and $\{\vartheta(I)\}$ is the $\mathbb{M}_p$-dual basis. We fix a bidegree (i,j) and argue with a picture:



Here, the horizontal axis represents degrees and the vertical axis represents weights. An element in $(\prod_I \mathbb{M}_p\{\vartheta(I)\})^{i,j}$, represented by the triangle, will only have non-zero components such as the ones represented by the bullet which lies on or below the dashed line of slope $1/2(p-1)$, above the degree-axis and of degree $\leq i$. Since there are only finitely many of these, we get the wanted isomorphism

$$\bigoplus_I \mathbb{M}_p\{\vartheta(I)\} \cong \prod_I \mathbb{M}_p\{\vartheta(I)\}.$$

Similarly, we check that $\bigoplus_I (A \otimes_{\mathbb{M}_p} \mathbb{M}_p\{P^I\})^\vee \cong \prod_I (A \otimes_{\mathbb{M}_p} \mathbb{M}_p\{P^I\})^\vee$, or equivalently $\bigoplus_I (A\{P^I\})^\vee \cong \prod_I (A\{P^I\})^\vee$. A is concentrated inside the first quadrant of the diagram above, so the elements of the basis that contribute to any given bidegree (i',j') lie within the rectangle with the origin and (i',j') diagonally opposite to each other, so there can only be finitely many. □

$A^\vee$ can be given more structure: The multiplication of A

$$\mu : A \otimes_{\mathbb{Z}/p} A \longrightarrow A$$

satisfies $\mu(a\mathsf{h} \otimes a') = \mu(a \otimes \mathsf{h}a')$ for all $\mathsf{h} \in \mathbb{M}_p$, so we obtain a map

$$\underline{\mu} : A \otimes_{\mathbb{M}_p} A \longrightarrow A.$$

Taking the $\mathbb{M}_p$-dual map and composing with the inverse of θ (since θ is an isomorphism in this case) we get a morphism of bigraded, commutative $\mathbb{Z}/p$-algebras ψ and a commutative diagram

$$\begin{array}{ccc} A^\vee & \xrightarrow{\psi} & A^\vee \otimes_{\mathbb{M}_p} A^\vee \\ & \underline{\mu}^\vee \searrow \quad \nearrow \theta^{-1} & \\ & (A \otimes_{\mathbb{M}_p} A)^\vee & \end{array}.$$

The fact that ψ is multiplicative is lemma 12.10 in [41]. By dualizing the inclusion $\eta : \mathbb{M}_p \rightarrowtail A$ we get a morphism of bigraded, commutative $\mathbb{Z}/p$-algebras $\varepsilon : A^\vee \longrightarrow \mathbb{M}_p$.

In contrast to the situation for the dual of the classical Steenrod algebra, there is more than one unit: First, let

$$\eta_L : \mathbb{M}_p \longrightarrow A^\vee$$

be the association given by sending $\mathsf{h} \in \mathbb{M}_p$ to the morphism $a \mapsto \mathsf{h}a(1_{\mathbb{M}_p})$ with $a \in A$.

Secondly, there is another construction one may consider. Following the work of John Milnor on the dual of the classical Steenrod algebra in [29], we can construct a morphism of rings the following way: For a smooth scheme X over F, define (see page 44 in [41])

$$\lambda_X : H^{*,*}(X; \mathbb{Z}/p) \longrightarrow H^{*,*}(X; \mathbb{Z}/p) \otimes_l A^\vee$$

by letting $\lambda_X(\mathsf{h}) = \sum_I P^I(\mathsf{h}) \otimes \vartheta(I)$ where the I's are admissible. This sum is finite since $H^{a,b}(X; \mathbb{Z}/p) = 0$ when $a > b + \dim(X)$ (theorem 3.6 in [28]) so only finitely many of the $P^I(\mathsf{h})$ are nonzero. When $X = \mathrm{Spec}(F)$, we get a morphism of $\mathbb{Z}/p$-algebras

$$\eta_R : \mathbb{M}_p \longrightarrow A^\vee$$

taking

$$\mathsf{h} \mapsto \sum_I P^I(\mathsf{h})\vartheta(I).$$

This is the right unit of $A^\vee$. Because of all the structure $A^\vee$ has, the following definitions are natural (see Ravenels book, [32] appendix 1):

Definition 3.1.22. *Let K be a commutative ring. A* (bi)*graded commutative bialgebroid over K is a pair (D, Γ) of* (bi)*graded commutative K-algebras with structure maps such that for any other* (bi)*graded commutative K-algebra E, the sets $\mathrm{Hom}_{K-alg}(D, E)$ and $\mathrm{Hom}_{K-alg}(\Gamma, E)$, morphisms of* (bi)*graded commutative K-algebras, are the objects and morphisms of a category. The structure maps are*

$$\eta_L : D \longrightarrow \Gamma$$

$$\eta_R : D \longrightarrow \Gamma$$

$$\psi : \Gamma \longrightarrow \Gamma \otimes_D \Gamma$$

$$\varepsilon : \Gamma \longrightarrow D$$

where ψ, ε, η_L and η_R are morphisms of (bi)*graded K-algebras. Γ is a D-bimodule with its left D-module structure given by η_L and right D-module structure given by η_R. These maps must satisfy*

- $\varepsilon\eta_L = 1_D$ *and* $\varepsilon\eta_R = 1_D$.
- $(1_\Gamma \otimes \varepsilon)\psi = 1_\Gamma$ *and* $(\varepsilon \otimes 1_\Gamma)\psi = 1_\Gamma$
- $(1_\Gamma \otimes \psi)\psi = (\psi \otimes 1_\Gamma)\psi$

Definition 3.1.23. *A morphism in the category of (bi)graded commutative bialgebroids is a pair of morphisms of* (bi)*graded K-algebras*

$$(\delta, \gamma) : (D, \Gamma) \longrightarrow (D', \Gamma')$$

making the diagrams

$$\begin{array}{ccc} D & \xrightarrow{\eta_?} & \Gamma \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \gamma} \\ D' & \xrightarrow{\eta_?} & \Gamma' \end{array}$$

(*where ? may be replaced with L or R*),

$$\begin{array}{ccc} \Gamma & \xrightarrow{\psi} & \Gamma \otimes_D \Gamma \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma\otimes\gamma} \\ \Gamma' & \xrightarrow{\psi} & \Gamma' \otimes_{D'} \Gamma' \end{array}$$

and

$$\begin{array}{ccc} \Gamma & \xrightarrow{\varepsilon} & D \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} \\ \Gamma' & \xrightarrow{\varepsilon} & D' \end{array}$$

commute.

Remark 3.1.24. The need for these definitions lies in the fact that the existence of a conjugation χ, which is a part of the definition of a Hopf algebroid, is troublesome. It must satisfy $\chi \circ \eta_L = \eta_R$ which is seen to be true here, but we must also have $\chi \circ \eta_R = \eta_L$ which is unclear for all of A^{bist}. Thus we limit ourselves to consider the structures in the definition given above. We should mention that it is possible to define χ for A as discussed in section 4.3 of [20].

Definition 3.1.25. *Let (D, Γ) be a (bi)graded commutative bialgebroid over K. Then a (bi)graded left comodule M over Γ is a (bi)graded left D-module together with a (bi)graded D-linear morphism $\psi : M \longrightarrow \Gamma \otimes_D M$ with ψ counital and coassociative. Elements $m \in M$ such that $\psi(m) = 1 \otimes m$ are said to be left primitive. One can define (bi)graded right comodules in an entirely similar way. A comodule algebra M is a comodule which is also a commutative associative D-algebra such that the structure map ψ is an algebra map.*

Definition 3.1.26. *Let (D, Γ) be a (bi)graded commutative bialgebroid over K and let I be a (bi)graded ideal in Γ. Then I is a (bi)graded bialgebroid ideal if*

1. *$\psi(I) \subset \operatorname{Ker}(\pi \otimes \pi)$ where $\pi \otimes \pi : \Gamma \otimes_D \Gamma \longrightarrow \Gamma/I \otimes_D \Gamma/I$ is the obvious morphism, and*

2. *$\varepsilon(I) = 0$.*

Lemma 3.1.27. *If (D, Γ) be a (bi)graded commutative bialgebroid over K and let I be a (bi)graded bialgebroid ideal. Then $(D, \Gamma/I)$ is a (bi)graded commutative bialgebroid and $(1_D, \pi) : (D, \Gamma) \longrightarrow (D, \Gamma/I)$ is a morphism of commutative bialgebroids.*

Proof. One defines $\eta'_L := \eta_L \circ \pi$ and $\eta'_R := \eta_R \circ \pi$, both morphisms of K-algebras $D \longrightarrow \Gamma/I$. These obviously make the following diagram commute:

$$\begin{array}{ccc} \Gamma & \xrightarrow{\pi} & \Gamma/I \\ {\scriptstyle \eta_?}\nwarrow & & \nearrow{\scriptstyle \eta'_?} \\ & D & \end{array} .$$

The fact that Γ/I inherits the structure of a K-algebra is general theory and the assumption $\varepsilon(I) = 0$ induces a morphism of K-algebras $\varepsilon' : \Gamma/I \longrightarrow D$ with a commutative diagram similar to the one above. Similarly, the composite $\Gamma \xrightarrow{\psi} \Gamma \otimes_D \Gamma \xrightarrow{\pi \otimes \pi} \Gamma/I \otimes_D \Gamma/I$ and the condition $\psi(I) \subset \operatorname{Ker}(\pi \otimes \pi)$ induces a morphism of K-algebras $\psi' : \Gamma/I \longrightarrow \Gamma/I \otimes_D \Gamma/I$ making the right diagrams commute. $\square$

In motivic homotopy theory, one can construct the classifying space, BG, of a linear algebraic group G over F: Given a faithful representation

$$G \to GL(V),$$

we can form the affine space

$$\mathbb{A}(V) := \mathrm{Spec}(Sym(V^\vee))$$

on which G acts. Let $U_i \subset \mathbb{A}(V)^i$ be the open subset where G acts freely. There are closed immersions $U_i \to U_{i+1}$ and we define

$$BG := \mathrm{colim}_i\, U_i/G$$

where the quotients are taken in the category of sheaves. There are other candidates for the notion of classifying space as described in [31], section 4. The construction described above serves as a geometric approximation to the models defined in a more topological way. The different models are however equivalent in $H_\bullet(F)$ in the cases considered here.

To calculate the dual motivic Steenrod algebra we will have to calculate our morphism of rings

$$\lambda_X : H^{*,*}(X;\mathbb{Z}/p) \longrightarrow H^{*,*}(X;\mathbb{Z}/p) \otimes_l A^\vee$$

with $X = B\mu_p$. Let λ be the resulting morphism. Voevodsky has the following result:

Theorem 3.1.28. *Let F be a field of characteristic 0. Then there are classes $u \in H^{1,1}(B\mu_p;\mathbb{Z}/p)$ and $v \in H^{2,1}(B\mu_p;\mathbb{Z}/p)$ and isomorphisms*

$$H^{*,*}(B\mu_2;\mathbb{Z}/2) \cong \mathbb{M}_2^{*,*}[u,v]/(u^2+\rho u+\tau v)$$

and

$$H^{*,*}(B\mu_p;\mathbb{Z}/p) \cong \mathbb{M}_p^{*,*}[u,v]/(u^2)$$

when p is odd.

Proof. These are special cases of theorem 6.10 in [41]. □

In [41] section 12, Voevodsky follows Milnors computation in [29] and shows that there are unique elements

$$\tau_i \in (A^\vee)_{2p^i-1,p^i-1}$$

and

$$\xi_j \in (A^\vee)_{2(p^j-1),p^j-1}$$

such that

$$\lambda(u) = u \otimes 1 + \sum_{i=0}^{\infty} v^{p^i} \otimes \tau_i$$

and

$$\lambda(v) = \sum_{j=0}^{\infty} v^{p^j} \otimes \xi_j.$$

Now, define

$$\Gamma_p := \mathbb{M}_p \left[\tau_i, \xi_j \mid i \geq 0, j \geq 1\right] / (\tau_i^2)$$

when p is odd and

$$\Gamma_2 := \mathbb{M}_2 \left[\tau_i, \xi_j \mid i \geq 0, j \geq 1\right] / (\tau_i^2 + \tau\xi_{i+1} + \rho\tau_{i+1} + \rho\tau_0\xi_{i+1}).$$

In both cases there is a morphism of bigraded, commutative algebras

$$\mathbb{M}_p \rightarrowtail \Gamma_p$$

over $\mathbb{Z}/p$ given by the inclusion of $\mathbb{M}_p$ into $\mathbb{M}_p \left[\tau_i, \xi_j \mid i \geq 0, j \geq 1\right]$ followed by reduction modulo the relevant ideal. It will be denoted η_L^p.

Consider the pair $(\mathbb{M}_p, \Gamma_p)$ of bigraded commutative algebras over $\mathbb{Z}/p$. We define a morphism

$$\psi_{\Gamma_p} : \Gamma_p \longrightarrow \Gamma_p \otimes_{\mathbb{M}_p} \Gamma_p$$

of $\mathbb{Z}/p$-algebras taking

$$\psi_{\Gamma_p}(\tau_k) = \tau_k \otimes 1 + \sum_{i=0}^{k} \xi_{k-i}^{p^i} \otimes \tau_i$$

$$\psi_{\Gamma_p}(\xi_l) = \sum_{j=0}^{l} \xi_{l-j}^{p^j} \otimes \xi_j$$

with the convention that $\xi_0 = 1$ in these formulae.

$$\varepsilon_{\Gamma_p} : \Gamma_p \longrightarrow \mathbb{M}_p$$

is defined by sending all τ_i's and ξ_j's with the exception of ξ_0 to zero. Together with the usual algebra structures over $\mathbb{Z}/p$ these maps are very close to giving us a bialgebroid.

Theorem 3.1.29. *$(\mathbb{M}_p, \Gamma_p)$ can be given the structure of a bigraded, commutative bialgebroid over $\mathbb{Z}/p$. As such, the pair $(\mathbb{M}_p, A^\vee)$ is isomorphic to $(\mathbb{M}_p, \Gamma_p)$ for any rational prime p.*

Proof. Following Milnor's paper [29], Voevodsky ([41], theorem 12.6) proves that there is an isomorphism of bigraded commutative $\mathbb{Z}/p$-algebras

$$\iota : \Gamma_p \longrightarrow A^\vee.$$

By construction, η_L, ψ and ε are morphisms of $\mathbb{Z}/p$-algebras and as such, they must take the same values as the corresponding morphisms η_L^p, ψ_{Γ_p} and ε_{Γ_p}. For $(\mathbb{M}_p, \Gamma_p)$ to be a bialgebroid of the wanted type, we define η_R^p to be the composite $\eta_R \circ \iota$. This morphism must take the same values as η_R so we are done. □

Remark 3.1.30. From this point on we identify Γ_p and $A^\vee$ under this isomorphism.

3.2 Finitely generated subalgebras of A

Our motivation for this section will be to extend the work of Adams, Gunawardena and Miller on the classical Steenrod algebra using a construction of Singer (see [1] for all this). Starting with a left A-module M, there is an algebraic object $R_+(M)$ we will study. Its construction needs some preliminary work before being defined.

Definition 3.2.1. *For $n \geq 0$, define the ideals $I(n) \subset A^\vee$ where*

$$I(n) := (\tau_{n+1}, \tau_{n+2}, \dots, \xi_1^{p^n}, \xi_2^{p^{n-1}}, \dots, \xi_n^p, \xi_{n+1}, \xi_{n+2}, \dots).$$

We define

$$A^\vee(n) := A^\vee / I(n)$$

When $n = -1$, we let $A^\vee(-1) := \mathbb{M}_p$.

The ideals $I(n)$ and algebras $A^\vee(n)$ are direct analogs of the ones found in chapter 2 and 6 of [16]. $A^\vee(n)$ is a free left module over $\mathbb{M}_p$ with a basis consisting of monomials

$$\xi^I \tau^J = \xi_1^{i_1} \cdots \xi_n^{i_n} \tau_0^{j_0} \cdots \tau_n^{j_n}$$

with $I := (i_1, \dots, i_n)$, $J := (j_0, \dots, j_n)$, $0 \leq i_k < p^{n+1-k}$ and $0 \leq j_l \leq 1$. We wish to study quotients of this sort and descend the bialgebroid structure of $A^\vee$ along the projection $\pi : A^\vee \longrightarrow A^\vee(n)$:

Lemma 3.2.2. *For each $n \geq 0$, $I(n)$ is a bigraded bialgebroid ideal.*

Proof. First, $\varepsilon(I(n)) = 0$ by definition. We want to see that $(\pi \otimes \pi)\psi(I(n)) = 0$. Since ψ is a $\mathbb{Z}/p$-algebra homomorphism, it suffices to check this on the generators of the ideal. We have

$$\psi(\tau_k) = \tau_k \otimes 1 + \sum_{i=0}^{k} \xi_{k-i}^{p^i} \otimes \tau_i.$$

The fact that $k \geq n+1$ implies that all the terms of the sum are in the image of $I(n) \otimes_{\mathbb{M}_p} A^\vee \subset A^\vee \otimes_{\mathbb{M}_p} A^\vee$ except the one where $k - i = 0$, but this is an element in the image of $A^\vee \otimes_{\mathbb{M}_p} I(n) \subset A^\vee \otimes_{\mathbb{M}_p} A^\vee$ (this is an injection since $\mathrm{Tor}_{1,(*,*)}^{\mathbb{M}_p}(A^\vee, A^\vee(n)) \cong 0$ in the relevant long exact sequence). Thus all these elements are sent to zero by $\pi \otimes \pi$. To conclude we perform the same check for the other generators of the ideal. We have:

$$\psi(\xi_l^{p^s}) = \sum_{i=0}^{l} \xi_{l-j}^{p^{j+s}} \otimes \xi_j^{p^s}$$

Again, if $l + s \geq n+1$, the terms are all in the image of $I(n) \otimes_{\mathbb{M}_p} A^\vee \subset A^\vee \otimes_{\mathbb{M}_p} A^\vee$ except when $l - j = 0$, and for this one index, we have an element of $A^\vee \otimes_{\mathbb{M}_p} I(n) \subset A^\vee \otimes_{\mathbb{M}_p} A^\vee$. ☐

Remark 3.2.3. For $n \geq 0$, the last result and lemma 3.1.27 imply that we have morphisms of $\mathbb{Z}/p$-algebras

$$\phi^n : A^\vee(n) \otimes_{\mathbb{Z}/p} A^\vee(n) \longrightarrow A^\vee(n)$$

and

$$\psi^n : A^\vee(n) \longrightarrow A^\vee(n) \otimes_{\mathbb{M}_p} A^\vee(n),$$

and for each n, these form the structure maps of a bigraded commutative bialgebroid $(\mathbb{M}_p, A^\vee(n))$. The pair

$$(\mathbb{M}_p, \pi) : (\mathbb{M}_p, A^\vee) \longrightarrow (\mathbb{M}_p, A^\vee(n))$$

forms a morphism in this category.

There are other ideals in $A^\vee$ relevant to our construction:

Definition 3.2.4. *For $n \geq 0$, define ideals $J(n) \subset A^\vee$ where*

$$J(n) := (\tau_{n+1}, \tau_{n+2}, \dots, \xi_2^{p^{n-1}}, \dots, \xi_n^p, \xi_{n+1}, \xi_{n+2}, \dots).$$

We let

$$C^\vee(n) := A^\vee / J(n)$$

in addition to

$$B^\vee(n) := C^\vee(n) \left[\xi_1^{-1}\right].$$

The ideals $J(n)$ and associated modules $B^\vee(n)$ are defined in the exact same manner as the ones found in [1] section 2.

Observe that for the $\xi_r^{p^s}$ in these ideals, we mean to take the ones where $n+1 \leq r+s$ with $2 \leq r$, so for $n=0$, ξ_1 is not in $J(0)$. As basic examples to keep in mind, we get $C^\vee(0) \cong \bigoplus \mathbb{M}_p\{\xi_1^k, \tau_0\xi_1^k\}$ for $k \geq 0$, and $B^\vee(0) \cong \bigoplus \mathbb{M}_p\{\xi_1^k, \tau_0\xi_1^k\}$ for $k \in \mathbb{Z}$.

$C^\vee(n)$ and $B^\vee(n)$ are free left $\mathbb{M}_p$-modules and have monomial bases consisting of the elements

$$\xi^I \tau^J = \xi_1^{i_1} \cdots \xi_n^{i_n} \tau_0^{j_0} \cdots \tau_n^{j_n}$$

with $I := (i_1, \dots, i_n)$, $J := (j_0, \dots, j_n)$, $0 \leq i_1$, $0 \leq i_l < p^{n+1-l}$ for $1 < l \leq n$ and $0 \leq j_l \leq 1$, and

$$\xi^I \tau^J = \xi_1^{i_1} \cdots \xi_n^{i_n} \tau_0^{j_0} \cdots \tau_n^{j_n}$$

with $I := (i_1, \dots, i_n)$, $J := (j_0, \dots, j_n)$, $i_1 \in \mathbb{Z}$, $0 \leq i_l < p^{n+1-l}$ for $1 < l \leq n$ and $0 \leq j_l \leq 1$ respectively.

We will see that $C^\vee(n)$ and $B^\vee(n)$ both carry more structure:

Lemma 3.2.5. *$\psi(J(n))$ lies in the sum of the images of $I(n) \otimes_{\mathbb{M}_p} A^\vee$ and $A^\vee \otimes_{\mathbb{M}_p} J(n)$ in $A^\vee \otimes_{\mathbb{M}_p} A^\vee$.*

Proof. We have

$$\psi(\tau_k) = \tau_k \otimes 1 + \sum_i \xi_{k-i}^{p^i} \otimes \tau_i.$$

If $k \geq n+1$, then $\tau_k \otimes 1 \in I(n) \otimes_{\mathbb{M}_p} A^\vee$ and $\xi_{k-i}^{p^i} \in I(n)$ except possibly when $i = k$. But for this case one has $1 \otimes \tau_k \in A^\vee \otimes_{\mathbb{M}_p} J(n)$ so our result holds for these generators. Similarly we have

$$\psi(\xi_l^{p^s}) = \sum_j \xi_{l-j}^{p^{j+s}} \otimes \xi_j^{p^s}$$

Here, $l + s \geq n+1$ so $\xi_{l-j}^{p^{j+s}} \in I(n)$ for $j \neq l$. For this single index, we get the element $1 \otimes \xi_l^{p^s}$ which lies in $A^\vee \otimes_{\mathbb{M}_p} J(n)$. □

Lemma 3.2.6. *$\psi(J(n))$ lies in the sum of the images of $J(n) \otimes_{\mathbb{M}_p} A^\vee$ and $A^\vee \otimes_{\mathbb{M}_p} I(n-1)$ in $A^\vee \otimes_{\mathbb{M}_p} A^\vee$.*

Proof. We have

$$\psi(\tau_k) = \tau_k \otimes 1 + \sum_i \xi_{k-i}^{p^i} \otimes \tau_i$$

$k \geq n+1$ implies that all terms are in the image of $J(n) \otimes_{\mathbb{M}_p} A^\vee \subset A^\vee \otimes_{\mathbb{M}_p} A^\vee$ except when $k - i = 1$ or $k - i = 0$ in which case these two terms lie in the image of $A^\vee \otimes_{\mathbb{M}_p} I(n-1) \subset A^\vee \otimes_{\mathbb{M}_p} A^\vee$. Similarly we have

$$\psi(\xi_l^{p^s}) = \sum_j \xi_{l-j}^{p^{j+s}} \otimes \xi_j^{p^s},$$

and here all terms lie in the image of $J(n) \otimes_{\mathbb{M}_p} A^\vee$ except when $l - j < 2$. For the two last terms we have elements in the image of
$A^\vee \otimes_{\mathbb{M}_p} I(n-1)$. □

The last two results imply that from the composites

$$A^\vee \xrightarrow{\psi} A^\vee \otimes_{\mathbb{M}_p} A^\vee \xrightarrow{\pi \otimes \pi} A^\vee(n) \otimes_{\mathbb{M}_p} C^\vee(n)$$

and

$$A^\vee \xrightarrow{\psi} A^\vee \otimes_{\mathbb{M}_p} A^\vee \xrightarrow{\pi \otimes \pi} C^\vee(n) \otimes_{\mathbb{M}_p} A^\vee(n-1),$$

we get induced $\mathbb{Z}/p$-algebra homomorphisms

$$\psi_l^n : C^\vee(n) \longrightarrow A^\vee(n) \otimes_{\mathbb{M}_p} C^\vee(n)$$

and

$$\psi_r^n : C^\vee(n) \longrightarrow C^\vee(n) \otimes_{\mathbb{M}_p} A^\vee(n-1),$$

making $C^\vee(n)$ a left comodule algebra over $A^\vee(n)$ and a right comodule algebra over $A^\vee(n-1)$.

We have diagrams

$$\begin{array}{ccc}
C^\vee(n) & \xrightarrow{\xi_1^{p^n}*} & C^\vee(n) \\
\downarrow{\scriptstyle \psi_l^n} & & \downarrow{\scriptstyle \psi_l^n} \\
A^\vee(n) \otimes_{\mathbb{M}_p} C^\vee(n) & \xrightarrow[A^\vee(n)\otimes(\xi_1^{p^n}*)]{} & A^\vee(n) \otimes_{\mathbb{M}_p} C^\vee(n)
\end{array}$$

and

$$\begin{array}{ccc}
C^\vee(n) & \xrightarrow{\xi_1^{p^n}*} & C^\vee(n) \\
\downarrow{\scriptstyle \psi_r^n} & & \downarrow{\scriptstyle \psi_r^n} \\
C^\vee(n) \otimes_{\mathbb{M}_p} A^\vee(n-1) & \xrightarrow[(\xi_1^{p^n}*)\otimes A^\vee(n-1)]{} & C^\vee(n) \otimes_{\mathbb{M}_p} A^\vee(n-1)
\end{array}$$

where $\xi_1^{p^n}*$ is the map that takes $c \mapsto \xi_1^{p^n} c$ for each element $c \in C^\vee(n)$.

Lemma 3.2.7. *These diagrams are commutative.*

Proof. Let $x \in A^\vee$. For the composite

$$A^\vee \xrightarrow{\psi} A^\vee \otimes_{\mathbb{M}_p} A^\vee \xrightarrow{\pi\otimes\pi} A^\vee(n) \otimes_{\mathbb{M}_p} C^\vee(n),$$

we have

$$(\pi\otimes\pi)(\psi(\xi_1^{p^n}x)) = (\pi\otimes\pi)(\psi(\xi_1^{p^n})\psi(x)) = (1\otimes\xi_1^{p^n})(\pi\otimes\pi)(\psi(x))$$

as can be explained by inspecting the expansion

$$\psi(\xi_1^{p^n}) = \xi_1^{p^n}\otimes 1 + 1\otimes\xi_1^{p^n}.$$

Here the first term lies in $I(n)\otimes_{\mathbb{M}_p} A^\vee$ and is projected to zero by $\pi\otimes\pi$ so the diagram for ψ_l^n must commute. Similarly, from the inspection of the composite

$$A^\vee \xrightarrow{\psi} A^\vee \otimes_{\mathbb{M}_p} A^\vee \xrightarrow{\pi\otimes\pi} C^\vee(n) \otimes_{\mathbb{M}_p} A^\vee(n-1)$$

we conclude that

$$(\pi\otimes\pi)(\psi(\xi_1^{p^n}x)) = (\pi\otimes\pi)(\psi(\xi_1^{p^n})\psi(x)) = (\xi_1^{p^n}\otimes 1)(\pi\otimes\pi)(\psi(x))$$

since the term $1\otimes\xi_1^{p^n} \in A^\vee \otimes_{\mathbb{M}_p} I(n-1)$. Thus, the diagram for ψ_r^n must commute. ∎

In other words, multiplication by $\xi_1^{p^n}$ is a morphism of left $A^\vee(n)$-comodules *and* a morphism of right $A^\vee(n-1)$-comodules. We recall a result from commutative algebra:

Proposition 3.2.8. *Given a commutative ring R and a non-zerodivisor $r \in R$, let $S := \{1, r, r^2, \ldots\}$. Then*

$$S^{-1}R \cong \operatorname{colim}(R \xrightarrow{*r} R \xrightarrow{*r} R \longrightarrow \cdots).$$

Proof. We use the proof of theorem 3.2.2 in [44]. The set S can be partially ordered by divisibility and forms a filtered category with morphisms $\operatorname{Hom}_S(s_1, s_2) = \{s \in S \mid s_1 s = s_2\}$. If we let F be the functor from S to the category of R-modules defined by $F(s) = R$ and $F(s_1) \xrightarrow{*s} F(s_2)$, then we can show that $\operatorname{colim}_S F \cong S^{-1}R$: Define maps R-linear homomorphisms $F(s) \longrightarrow S^{-1}R$ by sending $1 \longmapsto \frac{1}{s}$. These form a compatible system of R-linear maps so there is an induced map $\operatorname{colim}_S F \longrightarrow S^{-1}R$. It is surjective since $\frac{a}{s}$ is in the image of $F(s) \longrightarrow S^{-1}R$. Since we assumed that r was a non-zerodivisor, the maps $F(s) \longrightarrow S^{-1}R$ are all injective. Filtered colimits are exact and so the map $\operatorname{colim}_S F \longrightarrow S^{-1}R$ is also injective. □

Remark 3.2.9. We quickly remark that since $S^{-1}M \cong S^{-1}R \otimes_R M$ and

$$(\operatorname{colim}_i M_i) \otimes_R N \cong \operatorname{colim}_i(M_i \otimes_R N)$$

for a filtered diagram $\{M_i\}$ of R-modules, the argument above implies that

$$S^{-1}M \cong \operatorname{colim}(M \xrightarrow{*r} M \xrightarrow{*r} M \longrightarrow \cdots).$$

The remark implies $B^\vee(n)$ can be regarded as a colimit of $C^\vee(n)$ under multiplication with $\xi_1^{p^n}$: There are diagrams

$$\begin{array}{ccc}
C^\vee(n) & \xrightarrow{\psi_l^n} & A^\vee(n) \otimes_{\mathbb{M}_p} C^\vee(n) \\
\downarrow{\scriptstyle \xi_1^{p^n}} & & \downarrow{\scriptstyle A^\vee(n) \otimes (\xi_1^{p^n} *)} \\
C^\vee(n) & \xrightarrow{\psi_l^n} & A^\vee(n) \otimes_{\mathbb{M}_p} C^\vee(n) \\
\downarrow{\scriptstyle \xi_1^{p^n}} & & \downarrow{\scriptstyle A^\vee(n) \otimes (\xi_1^{p^n} *)} \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
B^\vee(n) & \xrightarrow{\psi_l^n} & A^\vee(n) \otimes_{\mathbb{M}_p} B^\vee(n)
\end{array}$$

and

$$\begin{array}{ccc}
C^\vee(n) & \xrightarrow{\psi_r^n} & C^\vee(n) \otimes_{\mathbb{M}_p} A^\vee(n-1) \\
\downarrow{\scriptstyle \xi_1^{p^n}} & & \downarrow{\scriptstyle (\xi_1^{p^n} *) \otimes A^\vee(n-1)} \\
C^\vee(n) & \xrightarrow{\psi_r^n} & C^\vee(n) \otimes_{\mathbb{M}_p} A^\vee(n-1) \\
\downarrow{\scriptstyle \xi_1^{p^n}} & & \downarrow{\scriptstyle (\xi_1^{p^n} *) \otimes A^\vee(n-1)} \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
B^\vee(n) & \xrightarrow{\psi_r^n} & B^\vee(n) \otimes_{\mathbb{M}_p} A^\vee(n-1)
\end{array}$$

coming from the universal property of $B^\vee(n)$. Hence, $B^\vee(n)$ becomes a left $A^\vee(n)$-comodule and a right $A^\vee(n-1)$-comodule through the coactions given at each step in the colimit.

There are inclusions of ideals $(0) \subset J(n) \subset J(n-1)$, in turn inducing horizontal surjective ring maps

$$\begin{array}{ccccc}
A^\vee & \twoheadrightarrow & C^\vee(n) & \twoheadrightarrow & C^\vee(n-1) \\
& & \downarrow & & \downarrow \\
& & B^\vee(n) & \twoheadrightarrow & B^\vee(n-1)
\end{array}$$

At each stage, there are short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \dfrac{J(n-1)}{J(n)} & \longrightarrow & \dfrac{A^\vee}{J(n)} & \longrightarrow & \dfrac{A^\vee}{J(n-1)} & \longrightarrow & 0 \\
& & & & \| & & \| & & \\
& & & & C^\vee(n) & & C^\vee(n-1) & &
\end{array}$$

and

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\dfrac{J(n-1)}{J(n)})[\xi^{-1}] & \longrightarrow & (\dfrac{A^\vee}{J(n)})[\xi^{-1}] & \longrightarrow & (\dfrac{A^\vee}{J(n-1)})[\xi^{-1}] & \longrightarrow & 0 \\
& & & & \| & & \| & & \\
& & & & B^\vee(n) & & B^\vee(n-1) & &
\end{array}$$

The projection

$$C^\vee(n) \to C^\vee(n-1)$$

is a map of $A^\vee(n-1)$-$A^\vee(n-2)$-bicomodules where we consider $C^\vee(n)$ as a bicomodule via the maps

$$C^\vee(n) \longrightarrow A^\vee(n) \otimes_{\mathbb{M}_p} C^\vee(n) \xrightarrow{\pi \otimes C^\vee(n)} A^\vee(n-1) \otimes_{\mathbb{M}_p} C^\vee(n)$$

and

$$C^\vee(n) \longrightarrow C^\vee(n) \otimes_{\mathbb{M}_p} A^\vee(n-1) \xrightarrow{C^\vee(n)\otimes\pi} C^\vee(n) \otimes_{\mathbb{M}_p} A^\vee(n-2).$$

The structure maps and the maps of bicomodules induce the same structure and maps for $B^\vee(n)$ via the directed system defining it.

Definition 3.2.10. *By application of* $\mathrm{Hom}_{\mathbb{M}_p}(-, \mathbb{M}_p)$ *, we let*

$$A(n) := \mathrm{Hom}_{\mathbb{M}_p}(A^\vee(n), \mathbb{M}_p)$$

and likewise for $B(n)$ *and* $C(n)$*.*

This dualization gives $A(n)$ the structure of a left module over $\mathbb{M}_p$. In addition, $A(n)$ can be given the structure of a $\mathbb{Z}/p$-algebra by the composite

$$A(n) \otimes_{\mathbb{M}_p} A(n) \xrightarrow{\theta} (A^\vee(n) \otimes_{\mathbb{M}_p} A^\vee(n))^\vee \xrightarrow{(\psi^n)^\vee} A(n)$$

where we use the map θ from lemma 3.1.19. It is an isomorphism by the same reasoning that we used in lemma 3.1.21. We end up with a chain of $\mathbb{Z}/p$-algebras

$$\mathbb{M}_p = A(-1) \rightarrowtail A(0) \rightarrowtail \cdots \rightarrowtail A(n-1) \rightarrowtail A(n) \rightarrowtail \cdots \rightarrowtail A$$

which are finite dimensional over $\mathbb{M}_p$. Both $B(n)$ and $C(n)$ inherit structures of $A(n)$-$A(n-1)$ bimodules and we produce a diagram dual to the one above (with injective horizontal maps):

$$\begin{array}{ccccc} A & \leftarrowtail & C(n) & \leftarrowtail & C(n-1) \\ & & \uparrow & & \uparrow \\ & & B(n) & \leftarrowtail & B(n-1) \end{array}$$

We recall that $B^\vee(n)$ has a given monomial basis and let P^k and βP^k be $\mathbb{M}_p$-dual to (the classes) ξ_1^k and $\tau_0\xi_1^k$ in $B^\vee(n)$ for $k \in \mathbb{Z}$ and $n \geq 0$. The projection

$$A^\vee \longrightarrow C^\vee(n)$$

sends $\xi_1^k \longmapsto \xi_1^k + J(n)$ and $\tau_0\xi_1^k \longmapsto \tau_0\xi_1^k + J(n)$ ($k \in \mathbb{N} \cup 0$). At each stage, we saw that there was a projection

$$C^\vee(n) \longrightarrow C^\vee(n-1),$$

and this map sends $\xi_1^k + J(n) \longmapsto \xi_1^k + J(n-1)$ and $\tau_0\xi_1^k + J(n) \longmapsto \tau_0\xi_1^k + J(n-1)$. After localization, the morphism

$$B^\vee(n) \longrightarrow B^\vee(n-1)$$

takes $\xi_1^k + J(n) \longmapsto \xi_1^k + J(n-1)$ and $\tau_0\xi_1^k + J(n) \longmapsto \tau_0\xi_1^k + J(n-1)$ for all $k \in \mathbb{Z}$.

Lemma 3.2.11. *At each step, the map*

$$B(n-1) \longrightarrow B(n)$$

sends $P^k, \beta P^k \in B(n-1)$ *to* $P^k, \beta P^k \in B(n)$.

Proof. This is checked easily given the monomial generators: Since the class of $\xi^I \tau^J \in B^\vee(n)$ is sent to the class of $\xi^I \tau^J \in B^\vee(n-1)$ and $P^k(\xi^I \tau^J)$ (for $P^k \in B(n-1)$) is zero unless $\xi^I \tau^J = \xi_1^k$, the composition of the map

$$B^\vee(n) \longrightarrow B^\vee(n-1)$$

and $P^k \in B(n-1)$ takes the same values as $P^k \in B(n)$ on the monomial generators of $B^\vee(n)$. Hence they are equal. The same argument shows that the statement concerning βP^k holds true. □

Thus, the composition

$$B(0) \longrightarrow B(n) \longrightarrow A$$

obtained from dualizing

$$A^\vee \longrightarrow B^\vee(n) \longrightarrow B^\vee(0)$$

sends P^k and βP^k to P^k and βP^k when $k \geq 0$. If $k < 0$, the corresponding generators map to 0 since the morphism $B(n) \longrightarrow A$ factors through $C(n)$.

Now, one must be careful when considering generators from different sides. $B^\vee(0)$ is free as a left module over $\mathbb{M}_p$ and is generated by the set $\{\xi_1^k, \tau_0 \xi_1^k \mid k \in \mathbb{Z}\}$. In fact, more is true for these generators:

Lemma 3.2.12. *As a right module over* $\mathbb{M}_p$, $B^\vee(0)$ *is free on the generators*

$$\{\xi_1^k, \tau_0 \xi_1^k \mid k \in \mathbb{Z}\}.$$

Proof. We construct a map of right $\mathbb{M}_p$-modules

$$f : \{\xi_1^k \tau_0^\epsilon \mid k \in \mathbb{Z}, 0 \leq \epsilon \leq 1\}\mathbb{M}_p \longrightarrow B^\vee(0)$$

and use theorem 2.6 of [6]. For this we impose filtrations on both sides and construct a map of filtered modules. Secondly, we must verify that both filtrations are exhaustive and Hausdorff and that the filtration on

$$\{\xi_1^k, \tau_0 \xi_1^k \mid k \in \mathbb{Z}\}\mathbb{M}_p$$

renders it complete. One must also check that the induced map of filtration quotients is an isomorphism.

Filter $B^\vee(0) \cong \mathbb{M}_p\{\xi_1^k\tau_0^\epsilon \mid k \in \mathbb{Z}, 0 \leq \epsilon \leq 1\}$ and $\{\xi_1^k\tau_0^\epsilon \mid k \in \mathbb{Z}, 0 \leq \epsilon \leq 1\}\mathbb{M}_p$ by letting

$$F^d := \mathbb{M}_p\{\xi_1^k\tau_0^\epsilon \mid k + \epsilon \geq d\} \subset B^\vee(0)$$

and

$$\overline{F}^d := \{\xi_1^k\tau_0^\epsilon \mid k + \epsilon \geq d\}\mathbb{M}_p \subset \{\xi_1^k\tau_0^\epsilon \mid k \in \mathbb{Z}, 0 \leq \epsilon \leq 1\}\mathbb{M}_p,$$

both decreasing filtrations. They are evidently exhaustive. Since both modules are sums, no element can be of infinite filtration except the trivial one. Hence they are both Hausdorff. Define a map of right $\mathbb{M}_p$-modules

$$f : \{\xi_1^k\tau_0^\epsilon \mid k \in \mathbb{Z}, 0 \leq \epsilon \leq 1\}\mathbb{M}_p \longrightarrow B^\vee(0)$$

by sending

$$\xi_1^k\tau_0^\epsilon m \longmapsto \eta_R(m)\xi_1^k\tau_0^\epsilon,$$

where $m \in \mathbb{M}_p$ and $\eta_R(m)\xi_1^k\tau_0^\epsilon$ means the class obtained after multiplication in $A^\vee$ and application of the composition $A^\vee \longrightarrow C^\vee(0) \longrightarrow B^\vee(0)$. Since

$$\eta_R(m) = \lambda(m) = \sum_I P^I(m)\vartheta(I)$$

for admissble I, we see that $f(\xi_1^k\tau_0^\epsilon m) = m\xi_1^k\tau_0^\epsilon + \sum m'$ with elements in the sum having higher filtration than $\xi_1^k\tau_0^\epsilon m$. This implies that the map f induces on filtration quotients is an isomorphism. Now, we want to evaluate the functor $R\lim$ on the filtration on $\{\xi_1^k\tau_0^\epsilon \mid k \in \mathbb{Z}, 0 \leq \epsilon \leq 1\}\mathbb{M}_p$ to check for completeness. This is done one bidegree at a time. Fixing one such, $R\lim$ vanishes when d is large enough so by [6], proposition 2.2, the filtration is complete and we are done. □

Lemma 3.2.13. *$B(n)$ is a free left $A(n)$-module with generators*

$$\{P^k \mid k \in \mathbb{Z}, p^n \mid k\}$$

Proof. For a proof, we begin by considering the following diagram:

$$\begin{array}{ccc} A^\vee(n) \otimes_{\mathbb{M}_p} B^\vee(n) & \xleftarrow{\psi_l^n} & B^\vee(n) \\ \Big\downarrow{\scriptstyle A^\vee(n)\otimes\rho} & \swarrow & \\ A^\vee(n) \otimes_{\mathbb{M}_p} \bigoplus_{p^n|k} \mathbb{M}_p\{\xi_1^k\} & & \end{array}$$

Here, ρ is defined to be $B^\vee(n) \longrightarrow B^\vee(0) \longrightarrow \bigoplus_{p^n|k} \mathbb{M}_p\{\xi_1^k\}$, where the first map comes from our projective system of modules and the second one is the projection

$$B^\vee(0) \cong \bigoplus_{k\in\mathbb{Z}} \mathbb{M}_p\{\xi_1^k, \tau_0\xi_1^k\} \longrightarrow \bigoplus_{p^n|k} \mathbb{M}_p\{\xi_1^k\}.$$

The composition of ψ_l^n and a projection tensored with an identity, both of which are left $A^\vee(n)$-comodule maps, is also a map of left $A^\vee(n)$-comodules. We will show that the composition in the diagram is an isomorphism of $\mathbb{M}_p$-modules, and so also a left $A^\vee(n)$-comodule isomorphism.

Define an isomorphism of $\mathbb{M}_p$-modules

$$\alpha : A^\vee(n) \otimes_{\mathbb{M}_p} \bigoplus_{p^n|k} \mathbb{M}_p\{\xi_1^k\} \longrightarrow B^\vee(n)$$

by sending

$$\xi^{(i_1,\ldots,i_n)}\tau^{(j_0,\ldots,j_n)} \otimes_{\mathbb{M}_p} \xi_1^k \longmapsto \xi^{(i_1+k,\ldots,i_n)}\tau^{(j_0,\ldots,j_n)}.$$

An inverse may be given by sending

$$\xi^{(i_1',\ldots,i_n')}\tau^{(j_0',\ldots,j_n')} \longmapsto \xi^{(b,i_2',\ldots,i_n')}\tau^{(j_0',\ldots,j_n')} \otimes \xi_1^{ap^n}$$

where $i_1' = ap^n + b$ with $0 \le b < p^n$.

We expand our diagram to

$$\begin{array}{ccc}
A^\vee(n) \otimes_{\mathbb{M}_p} B^\vee(n) & \xleftarrow{\ \psi_l^n\ } & B^\vee(n) \\
\downarrow{\scriptstyle A^\vee(n)\otimes\rho} & \swarrow & \uparrow{\scriptstyle \alpha} \\
A^\vee(n) \otimes_{\mathbb{M}_p} \bigoplus_{p^n|k} \mathbb{M}_p\{\xi_1^k\} & \longleftarrow & A^\vee(n) \otimes_{\mathbb{M}_p} \bigoplus_{p^n|k} \mathbb{M}_p\{\xi_1^k\}
\end{array}$$

where the lower horizontal map is chosen so that the lower triangle commutes. One may filter $A^\vee(n) \otimes_{\mathbb{M}_p} \bigoplus_{p^n|k} \mathbb{M}_p\{\xi_1^k\}$: Define

$$F^d := \mathbb{M}_p\{\mathsf{m} = \mathsf{m}' \otimes \mathsf{m}'' \in A^\vee(n) \otimes_{\mathbb{M}_p} \bigoplus_{p^n|k} \mathbb{M}_p\{\xi_1^k\} \mid deg(\mathsf{m}'') \ge d\}$$

where $\mathsf{m}' \in A^\vee(n)$ and $\mathsf{m}'' \in \bigoplus_{p^n|k} \mathbb{M}_p\{\xi_1^k\}$ are ranging over the elements of their respective monomial bases as left modules over $\mathbb{M}_p$. It is an exhaustive, decreasing filtration which is obviously Hausdorff. The map ψ_l^n takes

$$\mathsf{m} = \xi^I\tau^J = \xi^{(i_1,i_2,\ldots)}\tau^{(j_0,j_1,\ldots)} \in B^\vee(n)$$

to the product

$$\psi_l^n(\xi^{(b,i_2,\ldots)})\psi_l^n(\tau^{(j_0,j_1,\ldots)})\psi_l^n(\xi_1^{ap^n})$$

with $i_1 = ap^n + b$ and $0 \le b < p^n$. Multiplying these together and applying $(A^\vee(n) \otimes \rho)$, we get a product of factors of the form

$$(\xi_1 \otimes 1 + 1 \otimes \xi_1)^b(\xi_2 \otimes 1 + \xi_1^p \otimes \xi_1)^{i_2} * \ldots * (\xi_n \otimes 1 + \xi_{n-1}^p \otimes \xi_1)^{i_n},$$

$$\tau^{(j_0,j_1,\ldots)} \otimes 1$$

and either

$$(\xi_1 \otimes 1 + 1 \otimes \xi_1)^{ap^n} = \sum \binom{a}{c} \xi_1^{cp^n} \otimes \xi_1^{(a-c)p^n}$$

when a is non-negative, or $1 \otimes \xi_1^{ap^n}$ when a is negative. This last property follows from the definition of the left $A^\vee(n)$-comodule structure on $B^\vee(n)$. It was formed by taking the colimit of the left $A^\vee(n)$-comodule structure on $C^\vee(n)$ under the multiplication by $\xi_1^{p^n}$. The element $\xi_1^{ap^n} \in B^\vee(n)$ is the image of $1 \in C^\vee(n)$ at the $-a - th$ level of the defining tower and the commutativity of the diagrams of towers defining $B^\vee(n)$ and
$A^\vee(n) \otimes_{\mathbb{M}_p} B^\vee(n)$ implies that

$$\psi_l^n : B^\vee(n) \longrightarrow A^\vee(n) \otimes_{\mathbb{M}_p} B^\vee(n)$$

satisfies $\psi_l^n(\xi_1^{ap^n}) = 1 \otimes \xi_1^{ap^n}$. The product of the elements above can be written

$$\xi^{(b,i_2,\ldots)}\tau^{(j_0,j_1,\ldots)} \otimes \xi_1^{ap^n} + \sum \underline{\mathsf{m}}' \otimes \underline{\mathsf{m}}''$$

with all the other terms having higher filtration so the induced map of filtration quotients is an isomorphism. Again, we want to evaluate the functor $R\lim$ on the given filtration to check for completeness. Working with bigraded modules, this is done one bidegree at a time. In a fixed bidegree, there are no elements of infinite filtration and by the Mittag-Leffler criterion the derived limit vanishes. By [6], proposition 2.2, the filtration is complete. Hence, the map we are dealing with is an isomorphism of left $\mathbb{M}_p$-modules. Dualizing the first diagram in this proof we get

$$\begin{array}{ccc}
A(n) \otimes_{\mathbb{M}_p} B(n) & \xrightarrow{\phi_l^n} & B(n) \\
\uparrow{\scriptstyle A(n)\otimes\iota} & \nearrow & \\
A(n) \otimes_{\mathbb{M}_p} \prod_{p^n|k} \mathbb{M}_p\{P^k\} & &
\end{array}$$

with ϕ_l^n defined to be the $\mathbb{M}_p$-dual of ψ_l^n composed with θ of lemma 3.1.19. However, the module $\prod_{p^n|k} \mathbb{M}_p\{P^k\}$ is isomorphic to $\bigoplus_{p^n|k} \mathbb{M}_p\{P^k\}$ since the generators P^k all lie on a line of slope $1/2(p-1)$ in the degree-weight plane and will only contribute to finitely many bidegrees. All these facts taken together imply the diagonal arrow is an isomorphism of left $\mathbb{M}_p$-modules. □

Lemma 3.2.14. *$B(n)$ is a free right $A(n-1)$-module with generators*

$$\{P^k, \beta P^k \mid k \in \mathbb{Z}\}$$

Proof. The proof is similar to the one in the last lemma: Our diagram is now

$$\begin{array}{ccc} B^\vee(n)\otimes_{\mathbb{M}_p} A^\vee(n-1) & \xleftarrow{\psi_r^n} & B^\vee(n) \\ \Big\downarrow{\scriptstyle \pi\otimes A^\vee(n-1)} & \swarrow & \Big\uparrow{\scriptstyle\gamma} \\ B^\vee(0)\otimes_{\mathbb{M}_p} A^\vee(n-1) & \longleftarrow & B^\vee(0)\otimes_{\mathbb{M}_p} A^\vee(n-1) \end{array}$$

where π is the projection

$$(\frac{A^\vee}{J(n)})[\xi^{-1}] \longrightarrow (\frac{A^\vee}{J(0)})[\xi^{-1}].$$

There is an isomorphism of $\mathbb{M}_p$-modules

$$\gamma : B^\vee(0)\otimes_{\mathbb{M}_p} A^\vee(n-1) \longrightarrow B^\vee(n)$$

defined on our monomial basis by sending

$$\xi_1^l\tau_0^\epsilon \otimes \xi^{(i_1,\ldots,i_{n-1})}\tau^{(j_0,\ldots,j_{n-1})} \longmapsto \xi^{(l',i_1,\ldots,i_{n-1})}\tau^{(\epsilon,j_0,\ldots,j_{n-1})}.$$

Here $l' = l - (\sum_{s=1}^{n-1} i_s p^s) - (\sum_{t=0}^{n-1} j_t p^t)$, the unique integer such that the association is degree-preserving.

We point out to the reader that this map shifts the exponents to the right with the modification in the power of ξ_1 given to obtain a degree preserving map.

The inverse of γ is given by sending

$$\xi^{(i'_1,\ldots,i'_n)}\tau^{(j'_0,\ldots,j'_n)} \longmapsto \xi_1^{l''}\tau_0^{j'_0} \otimes \xi^{(i'_2,\ldots,i'_n)}\tau^{(j'_1,\ldots,j'_n)}$$

with $l'' = i'_1 + (\sum_{s=2}^{n} i'_s p^{s-1}) + (\sum_{t=1}^{n} j'_t p^{t-1})$.

We filter $B^\vee(0)\otimes_{\mathbb{M}_p} A^\vee(n-1)$ by the increasing submodules

$$G_d := \mathbb{M}_p\{\mathsf{m} = \mathsf{m}'\otimes\mathsf{m}'' \in B^\vee(0)\otimes_{\mathbb{M}_p} A^\vee(n-1) \mid deg(\mathsf{m}') \le d\}.$$

Given

$$\gamma(\xi_1^l\tau_0^\epsilon \otimes \xi^{(i_1,\ldots,i_{n-1})}\tau^{(j_0,\ldots,j_{n-1})}) = \xi^{(l',i_1,\ldots,i_{n-1})}\tau^{(\epsilon,j_0,\ldots,j_{n-1})}$$

with $l' = l-(\sum i_s p^s)-(\sum j_t p^t)$, the application of $(\pi\otimes A^\vee(n-1))\psi_r^n$ results in the multiplication of elements of the form

$$(\tau_0^\epsilon\otimes 1 + 1\otimes\tau_0^\epsilon),$$

$$(\xi_1^{p^s}\otimes\xi_s + 1\otimes\xi_{s+1})^{i_s} = (\xi_1^{i_s p^s}\otimes\xi_s^{i_s} + \ldots + 1\otimes\xi_{s+1}^{i_s}),$$

$$(\xi_1^{p^t}\otimes\tau_t + 1\otimes\tau_{t+1})^{j_t} = (\xi_1^{j_t p^t}\otimes\tau_t^{j_t} + \ldots + 1\otimes\tau_{t+1}^{j_t})$$

and either

$$(\xi_1^{l'} \otimes 1 + \ldots + 1 \otimes \xi_1^{l'})$$

if l' is non-negative or

$$\sum \binom{b}{c} \xi_1^{-ap^n+b-c} \otimes \xi_1^c$$

where $l' = -ap^n + b$ with $0 \leq b < p^n$ if l' is negative.

In the end the product is of the form

$$\xi_1^{(l'+\sum i_s p^s + \sum j_t p^t)} \tau_0^{\epsilon} \otimes \xi^{(i_1,\ldots,i_{n-1})} \tau^{(j_0,\ldots,j_{n-1})} + \sum \underline{\mathsf{m}}' \otimes \underline{\mathsf{m}}''$$

which is equal to

$$\xi_1^l \tau_0^{\epsilon} \otimes \xi^{(i_1,\ldots,i_{n-1})} \tau^{(j_0,\ldots,j_{n-1})} + \sum \underline{\mathsf{m}}' \otimes \underline{\mathsf{m}}''.$$

Here, the terms of the sum are all of lower filtration which implies the matrix representing our composite is invertible with 1's on the diagonal, thus also invertible. Hence the map induced on filtration quotients is an isomorphism of $\mathbb{M}_p$-modules. The filtration on $B^\vee(0) \otimes_{\mathbb{M}_p} A^\vee(n-1)$ is evidently Hausdorff and completeness follows since we can evaluate $R\lim$ in each bidegree as we saw in the proof of the last lemma.Thus, we use theorem 2.6 of [6] to conclude that our map is an isomorphism of $\mathbb{M}_p$-modules.

Taking $\mathbb{M}_p$-duals in the diagram we started with and using the fact that $B^\vee(0)$ has a basis as in lemma 3.2.12 which has finitely many generators in each bidegree as a module over $\mathbb{M}_p$, we produce the diagram

$$\begin{array}{ccc} B(n) \otimes_{\mathbb{M}_p} A(n-1) & \xrightarrow{\phi_r^n} & B(n) \\ \uparrow{\scriptstyle \pi^\vee \otimes A(n-1)} & \nearrow & \\ B(0) \otimes_{\mathbb{M}_p} A(n-1) & & \end{array}$$

where we have used bounds on the bidegrees and the map θ of lemma 3.1.19 which is an isomorphism under these circumstances. All these facts taken together imply the diagonal arrow is an isomorphism of left $\mathbb{M}_p$-modules and we have proved our result. □

Lemma 3.2.15. *A is a free left $A(n)$-module.*

Proof. The reasoning here resembles the work in the two former lemmas. We want to check that $A^\vee$ is a free left $A^\vee(n)$-comodule. First, the comodule structure is given by the map

$$A^\vee \xrightarrow{\psi} A^\vee \otimes_{\mathbb{M}_p} A^\vee \xrightarrow{\pi \otimes A^\vee} A^\vee(n) \otimes_{\mathbb{M}_p} A^\vee.$$

There is an isomorphism of left $\mathbb{M}_p$-modules

$$\delta : A^\vee(n) \otimes_{\mathbb{M}_p} \bigoplus_{I\in S, J\in T} \mathbb{M}_p\{\xi^I\tau^J\} \longrightarrow A^\vee$$

where

$$S = \{I = (i_1, \ldots, i_q) : p^{n+1-k} \mid i_k \text{ when } 1 \le k \le n, i_k \ge 0 \text{ when } k \ge n+1\}$$

and

$$T = \{J = (0, \ldots, 0, j_{n+1}, \ldots, j_{q'}) : 0 \le j_k \le 1 \text{ for } k \ge n+1\}.$$

It is defined by sending

$$\xi^{I'}\tau^{J'} \otimes \xi^I\tau^J \longmapsto \xi^{I''}\tau^{J''}.$$

Here , $I' := (i'_1, \ldots, i'_n)$ with $i'_k < p^{n+1-k}$ when $1 \le k \le n$. Also, $J' := (j_0, \ldots, j_n)$ with $0 \le j'_k \le 1$. Finally, the index sets $I'' = I + I'$ and $J'' = J + J'$ with $\xi^{I''}\tau^{J''}$ are gotten by multiplying monomials so that

$$\xi^{I''}\tau^{J''} = \xi^I\tau^J\xi^{I'}\tau^{J'} = \xi^I\xi^{I'}\tau^J\tau^{J'}.$$

The diagram of choice is now

$$\begin{array}{ccc} A^\vee(n) \otimes_{\mathbb{M}_p} A^\vee & \xleftarrow{(\pi\otimes A^\vee)\psi} & A^\vee \\ \big\downarrow{\scriptstyle A^\vee(n)\otimes\pi} & \swarrow & \\ A^\vee(n) \otimes_{\mathbb{M}_p} \bigoplus_{I\in S,J\in T} \mathbb{M}_p\{\xi^I\tau^J\} & & \end{array}$$

with π in the vertical, left-hand map being the projection taking all monomials except the ones specified by S and T to zero. Expanding the diagram to

$$\begin{array}{ccc} A^\vee(n) \otimes_{\mathbb{M}_p} A^\vee & \xleftarrow{(\pi\otimes A^\vee)\psi} & A^\vee \\ \big\downarrow{\scriptstyle A^\vee(n)\otimes\pi} & \swarrow & \big\uparrow{\scriptstyle\delta} \\ A^\vee(n) \otimes_{\mathbb{M}_p} \bigoplus_{I\in S,J\in T} \mathbb{M}_p\{\xi^I\tau^J\} & \longleftarrow & A^\vee(n) \otimes_{\mathbb{M}_p} \bigoplus_{I\in S,J\in T} \mathbb{M}_p\{\xi^I\tau^J\}, \end{array}$$

we go on and filter the right-hand tensor factor by letting

$$F_d := \mathbb{M}_p\{\mathsf{m} = \mathsf{m}' \otimes \mathsf{m}'' \in A^\vee(n) \otimes_{\mathbb{M}_p} \bigoplus_{I\in S, J\in T} \mathbb{M}_p\{\xi^I\tau^J\} \mid deg(\mathsf{m}'') \le d\}.$$

We start with a monomial $\xi^{I''}\tau^{J''} = \xi^{I'}\tau^{J'}\xi^{I}\tau^{J} \in A^\vee$ with sequences of exponents as described above. The application of ψ leads us to the multiplication of expressions

$$(\xi_k \otimes 1 + \sum_{l=1}^{k-1} \xi_{k-l}^{p^l} \otimes \xi_l + 1 \otimes \xi_k)^{i_k}$$

and

$$(\tau_k \otimes 1 + \sum_{l=1}^{k-1} \xi_{k-l}^{p^k} \otimes \tau_l + 1 \otimes \tau_k)^{j_k}.$$

Looking at $\psi(\xi^{I'}\tau^{J'})$, we see that the map $A^\vee(n) \otimes \pi$ sends this element to $\xi^{I'}\tau^{J'} \otimes 1$ because of the upper bounds on the exponents in the expressions resulting from the application of ψ. In any case, the expression we obtain after mapping to $A^\vee(n) \otimes_{\mathbb{M}_p} \bigoplus_{I\in S, J\in T} \mathbb{M}_p\{\xi^I\tau^J\}$ has the form

$$\xi^{I'}\tau^{J'} \otimes \xi^I\tau^J + \sum \mathsf{m}' \otimes \mathsf{m}''$$

where the monomials m'' are of lower filtration than $\xi^I\tau^J$. We conclude as before that our map of interest is an isomorphism of left $\mathbb{M}_p$-modules since this is the case for the lower horizontal map. Taking $\mathbb{M}_p$-duals in the diagram we started with we produce the diagram

$$\begin{array}{ccc} A(n) \otimes_{\mathbb{M}_p} A & \longrightarrow & A \\ \uparrow & \nearrow & \\ A(n) \otimes_{\mathbb{M}_p} \bigoplus_{I\in S, J\in T} \mathbb{M}_p\{\xi^I\tau^J\}^\vee & & \end{array}$$

where we have used the fact that $\bigoplus_{I\in S, J\in T} \mathbb{M}_p\{\xi^I\tau^J\}$ is bounded below and of finite type and the map θ of lemma 3.1.19 which is an isomorphism. Hence the diagonal arrow is an isomorphism of left $\mathbb{M}_p$-modules and we have proved our result. □

3.3 The motivic Singer construction

Recall that there is a directed system

$$B(0) \rightarrowtail \cdots \rightarrowtail B(n) \rightarrowtail B(n+1) \rightarrowtail \cdots.$$

We have also seen, in lemma 3.2.14, that as a right $A(n-1)$-module, $B(n)$ is free on the generators

$$\{P^k, \beta P^k \mid k \in \mathbb{Z}\}.$$

By lemma 3.2.11, the generators $\{P^k, \beta P^k\} \in B(0)$ are sent to $\{P^k, \beta P^k\} \in B(n)$ for $n > 0$. Focussing on the interaction between the stages in this tower; suppose we are given a bigraded left $A(n)$-module M. We can form the composite morphism of left $A(n)$-modules

$$B(n) \otimes_{A(n-1)} M \longrightarrow B(n+1) \otimes_{A(n-1)} M \longrightarrow B(n+1) \otimes_{A(n)} M$$

coming from functoriality in changes of base and in the left-hand tensor factor. Since both ends of the composite are isomorphic to $B(0) \otimes_{\mathbb{M}_p} M$ as bigraded left modules over $\mathbb{M}_p$ and generators are sent to generators, this is an isomorphism over $\mathbb{M}_p$.

An important definition arises:

Definition 3.3.1. *For a bigraded left A-module M, we define*

$$R_+(M) := \operatorname*{colim}_{n \to \infty} B(n) \otimes_{A(n-1)} M,$$

the motivic Singer construction.

The Singer construction appears in [37] and is also disussed in [1] where this section finds most of it inspiration.

As we are taking M to be a module over A from the start and evaluating the colimit, $R_+(M)$ can be given the structure of an A-module too. For $n \geq 0$ there are compositions

$$B(n) \otimes_{A(n-1)} M \longrightarrow A \otimes_{A(n-1)} M \longrightarrow A \otimes_A M \cong M$$

that fit into commutative diagrams

$$\begin{array}{ccc} B(n) \otimes_{A(n-1)} M & \longrightarrow & M \\ \downarrow \cong & \nearrow & \\ B(n+1) \otimes_{A(n)} M & & \end{array}$$

Definition 3.3.2. *Taking the colimit in this diagram we define*

$$\epsilon : R_+(M) \longrightarrow M$$

to be the resulting morphism of bigraded left A-modules.

For the applications we have in mind the map ϵ has an important property as observed in [1]:

Definition 3.3.3. *A map of bigraded left A-modules $M \longrightarrow N$ is said to be a Tor-equivalence if the induced map*

$$\operatorname{Tor}^A_{*,(*,*)}(\mathbb{M}_p, M) \longrightarrow \operatorname{Tor}^A_{*,(*,*)}(\mathbb{M}_p, N)$$

is an isomorphism.

Proposition 3.3.4. *Given a Tor-equivalence $M \longrightarrow N$ then the induced maps*

$$\mathrm{Tor}^A_{*,(*,*)}(K, M) \longrightarrow \mathrm{Tor}^A_{*,(*,*)}(K, N)$$

and

$$\mathrm{Ext}_A^{*,(*,*)}(N, L) \longrightarrow \mathrm{Ext}_A^{*,(*,*)}(M, L)$$

are isomorphisms if we assume that

- K can be written as a filtered colimit of submodules that are free and of finite rank over $\mathbb{M}_p$,
- L is isomorphic to the $\mathbb{M}_p$-dual of such a module K.

Proof. To prove the first part, one begins with looking at modules K that are isomorphic to a free $\mathbb{M}_p$-module of finite rank. $\mathrm{Tor}^A_{*,(*,*)}(-, M)$ commutes with finite sums and using the 5-lemma and the split exact sequence

$$0 \to \mathbb{M}_p \to K \to K' \to 0$$

establishes the first isomorphism for such choices of modules. Since $\mathrm{Tor}^A_{*,(*,*)}(-, M)$ commutes with filtered colimits, the first isomorphism follows. For the second one, we want to use the isomorphism

$$\mathrm{Tor}^A_{*,(*,*)}(K, M)^\vee \cong \mathrm{Ext}_A^{*,(*,*)}(M, K^\vee)$$

which holds under the stated conditions of proposition 5.1 chapter VI in [10]. If this holds then the map

$$\mathrm{Ext}_A^{*,(*,*)}(N, L) \longrightarrow \mathrm{Ext}_A^{*,(*,*)}(M, L)$$

can be written as the dual of

$$\mathrm{Tor}^A_{*,(*,*)}(K, M) \longrightarrow \mathrm{Tor}^A_{*,(*,*)}(K, N).$$

□

At this point we should clarify the relationship between homological algebra over A and $A(n)$. We have a directed system

$$\begin{array}{ccc} A(n) & \rightarrowtail & A \\ \downarrow & \nearrow & \\ A(n+1) & & \end{array}$$

to work with and saw, in lemma 3.2.15, that A is a free left $A(n)$-module. Given a free resolution $F_* \longrightarrow N \longrightarrow 0$ of a bigraded left module N over A, this also gives a free resolution of N over $A(n)$. Functoriality in the ground ring then gives us an isomorphism

$$\operatorname*{colim}_{n\to\infty} \operatorname{Tor}^{A(n)}_{*,(*,*)}(K,N) \cong \operatorname*{colim}_{n\to\infty} H_{*,(*,*)}(K \otimes_{A(n)} F_*),$$

with K being a right module over A (and so also a right module over $A(n)$). On the other hand, we also have

$$\operatorname*{colim}_{n\to\infty} H_{*,(*,*)}(K \otimes_{A(n)} F_*) \cong H_{*,(*,*)}(\operatorname*{colim}_{n\to\infty} K \otimes_{A(n)} F_*) \cong H_{*,(*,*)}(K \otimes_A F_*)$$

since homology commutes with directed colimits, two colimits commute with each other and the tensor product being a coequalizer. Thus,

$$\operatorname*{colim}_{n\to\infty} \operatorname{Tor}^{A(n)}_{*,(*,*)}(K,N) \cong \operatorname{Tor}^{A}_{*,(*,*)}(K,N).$$

This reduction comes to our aid in the next result:

Lemma 3.3.5. *If M is a free A-module, then $R_+(M)$ is flat as an A-module and*

$$\mathbb{M}_p \otimes_A R_+(M) \xrightarrow{\mathbb{M}_p \otimes \epsilon} \mathbb{M}_p \otimes_A M$$

is an isomorphism.

Proof. If M is a free A-module, then it is free over $A(n-1)$. This implies that $B(n) \otimes_{A(n-1)} M$ is isomorphic to a direct sum of $B(n)$'s. Since $B(n)$ is free over $A(n)$ by 3.2.13, $B(n) \otimes_{A(n-1)} M$ is free over $A(n)$ and hence also flat over $A(n)$. This property passes to the colimit. But then

$$\operatorname{Tor}^{A}_{s,(*,*)}(K, R_+(M)) \cong \operatorname*{colim}_{n\to\infty} \operatorname{Tor}^{A(n)}_{s,(*,*)}(K, R_+(M)) \cong 0$$

for any right A-module K and $s > 0$. A standard result from homological algebra, for which the reader may consult theorem 8.6 in chapter 5 of [27], then yields the proof of our first claim.

For the second claim we observe that it will be sufficient to prove the case where $M = A$. This is because direct sums commute with tensor products and directed colimits from which $R_+(-)$ is built. Now, we saw that $B(n)$ was free as a left module over $A(n)$ on generators $\{P^{ap^n} \mid a \in \mathbb{Z}\}$ in 3.2.13. Tensoring with $\mathbb{M}_p$ on the left, the steps in the directed system $\{B(n) \otimes_{A(n-1)} A\}$ become

$$\mathbb{M}_p \otimes_{A(n)} B(n) \otimes_{A(n-1)} A \longrightarrow \mathbb{M}_p \otimes_{A(n+1)} B(n+1) \otimes_{A(n)} A.$$

By lemma 3.2.11 the composition

$$\begin{array}{ccc} \mathbb{M}_p \otimes_{A(n)} B(n) & \longrightarrow & \mathbb{M}_p \otimes_{A(n+1)} B(n+1) \\ \| & & \| \\ \mathbb{M}_p\{P^k : p^n \mid k\} & & \mathbb{M}_p\{P^l : p^{n+1} \mid l\} \end{array}$$

sends

$$P^k \longmapsto \sum_{p^{n+1}|l} \mathsf{h}_l P^l$$

with $\mathrm{bideg}(\mathsf{h}_l) = \mathrm{bideg}(P^k) - \mathrm{bideg}(P^l)$, an integral multiple of $(2p-2, 1)$. This can only happen if $l = k$ since $\mathbb{M}_p^{i,j}$ is only nonzero for $i, j \geq 0$ and $j \geq i$ with our restrictions on the ground field. Passing to the colimit over n, k must be divisible by p^n for all n with the only possibility $k = 0$. Hence $\mathbb{M}_p \otimes_A R_+(A)$ has but one base element, corresponding to $k = 0$, so we get an isomorphism

$$\mathbb{M}_p \otimes_A R_+(A) \xrightarrow{\mathbb{M}_p \otimes \epsilon} \mathbb{M}_p \otimes_A A \cong \mathbb{M}_p.$$

□

Theorem 3.3.6. *Let M be an A-module. The map $\epsilon : R_+(M) \longrightarrow M$ is a Tor-equivalence.*

Proof. Let

$$\cdots \longrightarrow F_s \longrightarrow F_{s-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M$$

be a free resolution of M over A. By application of $R_+(-)$, we obtain the commutative diagram

$$\begin{array}{ccccccccc} \cdots \longrightarrow & R_+(F_s) & \longrightarrow & R_+(F_{s-1}) & \longrightarrow \cdots \longrightarrow & R_+(F_0) & \longrightarrow & R_+(M) \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots \longrightarrow & F_s & \longrightarrow & F_{s-1} & \longrightarrow \cdots \longrightarrow & F_0 & \longrightarrow & M. \end{array}$$

Observe that $R_+(-)$ preserves exactness: At each stage in the defining colimit we tensor with a free right module (lemma 3.2.14), which is flat, and since directed colimits preserve exactness this is the case for $R_+(-)$. Next, we remember from lemma 3.3.5 that $R_+(-)$ takes free modules to flat ones so the upper row is a flat resolution of $R_+(M)$. It is a general fact from homological algebra that such a resolution may be used to compute $\mathrm{Tor}^A_{*,(*,*)}(\mathbb{M}_p, R_+(M))$, see e.g. [44], 3.2.8. Tensoring the commutative diagram with $\mathbb{M}_p$ we get another diagram

$$\begin{array}{ccccc} \cdots \longrightarrow & \mathbb{M}_p \otimes_A R_+(F_s) & \longrightarrow & \mathbb{M}_p \otimes_A R_+(F_{s-1}) & \longrightarrow \cdots \\ & \downarrow {\scriptstyle 1_{\mathbb{M}_p} \otimes \epsilon} & & \downarrow {\scriptstyle 1_{\mathbb{M}_p} \otimes \epsilon} & \\ \cdots \longrightarrow & \mathbb{M}_p \otimes_A F_s & \longrightarrow & \mathbb{M}_p \otimes_A F_{s-1} & \longrightarrow \cdots \end{array}$$

in which the vertical morphisms are isomorphisms. Thus we have isomorphisms

$$\mathrm{Tor}^A_{*,(*,*)}(\mathbb{M}_p, R_+(M)) \longrightarrow \mathrm{Tor}^A_{*,(*,*)}(\mathbb{M}_p, M)$$

and we are done. □

Chapter 4

Inverse limits of motivic spectra

4.1 Realization in the motivic stable category

We want to realize the algebraic construction $R_+(\mathbb{M}_p)$ as a module over A using constructions in $SH(F)$, the category of motivic spectra over F. This will involve using models for classifying spaces in $H_\bullet(F)$ following the classical work on Segals conjecture. In [1] it was observed that the Singer construction relates to the study of stable cohomotopy groups of the classifying space $B\mathbb{Z}/p$. In our work we will try to exploit the techniques developed in conjunction with these considerations to say something about the corresponding problem for the linear algebraic group μ_p in motivic homotopy theory. This will involve manipulating diagrams of motivic spectra. Throughout, the notation $H^{*,*}(X)$ will mean $H^{*,*}(X;\mathbb{Z}/p)$ unless something else is stated. Although we will comment on phenomena arising for odd p the specific tower constructed for the realization is only made for the case $p = 2$. There is probably a similar construction for the odd case but we leave that for future work.

4.1.1 Preliminaries

The projective spaces $\mathbb{P}^n$ $(n \geq 0)$ represent motivic spaces. Although fibration sequences are subtle in motivic homotopy theory, there are special cases where we know certain details. For example, there are associated fiber sequences

$$\mathbb{G}_m \longrightarrow \mathbb{A}^{n+1} \setminus 0 \longrightarrow \mathbb{P}^n$$

where the algebraic group $\mathbb{G}_m$ is defined by

$$\mathbb{G}_m := \mathrm{Spec}(F[x, x^{-1}]).$$

This can be found in [46], example 4.4.9. Each projective space $\mathbb{P}^n$ comes with a tautological line bundle, γ^1_n. It can be thought of as the closed subset of $\mathbb{A}^{n+1} \prod \mathbb{P}^n$ satisfying the equations

$$x_i y_j = x_j y_i$$

where $(x_0, \ldots, x_n)$ are the coordinates of $\mathbb{A}^{n+1}$ and $(y_0, \ldots, y_n)$ are the coordinates of $\mathbb{P}^n$. These bundles are related through the fact that for each inclusion

$$\iota : \mathbb{P}^n \longrightarrow \mathbb{P}^{n+1}$$

we have $\iota^* \gamma^1_{n+1} = \gamma^1_n$. Passing to the obvious colimit, one defines $\mathbb{P}^\infty := \mathrm{colim}_n \mathbb{P}^n$. Recall the algebraic group μ_p from definition 3.1.2. We can think of $\mathbb{P}^n$ as the quotient $(\mathbb{A}^{n+1} \backslash 0)/\mathbb{G}_m$ where we let $\mathbb{G}_m$ act diagonally. There is a closed inclusion $\mu_p \longrightarrow \mathbb{G}_m$ and we let μ_p act on $\mathbb{A}^n \backslash 0$ through the action of $\mathbb{G}_m$. Related to the projective spaces are the motivic lens spaces:

Definition 4.1.1. *For $n \geq 1$, the motivic lens space L^n is defined to be the motivic space represented by $(\mathbb{A}^n \setminus 0)/\mu_p$.*

The inclusions $\iota : \mathbb{A}^n \setminus 0 \longrightarrow \mathbb{A}^{n+1} \setminus 0$ sending $(a_1, \ldots, a_n)$ to $(a_1, \ldots, a_n, 0)$ induce inclusions

$$\iota : L^n \longrightarrow L^{n+1}.$$

Using the resulting directed diagram, we define L^∞ to be the colimit. The reader should note that in [31], L^∞ is denoted $B(\mu_p)_{gm}$. The following basic fact on these spaces will be needed later:

Lemma 4.1.2. *The motivic spaces L^n are represented by smooth schemes.*

Proof. As with projective spaces, we cover L^n with the subspaces V_i in the diagram

$$\begin{array}{ccc} \mathbb{A}^n \setminus 0 & \xrightarrow{/\mu_p} & L^n \\ \uparrow & & \uparrow \\ U_i = \mathbb{G}_m \times \mathbb{A}^{n-1} & \xrightarrow{\mu_p} & V_i \end{array}$$

where one of the coordinates is not equal to zero. We identify

$$V_i \cong (\mathbb{G}_m \times \mathbb{A}^n)/\mu_p \cong \mathrm{Spec}((F[x_1, \ldots, x_n, x_i^{-1}])^{\mu_p})$$

and we have

$$F[x_1, \ldots, x_n, x_i^{-1}]^{\mu_p} \cong F[\mathbf{m}]$$

where $\mathbf{m}$ ranges through the monomials $x_1^{e_1} \cdots x_n^{e_n}$ such that

$$e_1 + \cdots + e_n \equiv 0 (\mathrm{mod}\ p)$$

Smoothness can be checked locally and the algebra $F[\mathbf{m}]$ is smooth.

□

The proof of lemma 6.3 in [41] shows that L^n is isomorphic to

$$E((\gamma^1_{n-1})^{\otimes p}) \setminus 0 \downarrow \mathbb{P}^{n-1}),$$

the total space of the complement of the zero section of the line bundle $(\gamma^1_{n-1})^{\otimes p}$ over $\mathbb{P}^{n-1}$. The projection induces a map

$$f_n : L^n \longrightarrow \mathbb{P}^{n-1}.$$

Remark 4.1.3. The tautological line bundle γ^1_{n-1} over $\mathbb{P}^{n-1}$ may be pulled back to L^n using f_n and we use the notation γ^1_{n-1} for the resulting bundle as well.

To aid our work with fibre bundles over the spaces L^n we recollect more material from [31]:

Remark 4.1.4. One fact we will use is the following: The very construction of $H_\bullet(F)$ forces an algebraic vector bundle $\mathcal{E} \longrightarrow X$, where X is represented by a smooth scheme, to be an $\mathbb{A}^1$-homotopy equivalence (example 2.2, p.106 in [31]).

Given an algebraic vector bundle, we will also consider motivic Thom spaces:

Definition 4.1.5. *Assume $\mathcal{E} \longrightarrow X$ to be a vector bundle with X smooth over F. Then the Thom space of $\mathcal{E}$ over X, denoted $Th(\mathcal{E} \downarrow X)$ or $Th(\mathcal{E})$, is defined to be*

$$\frac{E(\mathcal{E} \downarrow X)}{E(\mathcal{E} \downarrow X) \setminus X}$$

where $E(\mathcal{E} \downarrow X)$ is the total space of the bundle into which X is embedded through the zero section.

The definition makes sense since we are working with simplicial sheaves where quotients such as this exist. Concerning this construction there is a result we will use later. It is often refered to as the "homotopy purity" isomorphism:

Theorem 4.1.6. *Assume X and Y are smooth schemes over F and that $\iota : X \longrightarrow Y$ is a smooth, closed embedding with normal bundle $N_{Y,X}$. Then there is an isomorphism in $H_\bullet(F)$:*

$$\rho_{Y,X} : \frac{Y}{Y \setminus X} \longrightarrow Th(N_{Y,X})$$

Proof. This theorem 2.23 on page 115 in [31]. □

Given a vector bundle $\mathcal{E}$ of rank n over a smooth scheme X, there is a class $t(\mathcal{E})$ in $\tilde{H}^{2n,n}(Th(\mathcal{E} \downarrow X); \mathbb{Z})$ called the *Thom class* of $\mathcal{E}$. See the discussion in chapter 4 of [41] for this. We can map this class to $\tilde{H}^{2n,n}(Th(\mathcal{E} \downarrow X))$ by using the long exact sequence in cohomology associated to the sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{*p} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/p \longrightarrow 0$$

and we will use the notation $t(\mathcal{E})$ for this class as well. There is a "Thom diagonal" map defined as the bottom map in the diagram

$$\begin{array}{ccccc} E(\mathcal{E}\downarrow X)\setminus X & & \longrightarrow & & X \times E(\mathcal{E}\downarrow X)\setminus X \\ \downarrow & & & & \downarrow \\ E(\mathcal{E}\downarrow X) & \longrightarrow & E(\mathcal{E}\downarrow X) \times E(\mathcal{E}\downarrow X) & \longrightarrow & X \times E(\mathcal{E}\downarrow X) \\ \downarrow & & & & \downarrow \\ Th(\mathcal{E}\downarrow X) & & \longrightarrow & & X_+ \wedge Th(\mathcal{E}\downarrow X). \end{array}$$

It lets us multiply elements $x \in H^{*,*}(X)$ with $t(\mathcal{E}) \in \tilde{H}^{2n,n}(Th(\mathcal{E} \downarrow X))$ upon the evaluation of cohomology. In the diagram, the left vertical sequence is the one defined by the inclusion of the complement of the zero section. The map

$$E(\mathcal{E} \downarrow X) \to E(\mathcal{E} \downarrow X) \times E(\mathcal{E} \downarrow X)$$

is the diagonal and the map

$$E(\mathcal{E} \downarrow X) \times E(\mathcal{E} \downarrow X) \to X \times E(\mathcal{E} \downarrow X)$$

is induced by the projection of the bundle $\mathcal{E}$.

Naturally, there is a motivic Thom isomorphism:

Lemma 4.1.7. *For every motivic space $Y \in H_\bullet(F)$ there is an isomorphism*

$$\tilde{H}^{*,*}(Y \wedge X_+) \longrightarrow \tilde{H}^{*+2n,*+n}(Y \wedge Th(\mathcal{E} \downarrow X))$$

given by multiplication with $t(\mathcal{E})$ sending $y \in \tilde{H}^{*,*}(Y \wedge X_+)$ to $y \cup t(\mathcal{E})$.

Proof. Proposition 4.3 in [41]. □

Definition 4.1.8. *Using the zero section, we define the Euler class $e(\mathcal{E}) \in H^{2n,n}(X; \mathbb{Z})$ to be the restriction of $t(\mathcal{E})$ along $X_+ \longrightarrow Th(\mathcal{E})$.*

As with Thom classes, we will use the same notation for the class in $H^{2n,n}(X)$ resulting from a change of coefficients. Next, we will use the Gysin sequence for calculations:

Lemma 4.1.9. *Let $X \longrightarrow Y$ be the inclusion of a smooth, closed subscheme into a smooth scheme Y of codimension c. Then there is an associated long exact sequence*

$$\cdots \longrightarrow H^{*-2c,*-c}(Y) \xrightarrow{*e(N_{Y,X})} H^{*,*}(X) \longrightarrow H^{*,*}(X \setminus Y) \longrightarrow \cdots$$

of $\mathbb{M}_p$-modules.

Proof. See e.g. [28], theorem 15.15. □

We need to know the evaluation of motivic cohomology on both projective spaces and lens spaces. In the first case, this is entirely similar to the calculation made in classical algebraic topology:

Lemma 4.1.10. *We have $H^{*,*}(\mathbb{P}^n) \cong \mathbb{M}_p[v]/(v^{n+1})$ and $H^{*,*}(\mathbb{P}^\infty) \cong \mathbb{M}_p[v]$ where* $\mathrm{bideg}(v) = (2,1)$.

Proof. This is proved with an induction on n beginning with $H^{*,*}(\mathbb{P}^0) \cong \mathbb{M}_p$. The class v is the first Chern class $c_1(\gamma_0^1)$ living in $\mathbb{M}_p^{2,1}$ which is zero since we are working over a field of characteristic zero.

Let us assume inductively that $H^{*,*}(\mathbb{P}^{n-1}) \cong \mathbb{M}_p[v]/(v^n)$ with $v = c_1(\gamma_{n-1}^1)$. The normal bundle of the smooth embedding

$$j : \mathbb{P}^{n-1} \longrightarrow \mathbb{P}^n$$

is isomorphic to the tautological bundle γ_{n-1}^1 and through the homotopy purity isomorphism, we can view $\mathbb{P}^n$ as the Thom space

$$\frac{E(\gamma_{n-1}^1)}{E(\gamma_{n-1}^1) \setminus \mathbb{P}^{n-1}}$$

in $H_\bullet(F)$. The Thom isomophism gives us an isomorphism

$$H^{*,*}(\mathbb{P}^{n-1}) \cong \tilde{H}^{*+2,*+1}(\mathbb{P}^n)$$

sending $a \in H^{*,*}(\mathbb{P}^{n-1})$ to $t(\gamma_n^1) * j^*(a)$. Here $*$ is the module map given by the Thom diagonal and this is equal to the cup product $t(\gamma_n^1) \cup j^*(a)$. From this we see that

$$c_1(\gamma_{n-1}^1)^i \mapsto t(\gamma_n^1) \cup c_1(\gamma_{n-1}^1)^i = t(\gamma_n^1) \cup j^*(t(\gamma_n^1)^i) = t(\gamma_n^1)^{i+1}.$$

Specifically, $t(\gamma_n^1)^{n+1}$ must be zero. Hence the identification $H^{*,*}(\mathbb{P}^n) \cong \mathbb{M}_p[v]/(v^{n+1})$ must hold. The result for $H^{*,*}(\mathbb{P}^\infty)$ can be seen from evaluating inverse limits and using that all the maps in this system are surjective so any derived limit must be zero. □

Lemma 4.1.11. *There are ring isomorphisms*

$$H^{*,*}(L^n) \cong \mathbb{M}_2[u,v]/(u^2+\rho u+\tau v, v^n)$$
$$H^{*,*}(L^\infty) \cong \mathbb{M}_2[u,v]/(u^2+\rho u+\tau v)$$

for $p=2$ *and*

$$H^{*,*}(L^n) \cong \mathbb{M}_p[u,v]/(u^2, v^n)$$
$$H^{*,*}(L^\infty) \cong \mathbb{M}_p[u,v]/(u^2)$$

for p *odd where* $\mathrm{bideg}(u) = (1,1)$ *and* $\mathrm{bideg}(v) = (2,1)$

Proof. This is theorem 6.10 in [41]. The proof of lemma 6.3 in that same paper shows that L^n is isomorphic to $E((\gamma^1_{n-1})^{\otimes p}) \setminus 0 \downarrow \mathbb{P}^{n-1})$, the total space of the complement of the zero section of the line bundle $(\gamma^1_{n-1})^{\otimes p}$ over $\mathbb{P}^{n-1}$. There is an associated cofiber sequence

$$L^n \longrightarrow E((\gamma^1_{n-1})^{\otimes p} \downarrow \mathbb{P}^{n-1}) \longrightarrow Th((\gamma^1_{n-1})^{\otimes p} \downarrow \mathbb{P}^{n-1})$$

inducing a Gysin long exact sequence of $\mathbb{M}_p$-modules. Lemma 4.5 of [41] implies that $e((\gamma^1_{n-1})^{\otimes p}) = pe(\gamma^1_{n-1})$ which is zero with $\mathbb{Z}/p$-coefficients so the Gysin sequence induces a short exact sequence

$$H^{*,*}(\mathbb{P}^{n-1}) \longrightarrow H^{*,*}(L^n) \longrightarrow H^{*-1,*-1}(\mathbb{P}^{n-1}).$$

From this we conclude that $H^{*,*}(L^n)$ has a basis consisting of elements v^i, uv^i which follows from the identification of the cohomology of $\mathbb{P}^{n-1}$. The description of u^2 can be found in the discussion leading up to lemma 6.8 in [41]. Evaluating the colimit over the system originating in the inclusions $L^n \longrightarrow L^{n+1}$ and using the surjectivity of the resulting maps in the inverse system of cohomology groups, we also cover the cases for $H^{*,*}(L^\infty)$. □

Let $p \neq char(F)$ and assume $\zeta \in F$ is a primitive p-th root of unity. Having chosen one such, we can construct an isomorphism $\mu_p \cong \mathbb{Z}/p$ which in turn leads to a weak equivalence of classifying spaces

$$L^\infty \cong B\mathbb{Z}/p.$$

Using this and the inclusion

$$\mathbb{Z}/p \hookrightarrow \Sigma_p$$

sending $1 \in \mathbb{Z}/p$ to the cycle $(1\cdots p) \in \Sigma_p$, we construct a map

$$p_\zeta : L^\infty \longrightarrow B\Sigma_p.$$

The group Σ_p acts on $\mathbb{A}^p$ fixing the line where all coordinates are equal. From the diagonal inclusion

$$\mathbb{A}^1 \xrightarrow{\Delta} \mathbb{A}^p$$

we may construct an inclusion of vector bundles $L \hookrightarrow \varepsilon^p$ over $B\Sigma_p$. The quotient bundle has fibers $\mathbb{A}^p/\mathbb{A}^1$ and is isomorphic to the sum $\bigoplus_{i=1}^{p-1}(L)^{\otimes i}$.

Definition 4.1.12. *Define*

$$d := e(\frac{\varepsilon^p}{L}) \in \tilde{H}^{2p-2,p-1}(B\Sigma_{p+}),$$

the Euler class of $\frac{\varepsilon^p}{L}$.

We will be needing some results of Voevodsky:

Lemma 4.1.13. *Assuming that* $p \neq char(F)$ *and that* F *contains a primitive* p*-th root of unity, the injective map*

$$p_\zeta^* : \tilde{H}^{*,*}(B\Sigma_{p+}) \longrightarrow \tilde{H}^{*,*}(L_+^\infty),$$

satisfies

$$p_\zeta^*(d) = -v^{p-1}.$$

Proof. This is lemma 6.13 in [41]. □

Theorem 4.1.14. *Assuming that* $p \neq char(F)$*, there exists a unique class*

$$c \in \tilde{H}^{2p-3,p-1}(B\Sigma_{p+})$$

such that $\beta(c) = d$.

Proof. This is theorem 6.14 in [41]. □

Lemma 4.1.15. *Assuming that* $p \neq char(F)$ *and that* F *contains a primitive* p*-th root of unity one has*

$$p_\zeta^*(c) = -uv^{p-2}.$$

Proof. This is lemma 6.15 in [41]. □

Theorem 4.1.16. *If* $p \neq char(F)$*, there are ring isomorphisms*

$$H^{*,*}(B\Sigma_2) \cong \mathbb{M}_2[c,d]/(c^2 + \rho c + \tau d)$$

and

$$H^{*,*}(B\Sigma_p) \cong \mathbb{M}_p[c,d]/(c^2)$$

for p *odd.*

Proof. These are special cases of theorem 6.16 in [41] where the first isomorphism is simply a restatement of lemma 4.1.11. □

Before we move on to the homotopical constructions we record what the motivic power operations do to the classes c, d, u and v: In section 5 of [41] a total power operation, which we denote using the letter P, is defined. It has many of the same properties possessed by the total operations for singular cohomology. In particular, it is a morphism of rings. In what follows we split our considerations for the two cases p odd or even.

Assuming p odd, we have

$$\tilde{H}^{*,*}(L_+^\infty) \cong \mathbb{M}_p[u,v]/(u^2)$$

and

$$\beta(u) = v \text{ and } \beta(v) = 0.$$

It follows from theorem 9.5 and lemma 9.9 in [41] that

$$P(u) = u.$$

From the same two results and lemma 9.8 in the same reference we also have $P(v) = v + v^p = v(1+v^{p-1})$. The multiplicativity of P the implies that we get $P(v^b) = v^b(1+v^{p-1})^b$. Expanding this we see that

$$P^a(v^b) = \binom{b}{a} v^{b+a(p-1)}.$$

Now, from the discussion of p_ζ we saw that $p_\zeta^*(d) = -v^{p-1}$, $p_\zeta^*(c) = -uv^{p-2}$ and $\beta(c) = d$. Since β satisfies $\beta^2 = 0$ we must have $\beta(d) = 0$. Using functoriality and the morphism p_ζ we get

$$\begin{array}{ccc} P(d) & \longmapsto & P(-v^{p-1}) = -P(v)^{p-1} \\ & & \| \\ d(1-d)^{p-1} & \longmapsto & -v^{p-1}(1+v^{p-1})^{p-1}. \end{array}$$

From this and the fact that p_ζ^* is injective we deduce that $P(d) = d(1-d)^{p-1}$, $P(d^b) = d^b(1-d)^{b(p-1)}$ and finally the following lemma:

Lemma 4.1.17. *Assuming p odd, we have*

$$P^a(d^b) = (-1)^a \binom{b(p-1)}{a} d^{a+b}.$$

Similarly we have

$$\begin{array}{ccc} P(c) & \longmapsto & P(-uv^{p-2}) = -P(c)P(v)^{p-2} \\ & & \| \\ c(1-d)^{p-2} & \longmapsto & -uv^{p-2}(1+v^{p-1})^{p-2} \end{array}$$

from which we deduce that

$$P(c) = c(1-d)^{p-2}.$$

From this we deduce the following lemma

Lemma 4.1.18. *Assuming p odd, we have*

$$P^a(cd^{b-1}) = (-1)^a \binom{b(p-1)-1}{a} cd^{a+b-1}$$

Proof. This follows by considering

$$P(cd^{b-1}) = c(1-d)^{p-2} d^{b-1}(1-d)^{(b-1)(p-1)} = cd^{b-1}(1-d)^{b(p-1)-1}$$

and expanding. □

Now, assuming $p = 2$ we repeat the reasoning above: We have

$$\tilde{H}^{*,*}(L_+^\infty) \cong \mathbb{M}_2[u,v]/(u^2 + \rho u + \tau v).$$

For the class u, the Steenrod squares satisfy $Sq^0(u) = u$, $Sq^1(u) = \beta(u) = v$ and $Sq^i(u) = 0$ for $i \geq 2$ from instability (a consequence of lemma 9.9 in [31]). For v we have $Sq^0(v) = v$, $Sq^1(v) = \beta(v) = 0$ since $\beta^2 = 0$, $Sq^2(v) = v^2$ and $Sq^i(u) = 0$ for $i \geq 3$ from instability.

Lemma 4.1.19. *We have*

$$Sq^{2i}(v^k) = \binom{2k}{2i} v^{k+i}, Sq^{2i+1}(v^k) = 0, Sq^{2i}(uv^k) = \binom{2k}{2i} uv^{k+i}$$

$$\text{and } Sq^{2i+1}(uv^k) = \binom{2k}{2i} v^{k+i+1}.$$

Proof. This follows by using the Cartan formula and induction on k. □

4.1.2 The homotopical construction

Moving further, there are spectra we will use for the construction of a model for $R_+(\mathbb{M}_p)$. Recall from the introduction that there were towers of spectra used for calculating the stable homotopy groups of $\mathbb{R}P^\infty_{-\infty}$ using the Adams spectral sequence. This tower may also be realized in another way.

Consider bundles over $\mathbb{R}P^n$, for $n \geq 0$. A point in $\mathbb{R}P^n$ is a line L in $\mathbb{R}^{n+1}$. We get two bundles over $\mathbb{R}P^n$, one line bundle with fiber L and one trivial $(n+1)$-bundle with fiber $\mathbb{R}^{n+1}$, over that point. Call these γ_n^1 (the tautological bundle) and ϵ^{n+1}. The inclusion $L \subset \mathbb{R}^{n+1}$ defines an embedding of bundles $\gamma_n^1 \to \epsilon^{n+1}$.

In topology, we can form orthogonal complements. Let ζ_n be the n-bundle over $\mathbb{R}P^n$ with fiber $L^\perp \subset \mathbb{R}^{n+1}$ over L, so that $\gamma_n^1 \oplus \zeta_n \cong \epsilon^{n+1}$.

Taking k copies of these bundles, we get an embedding

$$k\gamma_n^1 \to k\epsilon^{n+1} = \epsilon^{k(n+1)}$$

of bundles over $\mathbb{R}P^n$. The sum $k\zeta_n$ is then an orthogonal complement, so that

$$k\gamma_n^1 \oplus k\zeta_n \cong k\epsilon^{n+1}.$$

In lemma 4.3 of [4], Atiyah shows that there is a homeomorphism

$$Th(k\gamma_n^1) \cong \mathbb{R}P_k^{n+k} = \mathbb{R}P^{n+k}/\mathbb{R}P^{k-1}$$

for $k \geq 0$. Using the Thom isomorphism, we get $H^*(\mathbb{R}P^n) \cong \tilde{H}^{*+k}(\mathbb{R}P_k^{n+k})$ where $H^*(-)$ is cohomology with $\mathbb{Z}/2$ coefficients. We have $H^*(\mathbb{R}P^n) = P(x)/(x^{n+1}) = \mathbb{Z}/2\{x^i \mid 0 \leq i \leq n\}$ where $x = w_1(\gamma_n^1)$ is the first Stiefel Whitney class of γ_n^1. The quotient map $\mathbb{R}P^{n+k} \to \mathbb{R}P_k^{n+k}$ induces an inclusion

$$\tilde{H}^*(\mathbb{R}P_k^{n+k}) \to H^*(\mathbb{R}P^{n+k})$$

that takes $\tilde{H}^*(\mathbb{R}P_k^{n+k})$ isomorphically to $\mathbb{Z}/2\{x^j \mid k \leq j \leq n+k\}$. Under this identification, the Thom isomorphism is given by multiplication by x^k, taking x^i to x^{i+k}.

The formal relation $k\zeta_n = k\epsilon^{n+1} - k\gamma_n^1$ in $KO(\mathbb{R}P^n)$ suggests that $k\zeta_n$ takes the role of $-k\gamma^1$ plus a trivial bundle. Given a vector bundle ξ and a trivial bundle ϵ^m, we get a homeomorphism $Th(\epsilon^m \oplus \xi) \cong \Sigma^m Th(\xi)$. In our situation we want to extend this so we define

$$Th(-k\gamma_n^1) := \Sigma^{-k(n+1)} Th(k\zeta_n)$$

as a spectrum so that

$$Th(k\zeta_n) \cong Th(k\epsilon^{n+1} - k\gamma_n^1) \cong \Sigma^{k(n+1)} Th(-k\gamma_n^1).$$

This means that $Th(-k\gamma_n^1)$ is a spectrum with $k(n+1)$-th space $Th(k\zeta_n)$, and more generally given at level $k(n+1)+\ell$ by $\Sigma^\ell Th(k\zeta_n)$ for $\ell \geq 0$, while the spaces for $\ell < 0$ are set equal to the base point $*$. We also use the suggestive notation

$$\mathbb{R}P_{-k}^{n-k} := Th(-k\gamma_n^1)$$

(as a spectrum) for $k > 0$. There is a Thom isomorphism $H^*(\mathbb{R}P^n) \cong H^{*-k}(\mathbb{R}P_{-k}^{n-k})$. We have an isomorphism

$$H^{*-k}(\mathbb{R}P_{-k}^{n-k}) \cong \mathbb{Z}/2\{x^j \mid -k \leq j \leq n-k\}$$

if we continue to write the Thom isomorphism as formal multiplication by x^{-k}.

Now we let k grow. The inclusions

$$k\gamma_n^1 \to (k+1)\gamma_n^1$$

obtained by adding $k\gamma_n^1$ to the inclusion

$$0 \to \gamma_n^1$$

induce maps of Thom complexes

$$\mathbb{R}P_k^{n+k} = Th(k\gamma_n^1) \to Th((k+1)\gamma_n^1) = \mathbb{R}P_{k+1}^{n+k+1}$$

for $k \geq 0$. Under the Thom isomorphisms, the induced map in cohomology corresponds to the map

$$H^{*-k-1}(\mathbb{R}P^n) \to H^{*-k}(\mathbb{R}P^n)$$

given by multiplication by the characteristic class $x \in H^1(\mathbb{R}P^n)$. The formal relations

$$(k+1)\zeta_n = (k+1)\epsilon^{n+1} - (k+1)\gamma_n^1$$

and

$$k\zeta_n + \epsilon^{n+1} = (k+1)\epsilon^{n+1} - k\gamma_n$$

in $KO(\mathbb{R}P^n)$ suggest that the inclusion

$$(k+1)\zeta_n \to k\zeta_n + \epsilon^{n+1}$$

obtained by adding $k\zeta_n$ to the inclusion

$$\zeta_n \to \epsilon^{n+1}$$

takes the role of that inclusion after adding a trivial bundle. We define the map of spectra

$$\mathbb{R}P_{-k-1}^{n-k-1} = Th(-(k+1)\gamma_n^1) \to Th(-k\gamma_n^1) = \mathbb{R}P_{-k}^{n-k}$$

to be the map of spectra

$$\Sigma^{-(k+1)(n+1)}Th((k+1)\zeta_n) \to \Sigma^{-(k+1)(n+1)}Th(k\zeta_n \oplus \epsilon^{n+1}) = \Sigma^{-k(n+1)}Th(k\zeta_n)$$

given at level $(k+1)(n+1)$ by the map

$$Th((k+1)\zeta_n) \to Th(k\zeta_n \oplus \epsilon^{n+1})$$

induced by the bundle inclusion

$$(k+1)\zeta_n \to k\zeta_n \oplus \epsilon^{n+1}.$$

The map in cohomology $H^*(\mathbb{R}P^{n-k}_{-k}) \to H^*(\mathbb{R}P^{n-k-1}_{-k-1})$ corresponds under the Thom isomorphism to the map

$$H^{*+k}(\mathbb{R}P^n) \to H^{*+k+1}(\mathbb{R}P^n)$$

given by multiplication by x.

Next we let n grow. The inclusion $\mathbb{R}^{n+1} \subset \mathbb{R}^{n+2}$ defines a map $i : \mathbb{R}P^n \to \mathbb{R}P^{n+1}$, covered by bundle maps $\gamma^1_n \to \gamma^1_{n+1}$ and $\epsilon^{n+1} \oplus \epsilon^1 \to \epsilon^{n+2}$, and isomorphisms $\gamma^1_n \cong i^*\gamma^1_{n+1}$ and $\epsilon^{n+1} \oplus \epsilon^1 \cong i^*(\epsilon^{n+2})$ over $\mathbb{R}P^n$. These are compatible with the inclusions $\gamma^1_n \to \epsilon^{n+1}$ and $\gamma^1_{n+1} \to \epsilon^{n+2}$. They determine a bundle map $\zeta_n \oplus \epsilon^1 \to \zeta_{n+1}$ of orthogonal complements, and an isomorphism $\zeta_n \oplus \epsilon^1 \cong i^*\zeta_{n+1}$. Taking the direct sum of k copies of these maps, we get a bundle map $k\zeta_n \oplus k\epsilon^1 \to k\zeta_{n+1}$ covering $i : \mathbb{R}P^n \to \mathbb{R}P^{n+1}$, and an isomorphism $k\zeta_n \oplus k\epsilon^1 \cong i^*k\zeta_{n+1}$. We define the map of spectra

$$\mathbb{R}P^{n-k}_{-k} = Th(-k\gamma^1_n) \to Th(-k\gamma^1_{n+1}) = \mathbb{R}P^{n+1-k}_{-k}$$

to be the map of spectra

$$\Sigma^{-k(n+1)}Th(k\zeta_n) \to \Sigma^{-k(n+2)}Th(k\zeta_{n+1})$$

given at level $k(n+2)$ by the map

$$\Sigma^k Th(k\zeta_n) \cong Th(k\zeta_n \oplus k\epsilon^1) \to Th(k\zeta_{n+1})$$

induced by the bundle map $k\zeta_n \oplus k\epsilon^1 \to k\zeta_{n+1}$. The map in cohomology $H^*(\mathbb{R}P^{n+1-k}_{-k}) \to H^*(\mathbb{R}P^{n-k}_{-k})$ corresponds under the Thom isomorphism to the surjective homomorphism $H^{*+k}(\mathbb{R}P^{n+1}) \to H^{*+k}(\mathbb{R}P^n)$ that sends x to x but takes x^{n+1} to 0.

Lemma 4.1.20. *The diagram*

$$\begin{array}{ccc} \mathbb{R}P^{n-k-1}_{-k-1} & \longrightarrow & \mathbb{R}P^{n-k}_{-k-1} \\ \downarrow & & \downarrow \\ \mathbb{R}P^{n-k}_{-k} & \longrightarrow & \mathbb{R}P^{n+1-k}_{-k} \end{array}$$

commutes. It induces the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}/2\{x^j \mid -k-1 \le j \le n-k-1\} & \longleftarrow & \mathbb{Z}/2\{x^j \mid -k-1 \le j \le n-k\} \\ \uparrow & & \uparrow \\ \mathbb{Z}/2\{x^j \mid -k \le j \le n-k\} & \longleftarrow & \mathbb{Z}/2\{x^j \mid -k \le j \le n+1-k\} \end{array}$$

in cohomology, where each homomorphism maps x^j to x^j, if defined, and otherwise to 0.

Proof. Over $\mathbb{R}P^n$, start with the inclusion $(k+1)\zeta_n \hookrightarrow k\zeta_n \oplus \epsilon^{n+1}$ where the last ζ_n is included in its defining trivial bundle ϵ^{n+1}. Then add the sum $(k+1)\epsilon^1$ so we get an inclusion

$$(k+1)\zeta_n \oplus (k+1)\epsilon^1 \hookrightarrow k\zeta_n \oplus \epsilon^{n+1} \oplus (k+1)\epsilon^1.$$

Using the bundle map $k\zeta_n \oplus k\epsilon^1 \to k\zeta_{n+1}$ covering $i : \mathbb{R}P^n \to \mathbb{R}P^{n+1}$ and the isomorphism $k\zeta_n \oplus k\epsilon^1 \cong i^* k\zeta_{n+1}$, we get a commutative diagram

$$\begin{array}{ccc}
(k+1)\zeta_n \oplus (k+1)\epsilon^1 & \longrightarrow & (k+1)\zeta_{n+1} \\
\downarrow & & \downarrow \\
k\zeta_n \oplus \epsilon^{n+1} \oplus (k+1)\epsilon^1 & & k\zeta_{n+1} \oplus \epsilon^{n+2} \\
& \searrow & \| \\
& & k\zeta_{n+1} \oplus \epsilon^{n+1} \oplus \epsilon^1
\end{array}$$

where the isomorphism $k\zeta_{n+1} \oplus \epsilon^{n+2} \cong k\zeta_{n+1} \oplus \epsilon^{n+1} \oplus \epsilon^1$ is taking place inside $(k+1)\epsilon^{n+2}$ over $\mathbb{R}P^{n+1}$. In the end, this commutative diagram induces a commutative square of Thom spaces and we recover the stated square we were after. We have already seen the cohomological identifications and how the powers of x map when varying n and k. □

Momentarily fixing k, we now define the spectrum $\mathbb{R}P^\infty_{-k}$ as the homotopy colimit

$$\mathbb{R}P^\infty_{-k} = \operatorname{hocolim}_n \mathbb{R}P^{n-k}_{-k}$$

of the diagram of spectra

$$\mathbb{R}P^{-k}_{-k} \to \cdots \to \mathbb{R}P^{n-k}_{-k} \to \mathbb{R}P^{n+1-k}_{-k} \to \dots .$$

We obtain an isomorphism

$$H^*(\mathbb{R}P^\infty_{-k}) \cong \lim_n H^*(\mathbb{R}P^{n-k}_{-k}) \cong \mathbb{Z}/2\{x^j \mid -k \le j\}\,.$$

We note that each $\mathbb{R}P^\infty_{-k}$ is bounded below and of finite type in the sense that $\pi_*(\mathbb{R}P^\infty_{-k})$ is bounded below and $H_*(\mathbb{R}P^\infty_{-k})$ is a finite dimensional vector space over $\mathbb{Z}/2$ in each degree. The spectrum $\mathbb{R}P^\infty_0$ is the suspension spectrum of $\mathbb{R}P^\infty_+$.

Letting k vary, again, the vertical arrows in the diagram of the lemma induce a diagram of spectra

$$\cdots \to \mathbb{R}P^\infty_{-k-1} \to \mathbb{R}P^\infty_{-k} \to \cdots \to \mathbb{R}P^\infty_0\,.$$

We define $\mathbb{R}P^\infty_{-\infty}$ as the homotopy limit

$$\mathbb{R}P^\infty_{-\infty} = \operatorname{holim}_k \mathbb{R}P^\infty_{-k}$$

of this diagram. Its continuous cohomology is defined to be the colimit

$$H_c^*(\mathbb{R}P_{-\infty}^{\infty}) := \mathrm{colim}_k\, H^*(\mathbb{R}P_{-k}^{\infty}) \cong \mathbb{Z}/2\{x^j \mid j \in \mathbb{Z}\} = P(x, x^{-1})\,.$$

The map $\mathbb{R}P_{-\infty}^{\infty} \to \mathbb{R}P_0^{\infty}$ induces the usual homomorphism $P(x) \to P(x, x^{-1})$ in cohomology.

With all this in mind, the constructions seen here are the ones that generalize to the motivic setting although there is a slight twist to the motivic substitute for $\mathbb{R}P_{-l}^{\infty}$ since we do not have a notion of orthogonality for motivic spaces. We start with considering bundles over L^n for $n \geq 0$. From the beginning of this chapter we remember the bundle γ_{n-1}^1 as the pullback of γ_{n-1}^1 from $\mathbb{P}^{n-1}$. It can be viewed as $(\mathbb{A}^n \backslash 0) \times_{\mu_p} \mathbb{A}^1$ over L^n where we identify $(\lambda x_1, \dots, \lambda x_n, y)$ and $(x_1, \dots, x_n, \lambda y)$ where $(x_1, \dots, x_n, y) \in (\mathbb{A}^n \setminus 0) \times \mathbb{A}^1$ and $\lambda \in \mu_p$. From the inclusion $\mathbb{A}^1 \subset \mathbb{A}^n$ we can construct an embedding of γ_{n-1}^1 into the trivial vector bundle ε^n over L^n by sending

$$(x_1, \dots, x_n, y) \mapsto (x_1, \dots, x_n, x_1 y, \dots, x_n y) \in L^n \times \mathbb{A}^n.$$

In our situation, we cannot form orthogonal complements in order to extend the definition of $Th(-k\gamma_n^1)$ from the topological situation. Instead, we consider ordinary complements.

Definition 4.1.21. Let $\eta \hookrightarrow \xi$ be an inclusion of vector bundles. Then we define

$$Th(\xi, \eta) := \frac{E(\xi)}{E(\xi) \setminus E(\eta)}$$

Lemma 4.1.22. *Given an isomorphism of vector bundles $\eta \oplus \zeta \cong \xi$ over a smooth scheme X, then we have $\mathbb{A}^1$-homotopy equivalences*

$$E(\zeta) \longrightarrow E(\xi)$$

and

$$E(\zeta) \setminus 0 \longrightarrow E(\xi) \setminus E(\eta),$$

hence the map

$$Th(\zeta) \longrightarrow Th(\xi, \eta)$$

is also such an equivalence.

Proof. It is obvious that ξ is a vector bundle over ζ and by remark 4.1.4 it is an $\mathbb{A}^1$-homotopy equivalence. As for the other case, the fiber bundle $E(\xi) \setminus E(\eta)$ is locally of the form $\mathbb{A}^n \setminus \mathbb{A}^m \cong \mathbb{A}^m \times (\mathbb{A}^{n-m} \setminus 0)$ so from the same remark, we have an $\mathbb{A}^1$-homotopy equivalence

$$E(\zeta) \setminus 0 \longrightarrow E(\xi) \setminus E(\eta).$$

□

Consider again the embedding of γ^1_{n-1} into the trivial vector bundle ε^n over L^n. We can form the k-fold sum and define the relevant spectrum with which to work:

Definition 4.1.23. *For $n \geq 0$ and $k \geq 0$, let $\underline{L}^{n-k}_{-k}$ be the motivic spectrum*

$$\underline{L}^{n-k}_{-k} = \Sigma_T^{-kn} Th(k\varepsilon^n, k\gamma^1_{n-1})$$

This means that $\underline{L}^{n-k}_{-k}$ is a motivic spectrum with kn-th space

$$\frac{E(k\varepsilon^n \downarrow L^n)}{E(k\varepsilon^n \downarrow L^n) \setminus E(k\gamma^1 \downarrow L^n)},$$

and more generally given at level $kn+\ell$ by $\Sigma^\ell_T Th(k\varepsilon^n, k\gamma^1_{n-1})$ for $\ell \geq 0$, while the spaces for $\ell < 0$ are set equal to the base point $*$.

We identify the cohomology groups of $\underline{L}^{n-k}_{-k}$:

Proposition 4.1.24. *As modules over $\mathbb{M}_2$ we have an isomorphism*

$$H^{*,*}(\underline{L}^{n-k}_{-k}) \cong \Sigma^{-(2k,k)} \mathbb{M}_2[u,v]/(u^2 + \rho u + \tau v, v^n)$$

where $\operatorname{bideg}(u) = (1,1)$ *and* $\operatorname{bideg}(v) = (2,1)$.

Proof. We saw the structure of the cohomology of Lens spaces as modules over $\mathbb{M}_p$ in lemma 4.1.11 and we had

$$H^{*,*}(L^n) \cong \mathbb{M}_2[u,v]/(u^2 + \rho u + \tau v, v^n).$$

Also, there is a chain of isomorphisms

$$\begin{array}{ccc} H^{*,*}\left(\frac{E(k\varepsilon^n)}{E(k\varepsilon^n)\setminus E(k\gamma^1_{n-1})}\right) & \xleftarrow{\cong} & H^{*,*}(Th(N_{k\varepsilon^n, k\gamma^1_{n-1}})) \\ & & \uparrow {\scriptstyle *t(N_{k\varepsilon^n, k\gamma^1_{n-1}})} \\ H^{*-2k(n-1),*-k(n-1)}(L^n) & \xrightarrow{\cong} & H^{*-2k(n-1),*-k(n-1)}(E(k\gamma^1_{n-1})) \end{array}$$

given by the purity equivalence, the Thom isomorphism and the fact that locally affine maps are $\mathbb{A}^1$ homotopy equivalences. Hence, given the suspension isomorphism

$$H^{*,*}(\underline{L}^{n-k}_{-k}) \cong H^{*+2kn,*+kn}(Th(k\varepsilon^n, k\gamma^1_{n-1})),$$

the result follows. □

Now we let k grow and for this we need to study the inclusions

$$E((k-1)\gamma^1_{n-1}) \subset E(k\gamma^1_{n-1})$$

over L^n. The following result of Voevodsky will be crucial.

Lemma 4.1.25. *Let $Z \subset Y \subset X$ be smooth embeddings of closed subschemes. There is an induced map of motivic spaces*

$$\pi : \frac{X}{X \setminus Y} \longrightarrow \frac{X}{X \setminus Z}.$$

For the induced map on motivic cohomology we have

$$\pi^*(a_{X,Z}) = \rho^*_{X,Y}(t(N_{X,Y}) * a_{Y,Z})$$

where $a_{X,Z}$ is the image of the Thom class $t(N_{X,Z})$ under $\rho^*_{X,Z}$ and $a_{Y,Z} \in \tilde{H}^{2c(Y,Z),c(Y,Z)}(Y)$ corresponds to the Thom class

$$t(N_{Y,Z}) \in \tilde{H}^{2c(Y,Z),c(Y,Z)}(Th(N_{Y,Z}))$$

under the map

$$Y \longrightarrow \frac{Y}{Y \setminus Z} \longrightarrow Th(N_{Y,Z})$$

and $c(Y,Z)$ is the codimension of Z in Y.

Proof. This is lemma 2.4 of [40]. The product $t(N_{X,Y}) * a_{Y,Z} \in H^{*,*}(Y)$ is formed using the Thom diagonal

$$Th(N_{Y,Z}) \longrightarrow Y_+ \wedge Th(N_{Y,Z})$$

and evaluating cohomology. □

Now, from the inclusions

$$E((k-1)\gamma^1_{n-1}) \subset E(k\gamma^1_{n-1}) \subset E(k\varepsilon^n)$$

we go on to form

$$\begin{array}{ccc}
E(k\varepsilon^n \downarrow L^n) \setminus E(k\gamma^1 \downarrow L^n) & \longrightarrow & E(k\varepsilon^n \downarrow L^n) \setminus E((k-1)\gamma^1 \downarrow L^n) \\
\downarrow & & \downarrow \\
E(k\varepsilon^n \downarrow L^n) & = & E(k\varepsilon^n \downarrow L^n) \\
\downarrow & & \downarrow \\
Th(k\varepsilon^n, k\gamma^1_{n-1}) & \dashrightarrow & Th(k\varepsilon^n, (k-1)\gamma^1_{n-1}) \\
 & & \downarrow g \\
 & & \Sigma^n_T Th((k-1)\varepsilon^n, (k-1)\gamma^1_{n-1})
\end{array}$$

Lemma 4.1.26. *g is an isomorphism in $H_\bullet(F)$.*

Proof. Let us consider the following situation: Let $A \hookrightarrow X$ and $B \hookrightarrow Y$ be closed immersions in Sm/F. Then there is a Nisnevich elementary distinguished square (see [31] for the definition)

$$\begin{array}{ccc} (X \setminus A) \times (Y \setminus B) & \longrightarrow & X \times (Y \setminus B) \\ \downarrow & & \downarrow \\ (X \setminus A) \times Y & \longrightarrow & (X \times Y) \setminus (A \times B) \end{array}$$

coming from the cover of $(X \times Y) \setminus (A \times B)$. Given this, there is an isomorphism

$$(X \times Y) \setminus (A \times B) \cong (X \times (Y \setminus B)) \bigcup_{(X \setminus A) \times (Y \setminus B)} ((X \setminus A) \times Y)$$

since such squares are pushouts (lemma 1.6 on page 98 in [31]). Hence we get

$$\frac{X \times Y}{(X \times (Y \setminus B)) \bigcup_{(X \setminus A) \times (Y \setminus B)} ((X \setminus A) \times Y)} \cong \frac{X \times Y}{(X \times Y) \setminus (A \times B)}.$$

Next, we let $X = E((k-1)\varepsilon^n \downarrow L^n)$, $Y = \mathbb{A}^n$, $A = E((k-1)\gamma^1 \downarrow L^n)$ and B be the origin. To conclude, we use the isomorphism

$$E((k-1)\varepsilon^n \downarrow L^n) \times \mathbb{A}^n \cong E(k\varepsilon^n \downarrow L^n)$$

and the identification

$$\Sigma_T^n \frac{E((k-1)\varepsilon^n \downarrow L^n)}{E((k-1)\varepsilon^n \downarrow L^n) \setminus E((k-1)\gamma^1 \downarrow L^n)} = \frac{E((k-1)\varepsilon^n \downarrow L^n)}{E((k-1)\varepsilon^n \downarrow L^n) \setminus E((k-1)\gamma^1 \downarrow L^n)} \wedge \frac{\mathbb{A}^n}{\mathbb{A}^n \setminus 0}.$$

□

In the end, the map

$$\underline{L}_{-k}^{n-k} \longrightarrow \underline{L}_{-k+1}^{n-k+1}$$

is then

$$\Sigma_T^{-kn} Th(k\varepsilon^n, k\gamma_{n-1}^1) \to \Sigma_T^{-kn} Th(k\varepsilon^n, (k-1)\gamma_{n-1}^1) \cong \Sigma_T^{-(k-1)n} Th((k-1)\varepsilon^n, (k-1)\gamma_{n-1}^1)$$

given at level kn by the map

$$Th(k\varepsilon^n, k\gamma_{n-1}^1) \to Th(k\varepsilon^n, (k-1)\gamma_{n-1}^1)$$

induced by the inclusion $E((k-1)\gamma^1) \subset E(k\gamma^1)$ over L^n.

The map in motivic cohomology can be decribed using lemma 4.1.25:

Proposition 4.1.27. *The associated homomorphism of $\mathbb{M}_2$-modules*

$$H^{*,*}(\underline{L}^{n-k+1}_{-k+1}) \longrightarrow H^{*,*}(\underline{L}^{n-k}_{-k})$$

takes the classes v^i and uv^i to v^{i+1} and uv^{i+1}.

Proof. The map $\underline{L}^{n-k}_{-k} \longrightarrow \underline{L}^{n-k+1}_{-k+1}$ was given by the composite

$$Th(k\varepsilon^n, k\gamma^1_{n-1}) \dashrightarrow Th(k\varepsilon^n, (k-1)\gamma^1_{n-1}).$$

To see the effect on cohomology we study the diagram

$$\begin{array}{ccc}
H^{*,*}(Th(k\varepsilon^n, k\gamma^1_{n-1})) & \xleftarrow{\pi^*} & H^{*,*}(Th(k\varepsilon^n, (k-1)\gamma^1_{n-1})) \\
\uparrow {\scriptstyle *t(N_{k\varepsilon^n, k\gamma^1_{n-1}})} & & \uparrow {\scriptstyle *t(N_{k\varepsilon^n, (k-1)\gamma^1_{n-1}})} \\
\Sigma^{-(2k(n-1),k(n-1))} H^{*,*}(E(k\gamma^1_{n-1})) & & \Sigma^{-(2k(n-1)+1),k(n-1)+1)} H^{*,*}(E((k-1)\gamma^1_{n-1})) \\
\uparrow \cong & & \uparrow \cong \\
\Sigma^{-(2k(n-1),k(n-1))} H^{*,*}(L^n) & & \Sigma^{-(2k(n-1)+1),k(n-1)+1)} H^{*,*}(L^n)
\end{array}$$

Here the upper vertical maps arise from the Thom isomorphism, the lower vertical ones come from the fact that vector bundles are $\mathbb{A}^1$-homotopy equivalences and π is the map from lemma 4.1.25. From that result we know that

$$\pi^*(a_{k\varepsilon^n,(k-1)\gamma^1_{n-1}}) = \rho^*_{k\varepsilon^n, k\gamma^1_{n-1}}(t(N_{k\varepsilon^n, k\gamma^1_n}) * a_{k\gamma^1,(k-1)\gamma^1_n}).$$

The functoriality of the classes involved means that all classes being multiplied with a class of bidegree $(2,1)$ corresponding to the class $a_{k\gamma^1_{n-1},(k-1)\gamma^1_{n-1}} \in \tilde{H}^{2,1}(k\gamma^1_{n-1})$, in turn corresponding to the Thom class

$$t(N_{k\gamma^1_n,(k-1)\gamma^1_{n-1}}) \in \tilde{H}^{2,1}(Th(N_{k\gamma^1_n,(k-1)\gamma^1_{n-1}}))$$

under the map

$$E(k\gamma^1_{n-1}) \longrightarrow \frac{E(k\gamma^1_{n-1})}{E(k\gamma^1_{n-1}) \setminus E((k-1)\gamma^1_{n-1})} \longrightarrow N_{k\gamma^1_{n-1},(k-1)\gamma^1_{n-1}}.$$

□

Next we let n grow. The inclusion $\mathbb{A}^n \subset \mathbb{A}^{n+1}$ defines a map

$$\iota : L^n \to L^{n+1},$$

covered by bundle maps $\gamma^1_{n-1} \to \gamma^1_n$ and $\varepsilon^n \oplus \varepsilon^1 \to \varepsilon^{n+1}$, and we have isomorphisms $\gamma^1_{n-1} \cong \iota^*\gamma^1_n$ and $\varepsilon^n \oplus \varepsilon^1 \cong \iota^*(\varepsilon^{n+1})$ over L^n. These are compatible with the inclusions $\gamma^1_{n-1} \to \varepsilon^n$ and $\gamma^1_n \to \varepsilon^{n+1}$.

Now, we consider the inclusions of bundles

$$E(\gamma_n^1) \subset E(k\varepsilon^{n+1})$$

over L^{n+1}. If we pull both of these bundles back over ι, the bundle $k\varepsilon^{n+1}$ splits to become $k\varepsilon^n \oplus k\varepsilon^1$ and since we have $k\gamma_{n-1}^1 \cong \iota^* k\gamma_n^1$, we get an induced map

$$Th(k\varepsilon^n \oplus k\varepsilon^1, k\gamma_{n-1}^1) \to Th(k\varepsilon^{n+1}, k\gamma_n^1)$$

covering

$$\iota : L^n \to L^{n+1}.$$

Lemma 4.1.28. *In $H_\bullet(F)$, there is an isomorphism*

$$Th(k\varepsilon^n \oplus k\varepsilon^1, k\gamma_{n-1}^1) \cong \Sigma_T^k Th(k\varepsilon^n, k\gamma_{n-1}^1).$$

Proof. This follows as in lemma 4.1.26: We use the diagram

$$\begin{array}{ccc} (X \setminus A) \times (Y \setminus B) & \longrightarrow & X \times (Y \setminus B) \\ \downarrow & & \downarrow \\ (X \setminus A) \times Y & \longrightarrow & (X \times Y) \setminus (A \times B) \end{array}$$

where we let $X = E(k\varepsilon^n \downarrow L^n)$, $Y = \mathbb{A}^k$, $A = E(k\gamma^1 \downarrow L^n)$ and B be the origin. Again, we use the identification

$$\Sigma_T^k \frac{E(k\varepsilon^n \downarrow L^n)}{E(k\varepsilon^n \downarrow L^n) \setminus E(k\gamma^1 \downarrow L^n)} = \frac{E(k\varepsilon^n \downarrow L^n)}{E(k\varepsilon^n \downarrow L^n) \setminus E(k\gamma^1 \downarrow L^n)} \wedge \frac{\mathbb{A}^k}{\mathbb{A}^k \setminus 0}$$

and the isomorphism

$$\frac{X \times Y}{(X \times (Y \setminus B)) \bigcup_{(X \setminus A) \times (Y \setminus B)} ((X \setminus A) \times Y)} \cong \frac{X \times Y}{(X \times Y) \setminus (A \times B)}.$$

□

Hence the map

$$\underline{L}_{-k}^{n-k} \longrightarrow \underline{L}_{-k}^{n-k+1}$$

is given at level $k(n+1)$ by the map

$$\Sigma_T^k Th(k\varepsilon^n, k\gamma_{n-1}^1) \cong Th(k\varepsilon^n \oplus k\varepsilon^1, k\gamma_{n-1}^1) \to Th(k\varepsilon^{n+1}, k\gamma_n^1).$$

Proposition 4.1.29. *The associated homomorphism of $\mathbb{M}_2$-modules*

$$H^{*,*}(\underline{L}_{-k}^{n-k+1}) \longrightarrow H^{*,*}(\underline{L}_{-k}^{n-k})$$

takes the classes v^j to v^j and uv^j to uv^j when $j \leq n$ but takes v^{n+1} and uv^{n+1} to 0.

Proof. To see the effect on cohomology we study the diagram

$$\begin{array}{ccc}
H^{*,*}(\Sigma_T^k Th(k\varepsilon^n, k\gamma^1_{n-1})) & \longleftarrow & H^{*,*}(Th(k\varepsilon^{n+1}, k\gamma^1_n)) \\
\uparrow & & \uparrow \\
\Sigma^{-kn(2,1)}H^{*,*}(E(k\gamma^1_{n-1})) & \longleftarrow & \Sigma^{-kn(2,1)}H^{*,*}(E(k\gamma^1_n)) \\
\uparrow & & \uparrow \\
\Sigma^{-kn(2,1)}H^{*,*}(L^n) & \longleftarrow & \Sigma^{-kn(2,1)}H^{*,*}(L^{n+1})
\end{array}$$

We explain the maps: The lower horizontal map is the standard projection $\Sigma^{-kn(2,1)}H^{*,*}(L^{n+1}) \to \Sigma^{-kn(2,1)}H^{*,*}(L^n)$ that sends v^j to v^j and uv^j to uv^j when $j \leq n$ but takes v^{n+1} and uv^{n+1} to 0. The upper vertical maps result from the Thom isomorphim although the left map is composed with the obvious suspension isomorphism. The lower vertical maps are the isomorphisms induced from the $\mathbb{A}^1$-homotopies coming from the bundles $k\gamma^1_n$ and $k\gamma^1_{n-1}$. Commutativity follows since

$$\Sigma_T^k Th(k\varepsilon^n, k\gamma^1_{n-1}) \cong Th(k\varepsilon^n \oplus k\varepsilon^1, k\gamma^1_{n-1}) \to Th(k\varepsilon^{n+1}, k\gamma^1_n)$$

covers

$$\iota : L^n \to L^{n+1}.$$

□

Lemma 4.1.30. *The diagram*

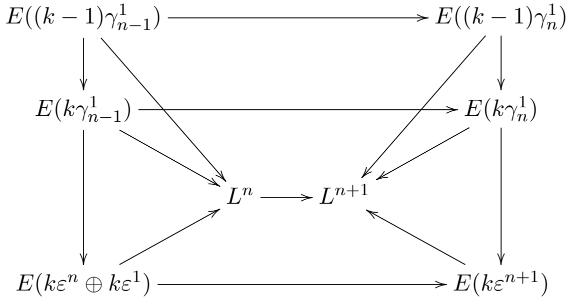
$$\begin{array}{ccc}
\underline{L}^{n-k}_{-k} & \longrightarrow & \underline{L}^{n-k+1}_{-k} \\
\downarrow & & \downarrow \\
\underline{L}^{n-k+1}_{-k+1} & \longrightarrow & \underline{L}^{n-k+2}_{-k+1}
\end{array}$$

commutes. It induces the commutative diagram

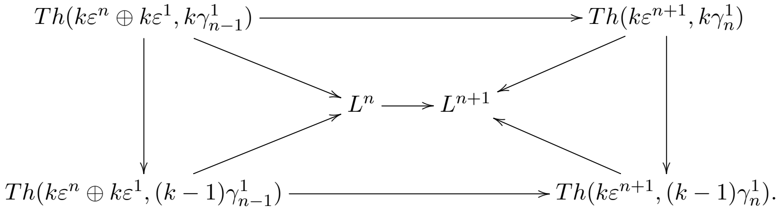
$$\begin{array}{ccc}
\Sigma^{-(2k,k)}\mathbb{M}_2[u,v]/(u^2+\rho u+\tau v, v^n) & \longleftarrow & \Sigma^{-(2k,k)}\mathbb{M}_2[u,v]/(u^2+\rho u+\tau v, v^{n+1}) \\
\uparrow & & \uparrow \\
\Sigma^{-(2k-2,k-1)}\mathbb{M}_2[u,v]/(u^2+\rho u+\tau v, v^n) & \longleftarrow & \Sigma^{-(2k-2,k-1)}\mathbb{M}_2[u,v]/(u^2+\rho u+\tau v, v^{n+1})
\end{array}$$

in cohomology, where each homomorphism maps v^j to v^j and uv^j to uv^j, if defined, and otherwise to 0.

Proof. We start with the commutative diagram



where the bundles on the left-hand side are the ones arising from pulling back the bundles on the right-hand side of the inclusion $L^n \to L^{n+1}$. It induces a commutative diagram



Since we have

$$Th(k\varepsilon^n \oplus k\varepsilon^1, k\gamma^1_{n-1}) \cong \Sigma^k_T Th(k\varepsilon^n, k\gamma^1_{n-1}),$$

$$Th(k\varepsilon^n \oplus k\varepsilon^1, (k-1)\gamma^1_{n-1}) \cong \Sigma^{n+k}_T Th((k-1)\varepsilon^n, k\gamma^1_{n-1})$$

and

$$Th(k\varepsilon^{n+1}, (k-1)\gamma^1_n) \cong \Sigma^{n+1}_T Th((k-1)\varepsilon^{n+1}, k\gamma^1_{n-1})$$

by lemmas 4.1.26 and 4.1.28, we get a commutative diagram of spectra

$$\begin{array}{ccc}
\Sigma_T^{-k(n+1)} \underline{L}^{n-k}_{-k} & \longrightarrow & \Sigma_T^{-k(n+1)} \underline{L}^{n-k+1}_{-k} \\
\downarrow & & \downarrow \\
\Sigma_T^{-k(n+1)} \underline{L}^{n-k+1}_{-k+1} & \longrightarrow & \Sigma_T^{-k(n+1)} \underline{L}^{n-k+2}_{-k+1}.
\end{array}$$

The induced maps after evaluating cohomology were described in lemmas 4.1.27 and 4.1.29. □

Momentarily fixing k, we now define the spectrum $\underline{L}^{\infty}_{-k}$ as the homotopy colimit

$$\underline{L}^{\infty}_{-k} = \operatorname{hocolim}_n \underline{L}^{n-k}_{-k}$$

of the diagram of spectra

$$\underline{L}^{-k}_{-k} \to \cdots \to \underline{L}^{n-k}_{-k} \to \underline{L}^{n+1-k}_{-k} \to \cdots .$$

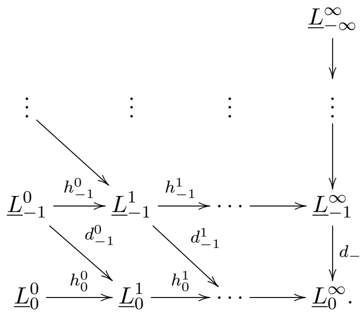
The spectrum $\underline{L}^{\infty}_{0}$ is the suspension spectrum of L^{∞}_{+}. Letting k vary, again, the vertical arrows in the diagram of the lemma induce a diagram of spectra

$$\cdots \to \underline{L}^{\infty}_{-k-1} \to \underline{L}^{\infty}_{-k} \to \cdots \to \underline{L}^{\infty}_{0} .$$

We define $\underline{L}^{\infty}_{-\infty}$ as the homotopy limit

$$\underline{L}^{\infty}_{-\infty} = \operatorname{holim}_k \underline{L}^{\infty}_{-k}$$

of this diagram. All these spectra and the maps between them can be represented in the following diagram:



Definition 4.1.31. *Given an inverse tower of motivic spectra*

$$Y \longrightarrow \cdots \longrightarrow Y_{-s-1} \longrightarrow Y_{-s} \longrightarrow \cdots \longrightarrow Y_0$$

with $Y := \operatorname{holim}_n Y_n$, *we define the continous cohomology of* Y *to be*

$$H^{*,*}_c(Y) := \operatorname{colim}_n H^{*,*}(Y_n).$$

Proposition 4.1.32. *As modules over* $\mathbb{M}_2$ *we have*

$$H^{*,*}(\underline{L}^{\infty}_{-k}) \cong \lim_n H^{*,*}(\underline{L}^{n-k}_{-k}) \cong \Sigma^{-(2k,k)}\mathbb{M}_2[u,v]/(u^2+\rho u+\tau v) .$$

and

$$H^{*,*}_c(\underline{L}^{\infty}_{-\infty}) \cong \mathbb{M}_2[u,v,v^{-1}]/(u^2+\rho u+\tau v),$$

where $\operatorname{bideg}(u) = (1,1)$ *and* $\operatorname{bideg}(v) = (2,1)$.

Proof. These are consequences of lemma 4.1.30. □

We note that each $\underline{L}^{\infty}_{-k}$ is bounded below and of finite type in the sense that $\pi_{*,*}(\underline{L}^{\infty}_{-k})$ is bounded below in topological degrees and $H_{*,*}(\underline{L}^{\infty}_{-k})$ is finite dimensional over $\mathbb{M}_2$ in each degree.

Before stating the last result of this section, we discuss the action of A on

$$R_+(\mathbb{M}_p) := \operatorname*{colim}_{n\to\infty} B(n) \otimes_{A(n-1)} \mathbb{M}_p \cong \mathbb{M}_p\{\beta^{\epsilon}P^k : \epsilon \in \{0,1\}, k \in \mathbb{Z}\}.$$

Although the case where p is odd is not used in our conclusions we show that these cases work out algebraically in case a suitable tower of spectra should arise at a later point in time. A little lemma is needed for these calculations.

Lemma 4.1.33. *Given a a prime p and assume $a < p^n$. The function sending $z \in \mathbb{Z}$ to $\underline{\binom{z}{a}} \in \mathbb{Z}/p$ is periodic with period p^n.*

Proof. We write $(1+x)^z = \Sigma\binom{z}{a}x^a$. Since $(1+x)^{p^n} = 1 + x^{p^n}$ in $\mathbb{Z}/p[x]$, we get $(1+x)^{z+p^n} = (1+x)^z(1+x^{p^n})$. Given that $a < p^n$, the coefficients of x^a in the expressions $(1+x)^z$ and $(1+x)^z(1+x^{p^n})$ are the same and hence

$$\binom{z+p^n}{a} \equiv \binom{z}{a} (\text{mod } p).$$

□

First, we assume $b \geq 0$. Given any $P^a \in A(n) \subset A$, that is $0 \leq a < p^n$, its action on

$$P^b \otimes 1 \in B(n) \otimes_{A(n-1)} \mathbb{M}_p \cong R_+(\mathbb{M}_p)$$

is given by $P^a(P^b \otimes 1) = P^aP^b \otimes 1$. We use the Adem relations to expand the product P^aP^b and the tensor splits as the sum

$$P^aP^b \otimes 1 = \sum_{i=0}^{\lfloor a/p \rfloor} (-1)^{a+i}\binom{(p-1)(b-i)-1}{a-pi} P^{a+b-i} \otimes P^i(1)$$

where P^i can be moved across the tensor since $i \leq \lfloor a/p \rfloor < p^{n-1}$. Now, $P^i(1) = 0$ except when $i = 0$, hence we get

$$P^aP^b \otimes 1 = (-1)^a\binom{(p-1)b-1}{a} P^{a+b} \otimes 1$$

Similarly, we have $P^a(\beta P^b \otimes 1) = P^a\beta P^b \otimes 1$ leading to the expansion

$$\begin{aligned} P^a\beta P^b \otimes 1 = &\sum_{i=0}^{\lfloor a/p \rfloor} (-1)^{a+i} \binom{(p-1)(b-i)}{a-pi} \beta P^{a+b-i} \otimes P^i(1) \\ &+ \sum_{i=0}^{\lfloor (a-1)/p \rfloor} (-1)^{a+i-1} \binom{(p-1)(b-i)-1}{a-pi-1} P^{a+b-i} \otimes \beta P^i(1) \end{aligned}$$

with the only nonzero summand being the one where $i = 0$. Thus we have

$$P^a\beta P^b \otimes 1 = (-1)^a \binom{(p-1)b}{a} \beta P^{a+b} \otimes 1.$$

Our next task is to consider negative b. If $b+kp^n \geq 0$, the element $\beta^\epsilon P^b \otimes 1 \in B(n) \otimes_{A(n-1)} \mathbb{M}_p$ maps to $\beta^\epsilon P^{b+kp^n} \otimes 1 \in C(n) \otimes_{A(n-1)} \mathbb{M}_p$ in the defining inverse system

$$B(n) \otimes_{A(n-1)} \mathbb{M}_p \to \cdots C(n) \otimes_{A(n-1)} \mathbb{M}_p \to \cdots \to C(n) \otimes_{A(n-1)} \mathbb{M}_p.$$

Hence $P^a(\beta^\epsilon P^b \otimes 1)$ maps to $P^a(\beta^\epsilon P^{b+kp^n} \otimes 1) \in C(n) \otimes_{A(n-1)} \mathbb{M}_p$ for $b + kp^n \geq 0$. Here we may use the formulas for positive b as we did above and since the binomial coefficients involved are periodic as in lemma 4.1.33, we see that the expressions $P^a(P^b \otimes 1)$ and

$$(-1)^a \binom{(p-1)b-1}{a} P^{a+b} \otimes 1$$

have the same image in $C(n) \otimes_{A(n-1)} \mathbb{M}_p$ for $b + kp^n \geq 0$ and hence they must be the same element in $R_+(\mathbb{M}_p)$. The same reasoning shows that

$$P^a\beta P^b \otimes 1 = (-1)^a \binom{(p-1)b}{a} \beta P^{a+b} \otimes 1$$

is also valid for negative b. Summarizing, we get the following result:

Proposition 4.1.34. *Let p be an odd prime. Then we have*

$$P^a P^b \otimes 1 = (-1)^a \binom{(p-1)b-1}{a} P^{a+b} \otimes 1$$

and

$$P^a\beta P^b \otimes 1 = (-1)^a \binom{(p-1)b}{a} \beta P^{a+b} \otimes 1$$

for all $b \in \mathbb{Z}$.

For $p = 2$ we argue as before: Taking $Sq^a \in A(n) \subset A$, its action on

$$Sq^b \otimes 1 \in B(n) \otimes_{A(n-1)} \mathbb{M}_2 \cong R_+(\mathbb{M}_2)$$

is given by $Sq^a(Sq^b \otimes 1) = Sq^a Sq^b \otimes 1$. We use the Adem relations to expand the product and the tensor splits as the sum

$$Sq^a Sq^b \otimes 1 = \sum_{i=0}^{\lfloor a/2 \rfloor} \binom{b-1-i}{a-2i} Sq^{a+b-i} \otimes Sq^i(1)$$

if a is odd and

$$\begin{aligned} Sq^a Sq^b \otimes 1 = & \sum_{i=0}^{\lfloor a/2 \rfloor} \tau^{\varepsilon_i} \binom{b-1-i}{a-2i} Sq^{a+b-i} Sq^i(1) \\ & + \rho \sum_{i=1, i \equiv b(2)}^{\lfloor a/2 \rfloor} \binom{b-1-i}{a-2i} Sq^{a+b-i-1} Sq^i(1) \end{aligned}$$

if a is even. Here we remind ourselves that we took

$$\varepsilon_i = \begin{cases} 1 & \text{if } b \text{ is even and } i \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

We have $Sq^i(1) = 0$ except for when $i = 0$, hence we get

$$Sq^a Sq^b \otimes 1 = \binom{b-1}{a} Sq^{a+b} \otimes 1.$$

The discussion of the action of A on $Sq^b \otimes 1$ for negative b is entirely the same as in the odd case and the formula

$$Sq^a Sq^b \otimes 1 = \binom{b-1}{a} Sq^{a+b} \otimes 1$$

is valid in these cases as well.

Proposition 4.1.35. *For $p = 2$ we have*

$$Sq^a Sq^b \otimes 1 = \binom{b-1}{a} Sq^{a+b} \otimes 1$$

for all $b \in \mathbb{Z}$.

Before we state the next proposition we need a simple observation on binomial coefficients modulo 2:

Lemma 4.1.36. *If m is even and n is odd then we have $\binom{m}{n} \equiv 0$ modulo 2.*

Proof. We have $\binom{m}{n} = \frac{m}{n}\binom{m-1}{n-1}$ and so $n\binom{m}{n} = m\binom{m-1}{n-1}$. Since m is even it cannot divide n and hence $\binom{m}{n}$ must be even. □

Proposition 4.1.37. *There is an A-module isomorphism*

$$R_+(\mathbb{M}_2) \cong \Sigma^{1,0} H_c^{*,*}(\underline{L}_{-\infty}^{\infty}) \cong \Sigma^{1,0}\mathbb{M}_2[u, v, v^{-1}]/(u^2 + \rho u + \tau v).$$

Proof. There is an isomorphism of $\mathbb{M}_2$-modules

$$\varphi : R_+(\mathbb{M}_2) \longrightarrow \Sigma^{1,0} H_c^{*,*}(\underline{L}_{-\infty}^{\infty})$$

sending

$$Sq^{2k} \otimes 1 \mapsto \Sigma^{1,0} uv^{k-1}$$

and

$$Sq^{2k+1} \otimes 1 \mapsto \Sigma^{1,0} v^k.$$

We will check that it is A-linear: First we deal with $Sq^b \otimes 1 \in R_+(\mathbb{M}_2)$ with b non-negative. There are commutative diagrams

$$\begin{array}{ccc}
Sq^{2k} \otimes 1 & \xrightarrow{\varphi} & \Sigma^{1,0} uv^{k-1} \\
\downarrow{\scriptstyle Sq^{2a}{}_*} & & \downarrow{\scriptstyle Sq^{2a}{}_*} \\
\binom{2k-1}{2a} Sq^{2k+2a} & & \binom{2(k-1)}{2a} \Sigma^{1,0} uv^{k-1+a} \\
& \searrow^{\varphi} & \Vert \\
& & \binom{2k-1}{2a} \Sigma^{1,0} uv^{k-1+a}
\end{array}$$

$$\begin{array}{ccc}
Sq^{2k+1} \otimes 1 & \xrightarrow{\varphi} & \Sigma^{1,0} v^{k} \\
\downarrow{\scriptstyle Sq^{2a}{}_*} & & \downarrow{\scriptstyle Sq^{2a}{}_*} \\
\binom{2k}{2a} Sq^{2k+2a+1} & \xrightarrow{\varphi} & \binom{2k}{2a} \Sigma^{1,0} v^{k+a},
\end{array}$$

$$\begin{array}{ccc}
Sq^{2k} \otimes 1 & \xrightarrow{\varphi} & \Sigma^{1,0} uv^{k-1} \\
\downarrow{\scriptstyle Sq^{2a+1}{}_*} & & \downarrow{\scriptstyle Sq^{2a+1}{}_*} \\
\binom{2k-1}{2a+1} Sq^{2k+2a+1} & & \binom{2(k-1)}{2a} \Sigma^{1,0} v^{k+a} \\
& \searrow^{\varphi} & \Vert \\
& & \binom{2k-1}{2a+1} \Sigma^{1,0} v^{k+a}
\end{array}$$

and

$$\begin{array}{ccc}
Sq^{2k+1}\otimes 1 & \xrightarrow{\varphi} & \Sigma^{1,0}v^k \\
{\scriptstyle Sq^{2a+1}{}_*}\downarrow & & \downarrow{\scriptstyle Sq^{2a+1}{}_*} \\
\binom{2k}{2a+1}Sq^{2k+2a+2} & & 0 \\
& \searrow^{\varphi} & \Vert \\
& & \binom{2k}{2a+1}\Sigma^{1,0}uv^{k+a}
\end{array}$$

The equalities can be seen from the fact that $\binom{2k-1}{2a} = \binom{2k-2}{2a} + \binom{2k-2}{2a-1}$, $\binom{2k-1}{2a+1} = \binom{2k-2}{2a+1} + \binom{2k-2}{2a}$. $\binom{2k-2}{2a-1}$, $\binom{2k-2}{2a+1}$ and $\binom{2k}{2a+1}$ are all divisible by 2 from the lemma beforehand.

We can extend φ to elements $Sq^b \otimes 1 \in R_+(\mathbb{M}_2)$ with b negative: The action of A on the elements $u^\epsilon v^b$ is determined by the identification in proposition 3.2.8 and using the formulas for positive squaring operations. For example, the action of Sq^{2a} on $u^\epsilon v^b$ is $\binom{2(b+k)}{2a}v^{b+k+a} = \binom{2b}{2a}v^{b+k+a}$ or $\binom{2(b+k)}{2a}uv^{b+k+a} = \binom{2b}{2a}uv^{b+k+a}$ depending on ϵ where $b+k \geq 0$. We saw the A-action on $R_+(\mathbb{M}_2)$ in the discussion before this lemma and how it was extended to negative b. From this we conclude that the isomorphism follows from the $Sq^b \otimes 1 \in R_+(\mathbb{M}_2)$ with b positive. □

4.2 The motivic Adams spectral sequence

This section recapitulates current knowledge of the convergence of the motivic Adams spectral sequence following [14] and [21]. As in the classical situation, there are two ways to set it up, a homological and a cohomological spectral sequence. They both converge to the homotopy groups of the same object in $SH(F)$ under finite type assumptions as discussed in [14] proposition 7.14. The homological spectral sequence is formed by constructing an exact couple: We follow chapter 2 of [32]. First, let E be a ring-spectrum over F, that is a ring-object in $SH(F)$, and take $X \in SH(F)$. We will assume that E is flat which happens if $E \wedge E$ is equivalent to a wedge of suspensions of E. In lemma 7.3 of [14], it is shown that $H(\mathbb{Z}/p)$ has this property. An E-Adams resolution of X is a diagram

$$\begin{array}{ccccccccc}
X & = & X_0 & \leftarrow & X_1 & \leftarrow & X_2 & \leftarrow & \cdots \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & K_0 & & K_1 & & K_2 & &
\end{array}$$

where the following conditions are met (proposition 2.2.1 in [32]):

- X_{s+1} is the fiber of the map $X_s \to K_s$;

- $E \wedge X_s$ is a retract of $E \wedge K_s$;
- K_s is a retract of $E \wedge K_s$;
- $\mathrm{Ext}^{t,(u,*)}_{E_{*,*}(E)}(E_*(S^{0,0}), E_{*,*}(K_s))$ is isomorphic to $\pi_{u,*}(K_s)$ when $t = 0$ and 0 when $t < 0$. Here Ext is taken in the category of comodules over $E_{*,*}(E)$.

Specifically, such resolutions can be constructed in the following way: Starting with the unit map $S \to E$ one forms the homotopy fiber, $\overline{E}$ say. There is an induced fiber sequence

$$\overline{E}^{s+1} \to \overline{E}^{s} \to E \wedge \overline{E}^{s}.$$

Smashing these sequences with X we obtain the tower above by setting $X_s := \overline{E}^s \wedge X$ and $K_s := E \wedge X_s$. The fiber sequences $X_{s+1} \longrightarrow X_s \longrightarrow K_s$ induce long exact sequences

$$\cdots \longrightarrow \pi_{(m,n)}(X_{s+1}) \longrightarrow \pi_{(m,n)}(X_s) \longrightarrow \pi_{(m,n)}(K_s) \longrightarrow \cdots ,$$

and defining $E_1^{s,(m,n)} := \pi_{(m-s,n)}(K_s)$ and $D_1^{s,(m,n)} := \pi_{(m-s,n)}(X_s)$, we produce an exact couple

$$\begin{array}{ccc} D_1 & \xrightarrow{\ i_1\ } & D_1 \\ & {\scriptstyle k_1}\nwarrow \quad \swarrow{\scriptstyle j_1} & \\ & E_1 & \end{array}$$

where i_1 is induced by $X_{s+1} \longrightarrow X_s$, j_1 is induced by $X_s \longrightarrow K_s$ and k_1 is the connecting morphism. In the language of section 7 in [6], this is a half-plane spectral sequence with entering differentials. For conditional convergence one modifies the tower so that the homotopy limit is trivial: Starting with the compositions

$$\overline{E}^{s} \to \overline{E}^{s-1} \to \cdots \to S$$

and defining C_{s-1} to be the cofiber of the map $\overline{E}^s \longrightarrow S$, we get induced maps $C_s \longrightarrow C_{s-1}$. From the tower constructed above, another tower of fibrations then results:

$$\begin{array}{ccccc} \bullet = & C_{-1} \wedge X & \longleftarrow & C_0 \wedge X & \longleftarrow \cdots \\ & \downarrow & & \downarrow & \\ & \Sigma^{1,0} E \wedge X & & \Sigma^{1,0} E \wedge \overline{E} \wedge X & \end{array}$$

and we let $X^{\wedge}_E$ be the homotopy limit of this tower. It is called the *E-nilpotent completion* of X, a definition found in [7] proposition 5.5. The

Adams spectral sequence converges conditionally to $\lim_s \pi_*(C_s \wedge X)$ and under the condition $R\lim E_\infty^{*;(*,*)} \cong 0$, strongly to $\pi_{*,*}(X^\wedge_E)$. The reader may consult [6], [2] section 15 part 3 or [32] chapter 2 for the construction, details on the E_2-term and its convergence. For our purposes, we take $E = H(\mathbb{Z}/p)$, the mod p motivic Eilenberg-MacLane spectrum.

Work on the convergence of this spectral sequence has focused on a particular class of motivic spectra, namely the *cellular* ones. See [13] for the basic results. The concept of cellularity is relevant to any pointed model category with examples being $H_\bullet(F)$ and $SH(F)$.

Definition 4.2.1. *Let $A := \{S^{k,l} \mid k \geq l \geq 0\}$. Then the class of unstably cellular spaces is the smallest class in $H_\bullet(F)$ such that*

- *all objects in A are unstably cellular;*
- *any object weakly equivalent to an unstably cellular object is unstably cellular;*
- *the homotopy colimit of any diagram $C : I \longrightarrow H_\bullet(F)$ such that each C_i is unstably cellular is unstably cellular.*

Definition 4.2.2. *Let $B := \{S^{k,l} \mid k, l \in \mathbb{Z}\}$. Then the class of cellular spectra is the smallest class in $SH(F)$ such that*

- *all objects in B are cellular;*
- *any object weakly equivalent to a cellular object is cellular;*
- *the homotopy colimit of any diagram $D : I \longrightarrow SH(F)$ such that each D_i is cellular is cellular.*

Remark 4.2.3. Following the conventions in [13], we say that a motivic space X is stably cellular if the suspension spectrum of X is cellular in $SH(F)$. In lemma 3.1 of that same article, it is shown that unstably cellular spaces are also stably cellular.

In $H_\bullet(F)$ and $SH(F)$, attaching cells to a space or spectrum X means taking the cofiber of a map of the form

$$\bigvee_i S^{k_i,l_i} \longrightarrow X.$$

where $k_i \geq l_i \geq 0$ for motivic spaces. Starting with a point and iterating we produce unstably cellular spaces and cellular spectra by lemma 2.2 in [13].

Definition 4.2.4. *A motivic cell spectrum X is of* finite type *if there exists a $k \in \mathbb{Z}$ such that X has no cells in bidegree $(k' + l, l)$ for each $k' < k$, and only finitely many cells in bidegrees $(k'' + l, l)$ for any $k'' \in \mathbb{Z}$.*

The motivic Adams spectral sequence does not always converge to the p-completion of the relevant spectrum so we need more theory to cover the cases that may arise. Recall that for a field F, we can define $GW(F)$, the Grothendieck Witt-ring of isomorphism classes of nonsingular quadratic forms over F. For the definition and the basic results one may consult chapter one of [15].

Definition 4.2.5. *The Milnor-Witt K-theory of F is the graded associative ring $K_*^{MW}(F)$ generated by the symbols $[u]$, for each unit $u \in F^\times$, of degree 1, and one symbol η of degree -1 subject to the following relations:*

- *For each $a \in F^\times \setminus 1$, we have $[a][1-a] = 0$*
- *For each pair $(a, b) \in (F^\times)^2$, we have $[ab] = [a] + [b] + \eta[a][b]$*
- *For each $u \in F^\times$, we have $[u]\eta = \eta[u]$*
- *For $h := \eta[-1] + 2$, we have $\eta h = 0$*

We recall some results of Fabien Morel:

Theorem 4.2.6. *Let F be a perfect field. Then there is a ring isomorphism*

$$K_*^{MW}(F) \to \pi_{*,*}(S^{0,0}).$$

Proof. This is theorem 6.2.1 in [30]. □

Theorem 4.2.7. *Let F be a perfect field. Then there is an isomorphism*

$$GW(F) \to [S^{0,0}, S^{0,0}]_{SH(F)}.$$

Proof. This is theorem 6.2.2 in [30]. □

There algebraic Hopf map

$$\mathbb{A}^2 \setminus 0 \to \mathbb{P}^1$$

in $H_\bullet(F)$ sending $(x, y) \mapsto [x : y]$ represents a class $\eta \in \pi_{1,1}(S^{0,0})$ after passing to the map induced on suspension spectra. Let X be a motivic spectrum over F. If we define X/p^n to be the homotopy cofiber in the sequence

$$X \xrightarrow{*p^n} X \longrightarrow X/p^n,$$

then these form an inverse system and we define

$$\widehat{X} := \operatorname{holim}_n X/p^n.$$

Similarly, there is an inverse system $X/(p^n, \eta^n)$, and we define

$$\overbrace{X} := \operatorname{holim}_n X/(p^n, \eta^n).$$

These will be refered to as the p-completion and the (p, η)-completion of X respectively.

Finally, we state the basic result on the convergence of the motivic Adams spectral sequence:

Theorem 4.2.8 ([21]). *Let F be of characteristic 0 and $X \in SH(F)$ be cellular of finite type. There is a map*

$$comp : X \longrightarrow X^{\wedge}_{H(\mathbb{Z}/p)}$$

and under the conditions on F and X this map is a completion at (p, η) meaning that

$$X^{\wedge}_{H(\mathbb{Z}/p)} \simeq \widehat{X}.$$

If $p > 2$ and $cd_p(F) < \infty$, or $p = 2$ and $cd_2(F[i]) < \infty$ then it is also a completion at p meaning that $X^{\wedge}_{H(\mathbb{Z}/p)} \simeq \widehat{X}$ is an equivalence.

Proof. This is theorem 1 in [21]. □

Corollary 4.2.1. *Under the assumptions of the last theorem, the motivic Adams spectral sequence converges strongly to the homotopy groups of the respective completion of X with E_2-term*

$$\mathrm{Cotor}^{A^\vee}_{*,*}(\mathbb{M}_p, H_{*,*}(X))$$

Proof. Corollary 3 in [21]. □

Remark 4.2.9. We remark that

$$\mathrm{Cotor}^{A^\vee}_{*,*}(\mathbb{M}_p, H_{*,*}(X))$$

is isomorphic to

$$\mathrm{Ext}^{*,*}_A(H^{*,*}(X), \mathbb{M}_p)$$

if $H^{*,*}(X)$ is free and of finite type as a module over $\mathbb{M}_p$. This follows by dualization: $\mathrm{Cotor}^{A^\vee}_{*,*}(\mathbb{M}_p, H_{*,*}(X))$ is calculated using the cobar resolution (see e.g. [32], appendix 1) with terms

$$\mathbb{M}_p \otimes_{\mathbb{M}_p} (A^\vee)^{\otimes s} \otimes_{\mathbb{M}_p} H_{*,*}(X)$$

and

$$\mathbb{M}_p \Box_{A^\vee} H_{*,*}(X)$$

as its 0-th term. Since both $H_{*,*}(X)$ and $A^\vee$ are free over $\mathbb{M}_p$ and of finite type, the $\mathbb{M}_p$-dual complex is a resolution of

$$(\mathbb{M}_p \Box_{A^\vee} H_{*,*}(X))^\vee \cong \mathbb{M}_p^\vee \otimes_A H_{*,*}(X)^\vee \cong \mathbb{M}_p^\vee \otimes_A H^{*,*}(X)$$

with homology groups $\mathrm{Tor}^A_{*,*}(\mathbb{M}_p, H^{*,*}(X))$ and so

$$\mathrm{Cotor}^{A^\vee}_{*,*}(\mathbb{M}_p, H_{*,*}(X)) \cong (\mathrm{Tor}^A_{*,*}(\mathbb{M}_p, H^{*,*}(X))^\vee.$$

In dualizing we need to know that $H_{*,*}(X)^\vee \cong H^{*,*}(X)$ which also follows by the assumptions on $H_{*,*}(X)$ and the discussion leading up to lemma 7.13 in [14]. From proposition 5.3, chapter 6 in [10], there is an isomorphism

$$\mathrm{Tor}^A_{*,*}(\mathbb{M}_p^\vee, H^{*,*}(X)) \cong \mathrm{Ext}^{*,*}_A(H^{*,*}(X), \mathbb{M}_p)^\vee$$

given that A is of finite type. Hence our isomorphism follows.

We turn to two handy results:

Lemma 4.2.10. *Let*

$$X \longrightarrow Y \longrightarrow Z$$

be a homotopy cofiber sequence in $SH(F)$. If any two of the three spectra in the sequence are cellular, then so is the third.

Proof. This is a special case of lemma 2.5 in [13]. We need to check that the motivic desuspension of every object in $B := \{S^{k,l} \mid k, l \in \mathbb{Z}\}$ is equivalent to an object in B which is evident. □

Lemma 4.2.11. *Let*

$$X \longrightarrow Y \longrightarrow Z$$

be a homotopy cofiber sequence in $SH(F)$ with all three spectra being cellular. If any two of the three spectra in the sequence are of finite type, then so is the third.

Proof. Let X, Y be of finite type and $f : X \longrightarrow Y$. Then $Z \cong Cf$, the mapping cone. In $SH(F)$, a pointed simplicial model category, we have $Cf := CX \cup_f Y$. This complex has a cell in bidegree $(a+1, b)$ for any cell of bidegree (a, b) in X and a cell of bidegree (a', b') for any cell (a', b') in Y and is then of finite type. There are induced cofiber sequences

$$Y \longrightarrow Z \longrightarrow \Sigma^{1,0}X$$

and

$$Z \longrightarrow \Sigma^{1,0}X \longrightarrow \Sigma^{1,0}Y$$

so assumptions of finite type for Y and Z, or X (and a fortiori $\Sigma^{1,0}X$) and Z, implies the finite type of $\Sigma^{1,0}X$ and $\Sigma^{1,0}Y$ by the same argument. Taking desuspensions then produces cell spectra with cells in one degree less than $\Sigma X^{1,0}$ and $\Sigma^{1,0}Y$. □

Remark 4.2.12. The objects $\mathbb{P}^n$ are cellular of finite type for $n \geq 0$: There are cofiber sequences

$$\mathbb{P}^{n-1} \longrightarrow \mathbb{P}^n \longrightarrow S^{2n,n}$$

in $H_\bullet(F)$ so the corresponding sequence in $SH(F)$ together with the fact that stable cellularity has a 2 out of 3-property in cofiber sequences as we saw in lemma 4.2.10. An induction on the dimension of the projective spaces then shows cellularity for these objects. Since cellularity is preserved under homotopy colimits we derive this property for $\mathbb{P}^\infty$. We just saw that the set of finite type motivic spectra also has the 2 out of 3-property for cofiber sequences so we can show that $\mathbb{P}^n$ is of finite type by the same argument. From the cofiber sequence above and the dimensions of the cells we are attaching, the motivic space $\mathbb{P}^\infty$ is contained in this class as well.

Lemma 4.2.13. *If X and Y are stably cellular objects of $H_\bullet(F)$ then so is $X \times Y$. In addition, if both are of finite type, then so is the product.*

Proof. The first part is proved in lemma 3.6 of [13]. From the reasoning in that same result, cellularity is proved by using the unstable cofiber sequence

$$X \vee Y \longrightarrow X \times Y \longrightarrow X \wedge Y$$

and showing that both $X \vee Y$ and $X \wedge Y$ are cellular. If both X and Y are of finite type then $X \vee Y$ will be too since we have a cofiber sequence

$$X \longrightarrow X \vee Y \longrightarrow Y.$$

The argument in lemma 3.3 in [13] showing that $X \wedge Y$ is cellular boils down to showing that it can be written as a homotopy colimit of smash products of the spheres in X and Y so if both of them are of finite type then so is $X \wedge Y$. □

Definition 4.2.14. *Let $\{U_\alpha\}$ be a Zariski-cover of a scheme X. We say that the cover is* completely (stably) cellular *if each intersection $U_{\alpha_1 \cdots \alpha_n} := U_{\alpha_1} \cap \cdots \cap U_{\alpha_n}$ is (stably) cellular.*

Lemma 4.2.15. *Let X be a scheme and $U_* \longrightarrow X$ be a hypercover in $H_\bullet(F)$. If each U_n is stably cellular, then so is X. If each U_n is unstably cellular, then so is X.*

Proof. Lemma 3.9 in [13]. For the definition of a hypercover, the reader may see definition 4.1 in [12]. □

Proposition 4.2.16. *Let X be an n-dimensional cubical diagram in $H_\bullet(F)$ or $SH(F)$ indexed on the set $\{1, ..., n\}$ with $X_{\{1,...,n\}}$ being the homotopy colimit. If all the other vertices in the cube are cellular and of finite type, all morphisms are cofibrations between cofibrant objects then $X_{\{1,...,n\}}$ is also cellular of finite type.*

Proof. Cellularity is immediate. Let S be a subset of $\{1, ..., n\}$. It corresponds to a vertex a_S in the cube with coordinates $(a_1, ..., a_n)$ where $a_i = 0$ if $i \notin S$ and where $a_i = 1$ if $i \in S$. Assume that there is a motivic space (or spectrum) $X(a_S) = X_S$ at corner a_S for every subset S and that

$$X_{\{1,...,n\}} = \text{hocolim}_S X_S$$

is situated at corner $a_{\{1,...,n\}} = (1, ..., 1)$. By the cofibrancy assumptions we may assume that

$$\text{colim}_{T<S} X_T \to X_S$$

is a cofibration for each $T \in 1, ..., n$ so the homotopy colimit and the colimit of the diagram are equivalent. Hence $X_{\{1,...,n\}} = \text{colim}_S X_S$. At this point the cube has 2^n corners and we extend it so that there are 3^n of them with coordinates $(a_1, ..., a_n)$ where $0 \leq a_i \leq 2$ for $1 \leq i \leq n$ and such that

$$X(a_1, .., 0, .., a_n) \to X(a_1, .., 1, .., a_n) \to X(a_1, .., 2, .., a_n)$$

is a cofiber sequence for each subsequence $(a_1, ..., a_{i-1}, a_{i+1}, ..., a_n)$ and $1 \leq i \leq n$. We remark that $X(2, ..., 2)$ is the zero-object in either category. Now assume that X_S is of finite type for each of the $2^n - 1$ possible $S \in \{1, ..., n\}$ in the original cube. By the two out of three property, $X(a_1, ..., a_n)$ is of finite type if one of the coordinates is 0.

From the cofiber sequence

$$X(2, .., 2, 0) \to X(2, .., 2, 1) \to X(2, .., 2, 2)$$

it follows that $X(2, .., 2, 1)$ is of finite type. As a consequence, the cofiber sequence

$$X(2, .., 0, 1) \to X(2, .., 1, 1) \to X(2, .., 2, 1)$$

then implies that $X(2, .., 1, 1)$ is of finite type. Using the cofiber sequences recursively, we end up with the sequence

$$X(0, 1..., 1) \to X(1, 1, .., 1) \to X(2, 1, ..., 1)$$

from which it follows that $X(1, 1, .., 1)$ is of finite type and we have proved our result.

□

For the next result, we need to recall that an algebraic fiber bundle with fiber F is a map

$$p : E \to B$$

such that B can be covered by Zariski opens so that p is locally of the form

$$U \times F \to U.$$

This is related to the discussion concerning lemma 3.9 in [13].

Proposition 4.2.17. *The motivic spaces L^n are stably cellular and of finite type for $n \geq 0$.*

Proof. We consider the covering by Zariski opens using $V_i := \mathbb{G}_m \times_{\mu_p} \mathbb{A}^{n-1}$ coming from the diagram

$$\begin{array}{ccc} \mathbb{A}^n \setminus 0 & \xrightarrow{/\mu_p} & L^n \\ \uparrow & & \uparrow \\ \mathbb{G}_m \times \mathbb{A}^{n-1} & \xrightarrow{/\mu_p} & V_i. \end{array}$$

where we let one of the coordinates be nonzero. Here, V_i is $\mathbb{A}^1$-homotopy equivalent (in $H_\bullet(F)$) to $\mathbb{G}_m/\mu_p \cong \mathbb{G}_m$ and more generally

$$V_{i_1 \cdots i_j} := V_{i_1} \cap \cdots \cap V_{i_n}$$

is homotopic to $(\mathbb{G}_m)^j/\mu_p \cong (\mathbb{G}_m^j)$: We identify

$$(\mathbb{G}_m)^j/\mu_p \cong \mathrm{Spec}(F[x_1^{\pm 1}, \ldots, x_j^{\pm 1}]^{\mu_p})$$

where μ_p acts on $F[x_1^{\pm 1}, \ldots, x_j^{\pm 1}]$ by multiplication in each variable. As an algebra over F it is isomorphic to $F[x_1^{\pm p}, (x_1^{p-1}x_2)^{\pm 1}, \ldots, (x_1^{p-1}x_j)^{\pm 1}]$ and so

$$\begin{array}{ccc} \mathrm{Spec}(F[x_1^{\pm 1}, \ldots, x_j^{\pm 1}]^{\mu_p}) & \xrightarrow{\cong} & \mathrm{Spec}(F[x_1^{\pm p}, (x_1^{p-1}x_2)^{\pm 1}, \ldots, (x_1^{p-1}x_j)^{\pm 1}]) \\ & & \downarrow \cong \\ & & (\mathbb{G}_m)^j. \end{array}$$

The variety $V_{i_1 \cdots i_j}$ is an affine bundle over $(\mathbb{G}_m)^j/\mu_p$ and given this relationship, they must be homotopy equivalent in $H_\bullet(F)$. $(\mathbb{G}_m)^j$ is stably cellular and of finite type by lemma 4.2.13 since $\mathbb{G}_m$ is the sphere $S^{1,1}$.

We recover L^n as the homotopy colimit of an n-cube consisting of corners

$$\{V_I := V_{i_1} \cap \cdots \cap V_{i_k} \mid I \subset \{1, \ldots, n\} \setminus \emptyset\}$$

with maps $V_I \longrightarrow V_J$ if $J \subset I$. Working with the injective objectwise model structure on these spaces, we may assume that all corners are cofibrant and that all maps are cofibrations. Thus, L^n is of stably cellular and of by finite type by proposition 4.2.16 since this holds for V_I. Cellularity is preserved under homotopy colimits and we derive this property for L^∞. □

We will verify the cellularity and finite type of motivic spectra such as the ones we dealt with in definition 4.1.23 and quote a result of Dugger and Isaksen.

Lemma 4.2.18. *If $p : E \longrightarrow B$ is an algebraic fiber bundle with fiber F such that F is stably cellular and B has a completely stably cellular cover that trivializes the bundle then E is stably cellular*

Proof. Lemma 3.9 in [13]. □

The diagrams of the former section had us looking at motivic spaces

$$\frac{E(k\varepsilon^n \downarrow L^n)}{E(k\varepsilon^n \downarrow L^n) \setminus E(k\gamma^1 \downarrow L^n)}$$

over L^n. If the result on the convergence of the Adams spectral sequence is to be of any use, these had better be cellular.

Lemma 4.2.19. *The spaces $\frac{E(k\varepsilon^n \downarrow L^n)}{E(k\varepsilon^n \downarrow L^n)) \setminus E(k\gamma^1 \downarrow L^n))}$ are stably cellular and of finite type.*

Proof. We use the covering $\{V_i\}_{1 \leq i \leq n}$. Over each of these the inclusion

$$E(k\gamma^1) \longrightarrow E(k\varepsilon^n)$$

is isomorphic to

$$E(k\mathbb{A}^1) \longrightarrow E(k\mathbb{A}^n)$$

which may be seen as follows: Over $(x_1, \ldots, x_n) \in V_1$, $E(\gamma^1)$ is given by the points $(y_1, \ldots, y_n)$ in $E(\mathbb{A}^n)$ such that $(y_1, \ldots, y_n) = y_1(1, x_2, \ldots, x_n)$. We send

$$(y_1, \ldots, y_n) \longmapsto (y_1, y_2 - y_1 x_2, \ldots, y_n - y_1 x_n) \in \mathbb{A}^1 \times \mathbf{0}$$

and conversely

$$(y_1, 0, \ldots, 0) \longmapsto (y_1, y_2 + y_1 x_2, \ldots, y_n + y_1 x_n) \in \mathbb{A}^n.$$

These formulae extend to the k-fold sum in an obvious way. We conclude that $E(k\varepsilon^n \downarrow L^n) \setminus E(k\gamma^1 \downarrow L^n)$ is isomorphic to $\mathbb{A}^{kn} \setminus \mathbb{A}^k$ locally. This space is equivalent to $\mathbb{A}^{k(n-1)} \setminus 0$ which in turn is equivalent to $S^{2k(n-1)-1,k(n-1)}$, a fact proved to be true in example 2.11 in [13]. Since the cover $\{V_i\}_{1 \leq i \leq n}$ satisfies the assumptions of lemma 4.2.15, $E(k\varepsilon^n \downarrow L^n) \setminus E(k\gamma^1 \downarrow L^n)$ is unstably cellular. In the end, $\frac{E(k\varepsilon^n \downarrow L^n)}{E(k\varepsilon^n \downarrow L^n) \setminus E(k\gamma^1 \downarrow L^n)}$ is a homotopy pushout which is then cellular from the definitions.

The finite type-property for these spectra needs to be verified. As we just saw, the bundle can be trivialized such that the fibers are obviously of finite type: They are equivalent to $S^{2k(n-1)-1,k(n-1)}$. Letting

$$p : \frac{E(k\varepsilon^n \downarrow L^n)}{E(k\varepsilon^n \downarrow L^n) \setminus E(k\gamma^1 \downarrow L^n)} \longrightarrow L^n$$

be the projection, we cover $\frac{E(k\varepsilon^n \downarrow L^n)}{E(k\varepsilon^n \downarrow L^n) \backslash E(k\gamma^1 \downarrow L^n)}$ using

$$V_i' := \{p^{-1}(V_i) \simeq V_i \times S^{2k(n-1)-1,k(n-1)}\}.$$

At this point we come back to the argument in proposition 4.2.16 using the cubical diagram arising from the covering $\{V_i'\}_{1 \leq i \leq n}$ and finite type follows in the exact same manner. This concludes the proof. □

Now that all spectra of the relevant types are known to be cellular and of finite type we return to the towers of our classifying spaces. Recall that we constructed towers

$$\underline{L}_{-\infty}^{\infty} \longrightarrow \cdots \longrightarrow \underline{L}_{-1}^{\infty} \longrightarrow \underline{L}^{\infty}.$$

At each finite stage, there is associated a motivic Adams spectral sequence. We want to get to the calculation of the homotopy of $\underline{L}_{-\infty}^{\infty}$. To do so, our considerations become more general: Suppose $\{Y_n\}$ is an inverse system of motivic spectra, each cellular and of finite type, and let $Y := \lim_n Y_n$. Then for each stage the Adams spectral sequence converges strongly as in corollary 4.2.1. We organize these spectra in a diagram which is lifted from [26] and can also be found in [11]:

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & Z_s & \longrightarrow & \cdots & \longrightarrow & Z_0 = Y \\
 & & \downarrow & & & & \downarrow \\
 & & \vdots & & & & \vdots \\
 & & \downarrow & & & & \downarrow \\
\cdots & \longrightarrow & Z_{n-1,s} & \longrightarrow & \cdots & \longrightarrow & Z_{n-1,0} = Y_{n-1} \\
 & & \downarrow & & & & \downarrow \\
\cdots & \longrightarrow & Z_{n,s} & \longrightarrow & \cdots & \longrightarrow & Z_{n,0} = Y_n.
\end{array}$$

Here, each row is an Adams resolution and at the top we have

$$Z_s := \operatorname{holim}_n Z_{n,s}.$$

We will need some additional conditions on our building blocks for further progress:

Lemma 4.2.20. *$H(\mathbb{Z}/p)$ is cellular and of finite type.*

Proof. This is lemma 6 in [21] or corollary 20 in [23]. The cellular structure is described in [23], section 3. □

We will also use the following result:

Proposition 4.2.21. *The spectra $H(\mathbb{Z}/p)$ are p, η-complete.*

Proof. For p-completeness, the maps

$$H(\mathbb{Z}/p) \xrightarrow{*p^n} H(\mathbb{Z}/p)$$

are 0 on homotopy groups and from the commutative diagram of cofiber sequences

$$\begin{array}{ccccccc}
H(\mathbb{Z}/p) & \xrightarrow{*p^{n+1}} & H(\mathbb{Z}/p) & \longrightarrow & H(\mathbb{Z}/p)/p^{n+1} & \longrightarrow & \Sigma^{1,0}H(\mathbb{Z}/p) \\
\downarrow{\scriptstyle *p} & & \| & & \downarrow & & \downarrow{\scriptstyle *p} \\
H(\mathbb{Z}/p) & \xrightarrow{*p^{n}} & H(\mathbb{Z}/p) & \longrightarrow & H(\mathbb{Z}/p)/p^{n} & \longrightarrow & \Sigma^{1,0}H(\mathbb{Z}/p)
\end{array}$$

we get an equivalence $H(\mathbb{Z}/p) \simeq \operatorname{holim}_n H(\mathbb{Z}/p)/p^n = \widehat{H(\mathbb{Z}/p)}$ by passing up the inverse tower. Next, concerning η-completeness, this element corresponds to a class in $\pi_{1,1}(S^{0,0})$. Furthermore, the map

$$H(\mathbb{Z}/p) \wedge \eta : H(\mathbb{Z}/p) \wedge S^{1,1} \longrightarrow H(\mathbb{Z}/p) \wedge S^{0,0} \cong H(\mathbb{Z}/p)$$

is homotopic to the trivial map. This follows from the fact that the same map represents a class in

$$H^{0,0}(H(\mathbb{Z}/p) \wedge S^{1,1}) \cong H^{-1,-1}(H(\mathbb{Z}/p))$$

with the last group being trivial at least when $char(F)$ is zero. Because of the triviality of this class, we have $H(\mathbb{Z}/p)/\eta \simeq H(\mathbb{Z}/p) \vee \Sigma^{1,0}H(\mathbb{Z}/p)$ and also $H(\mathbb{Z}/p)/\eta^n \simeq H(\mathbb{Z}/p) \vee \Sigma^{n+1,n}H(\mathbb{Z}/p)$ for all $n \geq 0$. The maps in the inverse system defining $\widehat{H(\mathbb{Z}/p)}$ are the identity maps on the $H(\mathbb{Z}/p)$ and nilpotent on the $\Sigma^{n+1,n}H(\mathbb{Z}/p)$-summands. Hence, following the same tower argument as before, $H(\mathbb{Z}/p)$ is also η-complete. The stated result follows by using the diagram below:

$$\begin{array}{ccccccc}
\Sigma^{2n,n}H(\mathbb{Z}/p) & \xrightarrow{*\eta^n} & H(\mathbb{Z}/p) & \longrightarrow & H(\mathbb{Z}/p)/\eta^n & \longrightarrow & \Sigma^{2n+1,n}H(\mathbb{Z}/p) \\
\downarrow{\scriptstyle *p^m} & & \downarrow{\scriptstyle *p^m} & & \downarrow{\scriptstyle *p^m} & & \downarrow{\scriptstyle *p^m} \\
\Sigma^{2n,n}H(\mathbb{Z}/p) & \xrightarrow{*\eta^n} & H(\mathbb{Z}/p) & \longrightarrow & H(\mathbb{Z}/p)/\eta^n & \longrightarrow & \Sigma^{2n+1,n}H(\mathbb{Z}/p) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma^{2n,n}H(\mathbb{Z}/p)/p^m & \longrightarrow & H(\mathbb{Z}/p)/p^m & \longrightarrow & H(\mathbb{Z}/p)/(p^m, \eta^n) & \longrightarrow & \Sigma^{2n+1,n}H(\mathbb{Z}/p)/p^m
\end{array}$$

□

For each row of the diagram concerning the $Z_{n,s}$, the corresponding spectral sequence is generated by the exact couple

$$\begin{array}{ccc} \pi_{*,*}(Z_{n+1,s}) & \longrightarrow & \pi_{*,*}(Z_{n,s}) \\ & \nwarrow \quad \swarrow & \\ & \pi_{*,*}(K_{n,s}). & \end{array}$$

Proposition 4.2.22. *Assume $\mathbb{M}_p$ to be noetherian and that it is finite dimensional as a module over $\mathbb{Z}/p$ in each bidegree. If we let*

$$E_r^{*,(*,*)}(Y) := \lim_n E_r^{*,(*,*)}(Y_n)$$

then the trigraded groups $\{E_r^{*,(*,*)}(Y)\}$ are the terms of a spectral sequence with E_2-term

$$E_2^{s,(t,*)}(Y) \cong \mathrm{Ext}_A^{s,(t,*)}(\operatorname{colim}_n H^{*,*}(Y_n), \mathbb{M}_p) \Longrightarrow \pi_{t-s,*}(\overbrace{Y})$$

If $p > 2$ and $cd_p(F) < \infty$, or when $p = 2$ and $cd_2(F[i]) < \infty$ then it converges strongly to the p-completion of Y.

Proof. This is an adaption of proposition 2.2 in [26]. For each s, $Z_{n,s}$ and $K_{n,s}$ are of finite type: Both $S^{0,0}$ and $H(\mathbb{Z}/p)$ are of finite type and the fiber sequence

$$\overline{H(\mathbb{Z}/p)} \longrightarrow S^{0,0} \longrightarrow H(\mathbb{Z}/p)$$

implies that $\overline{H(\mathbb{Z}/p)}$ is of finite type. An induction using the fiber sequence

$$\overline{H(\mathbb{Z}/p)}^{s+1} \longrightarrow \overline{H(\mathbb{Z}/p)}^{s} \longrightarrow H(\mathbb{Z}/p) \wedge \overline{H(\mathbb{Z}/p)}^{s}$$

followed by smashing with appropriate spectra $X_{n,s}$ (also of finite type) imply that $Z_{n,s} = \overline{H(\mathbb{Z}/p)}^{s} \wedge X_{n,s}$ and $K_{n,s} = H(\mathbb{Z}/p) \wedge Z_{n,s}$ are of finite type. At each n we will need both E_2 and E_∞-terms to be of finite type over $\mathbb{M}_p$ and for this we assume $\mathbb{M}_p$ to be noetherian so taking subquotients remains of finite type. Completing all rows with respect to (p, η), we get exact couples

$$\begin{array}{ccc} \pi_{*,*}(\overbrace{Z_{n+1,s}}) & \longrightarrow & \pi_{*,*}(\overbrace{Z_{n,s}}) \\ & \nwarrow \quad \swarrow & \\ & \pi_{*,*}(\overbrace{K_{n,s}}). & \end{array}$$

We saw that $\overbrace{K_{n,s}} \simeq K_{n,s}$ so these exact couples have the same E_r-terms as the ones we started with. All homotopy groups involved are compact

Hausdorff since the spectra $Z_{n,s}$ and $K_{n,s}$ are of finite type over $\mathbb{M}_p$ and $\mathbb{M}_p$ is finite dimensional in each bidegree over $\mathbb{Z}/p$. For this class of groups, filtered inverse limits are exact (see e.g. lemma 1.1.5 in [33]) so we pass to the homotopy limits at the top row and obtain the exact couple

$$\begin{array}{ccc} \pi_{*,*}(\overbrace{Z_s}) & \longrightarrow & \pi_{*,*}(\overbrace{Z_s}) \\ & \nwarrow \quad \swarrow & \\ & \lim_n \pi_{*,*}(K_{n,s}). & \end{array}$$

Fixing n we have conditional as well as strong convergence so

$$\lim_s \pi_{*,*}(Z_{n,s}) = 0 = R\lim_s \pi_{*,*}(Z_{n,s})$$

and

$$R\lim_r E_r^{s,*,*}(X_n) = 0.$$

We may interchange limits so

$$\lim_s \lim_n \pi_{*,*}(Z_{n,s}) \cong \lim_n \lim_s \pi_{*,*}(Z_{n,s}) = 0$$

Also, since derived inverse limits are exact, there is a diagram of short exact sequences

$$\begin{array}{ccccc} R\lim_s \lim_n \pi_{*,*}(Z_{n,s}) & \longrightarrow & R\lim_{n,s} \pi_{*,*}(Z_{n,s}) & \longrightarrow & \lim_s R\lim_n \pi_{*,*}(Z_{n,s}) \\ & & \| & & \\ R\lim_n \lim_s \pi_{*,*}(Z_{n,s}) & \longrightarrow & R\lim_{n,s} \pi_{*,*}(Z_{n,s}) & \longrightarrow & \lim_n R\lim_s \pi_{*,*}(Z_{n,s}) \end{array}$$

coming from the collapse of the spectral sequences in [35] theorem 3. Hence the limit spectral sequence is conditionally convergent. For strong convergence it is enough to consider the diagram

$$\begin{array}{ccccc} R\lim_r \lim_n E_r^{s,*,*}(X_n) & \longrightarrow & R\lim_{n,r} E_r^{s,*,*}(X_n) & \longrightarrow & \lim_r R\lim_n E_r^{s,*,*}(X_n) \\ & & \| & & \\ R\lim_n \lim_r E_r^{s,*,*}(X_n) & \longrightarrow & R\lim_{n,r} E_r^{s,*,*}(X_n) & \longrightarrow & \lim_n R\lim_r E_r^{s,*,*}(X_n) \end{array}$$

which is coming from [35]. From our assumptions, each $\lim_r E_r^{s,t,*}(X_n)$ is finite dimensional over $\mathbb{Z}/p$ so $R\lim_n \lim_r E_r^{s,t,*}(X_n) = 0$ so our spectral sequence also converges strongly. $\square$

This result has obvious implications for our tower

$$\underline{L}_{-\infty}^{\infty} \longrightarrow \cdots \longrightarrow \underline{L}_{-1}^{\infty} \longrightarrow \underline{L}^{\infty}.$$

and we state what we set out to prove from the beginning:

Theorem 4.2.23. *There is an inverse limit of Adams spectral sequences arising from the tower*

$$\underline{L}^{\infty}_{-\infty} \longrightarrow \cdots \longrightarrow \underline{L}^{\infty}_{-1} \longrightarrow \underline{L}^{\infty}.$$

If we let $E_r^{*,(*,*)}(\underline{L}^{\infty}_{-\infty}) := \operatorname{colim}_k E_r^{*,(*,*)}(\underline{L}^{\infty}_{-k})$ *then the trigraded groups* $\{E_r^{*,(*,*)}(\underline{L}^{\infty}_{-\infty})\}$ *are the terms of a spectral sequence with* E_2*-term*

$$E_2^{s,(t,*)}(\underline{L}^{\infty}_{-\infty}) \cong \operatorname{Ext}^{s,(t,*)}(\operatorname{colim}_k H^{*,*}(\underline{L}^{\infty}_{-k}), \mathbb{M}_p) \Longrightarrow \pi_{t-s,*}(\widehat{\underline{L}^{\infty}_{-\infty}})$$

If $cd_2(F[i]) < \infty$ *then it converges strongly to the* 2*-completion of* $\underline{L}^{\infty}_{-\infty}$*.*

Proof. Follows immediately from the last proposition and the finite type of the $\underline{L}^{\infty}_{-k}$. □

Theorem 4.2.24. *The* E_2*-term of the spectral sequence in the last theorem is isomorphic to*

$$E_2^{s,(t,*)}(\Sigma^{-1,0}R_+(\mathbb{M}_2), \mathbb{M}_2)$$

as an A*-module.*

Proof. This is a consequence of theorem 3.3.6 and proposition 4.1.37. □

Chapter 5

Concluding comments

Now that we have seen that the construction $R_+(\mathbb{M}_2)$ may be realized topologically as it was in the classical case, it is only natural that one should try to say something about what the conclusions may imply for a motivic version of the Segal conjecture for the case where we are dealing with μ_2. There are papers on equivariant spectra in the motivic setting ([9], [22]) but generalizations of Carlssons argument in [8] is not known to the author. One should speculate though.

There should be a tower of motivic spectra for dealing with μ_p for odd p but the author has not found a suitable candidate as of yet. Also, it would be interesting to see how the constructions in this thesis behaves under realization functors from motivic spaces to topological spaces. The computations in [14] suggest that one might find interesting information concerning the classical Adams spectral sequences by using this bridge. The author has not spent any time on these ideas so this might be another theme for future research.

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