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BSDES DRIVEN BY TIME-CHANGED LÉVY NOISES AND OPTIMAL CONTROL

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Abstract. We study backward stochastic differential equations (BSDE's) for time-changed Lévy noises when the time-change is independent of the Lévy process. We prove existence and uniqueness of the solution and we obtain an explicit formula for linear BSDE's and a comparison principle. BSDE's naturally appear in control problems. Here we prove a sufficient maximum principle for a general optimal control problem of a system driven by a time-changed Lévy noise. As an illustration we solve the mean-variance portfolio selection problem.

1. Introduction

We establish a framework for the study of backward stochastic differential equations (BSDE's) driven by a conditional Brownian motion and a doubly stochastic Poisson random field. Indeed the structure of these noises can be strongly related to the corresponding time-changed Browian motion and the time-changed Poisson random measure when the time-change is independent of the Brownian motion and Poisson field.

In the framework of the non-anticipating integration for martingale random fields, we prove the existence and uniqueness of the solution of a general BSDE of the form

$$\begin{aligned} Y_t &= \xi + \int_t^T g_s\big(\lambda_s, Y_s, \phi_s\big)\, ds - \int_t^T \int_{\mathbb{R}} \phi_s(z)\, \mu(ds, dz) \\ &= \xi + \int_t^T g_s\big(\lambda_s, Y_s, \phi_s\big)\, ds - \int_t^T \phi_s(0)\, dB_s - \int_t^T \int_{\mathbb{R}_0} \phi_s(z)\, \tilde{H}(ds, dz) \end{aligned} \tag{1.1}$$

where μ is the mixture of a conditional Brownian measure B on $[0,T]\times\{0\}$ and a centered doubly stochastic Poisson measure $\tilde{H}$ on $[0,T]\times\mathbb{R}_0$ ($\mathbb{R}_0 := \mathbb{R}\setminus\{0\}$). Namely

$$\mu(\Delta) := B\big(\Delta\cap[0,T]\times\{0\}\big) + \tilde{H}\big(\Delta\cap[0,T]\times\mathbb{R}_0\big), \tag{1.2}$$

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for any Borel measurable set Δ in $[0,T]\times\mathbb{R}$. Moreover we specifically study linear BSDE's achieving a closed form solution for the process Y and use this solution to obtain a comparison theorem.

These results rely strongly on the stochastic integral representation of square integrable random variables and martingales. In the language of time-change, we can formulate the result as follows: Given the time-change processes Λ^B and Λ^H, the complete filtered probability space $(\Omega,\mathcal{F},\mathbb{P},\mathbb{G})$ where $\mathbb{G}$ is the filtration generated by μ and the whole of Λ^B and Λ^H, any L^2-martingale M can be represented as

$$M_t = M_0 + \int_0^t \int_{\mathbb{R}} \phi_s(z)\,\mu(ds,dz) \tag{1.3}$$

where ϕ is proved to exist and M_0 is a random element measurable with respect to Λ^B and Λ^H.

In [DS13] a detailed study on the structure of the spaces generated by the measure $\tilde{H}$ is carried though achieving chaos decompositions via orthogonal polynomials and also integral representation results of type (1.3) in which the integrand is given in closed form via the non-anticipating stochastic derivative in first place and then via Clark-Ocone type formulae and anticipating stochastic derivatives. These results hold for very general choices of Λ^H also beyond the present paper. Here we give an alternative slimmer proof for representation (1.3) which will provide only existence of the integrand ϕ. This is enough for the study of (1.1).

We remark that (1.3) shows that martingales M of the type considered do not have a (full) predictable representation property as described in [BY78, Nua95, RY91] since the initial value M_0 is not a constant in general. Indeed the predictable representation property depends on the combination of integrator and the information flow. In [Di 07, Theorem 2.2] it is proved that the predictable property with respect to the class of random measures μ with independent values *if and only if* μ is given as a mixture of Gaussian and centered Poisson random measures.

The integration and the representation results are developed with respect to the filtration $\mathbb{G}$, the filtration generated by μ and *the entire history* of Λ^B and Λ^H. It is with respect to Λ^B and Λ^H that H and B have conditionally independent increments. From the point of view of modeling and applications $\mathbb{G}$ is not a natural choice of filtration since it includes "anticipating information", the future values of Λ^B and Λ^H. However we can still apply our results in the problems related to models where the reference filtration is $\mathbb{F}$, the smallest right-continouos filtration to which μ is adapted. Indeed we show sufficient conditions for solving an optimal control problem with a classical performance functional for both $\mathbb{G}$- and $\mathbb{F}$-predictable controls. This is achieved by projecting the results obtained for the $\mathbb{G}$-predictable case onto the $\mathbb{F}$-predictable one.

The framework proposed based on specific integral representation under $\mathbb{G}$ is a novel framework for problems related to time-changed processes. The work [Lim05] considers BSDE's with doubly stochastic Poisson processes, where the intensity of the doubly stochastic Poisson process depends on a Brownian motion in a specific way. Our setting does not overlap with that of [Lim05] due to a different relationship between the noises considered. Our BSDE also differs from another approach to BSDE's beyond Lévy processes, [CFS08, BS04, ØZ09, JMSS12, KXY10], where an extra martingale N is inserted to the backward stochastic differential equation so that Y attains the terminal condition $Y_t = \xi$ and Y_0 is a real number. BSDE's with random measures is discussed in [Jia00] assuming a martingale representation exist. We however prove the martingale representation and explicitly link the random measures, the martingale representation, and the conditions on the driver.

Taking a view towards applications we sketch some of the uses of the time-changed Lévy processes in mathematical finance and the relevance of our BSDE-framework. This is not meant as a comprehensive review. The time-changed Lévy processes occur in mathematical finance in the modeling of asset prices as follows:

$$\begin{aligned} dS_t &= S_{t\text{-}}\Big(\int_{\mathbb{R}} \psi_t(z)\,\mu(dt,dz) \\ &= S_{t\text{-}}\Big(\psi_t(0)\,dB_t + \int_{\mathbb{R}_0} \psi_t(z)\,\tilde{H}(dt,dz)\Big) \quad S_0 > 0. \end{aligned} \tag{1.4}$$

The well-known stochastic exponentiation model of [CGMY03, Section 4.3], where stocks are modeled as time-changed pure jump Lévy processes, can be described in our terminology as

$$S_t = \exp\Big\{\int_0^t\int_{\mathbb{R}_0} z\,\tilde{H}(ds,dz) - \int_0^t\int_{\mathbb{R}_0} \left[e^z - 1 - z\right]\lambda_s^H \nu(dz)ds\Big\} \tag{1.5}$$

which in differential form is

$$dS_t = S_{t\text{-}}\Big(\int_{\mathbb{R}_0} \left(e^z - 1\right)\tilde{H}(ds,dz)\Big). \tag{1.6}$$

Here the jump measure ν and time-change intensity λ^H determine the properties of S.

A popular class of stochastic volatility models with Brownian filtrations including [BNNS02, Hes93, HW87, SS91] is

$$dS_t = \rho S_{t\text{-}}\,dt + \sigma S_{t\text{-}}\lambda_t^B\,dW_t^{(1)} \tag{1.7}$$

$$d\lambda_t^B = M(\lambda_t^B)\,dt + K(\lambda_t^B)\,dW_t^{(2)} \tag{1.8}$$

where M and K are real functions, $\rho, \sigma \in \mathbb{R}$ and $W^{(1)}$ and $W^{(2)}$ are Brownian motions. Here S is the asset price and λ^B the stochastic volatility. Whenever $W^{(1)}$ and $W^{(2)}$ are *independent*, $B_t := \int_0^t \lambda_t^B \, dW_t^{(1)}$ is a conditional independent Brownian motion as in Definition 2.1 and our framework applies.

In credit risk, the jump times of the doubly stochastic Poisson process are used to signify the occurence of downwards abrupt price movements and default. A classical example [Lan98] is the case of an integer valued stochastic process H_t, $t \in [0, T]$, with $\nu(dz) = \mathbf{1}_{\{z=1\}}(z)$ and λ^H given. Then $\tilde{H}_t = H_t - \Lambda_t^H$. The default time τ is the first jump of H, ie $\tau = \inf_t\{H(t) > 0\}$. This is then used to model bonds or derivatives of the form $P\mathbf{1}_{\tau>T}$, where P is a random variable, so that $P\mathbf{1}_{\tau>T}$ is a payoff which is received only if there is no default. An example of type (1.4) is the zero coupon bond which can be modeled as

$$dS_t = S_{t\text{-}}\Big(\lambda_{t\text{-}}^H \, dt - \, d\tilde{H}_t\Big), \quad S_0 = 1, \quad \text{for } t \leq \tau.$$

To the best of our knowlegde, the present work is the first to detail BSDE's for time-changed Lévy processes in general form, which opens up for studies on risk measures and filtration-consistent expectations as in [Gia06, Roy06] via our comparison theorem. Moreover we explicitly treat general optimal control problems with time-changed Lévy processes, see e.g. (6.1), via the present BSDE. Indeed the BSDE can be used to investigate mean-variance hedging, utility maximization and optimal consumption problems for assets modeled as in (1.4) via Theorems 6.2 and 6.3. Utility maximization for time-changed Lévy processes is studied in [KMK10a, KMK10b] for the power utility. Mean-variance hedging (for stochastic volatility and credit risk) has been discussed in terms of affine models [KV09, KP11] and with BSDEs for general semi-martingales [BS04, JMSS12, KXY10]. However [BS04] only consider continuous semi-martingales, [JMSS12] requires a system of several BSDEs while [KXY10] requires a martingale representation result which is not true in our setting.

The present paper is organized as follows. In the next section the details about the noises considered and the integration framework are set into place. Section 3 is dedicated to the martingale representation type of result while section 4 deals with existence and uniqueness of the solution of the BSDE's (1.1). The study of explicit solutions of linear BSDE's and their applications to prove a comparison theorem is given in section 5. Finally we show a sufficient maximum principle in section 6 and we trace its use in some optimal control problems in section 7. There we study expected utility of the final wealth, for which we find a characterization of the optimal portfolio, and a mean-variance portfolio selection problem for which we give an explicit formula of the optimal portfolio.

2. The framework

2.1. **The random measures and their properties.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $X := [0,T] \times \mathbb{R}$, we will consider $X = \big([0,T] \cup \{0\}\big) \cup \big([0,T] \times \mathbb{R}_0\big)$, where $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$ and $T > 0$. Denote $\mathcal{B}_X$ the Borel σ-algebra on X. Throughout this presentation $\Delta \subset X$ denotes an element Δ in $\mathcal{B}_X$.

Let $\lambda := (\lambda^B, \lambda^H)$ be a two dimensional stochastic process such that each component λ^l, $l = B, H$, satisfies

i) $\lambda^l_t \geq 0$ $\mathbb{P}$-a.s. for all $t \in [0,T]$,
ii) $\lim_{h\to 0} \mathbb{P}\big(|\lambda^l_{t+h} - \lambda^l_t| \geq \epsilon\big) = 0$ for all $\epsilon > 0$ and almost all $t \in [0,T]$,
iii) $\mathbb{E}\big[\int_0^T \lambda^l_t \, dt\big] < \infty$,

We denote $\mathcal{L}$ as the space of all processes $\lambda := (\lambda^B, \lambda^H)$ satisfying i)-ii)-iii) above.

Define the random measure Λ on X by

$$\Lambda(\Delta) := \int_0^T \mathbf{1}_{\{(t,0)\in\Delta\}}(t)\, \lambda^B_t dt + \int_0^T \int_{\mathbb{R}_0} \mathbf{1}_{\Delta}(t,z)\, \nu(dz) \lambda^H_t dt, \tag{2.1}$$

as the mixture of measures on disjoint sets. Here ν is a deterministic, σ-finite measure on the Borel sets of $\mathbb{R}_0$ satisfying

$$\int_{\mathbb{R}_0} z^2 \, \nu(dz) < \infty.$$

We denote the σ-algebra generated by the values of Λ by $\mathcal{F}^\Lambda$. Furthermore, Λ^H denotes the restriction of Λ to $[0,T] \times \mathbb{R}_0$ and Λ^B the restriction of Λ to $[0,T] \times \{0\}$. Hence $\Lambda(\Delta) = \Lambda^B(\Delta \cap [0,T] \times \{0\}) + \Lambda^H(\Delta \cap [0,T] \times \mathbb{R}_0)$, $\Delta \subseteq X$. Here below we introduce the noises driving (1.1).

Definition 2.1. *B is a signed random measure on the Borel sets of $[0,T] \times \{0\}$ satisfying,*

A1) $\mathbb{P}\Big(B(\Delta) \leq x \,\Big|\, \mathcal{F}^\Lambda\Big) = \mathbb{P}\Big(B(\Delta) \leq x \,\Big|\, \Lambda^B(\Delta)\Big) = \Phi\big(\frac{x}{\sqrt{\Lambda^B(\Delta)}}\big)$, $x \in \mathbb{R}$, $\Delta \subseteq [0,T] \times \{0\}$,
A2) *$B(\Delta_1)$ and $B(\Delta_2)$ are conditionally independent given $\mathcal{F}^\Lambda$ whenever Δ_1 and Δ_2 are disjoint sets.*

Here Φ stands for the cumulative probability distribution function of a standard normal random variable.

H is a random measure on the Borel sets of $[0,T] \times \mathbb{R}_0$ satisfying

A3) $\mathbb{P}\Big(H(\Delta) = k \,\Big|\, \mathcal{F}^\Lambda\Big) = \mathbb{P}\Big(H(\Delta) = k \,\Big|\, \Lambda^H(\Delta)\Big) = \frac{\Lambda^H(\Delta)^k}{k!} e^{-\Lambda^H(\Delta)}$, $k \in \mathbb{N}$, $\Delta \subseteq [0,T] \times \mathbb{R}_0$,
A4) *$H(\Delta_1)$ and $H(\Delta_2)$ are conditionally independent given $\mathcal{F}^\Lambda$ whenever Δ_1 and Δ_2 are disjoint sets.*

Furthermore we assume that

A5) *B and H are conditionally independent given $\mathcal{F}^\Lambda$.*

Conditions A1) and A3) mean that conditional on Λ, B is a Gaussian random measure and H is Poisson a random measure. In particular, if λ^B and λ^H are deterministic then B is a Wiener process and H is a Poisson random random measure.

We refer to [Gri75] or [Kal97] for the existence of conditional distributions as in Definition 2.1.

Let $\tilde{H} := H - \Lambda^H$ be the signed random measure given by

$$\tilde{H}(\Delta) = H(\Delta) - \Lambda^H(\Delta), \quad \Delta \subset [0,T] \times \mathbb{R}_0.$$

Definition 2.2. *We define the signed random measure μ on the Borel subsets of X by*

$$\mu(\Delta) := B\Big(\Delta \cap [0,T] \times \{0\}\Big) + \tilde{H}\Big(\Delta \cap [0,T] \times \mathbb{R}_0\Big), \quad \Delta \subseteq X. \tag{2.2}$$

Clearly, from A1) we have that the conditional first moment of B is $\mathbb{E}\big[B(\Delta)\big|\mathcal{F}^\Lambda\big] = 0$ and from A3) the conditional first moment of H is $\mathbb{E}\big[H(\Delta)\big|\mathcal{F}^\Lambda\big] = \Lambda^H(\Delta)$ so that $\mathbb{E}\big[\tilde{H}(\Delta)\big|\mathcal{F}^\Lambda\big] = 0$. Thus

$$\mathbb{E}\big[\mu(\Delta)\big|\mathcal{F}^\Lambda\big] = 0. \tag{2.3}$$

The second conditional moments of B and $\tilde{H}$ are given by

$$\mathbb{E}\big[B(\Delta)^2\big|\mathcal{F}^\Lambda\big] = \Lambda^B(\Delta),$$
$$\mathbb{E}\big[\tilde{H}(\Delta)^2\big|\mathcal{F}^\Lambda\big] = \Lambda^H(\Delta).$$

By the conditional independence A2), A4) and A5) we have

$$\mathbb{E}\Big[\mu(\Delta)^2\Big|\mathcal{F}^\Lambda\Big] = \Lambda(\Delta)$$

and

$$\mathbb{E}\big[\mu(\Delta_1)\mu(\Delta_2)\big|\mathcal{F}^\Lambda\big] = \mathbb{E}\big[\mu(\Delta_1)\big|\mathcal{F}^\Lambda\big]\mathbb{E}\big[\mu(\Delta_2)\big|\mathcal{F}^\Lambda\big] = 0 \tag{2.4}$$

for Δ_1 and Δ_2 disjoint. Hence $\mu(\Delta_1)$ and $\mu(\Delta_2)$ are conditionally orthogonal.

The random measures B and H are related to a specific form of time-change for Brownian motion and pure jump Lévy process. More specifically define $B_t := B([0,t] \times \{0\})$, $\Lambda_t^B := \int_0^t \lambda_s^B\, ds$, $\eta_t := \int_0^t \int_{\mathbb{R}_0} z\, \tilde{H}(ds, dz)$ and $\hat{\Lambda}_t^H := \int_0^t \lambda_s^H\, ds$, for $t \in [0,T]$.

We can immediately see the role that the time-change processes Λ^B and $\hat{\Lambda}^H$ play, studying the characteristic function of B and η. In fact, from A1) and A3) we see that the conditional characteristic functions of B_t and η_t are

given by

$$\mathbb{E}\big[e^{icB_t}\big|\mathcal{F}^\Lambda\big] = \exp\big\{\int_0^t \frac{1}{2}c^2\,\lambda_s^B ds\big\} = \exp\big\{\frac{1}{2}c^2\Lambda_t^B\big\}, \quad c\in\mathbb{R} \tag{2.5}$$

$$\mathbb{E}\big[e^{ic\eta_t}\big|\mathcal{F}^\Lambda\big] = \exp\big\{\int_0^t\int_{\mathbb{R}_0}\big[e^{icz}-1-icz\big]\,\nu(dz)\lambda_t^H dt\big\}$$

$$= \exp\big\{\big(\int_{\mathbb{R}_0}\big[e^{icz}-1-icz\big]\,\nu(dz)\big)\,\hat{\Lambda}_t^H\big\}, \quad c\in\mathbb{R}. \tag{2.6}$$

Indeed there is a strong connection between the distributions of B and the Brownian motion and between η and a centered pure jump Lévy process with the same jump behavior. The relationship is based on a random distortion of the time scale. The following characterization is due to [Ser72, Theorem 3.1] (see also [Gri75]).

Theorem 2.3. *Let W_t, $t\in[0,T]$ be a Brownian motion and N_t, $t\in[0,T]$ be a centered pure jump Lévy process with Levy measure ν. Assume that both W and N are* independent *of Λ. Then B satisfies A1)-*(2.5) *and A2) if and only if, for any $t\geq 0$,*

$$B_t \overset{d}{=} W_{\Lambda_t^B},$$

and η satisfies A3)-(2.6) *and A4) if and only if, for any $t\geq 0$,*

$$\eta_t \overset{d}{=} N_{\hat{\Lambda}_t^H}.$$

In addition, B is infinitely divisible if Λ^B is infinitely divisible and η is infinitely divisible if $\hat{\Lambda}^H$ is infinitely divisible, see [BNMS06, Theorem 7.1].

2.2. **Stochastic non-anticipating integration.** Let us define $\mathbb{F}^\mu = \{\mathcal{F}_t^\mu,\ t\in[0,T]\}$ as the filtration generated by $\mu(\Delta)$, $\Delta\subset[0,t]\times\mathbb{R}$. In view of (2.2), A1), and A3) we can see, that for any $t\in[0,T]$,

$$\mathcal{F}_t^\mu = \mathcal{F}_t^B \vee \mathcal{F}_t^H \vee \mathcal{F}_t^\Lambda,$$

where $\mathcal{F}_t^B$ is generated by $B(\Delta\cap[0,T]\times\{0\})$, $\mathcal{F}_t^H$ by $H(\Delta\cap[0,T]\times\mathbb{R}_0)$, and $\mathcal{F}_t^\Lambda$ by $\Lambda(\Delta)$, $\Delta\in[0,t]\times\mathbb{R}$. This is an application of [Win01, Theorem 1] and [DS13, Theorem 2.8]. Set $\mathbb{F}=\{\mathcal{F}_t,\ t\in[0,T]\}$ where

$$\mathcal{F}_t = \bigcap_{r>t}\mathcal{F}_r^\mu$$

Furthermore, we set $\mathbb{G}=\{\mathcal{G}_t,\ t\in[0,T]\}$ where $\mathcal{G}_t=\mathcal{F}_t^\mu\vee\mathcal{F}^\Lambda$. Remark that $\mathcal{G}_T=\mathcal{F}_T$, $\mathcal{G}_0=\mathcal{F}^\Lambda$, while $\mathcal{F}_0^\mu$ is trivial. From now on we set $\mathcal{F}=\mathcal{F}_T$.

Lemma 2.4. *The filtration $\mathbb{G}$ is right-continouos.*

Proof. This can be shown using classical arguments from the Lévy case (as in e.g. [App04, Theorem 2.1.9]). □

For $\Delta \subset (t,T] \times \mathbb{R}$, the conditional independence A2), A4) means that

$$\mathbb{E}\big[\mu(\Delta)\,\big|\mathcal{G}_t\big] = \mathbb{E}\big[\mu(\Delta)\,\big|\mathcal{F}_t \vee \mathcal{F}^\Lambda\big] = \mathbb{E}\big[\mu(\Delta)\,\big|\mathcal{F}^\Lambda\big] = 0. \tag{2.7}$$

Thus μ has the martingale property with respect to $\mathbb{G}$ from (2.3). Hence μ is a martingale random field with respect to $\mathbb{G}$ in the sense of [DE10] since

- μ has a σ-finite variance measure
$$m(\Delta) := \mathbb{E}\big[\mu(\Delta)^2\big] = \mathbb{E}\big[\Lambda(\Delta)\big],$$
- μ is $\mathbb{G}$-adapted,
- μ has conditionally orthogonal values, if $\Delta_1, \Delta_2 \subset (t,T] \times \mathbb{R}$ such that $\Delta_1 \cap \Delta_2 = \emptyset$ then, combining the arguments in (2.4) and (2.7),

$$\mathbb{E}\Big[\mu(\Delta_1)\mu(\Delta_2)\,\Big|\mathcal{G}_t\Big] = \mathbb{E}\Big[\mu(\Delta_1)\,\Big|\mathcal{F}^\Lambda\Big]\mathbb{E}\Big[\mu(\Delta_2)\,\Big|\mathcal{F}^\Lambda\Big] = 0. \tag{2.8}$$

Denote $\mathcal{I}$ as the subspace of $L^2([0,T] \times \mathbb{R} \times \Omega, \mathcal{B}_X \times \mathbb{P}, \Lambda \times \mathbb{P})$ of the random fields admitting a $\mathbb{G}$-predictable modification, in particular

$$\|\phi\|_{\mathcal{I}} := \Big(\mathbb{E}\Big[\int_0^T \phi_s(0)^2\,\lambda_s^B ds + \int_0^T \int_{\mathbb{R}_0} \phi_s(z)^2\,\nu(dz)\lambda_s^H ds\Big]\Big)^{\frac{1}{2}} < \infty. \tag{2.9}$$

For any $\phi \in \mathcal{I}$ we define the (Itô type) non-anticipative stochastic integral $I : \mathcal{I} \Rightarrow L^2(\Omega, \mathcal{F}, \mathbb{P})$ by

$$I(\phi) := \int_0^T \phi_s(0)\,dB_s + \int_0^T \int_{\mathbb{R}_0} \phi_s(z)\,\tilde{H}(ds,dz).$$

We refer to [DE10] for details on the integration with respect to martingale random fields of the type discussed here. In particular, I is a linear isometric operator:

$$\Big(E\big[I(\phi)^2\big]\Big)^{\frac{1}{2}} = \|I(\phi)\|_{L^2(\Omega,\mathcal{F},\mathbb{P})} = \|\phi\|_{\mathcal{I}}. \tag{2.10}$$

Because of the structure of the filtration considered we have:

Lemma 2.5. *Consider $\xi \in L^2\big(\Omega, \mathcal{F}^\Lambda, \mathbb{P}\big)$ and $\phi \in \mathcal{I}$. Then*

$$\xi I(\phi) = I(\xi\phi),$$

whenever either side of the equality exists as an element in $L^2\big(\Omega, \mathcal{F}, \mathbb{P}\big)$.

Proof. Assume that ξ is bounded and $\phi \in \mathcal{I}$ is simple, i.e.

$$\phi_s(z,\omega) = \sum_{j=1}^J \phi_j(\omega)\mathbf{1}_{\Delta_j}(s,z),$$

where, for $j = 1, \dots J$, we have $\Delta_j = (d_j, u_j] \times Z_j$, $0 \le d_j \le u_j$, $Z_j \subseteq \mathbb{R}$. Then

$$\xi I(\phi) = \xi \sum_{j=1}^J \phi_j \mu(\Delta_j) = \sum_{j=1}^J \xi\phi_j\mu(\Delta_j) = I(\xi\phi),$$

where $\xi\phi_j$ is $\mathcal{G}_{d_j}$-measurable since ξ is $\mathcal{F}^\Lambda$-measurable. The general case follows by taking limits. □

Remark 2.6. The random field μ is also a martingale random field with respect to $\mathbb{F}$ and integration can be done as for $\mathbb{G}$. However, results such as Lemma 2.5 and the forthcoming representation would not hold. See also [DS13, Remark 4.4].

3. Integral and martingale representation theorems

In this section we prove an integral representation theorem for a random variable $\xi \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ in the setting described above. We freshly prove this result here for the sake of completeness. There are other similar results in the literature available. We refer for example to [JS03, Theorem III.4.34]. See Remark 3.4 for further details.

Recall that $\mathcal{G}_T = \mathcal{F}_T$. Here we remark that $\mathcal{F}_T = \sigma\{\mu(\Delta),\ \Delta \subseteq X\} = \sigma\{I(\phi),\ \phi \in \mathcal{I}\}$ (indeed $\mu(\Delta) = I(\mathbf{1}_\Delta)$). Denote $\mathcal{K} := \{\phi \in \mathcal{I} \big| \phi$ is $\mathcal{F}^\Lambda$-measurable, $\phi\mathbf{1}_{\mathbb{R}_0}$ is bounded a.e., and $\int_0^T \int_{\mathbb{R}} \phi_s(z)^2\, \Lambda(ds, dz)$ is a bounded random variable$\}$.

Lemma 3.1. *For any $\phi \in \mathcal{K}$ we have*

$$\exp\{I(\phi)\} \in L^2(\Omega, \mathcal{F}, \mathbb{P}), \qquad \textit{and}$$
$$\frac{\exp\{I(\phi)\}}{\mathbb{E}\big[\exp\{I(\phi)\}\big|\mathcal{F}^\Lambda\big]} \in L^2(\Omega, \mathcal{F}, \mathbb{P}).$$

Furthermore, the random variables $\{e^{I(\phi)}, \phi \in \mathcal{K}\}$ form a total subset of $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

Proof. The first claim is proved in [Yab07, Lemma 6], the second can be shown using arguments as in the proofs of [Yab07, Lemmas 4 and 6]. The last claim is proved in [Yab07, Lemma 9]. □

Lemma 3.2. *Assume $\phi \in \mathcal{K}$. Define, for $t \in [0, T]$,*

$$\zeta_t = \exp\Big\{ \int_0^t \phi_s(0)\, dB_s + \int_0^t \int_{\mathbb{R}_0} \phi_s(z)\, \tilde{H}(ds, dz) \Big\}.$$

Then the following representation holds:

$$\zeta_T = \mathbb{E}\big[\zeta_T \,\big|\mathcal{F}^\Lambda\big] + \int_0^T \Big[\mathbb{E}\big[\frac{\zeta_T}{\zeta_s}\,\big|\mathcal{F}^\Lambda\big]\zeta_{s\text{-}}\phi_s(0)\Big]\, dB_s$$
$$+ \int_0^T \int_{\mathbb{R}_0} \Big[\mathbb{E}\big[\frac{\zeta_T}{\zeta_s}\,\big|\mathcal{F}^\Lambda\big]\zeta_{s\text{-}}\big(e^{\phi_s(z)} - 1\big)\Big]\, \tilde{H}(ds, dz). \tag{3.1}$$

Note that the integrands in (3.1) are $\mathbb{G}$-predictable.

Proof. Let

$$
\begin{aligned}
(3.2)\quad Y_t &= \frac{\zeta_t}{\mathbb{E}\big[\zeta_t \,\big|\mathcal{F}^\Lambda\big]} \\
&= \exp\Big\{ \int_0^t \phi_s(0)\, dB_s + \int_0^t \int_{\mathbb{R}_0} \phi_s(z)\, \tilde{H}(ds,dz) - \int_0^t \frac{1}{2}\phi_s(0)^2\, \lambda_s^B ds \\
&\quad - \int_0^t \int_{\mathbb{R}_0} \big[e^{\phi_s(z)} - 1 - \phi_s(z)\big]\, \nu(dz)\lambda_s^H ds \Big\}.
\end{aligned}
$$

Note that both Y_t and ζ_t are elements of $L^2(\Omega, \mathcal{F}, \mathbb{P})$ by Lemma 3.1. By Itô's formula

$$
\begin{aligned}
(3.3)\qquad dY_t &= Y_{t\text{-}}\Big(\phi_t(0)\, dB_t + \int_{\mathbb{R}_0} \big(e^{\phi_t(z)} - 1\big)\, \tilde{H}(dt,dz)\Big), \\
Y_0 &= 1.
\end{aligned}
$$

Combining (3.2) and (3.3) the above equalities yields

$$
\begin{aligned}
\zeta_T &= \mathbb{E}\big[\zeta_T \,\big|\mathcal{F}^\Lambda\big] Y_T \\
&= \mathbb{E}\big[\zeta_T \,\big|\mathcal{F}^\Lambda\big]\Big(1 + \int_0^T Y_{s\text{-}}\phi_s(0)\, dB_s + \int_0^T \int_{\mathbb{R}_0} Y_{s\text{-}}\big(e^{\phi_s(z)} - 1\big)\, \tilde{H}(ds,dz)\Big) \\
&= \mathbb{E}\big[\zeta_T \,\big|\, \mathcal{F}^\Lambda\big] + \int_0^T \mathbb{E}\big[\zeta_T \,\big|\, \mathcal{F}^\Lambda\big] Y_{s\text{-}}\phi_s(0)\, dB_s \\
&\quad + \int_0^T \int_{\mathbb{R}_0} \Big[\mathbb{E}\big[\zeta_T \,\big|\, \mathcal{F}^\Lambda\big] Y_{s\text{-}}\big(e^{\phi_s(z)} - 1\big)\Big]\, \tilde{H}(ds,dz) \\
&= \mathbb{E}\big[\zeta_T \,\big|\, \mathcal{F}^\Lambda\big] + \int_0^T \mathbb{E}\big[\frac{\zeta_T}{\zeta_s} \,\big|\, \mathcal{F}^\Lambda\big] \zeta_{s\text{-}}\phi_s(0)\, dB_s \\
&\quad + \int_0^T \int_{\mathbb{R}_0} \Big[\mathbb{E}\big[\frac{\zeta_T}{\zeta_s} \,\big|\, \mathcal{F}^\Lambda\big] \zeta_{s\text{-}}\big(e^{\phi_s(z)} - 1\big)\Big]\, \tilde{H}(ds,dz)
\end{aligned}
$$

where we used Lemma 2.5 and the equations

$$
Y_s \mathbb{E}\big[\zeta_T \,\big|\mathcal{F}^\Lambda\big] = Y_s \mathbb{E}\big[\zeta_s \,\big|\, \mathcal{F}^\Lambda\big] \mathbb{E}\big[\frac{\zeta_T}{\zeta_s} \,\big|\mathcal{F}^\Lambda\big] = \zeta_s \mathbb{E}\big[\frac{\zeta_T}{\zeta_s} \,\big|\mathcal{F}^\Lambda\big].
$$

□

Theorem 3.3. *Assume $\xi \in L^2\big(\Omega, \mathcal{F}, \mathbb{P}\big)$. Then there exists a unique $\phi \in \mathcal{I}$ such that*

$$\xi = \mathbb{E}\big[\xi \,\big|\mathcal{F}^{\Lambda}\big] + \int_0^T \int_{\mathbb{R}} \phi_s(z)\, \mu(ds, dz). \tag{3.4}$$

Note that the two summands in (3.4) are orthogonal. Here $\mathbb{E}[\xi \,|\mathcal{F}^{\Lambda}]$ represents the stochastic component of ξ which cannot be recovered by integration on μ.

Proof. At first let $\xi = \zeta(T)$, where

$$\zeta(T) = \exp\Big\{ \int_0^T \int_{\mathbb{R}} \kappa_s(z)\, \mu(ds, dz) \Big\}.$$

From Lemma 3.2 the representation (3.4) holds in this case.

Consider a general $\xi \in L^2\big(\Omega, \mathcal{F}, \mathbb{P}\big)$. Then ξ can be approximated by a sequence of linear combinations of the form (3.4) by Lemma 3.1. Let $\{\xi_n\}_{n\geq 1}$ be such a sequence. Then, by (2.10), we have

$$\mathbb{E}\Big[\big(\xi_n - \xi_m\big)^2\Big] = \mathbb{E}\Big[\Big(\mathbb{E}\big[\xi_n - \xi_m \big|\mathcal{F}^{\Lambda}\big]\Big)^2 + \int_0^T \int_{\mathbb{R}} \big(\phi_s^{(n)}(z) - \phi_s^{(m)}(z)\big)^2\, \Lambda(ds, dz)\Big].$$

Thus $\{\phi^{(n)}\}_{n\geq 1}$ is a Cauchy-sequence in $\mathcal{I}$, which proves existence. To prove uniqueness, suppose

$$\begin{aligned} \xi &= \mathbb{E}\big[\xi \,\big|\mathcal{F}^{\Lambda}\big] + \int_0^T \int_{\mathbb{R}} \phi_s(z)\, \mu(ds, dz) \\ &= \mathbb{E}\big[\xi \,\big|\mathcal{F}^{\Lambda}\big] + \int_0^T \int_{\mathbb{R}} \psi_s(z)\, \mu(ds, dz). \end{aligned}$$

Then, using (2.10), $\mathbb{E}\big[\int_0^T \int_{\mathbb{R}} \big(\phi_s(z) - \psi_s(z)\big)^2\, \Lambda(ds, dz)\big] = 0$.

□

Remark 3.4. We have here chosen to prove the above result using classical arguments well established for integrators as the Brownian motion, see e.g. [Øks05, Section 4] and the Poisson random measure, see e.g. [Løk05]. The existence of such a representation is a topic of [JS03, Chapter 3]. There the result is obtained after a discussion on the solution of *the martingale problem* (see [JS03, Chapter 3]).

In [DS13] we have instead proven this result for $\tilde{H}$ using orthogonal polynomials and we have derived an *explicit formula* for the integrand ϕ using the non-anticipating derivative, see [DS13, Theorem 5.1]. This result holds for more general choices of Λ^H, but with an assumption on the moments.

There are other related results in the literature. In [Yab07, Proposition 41] the same representation is proved for a class of Malliavin differentiable random variables (à la Clark-Ocone type results).

If $\mathcal{F}_T^H$-measurable ξ are considered then representation is given in the general context of (marked) point processes, see for instance [Bré81, Theorem 4.12 and 8.8] or [Dav76, BVW75, Jac75]. Our result differs in the choice of filtration, which also leads to slightly different integrals. In [Bré81, Dav76, BVW75, Jac75] the integrator in the representation theorem are given by $H - \vartheta$ where ϑ is $\mathbb{F}^H$-predictable compensator of H. Our Λ^H is not $\mathbb{F}^H$-predictable.

Theorem 3.5. *Assume M_t, $t \in [0,T]$, is a $\mathbb{G}$-martingale. Then there exists a unique $\phi \in \mathcal{I}$ such that*

$$M_t = \mathbb{E}\big[M_T \,\big|\mathcal{F}^\Lambda\big] + \int_0^t \int_{\mathbb{R}} \phi_s(z)\, \mu(ds, dz), \quad t \in [0,T].$$

Proof. The proof follows classical arguments as in [Øks05, Theorem 4.3.4] using Theorem 3.3. □

4. BSDE: Existence and uniqueness of the solution

Hereafter we tackle directly the question of existence and uniqueness of the solution of (1.1):

$$Y_t = \xi + \int_t^T g_s\big(\lambda_s, Y_s, \phi_s\big)\, ds - \int_t^T \int_{\mathbb{R}} \phi_s(z)\, \mu(ds, dz), \quad t \in [0,T].$$

Indeed for the given terminal condition ξ and driver (or generator) g, a solution is given by the couple of $\mathbb{G}$-adapted processes (Y, ϕ) on $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the equation above. In the sequel we characterize explicitly the functional spaces in use and the elements of the BSDE to obtain a solution. In the following section we study explicitly the case when the driver g is linear.

Let S be the space of $\mathbb{G}$-adapted stochastic processes $Y(t,\omega)$, $t \in [0,T]$, $\omega \in \Omega$, such that

$$\|Y\|_S := \sqrt{\mathbb{E}\big[\sup_{0 \le t \le T} |Y_t|^2\big]} < \infty,$$

and $\mathcal{H}_2^{\mathcal{G}}$ be the space of $\mathbb{G}$-predictable stochastic processes $f(t,\omega)$, $t \in [0,T]$, $\omega \in \Omega$, such that

$$\mathbb{E}\Big[\int_0^T f_s^2\, ds\Big] < \infty.$$

Recall the definition of $\mathcal{I}$ in (2.9) and denote Φ the space of functions $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$|\phi(0)|^2 + \int_{\mathbb{R}_0} \phi(z)^2 \, \nu(dz) < \infty. \tag{4.1}$$

Definition 4.1. *We say that* (ξ, g) *are* standard parameters *when* $\xi \in L^2\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ *and* $g : [0,T] \times [0,\infty)^2 \times \mathbb{R} \times \Phi \times \Omega \to \mathbb{R}$ *such that* g *satisfies (for some* $K_g > 0$*)*

(4.2) $g.(\lambda, Y, \phi, \cdot)$ *is* $\mathbb{G}$*-adapted for all* $\lambda \in \mathcal{L}$, $Y \in S$, $\phi \in \mathcal{I}$,

(4.3) $g.(\lambda., 0, 0, \cdot) \in \mathcal{H}_2^{\mathcal{G}}$, *for all* $\lambda \in \mathcal{L}$

(4.4) $$\big|g_t\big((\lambda^B, \lambda^H), y_1, \phi^{(1)}\big) - g_t\big((\lambda^B, \lambda^H), y_2, \phi^{(2)}\big)\big| \le K_g\Big(\big|y_1 - y_2\big| + \big|\phi^{(1)}(0) - \phi^{(2)}(0)\big|\sqrt{\lambda^B} + \sqrt{\int_{\mathbb{R}_0} |\phi^{(1)}(z) - \phi^{(2)}(z)|^2 \, \nu(dz)}\sqrt{\lambda^H}\Big),$$

for all $(\lambda^B, \lambda^H) \in [0,\infty)^2, y_1, y_2 \in \mathbb{R}$, *and* $\phi^{(1)}, \phi^{(2)} \in \Phi$ $dt \times d\mathbb{P}$ *a.e.*

We recall the fundamental inequality $(a_1 + a_2 + \cdots + a_n)^2 \le n(a_1^2 + a_2^2 + \ldots a_n^2)$, for any $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \in \mathbb{R}$, playing an important role in the technical lemmas below.

Lemma 4.2. *Consider* $(Y, \phi), (U, \psi) \in S \times \mathcal{I}$. *Let* $g : [0,T] \times [0,\infty)^2 \times \mathbb{R} \times \Phi \times \Omega \to \mathbb{R}$ *satisfy* (4.3) *and* (4.4). *Then, for any* $t \in [0,T]$, *we have*

$$\mathbb{E}\Big[\Big(\int_t^T g_s(\lambda_s, Y_s, \phi_s) - g_s(\lambda_s, U_s, \psi_s) \, ds\Big)^2\Big] \le 3K_g^2(T-t)$$

$$\mathbb{E}\Big[\, (T-t) \sup_{t \le r \le T} |Y_r - U_r|^2 + \int_t^T \int_{\mathbb{R}} |\phi_s(z) - \psi_s(z)|^2 \Lambda(ds, dz)\Big] \tag{4.5}$$

and

$$\mathbb{E}\Big[\Big(\int_t^T \Big|g_s(\lambda_s, U_s, \psi_s)\Big| \, ds\Big)^2\Big] \le (T-t)\mathbb{E}\Big[2\int_t^T |g_s(\lambda_s, 0, 0)|^2 \, ds$$

$$+ 6K_g^2\Big((T-t) \sup_{t \le r \le T} |U_r|^2 + \int_t^T \int_{\mathbb{R}} |\psi_s(z)|^2 \, \Lambda(ds, dz)\Big)\Big]. \tag{4.6}$$

Proof. Let $t \in [0,T]$. Inequality (4.5) follows from the Lipschitz conditions (4.4):

$$
\begin{aligned}
&\mathbb{E}\Big[\Big(\int_t^T g_s(\lambda_s, Y_s, \phi_s) - g_s(\lambda_s, U_s, \psi_s)\, ds\Big)^2\Big] \\
&\quad \leq K_g^2\, \mathbb{E}\Big[\, \Big(\int_t^T |Y_s - U_s| + |\phi_s(0) - \psi_s(0)|\sqrt{\lambda_s^B} \\
&\qquad\qquad + \sqrt{\int_{\mathbb{R}_0} \big|\phi_s(z) - \psi_s(z)\big|^2 \nu(dz)}\sqrt{\lambda_s^H}\, ds\Big)^2\Big] \\
&\quad \leq 3K_g^2(T-t)\mathbb{E}\Big[\int_t^T \big|Y_s - U_s\big|^2 + \big|\phi_s(0) - \psi_s(0)\big|^2 \lambda_s^B \\
&\qquad\qquad + \int_{\mathbb{R}_0} \big|\phi_s(z) - \psi_s(z)\big|^2 \nu(dz)\lambda_s^H\, ds\Big] \\
&\quad \leq 3K_g^2(T-t)\mathbb{E}\Big[\, (T-t) \sup_{t \leq r \leq T} \big|Y_r - U_r\big|^2 \\
&\qquad\qquad + \int_t^T \int_{\mathbb{R}} \big|\phi_s(z) - \psi_s(z)\big|^2 \Lambda(ds, dz)\Big].
\end{aligned}
$$

For inequality (4.6) we have

$$
\begin{aligned}
&\mathbb{E}\Big[\Big(\int_t^T \Big|g_s(\lambda_s, U_s, \psi_s)\Big|\, ds\Big)^2\Big] \\
&\quad \leq (T-t)\mathbb{E}\Big[\int_t^T \big|g_s(\lambda_s, U_s, \psi_s)\big|^2\, ds\Big] \\
&\quad \leq (T-t)\mathbb{E}\Big[\int_t^T \Big(\Big|g_s(\lambda_s, 0, 0)\Big| + \Big|g_s(\lambda_s, U_s, \psi_s) - g_s(\lambda_s, 0, 0)\Big|\Big)^2\, ds\Big] \\
&\quad \leq 2(T-t)\mathbb{E}\Big[\int_t^T \Big|g_s(\lambda_s, 0, 0)\Big|^2 + \Big|g_s(\lambda_s, U_s, \psi_s) - g_s(\lambda_s, 0, 0)\Big|^2\, ds\Big]
\end{aligned}
$$

The result now follows from (4.4) by proceeding as in the proof of (4.5) above. □

Lemma 4.3. *Consider $U \in S$, $\psi, \phi \in \mathcal{I}$ and let (ξ, g) be standard parameters. Define a stochastic process Y_t, $t \in [0,T]$, by*

$$Y_t = \xi + \int_t^T g_s\big(\lambda_s, U_s, \psi_s\big)\, ds - \int_t^T \int_{\mathbb{R}} \phi_s(z)\, \mu(ds, dz). \tag{4.7}$$

Then $Y \in S$. In particular we have

$$\begin{aligned} \mathbb{E}\Big[\sup_{t\le r\le T} |Y_r|^2\Big] \le \mathbb{E}\Big[3\xi^2 + 3\Big(\int_t^T \Big|g_s\big(\lambda_s, U_s, \psi_s\big)\Big|\, ds\Big)^2 \\ + 30 \int_t^T \int_{\mathbb{R}} |\phi_s(z)|^2\, \Lambda(ds, dz)\Big]. \end{aligned} \tag{4.8}$$

Proof. Directly from (4.7), taking the square, we have

$$|Y_t|^2 \le 3\xi^2 + 3\Big(\int_t^T \Big|g_s\big(\lambda_s, U_s, \psi_s\big)\Big| ds\Big)^2 + 3\Big(\int_t^T \int_{\mathbb{R}} \phi_s(z)\, \mu(ds, dz)\Big)^2.$$

In the next step we take the supremum and obtain

$$\begin{aligned} \mathbb{E}\Big[\sup_{t\le r\le T} |Y_r|^2\Big] \le \mathbb{E}\Big[3\xi^2 + 3\Big(\int_t^T \big|g_s\big(\lambda_s, U_s, \psi_s\big)\big|\, ds\Big)^2\Big] \\ + \mathbb{E}\Big[\sup_{t\le r\le T} 3\Big(\int_r^T \int_{\mathbb{R}} \phi_s(z)\, \mu(ds, dz)\Big)^2\Big]. \end{aligned}$$

We have

$$\begin{aligned} &\mathbb{E}\Big[\sup_{t\le r\le T} \Big(\int_r^T \int_{\mathbb{R}} \phi_s(z)\, \mu(ds, dz)\Big)^2\Big] \\ &= \mathbb{E}\Big[\sup_{t\le r\le T} \Big(\int_t^T \int_{\mathbb{R}} \phi_s(z)\, \mu(ds, dz) - \int_t^r \int_{\mathbb{R}} \phi_s(z)\, \mu(ds, dz)\Big)^2\Big] \\ &\le \mathbb{E}\Big[2\Big(\int_t^T \int_{\mathbb{R}} \phi_s(z)\, \mu(ds, dz)\Big)^2 + 2 \sup_{t\le r\le T} \Big(\int_t^r \int_{\mathbb{R}} \phi_s(z)\, \mu(ds, dz)\Big)^2\Big] \\ &\le 10\mathbb{E}\Big[\int_t^T \int_{\mathbb{R}} \phi_s(z)^2\, \Lambda(ds, dz)\Big] \end{aligned}$$

by application of Doob's martingale inequality, see e.g. [App04, Theorem 2.1.5]. Equation (4.8) follows, and we conclude that $Y \in S$ by (4.6). $\square$

Now let (g, ξ) be standard parameters. Define the mapping

$$\Theta : S \times \mathcal{I} \to S \times \mathcal{I}, \quad \Theta(U, \psi) := (Y, \phi) \tag{4.9}$$

as follows. The component ϕ is given by Theorem 3.5 as the unique element in $\mathcal{I}$ that provides the stochastic integral representation

$$M_t = M_0 + \int_0^t \int_{\mathbb{R}} \phi_s(z)\, \mu(ds, dz), \quad t \in [0, T],$$

of the martingale

$$M_t = \mathbb{E}\big[\xi + \int_0^T g_s\big(\lambda_s, U_s, \psi_s\big)\, ds \,\big|\mathcal{G}_t\big], \quad t \in [0, T].$$

Note that $M_0 = \mathbb{E}\big[\xi + \int_0^T g_s\big(\lambda_s, U_s, \psi_s\big)\, ds \,\big|\mathcal{F}^\Lambda\big]$. The component Y in (4.9) is defined by

$$Y_t = \mathbb{E}\Big[\xi + \int_t^T g_s(\lambda_s, U_s, \psi_s)\, ds \,\Big|\mathcal{G}_t\Big], \quad t \in [0, T]. \tag{4.10}$$

Note that

$$\begin{aligned} Y_t &= M_t - \int_0^t g_s(\lambda_s, U_s, \psi_s)\, ds \\ &= M_0 + \int_0^t \int_{\mathbb{R}} \phi_s(z)\, \mu(ds, dz) - \int_0^t g_s(\lambda_s, U_s, \psi_s)\, ds. \end{aligned}$$

Since $Y_T = \xi$, we also have $Y_t = \xi - Y_T + Y_t$ so that

$$Y_t = \xi + \int_t^T g_s\big(\lambda_s, U_s, \psi_s\big)\, ds - \int_t^T \int_{\mathbb{R}} \phi_s(z)\, \mu(ds, dz). \tag{4.11}$$

Hence $Y \in S$ by Lemma 4.3 and the mapping (4.9) is well-defined.

We use the mapping Θ to prove that the BSDE of type (1.1) admits a unique solution for the given standard parameters (ξ, g).

Lemma 4.4. *Consider* $(U^{(1)}, \psi^{(1)}), (U^{(2)}, \psi^{(2)}) \in S \times \mathcal{I}$ *and define* $(Y^{(1)}, \phi^{(1)}) = \Theta(U^{(1)}, \psi^{(1)})$ *and* $(Y^{(2)}, \phi^{(2)}) = \Theta(U^{(2)}, \psi^{(2)})$. *Set* $\bar{U} = U^{(1)} - U^{(2)}$, $\bar{\psi} =$

$\psi^{(1)}-\psi^{(2)}$, $\bar{Y}=Y^{(1)}-Y^{(2)}$ *and* $\bar{\phi}=\phi^{(1)}-\phi^{(2)}$. *There exists a* $K>0$ *such that*

$$\mathbb{E}\left[\sup_{t\le r\le T}\left|\bar{Y}_r\right|^2+\int_t^T\int_{\mathbb{R}}\left|\bar{\phi}_s(z)\right|^2\Lambda(ds,dz)\right]$$

(4.12)

$$\le K(T-t)\mathbb{E}\Big[(T-t)\sup_{t\le r\le T}\left|\bar{U}_r\right|^2+\int_t^T\int_{\mathbb{R}}\left|\bar{\psi}_s(z)\right|^2\Lambda(ds,dz)\Big],\quad t\in[0,T].$$

Proof. From (4.11), for any $t\in[0,T]$, we have

$$\bar{Y}_t=\int_t^T g_s(\lambda_s,U_s^{(1)},\psi_s^{(1)})\,ds-\int_t^T g_s(\lambda_s,U_s^{(2)},\psi_s^{(2)})\,ds$$
$$-\int_t^T\int_{\mathbb{R}}\bar{\phi}_s(z)\,\mu(ds,dz).$$

Since

$$E\Big[\bar{Y}_t\int_t^T\int_{\mathbb{R}}\bar{\phi}_s(z)\,\mu(ds,dz)\Big]=E\Big[\bar{Y}_t\,\mathbb{E}\big[\int_t^T\int_{\mathbb{R}}\bar{\phi}_s(z)\,\mu(ds,dz)\big|\mathcal{G}_t\big]\Big]=0,$$

we have

$$\mathbb{E}\left[\Big(\bar{Y}_t+\int_t^T\int_{\mathbb{R}}\bar{\phi}_s(z)\,\mu(ds,dz)\Big)^2\right]=\mathbb{E}\Big[\left|\bar{Y}_t\right|^2+\int_t^T\int_{\mathbb{R}}\left|\bar{\phi}_s(z)\right|^2\Lambda(ds,dz)\Big]$$

(4.13)

$$=\mathbb{E}\Big[\Big(\int_t^T g_s(\lambda_s,U_s^{(1)},\psi_s^{(1)})\,ds-\int_t^T g_s(\lambda_s,U_s^{(2)},\psi_s^{(2)})\,ds\Big)^2\Big].$$

We apply (4.5) and obtain

$$\mathbb{E}\Big[\int_t^T\int_{\mathbb{R}}\left|\bar{\phi}_s(z)\right|^2\Lambda(ds,dz)\Big]\le\mathbb{E}\Big[\left|\bar{Y}_t\right|^2+\int_t^T\int_{\mathbb{R}}\left|\bar{\phi}_s(z)\right|^2\Lambda(ds,dz)\Big]$$

$$\le 3K_g^2(T-t)\mathbb{E}\Big[(T-t)\sup_{t\le r\le T}\left|\bar{U}_r\right|^2+\int_t^T\int_{\mathbb{R}}\left|\bar{\psi}_s(z)\right|^2\Lambda(ds,dz)\Big].\tag{4.14}$$

By (4.5), (4.8) and (4.14) we have

$$
\begin{aligned}
\mathbb{E}\Big[\sup_{t\le r\le T}|\bar{Y}_r|^2\Big] &\le \mathbb{E}\Big[0+3\Big(\int_t^T\Big|g_s\big(\lambda_s,U_s^{(1)},\psi_s^{(1)}\big)-g_s(\lambda_s,U_s^{(2)},\psi_s^{(2)})\Big|ds\Big)^2\\
&\quad+30\Big(\int_t^T\int_{\mathbb{R}}|\bar{\phi}_s(z)|^2\,\Lambda(ds,dz)\Big)\Big]\\
&\le\Big(9+90\Big)K_g^2(T-t)^2\mathbb{E}\Big[\sup_{t\le r\le T}|\bar{U}_r(z)|^2\Big]\\
&\quad+(9+90)K_g^2(T-t)\mathbb{E}\Big[\int_t^T\int_{\mathbb{R}}|\bar{\psi}_s(z)|^2\,\Lambda(ds,dz)\Big].
\end{aligned}
\tag{4.15}
$$

Combining (4.15) and (4.14) gives (4.12). □

The existence and uniqueness for the BSDE now follow from the above estimates:

Theorem 4.5. *Let (g,ξ) be standard parameters. Then there exists a unique couple $(Y,\phi)\in S\times\mathcal{I}$ such that*

$$
\begin{aligned}
Y_t &= \xi+\int_t^T g_s\big(\lambda_s,Y_s,\phi_s\big)\,ds-\int_t^T\int_{\mathbb{R}}\phi_s(z)\,\mu(ds,dz)\\
&= \xi+\int_t^T g_s\big(\lambda_s,Y_s,\phi_s\big)\,ds-\int_t^T\phi_s(0)\,dB_s-\int_t^T\int_{\mathbb{R}_0}\phi_s(z)\,\tilde{H}(ds,dz).
\end{aligned}
\tag{4.16}
$$

Proof. Let K be as in (4.12). Choose $t_1\in[0,T)$ such that $\max\big\{K(T-t_1)^2,K(T-t_1)\big\}<1$. Denote $S(u,v)$ as the space consisting of the elements of S equipped with the norm $\|Y\|^2_{S(u,v)}=\mathbb{E}\big[\sup_{u\le r\le v}|Y_r|^2\big]$ and $\mathcal{I}(u,v)$ as the space of the elements of $\mathcal{I}$ equipped with the norm $\|\phi\|^2_{\mathcal{I}(u,v)}=\mathbb{E}[\int_u^v\int_{\mathbb{R}}|\phi_s(z)|^2\,\Lambda(ds,dz)]$. From (4.12), Θ is a contraction on $S(t_1,T)\times\mathcal{I}(t_1,T)$, and thus there exists a unique $(Y^{(1)},\phi^{(1)})\in S(t_1,T)\times\mathcal{I}(t_1,T)$ such that $\Theta(Y^{(1)},\phi^{(1)})=(Y^{(1)},\phi^{(1)})$ on $[t_1,T]$, i.e.

$$
Y_t^{(1)}=\xi+\int_t^T g_s\big(\lambda_s,Y_s^{(1)},\phi_s^{(1)}\big)\,ds-\int_t^T\int_{\mathbb{R}}\phi_s^{(1)}(z)\,\mu(ds,dz),\quad t\in[t_1,T]
$$

Take $t_2 \in [0, t_1)$ so that $\max\{K(t_1 - t_2)^2, K(t_1 - t_2)\} < 1$. Next, $\tilde{\phi} \in \mathcal{I}(t_2, t_1)$ is given by Theorem 3.5, which is depending on $\tilde{U}$ and $\tilde{\psi}$, i.e.

$$E\Big[Y^{(1)}_{t_1} + \int_0^{t_1} g_s(\lambda_s, \tilde{U}_s, \tilde{\psi}_s)\, ds \,\Big|\mathcal{G}_t\Big] = E\Big[Y^{(1)}_t + \int_0^{t_1} g_s(\lambda_s, \tilde{U}_s, \tilde{\psi}_s)\, ds \,\Big|\mathcal{G}_{t_2}\Big] + \int_{t_2}^{t}\int_{\mathbb{R}} \tilde{\phi}_s(z)\, \mu(ds, dz), \quad t \in [t_2, t_1],$$

In addition, $\tilde{Y}_t$ is defined as

$$\tilde{Y}_t = E\Big[Y^{(1)}_{t_1} + \int_t^{t_1} g_s(\lambda_s, \tilde{U}_s, \tilde{\psi}_s)\, ds \,\Big|\mathcal{G}_t\Big], \quad t \in [t_2, t_1].$$

Then, $\tilde{\Theta}$ can be defined by $\tilde{\Theta}(\tilde{U}, \tilde{\psi}) = (\tilde{Y}, \tilde{\phi})$ for $(\tilde{U}, \tilde{\psi}) \in S(t_2, t_1) \times \mathcal{I}(t_2, t_1)$.

Following the same arguments as above we conclude that $\tilde{\Theta}$ is a contraction on $S(t_2, t_1) \times \mathcal{I}(t_2, t_1)$ so that there exists a unique element $(Y^{(2)}, \phi^{(2)}) \in S(t_2, t_1) \times \mathcal{I}(t_2, t_1)$ such that $(Y^{(2)}, \phi^{(2)}) = \tilde{\Theta}(Y^{(2)}, \phi^{(2)})$. Then we have

(4.17)
$$Y^{(2)}_t = Y^{(1)}_{t_1} + \int_t^{t_1} g_s\big(\lambda_s, Y^{(2)}_s, \phi^{(2)}_s\big)\, ds - \int_t^{t_1}\int_{\mathbb{R}} \phi^{(2)}_s(z)\, \mu(ds, dz), \quad t \in [t_2, t_1].$$

Now consider

$$\begin{aligned} Y_t &= Y^{(1)}_t \mathbf{1}_{t_1 < t \le T}(t) + Y^{(2)}_t \mathbf{1}_{t_2 < t \le t_1}(t), \quad t \in [t_2, T], \\ \phi_t &= \phi^{(1)}_t \mathbf{1}_{t_1 < t \le T}(t) + \phi^{(2)}_t \mathbf{1}_{t_2 < t \le t_1}(t), \quad t \in [t_2, T]. \end{aligned} \tag{4.18}$$

We can see that

$$Y_t = \xi + \int_t^T g_s\big(\lambda_s, Y_s, \phi_s\big)\, ds - \int_t^T\int_{\mathbb{R}} \phi_s(z)\, \mu(ds, dz), \quad \text{for } t \in [t_2, T]. \tag{4.19}$$

In fact, clearly (4.19) holds for $t \in [t_1, T]$. Assume $t \in (t_2, t_1]$, then

$$\begin{aligned} Y_t &= Y^{(1)}_{t_1} + \int_t^{t_1} g_s\big(\lambda_s, Y^{(2)}_s, \phi_s\big)\, ds - \int_t^{t_1}\int_{\mathbb{R}} \phi^{(2)}_s(z)\, \mu(ds, dz) \\ &= \xi + \int_{t_1}^{T} g_s\big(\lambda_s, Y^{(1)}_s, \phi_s\big)\, ds - \int_{t_1}^{T}\int_{\mathbb{R}} \phi^{(1)}_s(z)\, \mu(ds, dz) \\ &\quad + \int_t^{t_1} g_s\big(\lambda_s, Y^{(2)}_s, \phi_s\big)\, ds - \int_t^{t_1}\int_{\mathbb{R}} \phi^{(2)}_s(z)\, \mu(ds, dz) \end{aligned}$$

$$= \xi + \int_t^T g_s\big(\lambda_s, Y_s, \phi_s\big)\, ds - \int_t^T \int_{\mathbb{R}} \phi_s(z)\, \mu(ds, dz).$$

Proceed iteratively. Eventually, there is a step n such that $\max\big\{K(t_n - t_{n+1})^2, K(t_n - t_{n+1})\big\} < 1$ for $t_{n+1} = 0$ (here $t_0 = T$). Then we conclude and have found a (unique) couple $(Y, \phi) \in S(0,T) \times \mathcal{I}(0,T) = S \times \mathcal{I}$ such that (4.16) holds. □

Remark 4.6. The initial point Y_0 of the solution Y is *not* necessarily a (deterministic) constant. From the definition of $\mathbb{G}$ and (4.10), we see that Y_0 is a square integrable $\mathcal{F}^\Lambda$-measurable random variable. To be specific we have:

$$Y_0 = \mathbb{E}\Big[\xi + \int_0^T g_s(\lambda_s, Y_s, \phi_s)\, ds \,\Big|\, \mathcal{F}^\Lambda\Big].$$

5. Linear BSDE's and a comparison theorem

In the case of a linear driver the BSDE with Brownian motion or Lévy processes have an explicit representation. A similar represention holds in our case.

Theorem 5.1. *Assume we have the following BSDE:*

$$-dY_t = \Big[A_t Y_t + C_t + E_t(0)\phi_t(0)\sqrt{\lambda_t^B} + \int_{\mathbb{R}_0} E_t(z)\phi_t(z)\, \nu(dz)\sqrt{\lambda_t^H}\Big]\, dt$$

$$- \phi_t(0)\, dB_t - \int_{\mathbb{R}_0} \phi_t(z)\, \tilde{H}(dt, dz), \quad Y_T = \xi, \tag{5.1}$$

where the coefficients satisify

i) *A is a bounded stochastic process, there exists $K_A > 0$ such that $|A_t| \le K_A$ for all $t \in [0,T]$ $\mathbb{P}$-a.s.,*
ii) *$C \in \mathcal{H}_2^{\mathcal{G}}$,*
iii) *$E \in \mathcal{I}$,*
iv) *There exists $K_E > 0$ such that $0 \le E_t(z) < K_E z$ for $z \in \mathbb{R}_0$, and $|E_t(0)| < K_E$ $dt \times d\mathbb{P}$-a.e.*

Then (5.1) *has a unique solution (Y, ϕ) in $S \times \mathcal{I}$ and Y has representation*

$$Y_t = \mathbb{E}\Big[\xi\Gamma_T(t) + \int_t^T \Gamma_s(t) C_s\, ds \,\Big|\, \mathcal{G}_t\Big], \quad t \in [0,T],$$

where

$$\Gamma_s(t) = \exp\Big\{ \int_t^s A_u - \frac{1}{2}E_u(0)^2 \mathbf{1}_{\{\lambda_u^B \neq 0\}}\, du + \int_t^s E_u(0) \frac{\mathbf{1}_{\{\lambda_u^B \neq 0\}}}{\sqrt{\lambda_u^B}}\, dB_u$$
$$+ \int_t^s \int_{\mathbb{R}_0} \Big[\ln\big(1 + E_u(z) \frac{\mathbf{1}_{\{\lambda_u^H \neq 0\}}}{\sqrt{\lambda_u^H}}\big) - E_u(z) \frac{\mathbf{1}_{\{\lambda_u^H \neq 0\}}}{\sqrt{\lambda_u^H}} \Big] \nu(dz) \lambda_u^H\, du$$
$$+ \int_t^s \int_{\mathbb{R}_0} \ln\big(1 + E_u(z) \frac{\mathbf{1}_{\{\lambda_u^H \neq 0\}}}{\sqrt{\lambda_u^H}}\big)\, \tilde{H}(du, dz) \Big\}.$$

Note that $\Gamma_s(t) = \frac{\Gamma_s(0)}{\Gamma_t(0)}$.

Proof. The proof follows classical arguments, see e.g. [Pha09, Theorem 6.2.2]. Condition (4.3) is guaranteed by ii). From Hölder's inequality

$$\int_{\mathbb{R}_0} |E_t(z)\phi_t(z)|\, \nu(dz) \sqrt{\lambda_t^H} \leq \sqrt{\int_{\mathbb{R}_0} E_t^2(z)\, \nu(dz)} \sqrt{\int_{\mathbb{R}_0} \phi_t^2(z)\, \nu(dz)} \sqrt{\lambda_t^H}$$
$$\leq K_E \sqrt{\int_{\mathbb{R}_0} z^2\, \nu(dz)} \sqrt{\int_{\mathbb{R}_0} \phi_t^2(z)\, \nu(dz)} \sqrt{\lambda_t^H}, \tag{5.2}$$

so from i) and iv) we obtain (4.4). It follows from Theorem 4.5 that (5.1) has a unique solution $(Y, \phi) \in S \times \mathcal{I}$.

Denote $\Gamma_t = \Gamma_t(0)$. We have $\Gamma_0 = 1$ and Itô's formula gives us

$$d\Gamma_t = \Gamma_{t\text{-}} \Big(A_t\, dt + E_t(0) \frac{\mathbf{1}_{\{\lambda_t^B \neq 0\}}}{\sqrt{\lambda_t^B}}\, dB_t + \int_{\mathbb{R}_0} E_t(z) \frac{\mathbf{1}_{\{\lambda_t^H \neq 0\}}}{\sqrt{\lambda_t^H}}\, \tilde{H}(dt, dz) \Big). \tag{5.3}$$

Starting from (5.3),

$$\mathbb{E}\Big[|\Gamma_t|^2\Big] \leq 4\mathbb{E}\Big[1 + \Big(\int_0^t \Gamma_{s\text{-}} A_s\, ds \Big)^2 + \Big(\int_0^t \Gamma_{s\text{-}} E_s(0) \frac{\mathbf{1}_{\{\lambda_s^B \neq 0\}}}{\sqrt{\lambda_s^B}}\, dB_s \Big)^2$$
$$+ \Big(\int_0^t \int_{\mathbb{R}_0} \Gamma_{s\text{-}} E_s(z) \frac{\mathbf{1}_{\{\lambda_s^H \neq 0\}}}{\sqrt{\lambda_s^H}}\, \tilde{H}(ds, dz) \Big)^2 \Big]$$

$$
\begin{aligned}
&\leq 4\mathbb{E}\Big[1 + T\int_0^t |\Gamma_{s\text{-}}|^2 A_s^2\,ds + \int_0^t |\Gamma_{s\text{-}} E_s(0)|^2\,ds \\
&\qquad + \int_0^t \int_{\mathbb{R}_0} |\Gamma_{s\text{-}}|^2 K_E^2 z^2\,\nu(dz)\,ds\Big] \\
&\leq K_\Gamma \mathbb{E}\Big[1 + \int_0^t |\Gamma_{s\text{-}}|^2\,ds\Big]
\end{aligned}
$$

for some $K_\Gamma > 0$, since A and $E(0)$ are bounded and z^2 is integrable with respect to ν. We conclude that $\Gamma_t \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ for all $t \in [0,T]$ by Gronwall's inequality. By Itô's formula we have

$$
\begin{aligned}
d(Y_t\Gamma_t) &= \Gamma_{t\text{-}}\Big(\big[-A_tY_t - C_t - E_t(0)\phi_t(0)\sqrt{\lambda_t^B}\big]\,dt + \phi_t(0)\,dB_t \\
&\quad - \int_{\mathbb{R}_0} E_t(z)\phi_t(z)\,\nu(dz)\sqrt{\lambda_t^H}\,dt + \int_{\mathbb{R}_0} \phi_t(z)\,\tilde{H}(dt,dz)\Big) \\
&\quad + Y_{t\text{-}}\Gamma_{t\text{-}}\Big(A_t\,dt + E_t(0)\frac{\mathbf{1}_{\{\lambda_t^B\neq 0\}}}{\sqrt{\lambda_t^B}}\,dB_t + \int_{\mathbb{R}_0} E_t(z)\frac{\mathbf{1}_{\{\lambda_t^H\neq 0\}}}{\sqrt{\lambda_t^H}}\,\tilde{H}(dt,dz)\Big) \\
&\quad + \Gamma_{t\text{-}}\Big(E_t(0)\phi_t(0)\frac{\mathbf{1}_{\{\lambda_t^B\neq 0\}}}{\sqrt{\lambda_t^B}}\lambda_t^B\,dt + \int_{\mathbb{R}_0} E_t(z)\phi_t(z)\frac{\mathbf{1}_{\{\lambda_t^H\neq 0\}}}{\sqrt{\lambda_t^H}}\,H(dt,dz)\Big) \\
&= -\Gamma_{t\text{-}}C_t\,dt + \Big[\Gamma_{t\text{-}}\phi_t(0) + Y_{t\text{-}}E_t(0)\frac{\mathbf{1}_{\{\lambda_t^B\neq 0\}}}{\sqrt{\lambda_t^B}}\Big]\,dB_t + \int_{\mathbb{R}_0}\Big[\phi_t(z)\Gamma_{t\text{-}} \\
&\quad + Y_{t\text{-}}\Gamma_{t\text{-}}E_t(z)\frac{\mathbf{1}_{\{\lambda_t^H\neq 0\}}}{\sqrt{\lambda_t^H}} + \Gamma_{t\text{-}}\phi_t(z)E_t(z)\frac{\mathbf{1}_{\{\lambda_t^H\neq 0\}}}{\sqrt{\lambda_t^H}}\Big]\,\tilde{H}(dt,dz).
\end{aligned} \tag{5.4}
$$

Hence $Y_t\Gamma_t + \int_0^t \Gamma_s C_s\,ds$, $t \in [0,T]$, is a $\mathbb{G}$-martingale so that

$$
\begin{aligned}
Y_t\Gamma_t + \int_0^t \Gamma_s C_s\,ds &= \mathbb{E}\Big[Y_T\Gamma_T + \int_0^T \Gamma_s C_s\,ds\,\Big|\mathcal{G}_t\Big] \\
Y_t\Gamma_t &= \mathbb{E}\Big[\xi\Gamma_T + \int_t^T \Gamma_s C_s\,ds\,\Big|\mathcal{G}_t\Big] \\
Y_t &= \mathbb{E}\Big[\xi\Gamma_T^t + \int_t^T \Gamma_s(t) C_s\,ds\,\Big|\mathcal{G}_t\Big].
\end{aligned}
$$

(Recall that $\Gamma_s(t) = \frac{\Gamma_s}{\Gamma_t}$). □

Theorem 5.2. *Let $(g^{(1)}, \xi^{(1)})$ and $(g^{(2)}, \xi^{(2)})$ be two sets of standard parameters for the BSDE's with solutions $(Y^{(1)}, \phi^{(1)})$, $(Y^{(2)}, \phi^{(2)}) \in S \times \mathcal{I}$. Assume that*

$$g_t^{(2)}(\lambda, y, \phi, \omega) = f_t\Big(y, \phi(0)\kappa_t(0)\sqrt{\lambda^B}, \int_{\mathbb{R}_0} \phi(z)\kappa_t(z)\,\nu(dz)\sqrt{\lambda^H}, \omega\Big)$$

where $\kappa \in \mathcal{I}$ satisfies condition iv) from Theorem 5.1 and f is a function $f : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$ which satisfies, for some $K_f > 0$,

$$|f_t(y,b,h) - f_t(y',b',h')| \le K_f\Big(|y-y'| + |b-b'| + |h-h'|\Big), \tag{5.5}$$

$dt \times d\mathbb{P}$ a.e. and

$$\mathbb{E}\Big[\int_0^T |f_t(0,0,0)|^2\,dt\Big] < \infty.$$

If $\xi^{(1)} \le \xi^{(2)}$ $\mathbb{P}$-a.s. and $g_s^{(1)}(\lambda_s, Y_s^{(1)}, \phi_s^{(1)}) \le g_s^{(2)}(\lambda_s, Y_s^{(1)}, \phi_s^{(1)})$ $dt \times d\mathbb{P}$-a.e., then

$$Y_t^{(1)} \le Y_t^{(2)} \quad dt \times d\mathbb{P}\text{-a.e.}$$

It can be shown that $g^{(2)}$ does indeed satisfy conditions (4.2)-(4.3)-(4.4) in Definition 4.1, recall in particular (5.2).

Proof. Define $\bar{Y}_t := Y_t^{(2)} - Y_t^{(1)}$, $\bar{\phi}_t := \phi_t^{(2)} - \phi_t^{(1)}$,

$$\phi_t^{(2,H)}(z) := \phi_t^{(2)}\mathbf{1}_{\{z\neq 0\}} + \phi_t^{(1)}\mathbf{1}_{\{z=0\}},$$
$$\phi_t^{(2,B)}(z) := \phi_t^{(2)}\mathbf{1}_{\{z=0\}} + \phi_t^{(1)}\mathbf{1}_{\{z\neq 0\}},$$

and

$$C_t := g_t^{(2)}(\lambda_t, Y_t^{(1)}, \phi_t^{(1)}) - g_t^{(1)}(\lambda_t, Y_t^{(1)}, \phi_t^{(1)}),$$
$$A_t := \frac{g_t^{(2)}(\lambda_t, Y_t^{(2)}, \phi_t^{(1)}) - g_t^{(2)}(\lambda_t, Y_t^{(1)}, \phi_t^{(1)})}{\bar{Y}_t}\mathbf{1}_{\{\bar{Y}_t\neq 0\}},$$
$$D_t := \frac{g_t^{(2)}(\lambda_t, Y_t^{(2)}, \phi_t^{(2,H)}) - g_t^{(2)}(\lambda_t, Y_t^{(2)}, \phi_t^{(1)})}{\int_{\mathbb{R}_0} \kappa_t(z)\bar{\phi}_t(z)\,\nu(dz)\sqrt{\lambda_t^H}}\mathbf{1}_{\{\int_{\mathbb{R}_0} \kappa_t(z)\bar{\phi}_t(z)\,\nu(dz)\sqrt{\lambda_t^H}\neq 0\}},$$
$$F_t := \frac{g_t^{(2)}(\lambda_t, Y_t^{(2)}, \phi_t^{(2,B)}) - g_t^{(2)}(\lambda_t, Y_t^{(2)}, \phi_t^{(1)})}{\kappa_t(0)\bar{\phi}_t(0)\sqrt{\lambda_t^B}}\mathbf{1}_{\{\kappa_t(0)\bar{\phi}_t(0)\sqrt{\lambda_t^B}\neq 0\}}.$$

Then

$$-d\bar{Y}_t = \Big[A_t\bar{Y}_t + C_t + F_t\kappa_t(0)\bar{\phi}_t(0)\sqrt{\lambda_t^B} + D_t\int_{\mathbb{R}_0}\kappa_t(z)\bar{\phi}_t(z)\,\nu(dz)\sqrt{\lambda_t^H}\Big]dt$$
$$-\bar{\phi}_t(0)\,dB_t - \int_{\mathbb{R}_0}\bar{\phi}_t(z)\,\tilde{H}(dt,dz),$$

(5.6)
$$\bar{Y}_T = \xi_2 - \xi_1.$$

The processes A, D and F are bounded due to the Lipschitz condition (5.5), and $C \in \mathcal{H}_2^{\mathcal{G}}$ since it is a difference of functions in $\mathcal{H}_2^{\mathcal{G}}$. It follows that $F\kappa(0) + D\kappa(z)\mathbf{1}_{\mathbb{R}_0}(z)$ satisfies iv) in Theorem 5.1.

Thus the assumptions of Theorem 5.1 are satisfied. The BSDE in (5.6) has solution

$$\bar{Y}_t = \mathbb{E}\Big[\bar{\xi}\Gamma_T^t + \int_t^T \Gamma_s(t)C_s\,ds\,\Big|\mathcal{G}_t\Big]$$

which is positive a.s. since $\bar{\xi}$, Γ and C are all positive a.s.

□

6. Sufficient stochastic maximum principle

Here we show an application of the BSDE, proving sufficient conditions for an optimal control problem with both $\mathbb{G}$ and $\mathbb{F}$-predictable controls. This problem cannot be solved with dynamic programming methods since the state process is, in general, not Markovian. We consider the optimization problem associated to the performance functional

$$J(u) = \mathbb{E}\Big[\int_0^T f_t(\lambda_t, u_t, X_{t\text{-}})\,dt + l(X_T)\Big], \tag{6.1}$$

where $l(x,\omega)$, $x \in \mathbb{R}$, $\omega \in \Omega$ is a stochastic function concave and differentiable in x a.s. and $f_t(\lambda, u, x, \omega)$, $t \in [0,T]$, $\lambda \in [0,\infty)^2$, $u \in \mathcal{U}$, $x \in \mathbb{R}$, $\omega \in \Omega$ is a stochastic function differentiable in x for a.s. Here $\mathcal{U} \subseteq \mathbb{R}$ is a closed, convex set. The state process X_t, $t \in [0,T]$, has the form

$$dX_t = b_t(\lambda_t, u_t, X_{t\text{-}})\,dt + \int_{\mathbb{R}}\kappa_t(z, \lambda_t, u_t, X_{t\text{-}})\,\mu(dt,dz), \tag{6.2}$$
$$X_0 \in \mathbb{R},$$

where $b_t(\lambda, u, x)$ and $\kappa_t(z, \lambda, u, x)$, $t \in [0,T]$, $\lambda \in [0,\infty)^2$, $z \in \mathbb{R}$, $u \in \mathcal{U}$, $x \in \mathbb{R}$ are $\mathbb{F}$-adapted stochastic processes differentiable in x a.s. We denote these derivatives $\partial_x b$ and $\partial_x \kappa$ respectively. The stochastic process u_t, $t \in [0,T]$, is the control. We have the following definition

Definition 6.1. *The admissible controls are càglàd stochastic processes* $u : [0,T] \times \Omega \to \mathcal{U}$*, such that* X (6.2) *has a unique strong solution,*

$$\mathbb{E}\Big[\int_0^T |f_t(\lambda_t, u_t, X_{t\text{-}})|^2\, dt + |l(X_T)| + |\partial_x l(X_T)|^2\Big] < \infty, \tag{6.3}$$

and for some $K_1 > 0$ *we have*

$$\Big|\partial_x \kappa_t(0, \lambda_t, u_t, X_{t\text{-}})\Big|\sqrt{\lambda_t^B} \le K_1 \quad dt \times d\mathbb{P}\text{-}a.e, \tag{6.4}$$

$$\int_{\mathbb{R}_0} \big(\partial_x \kappa_t(z, \lambda_t, u_t, X_{t\text{-}})\big)^2 \nu(dz)\sqrt{\lambda_t^H} \le K_1 \quad dt \times d\mathbb{P}\text{-}a.e, \tag{6.5}$$

$$\big|\partial_x b_t(\lambda_t, u_t, X_{t\text{-}})\big| \le K_1 \quad dt \times d\mathbb{P}\text{-}a.e. \tag{6.6}$$

The admissible controls are either $\mathbb{G}$*-predictable or* $\mathbb{F}$*-predictable and we denote these sets as* $\mathcal{A}^{\mathbb{G}}$ *and* $\mathcal{A}^{\mathbb{F}}$ *respectively. The couple* (u, X) *is called an admissible pair.*

Naturally $\mathcal{A}^{\mathbb{F}} \subset \mathcal{A}^{\mathbb{G}}$. Remark also that X is $\mathbb{G}$-adapted if $u \in \mathcal{A}^{\mathbb{G}}$ and X is $\mathbb{F}$-adapted if $u \in \mathcal{A}^{\mathbb{F}}$. Given the performance functional J (6.1) we aim to find an optimal control depending on the information available:

$$\sup_{u \in \mathcal{A}^{\mathbb{G}}} J(u) \tag{6.7}$$

$$\sup_{u \in \mathcal{A}^{\mathbb{F}}} J(u). \tag{6.8}$$

For a detailed discussion on the existence of a strong solution to (6.2) we refer to [Jac79b, Jac79a]. However the following conditions are sufficient [Jac79b]: for u a càglàd stochastic process there exist a $K_2 > 0$ such that

$$\big|\kappa_t(0, \lambda_t, u_t, x) - \kappa_t(0, \lambda_t, u_t, x')\big| \le K_2\big|x - x'\big| \quad \mathbb{P}\text{-a.s.}, \tag{6.9}$$

$$\big|\kappa_t(z, \lambda_t, u_t, x) - \kappa_t(z, \lambda_t, u_t, x')\big| \le K_2|z||x - x'| \quad \text{for } z \neq 0\ \mathbb{P}\text{-a.s.}, \tag{6.10}$$

$$\big|b_t(\lambda_t, u_t, x) - b_t(\lambda_t, u_t, x')\big| \le K_2|x - x'| \quad \mathbb{P}\text{-a.s.}, \tag{6.11}$$

$$\int_0^T \int_{\mathbb{R}} |\kappa_s(z, \lambda_s, u_s, a)|^2\, \Lambda(ds, dz) < \infty \quad \mathbb{P}\text{-a.s.}, \tag{6.12}$$

$$\int_0^T \big|b_s(\lambda_s, u_s, a)\big|\, ds \le \infty \quad \mathbb{P}\text{-a.s.}, \tag{6.13}$$

for some $a \in \mathbb{R}$, all $t \in [0,T]$ and all $x, x' \in \mathbb{R}$.

We define the Hamiltonian, $\mathcal{H} : [0,T] \times [0,\infty)^2 \times \mathcal{U} \times \mathbb{R} \times \mathbb{R} \times \Phi \times \Omega \to \mathbb{R}$ (where Φ is defined in (4.1)), by

$$\mathcal{H}_t(\lambda, u, x, y, \phi) = f_t(\lambda, u, x) + b_t(\lambda, u, x)y + \kappa_t(0, \lambda, u, x)\phi(0)\lambda^B + \int_{\mathbb{R}_0} \kappa_t(z, \lambda, u, x)\phi(z)\, \lambda^H\, \nu(dz).$$

Corresponding to the admissible pair (u, X) is the couple (Y, ϕ), which is the solution to the BSDE of type (1.1)

$$\begin{aligned} Y_t &= \partial_x l(X_T) + \int_t^T \partial_x \mathcal{H}_s(\lambda, u_s, X_{s-}, Y_{s-}, \phi_s)\, ds - \int_t^T \int_{\mathbb{R}} \phi_s(z)\, \mu(ds, dz), \\ Y_T &= \partial_x l(X_T). \end{aligned} \tag{6.14}$$

Here $\partial_x \mathcal{H}_t = \frac{\partial}{\partial x}\mathcal{H}_t(\lambda, u, x, y, \phi)$ and we note that $\mathcal{H}$ is differentiable in x by the assumptions on f, g and κ. The above conditions, (6.3)-(6.4)-(6.5)-(6.6), ensure that the pair $\big(\partial_x \mathcal{H}, \partial_x l(X_T)\big)$ are standard parameters (Definition 4.1). By Theorem 4.5 the BSDE (6.14) has a unique solution (Y, ϕ).

In the sequel we set $\hat{b}_s = b_s(\lambda_s, \hat{u}_s, \hat{X}_{s-})$, etc. for the coefficients associated with the admissible pair $(\hat{u}, \hat{X})$ with solution $(\hat{Y}, \hat{\phi})$ of the adjoint equation (6.14). Set $b_s = b_s(\lambda_s, u_s, X_{s-})$ and so forth for the coefficents associated to another arbitrary, admissible pair (u, X). In addition $\hat{\mathcal{H}}_s(u, x) = \mathcal{H}_s(\lambda_s, u, x, \hat{Y}_{s-}, \hat{\phi}_s)$.

Theorem 6.2. *Let $\hat{u} \in \mathcal{A}^{\mathbb{G}}$. Assume that*

$$\mathbb{E}\Big[\int_0^{\cdot}\int_{\mathbb{R}} \big|\hat{Y}_{s-}\big(\hat{\kappa}_s(z) - \kappa_s(z)\big)\big|^2 + \big|\big(\hat{X}_{s-} - X_{s-}\big)\hat{\phi}_s(z)\big|^2\, \Lambda(ds, dz)\Big] < \infty \tag{6.15}$$

for all $u \in \mathcal{A}^{\mathbb{G}}$. If

$$h_t(x) = \max_{u \in \mathcal{U}} \mathcal{H}_t(\lambda_t, u, x, \hat{Y}_{t-}, \hat{\phi}_t) \tag{6.16}$$

exists and is a concave function in x for all $t \in [0,T]$ $\mathbb{P}$-a.s., and

$$\mathcal{H}_t(\lambda_t, \hat{u}_t, \hat{X}_{t-}, \hat{Y}_{t-}, \hat{\phi}_t) = h_t(\hat{X}_t) \tag{6.17}$$

for all $t \in [0,T]$, then $\hat{u}$ is optimal for (6.7) and $(\hat{u}, \hat{X})$ is an optimal pair.

Proof. We proceed as in [FØS04]. Recall that for l concave and differentiable we have $l(x_2) - l(x_1) \geq \partial_x l(x_2)(x_2 - x_1)$, $x_1, x_2 \in \mathbb{R}$. Thus, by Itô's formula, (6.15) and the fact that $\hat{X}_0 - X_0 = 0$, we have

$$\begin{aligned} &\mathbb{E}\Big[\, l(\hat{X}_T) - l(X_T)\Big] \\ &\geq \mathbb{E}\Big[\partial_x\, l(\hat{X}_T)\big(\hat{X}_T - X_T\big)\Big] \\ &= \mathbb{E}\Big[\, \hat{Y}_T\big(\hat{X}_T - X_T\big)\Big] \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\Big[\int_0^T -\big(\hat{X}_{s\text{-}} - X_{s\text{-}}\big)\partial_x \hat{\mathcal{H}}_s(\hat{u}_s, \hat{X}_{s\text{-}}) + \hat{Y}_{s\text{-}}\big(\hat{b}_s - b_s\big)\, ds \\
&\quad + \int_0^T \int_{\mathbb{R}} \Big\{\hat{Y}_{s\text{-}}\big(\hat{\kappa}_s(z) - \kappa_s(z)\big) + \big(\hat{X}_{s\text{-}} - X_{s\text{-}}\big)\hat{\phi}_s(z)\Big\}\, \mu(ds, dz) \\
&\quad + \int_0^T \big(\hat{\kappa}_s(0) - \kappa_s(0)\big)\hat{\phi}_s(0)\, \lambda_s^B\, ds + \int_0^T \int_{\mathbb{R}_0} \big(\hat{\kappa}_s(z) - \kappa_s(z)\big)\hat{\phi}_s(z)\, H(ds, dz)\Big] \\
&= \mathbb{E}\Big[\int_0^T -\big(\hat{X}_{s\text{-}} - X_{s\text{-}}\big)\partial_x \hat{\mathcal{H}}_s(\hat{u}_s, \hat{X}_{s\text{-}}) + \hat{Y}_{s\text{-}}\big(\hat{b}_s - b_s\big)\, ds \\
&\quad + \int_0^T \int_{\mathbb{R}} \big(\hat{\kappa}_s(z) - \kappa_s(z)\big)\hat{\phi}_s(z)\, \Lambda(ds, dz)\Big].
\end{aligned}
$$

We remark that ϕ is integrable with respect to $H \times \mathbb{P}$ by (6.10), Cauchy's inequality and (6.15). Furthermore, from the Hamiltonian, we have

$$
\begin{aligned}
&\mathbb{E}\Big[\int_0^T \big\{\hat{f}_s - f_s\big\}\, ds\Big] \\
&\quad = \mathbb{E}\Big[\int_0^T \Big\{\hat{\mathcal{H}}_s(\hat{u}_s, \hat{X}_{s\text{-}}) - \hat{\mathcal{H}}_s(u_s, X_{s\text{-}}) - \big(\hat{b}_s - b_s\big)\hat{Y}_{s\text{-}} \\
&\qquad - \big(\hat{\kappa}_s(0) - \kappa_s(0)\big)\hat{\phi}_s(0)\lambda_s^B - \int_{\mathbb{R}_0} \big(\hat{\kappa}_s(z) - \kappa_s(z)\big)\hat{\phi}_s(z)\, \nu(dz)\lambda_s^H\Big\}\, ds\Big] \\
&\quad = \mathbb{E}\Big[\int_0^T \Big\{\hat{\mathcal{H}}_s(\hat{u}_s, \hat{X}_{s\text{-}}) - \hat{\mathcal{H}}_s(u_s, X_{s\text{-}}) - \big(\hat{b}_s - b_s\big)\hat{Y}_{s\text{-}}\Big\}\, ds \\
&\qquad - \int_0^T \int_{\mathbb{R}} \big(\hat{\kappa}_s(z) - \kappa_s(z)\big)\hat{\phi}_s(z)\, \Lambda(ds, dz)\Big].
\end{aligned}
$$

Hence

$$
\begin{aligned}
J(\hat{u}) - J(u) \geq \mathbb{E}\Big[\int_0^T \Big\{&\hat{\mathcal{H}}_s(\hat{u}_s, \hat{X}_{s\text{-}}) \\
&- \hat{\mathcal{H}}_s(u_s, X_{s\text{-}}) - \big(\hat{X}_{s\text{-}} - X_{s\text{-}}\big)\partial_x \hat{\mathcal{H}}_s(\hat{u}_s, \hat{X}_{s\text{-}})\Big\}\, ds\Big]. \qquad (6.18)
\end{aligned}
$$

The integrand in (6.18) is non-negative $dt \times d\mathbb{P}$-a.e. by the maximality of $\hat{u}$ (6.17) and the concavity of h_t, see [SS87, page 108]. Hence $\hat{u}$ is also an optimal control by inequality (6.18). We sketch the last part of the argument for completeness. From (6.16) and (6.17) we have $h_t(\hat{X}_{t\text{-}}) = \hat{\mathcal{H}}_t(\hat{u}_t, \hat{X}_{t\text{-}})$. Thus

$$\hat{\mathcal{H}}_t(u,x) - \hat{\mathcal{H}}_t(\hat{u}_t, \hat{X}_{t\text{-}}) \leq h_t(x) - h_t(\hat{X}_{t\text{-}}), \quad \text{for all } (t,u,x). \tag{6.19}$$

To prove that the integrand in (6.18) is non-negative it is sufficient to show that almost surely

$$h_t(X_{t\text{-}}) - h_t(\hat{X}_{t\text{-}}) - \partial_x \hat{\mathcal{H}}_t(\hat{u}_t, \hat{X}_{t\text{-}})\big(X_{t\text{-}} - \hat{X}_{t\text{-}}\big) \leq 0. \tag{6.20}$$

Fix $t \in [0,T]$. Since $x \to h_t(x)$ is concave, it follows by a separating hyperplane argument that there exists $a \in \mathbb{R}$ such that

$$h_t(x) - h_t(\hat{X}_{t\text{-}}) - a(x - \hat{X}_{t\text{-}}) \leq 0, \quad \text{for all } x. \tag{6.21}$$

Define

$$\rho(x) := \hat{\mathcal{H}}_t(u_t, x) - \hat{\mathcal{H}}_t(\hat{u}_t, \hat{X}_{t\text{-}}) - a(x - \hat{X}_{t\text{-}}).$$

By (6.19) and (6.21) $\rho(x) \leq 0$ for all x. Clearly $\rho(\hat{X}_{t\text{-}}) = 0$. Hence $\partial_x \rho(\hat{X}_{t\text{-}}) = 0$ so that $\partial_x \hat{\mathcal{H}}_t(\hat{X}_{t\text{-}}, \hat{u}_t) = a$. Substituting into (6.21) we obtain (6.20). □

Recall that the solution of the BSDE (6.14) is $\mathbb{G}$-adapted. However, the other coefficients in (6.17) are $\mathbb{F}$-adapted whenever $u \in \mathcal{A}^{\mathbb{F}}$. We use this fact to find an optimal $\mathbb{F}$-predictable control via projections. We keep the notation used in the proof of Theorem 6.2.

Theorem 6.3. *Let $\hat{u} \in \mathcal{A}^{\mathbb{F}}$. Denote the corresponding state process as $\hat{X}$ with solution $(\hat{Y}, \hat{\phi})$ of the adjoint equation* (6.14)*. Assume* (6.15) *holds. Denote*

$$\begin{aligned}
\mathcal{H}_t^{\mathbb{F}}(\lambda_t, u, x, \hat{Y}_{t\text{-}}, \hat{\phi}_t) &:= \mathbb{E}\Big[\mathcal{H}_t(\lambda_t, u, x, \hat{Y}_{t\text{-}}, \hat{\phi}_t)\,\big|\mathcal{F}_t\Big] \\
&= f_t(\lambda_t, u, x) + b_t(\lambda_t, u, x)\mathbb{E}\big[\hat{Y}_{t\text{-}}\,\big|\mathcal{F}_t\big] + \kappa_t(0, \lambda_t, u, x)\mathbb{E}\big[\hat{\phi}_t(0)\,\big|\mathcal{F}_t\big] \\
&\quad + \int_{\mathbb{R}_0} \kappa_t(z, \lambda_t, u, x)\mathbb{E}\big[\phi_t(z)\,\big|\mathcal{F}_t\big]\,\lambda_t^H\,\nu(dz)
\end{aligned}$$

for all $t \in [0,T]$. If

$$h_t^{\mathbb{F}}(x) = \max_{u \in \mathcal{U}} \mathcal{H}_t^{\mathbb{F}}\big(\lambda_t, u, x, \hat{Y}_{t\text{-}}, \hat{\phi}_t\big) \tag{6.22}$$

exists and is a concave function in x for all $t \in [0,T]$, and

$$\mathcal{H}_t^{\mathbb{F}}(\lambda_t, \hat{u}_t, \hat{X}_t, \hat{Y}_{t\text{-}}, \hat{\phi}_t) = h_t^{\mathbb{F}}(\hat{X}_t), \tag{6.23}$$

then $(\hat{u}, \hat{X})$ is an optimal pair for (6.8)*.*

Proof. The arguments in the proof of Theorem 6.2 leading to

$$J(\hat{u})-J(u) \geq \mathbb{E}\Big[\int_0^T \hat{\mathcal{H}}_s(\hat{u}_s,\hat{X}_{s\text{-}})-\hat{\mathcal{H}}_s(u_s,X_{s\text{-}})-\big(\hat{X}_{s\text{-}}-X_{s\text{-}}\big)\partial_x\hat{\mathcal{H}}_s(\hat{u}_s,\hat{X}_{s\text{-}})\,ds\Big] \tag{6.24}$$

still hold. Since $\hat{u}$ and u are $\mathbb{F}$-predictable controls the only coefficients in the integrand in (6.24) that are not $\mathbb{F}$-adapted are the solution of the adjoint equation $(\hat{Y},\hat{\phi})$ so that

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T \hat{\mathcal{H}}_s(\hat{u}_s,\hat{X}_{s\text{-}})-\hat{\mathcal{H}}_s(u_s,X_{s\text{-}})-\partial_x\hat{\mathcal{H}}_s(\hat{u}_s,\hat{X}_{s\text{-}})\big(\hat{X}_{s\text{-}}-X_{s\text{-}}\big)\,ds\Big]\\
&=\mathbb{E}\Big[\int_0^T \hat{f}_s-f_s+\big(\hat{b}_s-b_s\big)\mathbb{E}\big[\hat{Y}_{s\text{-}}\,\big|\mathcal{F}_s\big]+\big(\hat{X}_{s\text{-}}-X_{s\text{-}}\big)\partial_x f_s+\partial_x b_s\mathbb{E}\big[\hat{Y}_{s\text{-}}\,\big|\mathcal{F}_s\big]\,ds\Big]\\
&\quad+\mathbb{E}\Big[\int_0^T\int_{\mathbb{R}}\Big\{\big(\hat{\kappa}_s(z)-\kappa_s(z)\big)\mathbb{E}\big[\hat{\phi}_s(z)\,\big|\mathcal{F}_s\big]\\
&\quad+\big(\hat{X}_{s\text{-}}-X_{s\text{-}}\big)\partial_x\hat{\kappa}_s(z)\mathbb{E}\big[\hat{\phi}_s(z)\,\big|\mathcal{F}_s\big]\Big\}\,\Lambda(ds,dz)\Big]\\
&=\mathbb{E}\Big[\int_0^T \hat{\mathcal{H}}^{\mathbb{F}}_s(\hat{u}_s,\hat{X}_{s\text{-}})-\hat{\mathcal{H}}^{\mathbb{F}}_s(u_s,X_{s\text{-}})-\partial_x\hat{\mathcal{H}}^{\mathbb{F}}_s(\hat{u}_s,\hat{X}_{s\text{-}})\big(\hat{X}_{s\text{-}}-X_{s\text{-}}\big)\,ds\Big].
\end{aligned} \tag{6.25}$$

The integrand in (6.25) is non-negative $dt\times d\mathbb{P}$-a.e. by the maximality of $\hat{u}$ (6.23) and the concavity of $h^{\mathbb{F}}_t$ (6.22). The argument is the same as in the proof of Theorem 6.2. □

For a study on necessary maximum principles in the case of time-changed Lévy noise we refer to [Sju13], where techniques of non-anticipating stochastic derivatives are used, see [DE10].

7. Optimal portfolio problems

Here we show how investments in financial assets can be modeled within our framework with a state process suitable for various optimization problems. First we setup the general framework, then consider the specific problems of mean-variance hedging and utility maximization.

We consider two assets, a risk free asset R and a risky asset S defined by

$$\begin{aligned}
dR_t &= \rho_t R_{t\text{-}}\,dt, && R_0=1,\\
dS_t &= \alpha_t S_{t\text{-}}\,dt+S_{t\text{-}}\int_{\mathbb{R}}\psi_t(z)\,\mu(dt,dz), && S_0>0.
\end{aligned}$$

Models of this type include (1.6) (the model from [CGMY03]) and (1.7). Here α and ρ are $\mathbb{F}$-adapted stochastic processes with $\alpha, \rho : [0,T] \times \Omega \to \mathbb{R}$ and $\psi \in \mathcal{I}$ is an $\mathbb{F}$-adapted random field. We assume that ρ is bounded. Let $z_t^{(R)}$ denote the units of R held at time t and $z_t^{(S)}$ the number of units of S held at time t. The wealth process X_t, $t \in [0,T]$, is the value of the assets held,

$$X_t = z_t^{(R)} R_t + z_t^{(S)} S_t. \tag{7.1}$$

We assume that the portfolio is self-financing, i.e. that

$$dX_t = z_t^{(R)} dR_t + z_t^{(S)} dS_t. \tag{7.2}$$

Let $u_t = z_t^{(S)} S_t$, $t \in [0,T]$, denote the amount of wealth invested in the risky asset S. By (7.1) and (7.2) the wealth equation is given by

$$dX_t = \big[\rho_t X_t + (\alpha_t - \rho_t) u_t\big]\, dt + u_t \int_{\mathbb{R}} \psi_t(z)\, \mu(dt, dz). \tag{7.3}$$

Clearly assumptions (6.4), (6.5) and (6.6) are satisfied. We assume that (7.3) admits a strong solution. For this we see that the sufficient conditions (6.9), (6.10), (6.11) are satisfied and together with

$$\int_0^T |\alpha_s - \rho_s||u_s|\, ds < \infty$$

$$\int_0^T \int_{\mathbb{R}_0} |u_s \psi_s(z)|^2\, \Lambda(ds, dz) < \infty$$

$\mathbb{P}$-a.s. also (6.12) and (6.13). The SDE (7.3) is of Ornstein-Uhlenbeck type and has solution

$$X_t = e^{\int_0^t \rho_r\, dr} \Big(X_0 + \int_0^t e^{-\int_0^s \rho_r\, dr} (\alpha_s - \rho_s) u_s\, ds$$

$$+ \int_0^t \int_{\mathbb{R}} e^{-\int_0^s \rho_r\, dr} u_s \psi_s(z)\, \mu(ds, dz) \Big). \tag{7.4}$$

We consider portfolio problems of type (6.7)-(6.8) associated

$$J(u) = \mathbb{E}\big[l(X_T)\big] \tag{7.5}$$

with l as in (6.1). Problems of this type include utility maximization and mean-variance portfolio selection. Hedging problems are also included, since l is a function of both ω and X we can consider e.g. the mean-variance hedge

by $l(X_T) = -(X_T - Z)^2$ for a square integrable random variable Z. The Hamiltonian for this class of problems is

$$\begin{aligned}\mathcal{H}_t(\lambda, u, x, y, \phi) &= \big[\rho_t x + (\alpha_t - \rho_t)u\big]y + u\psi_t(0)\phi(0)\lambda_t^B \\ &\quad + \int_{\mathbb{R}_0} \big[u\psi_t(z)\phi(z)\big]\,\lambda_t^H \nu(dz)\end{aligned} \tag{7.6}$$

and the associated BSDE is given by

$$Y_t = \partial_x l(X_T) + \int_t^T Y_{s\text{-}}\rho_s\,ds - \int_t^T \int_{\mathbb{R}} \phi_s(z)\,\mu(ds, dz). \tag{7.7}$$

By Theorem 5.1 we also have the representation

$$Y_t = \mathbb{E}\Big[\partial_x l(X_T)\exp\big\{\int_t^T \rho_s\,ds\big\}\Big|\mathcal{G}_t\Big]. \tag{7.8}$$

7.1. **The mean-variance portfolio selection.** Here we discuss mean-variance portfolio selection starting from an initial wealth $x \in \mathbb{R}$, i.e. solve $\inf_u \mathbb{E}\big[(X_T - E[X_T])^2\big]$ with $E[X_T] = k$ for some $k \in \mathbb{R}$ and controls taking values in $\mathcal{U} = \mathbb{R}$. For notational convenience we consider the equivalent formulation $J(u) = \mathbb{E}\big[-\frac{1}{2}\big(X_T - k\big)^2\big]$ and want to find

$$\sup_{u\in\mathcal{A}^{\mathbb{F}}} J(u) = \sup_{u\in\mathcal{A}^{\mathbb{F}}} \mathbb{E}\Big[-\frac{1}{2}\big(X_T - k\big)^2\Big]. \tag{7.9}$$

To solve this problem we first consider the optimization on $u \in \mathcal{A}^{\mathbb{G}}$ with deterministic coefficients and apply Theorem 6.2. To avoid trivial solutions we assume $\alpha_t > \rho_t$ $dt \times d\mathbb{P}$ a.e.

Theorem 7.1. *Assume that ρ and α are deterministic. Consider the feedback control $\hat{u}^{\mathbb{G}} \in \mathcal{A}^{\mathbb{G}}$ given by*

$$\hat{u}_t^{\mathbb{G}} = \frac{-\big(\alpha_t - \rho_t\big)\big(A_t\hat{X}_t + C_t\big)}{A_t\big(|\psi_t(0)|^2\lambda_t^B + \int_{\mathbb{R}_0}|\psi_t(z)|^2\,\lambda_t^H\,\nu(dz)\big)},$$

where $\hat{X}$ refers to (7.4) *with $\hat{u}^{\mathbb{G}}$ and*

$$A_t = -\exp\Big\{-\int_t^T \frac{(\alpha_s - \rho_s)^2}{|\psi_s(0)|^2\lambda_s^B + \int_{\mathbb{R}_0}|\psi_s(z)|^2\,\lambda_s^H\,\nu(dz)} - 2\rho_s\,ds\Big\} \tag{7.10}$$

$$C_t = k\exp\Big\{-\int_t^T \frac{(\alpha_s - \rho_s)^2}{|\psi_s(0)|^2\lambda_s^B + \int_{\mathbb{R}_0}|\psi_s(z)|^2\,\lambda_s^H\,\nu(dz)} - \rho_s\,ds\Big\}. \tag{7.11}$$

If (6.15) *holds then $\hat{u}^{\mathbb{G}}$ is optimal for $\sup_{u\in\mathcal{A}^{\mathbb{G}}} J(u)$.*

Remark that, from (7.10)-(7.11), the processes A and C depend on future values of λ^B and λ^H, hence they are $\mathbb{G}$-adapted, but in general not $\mathbb{F}$-adapted. Observe that equation (7.8) gives the useful characterization of the adjoint equation

$$\begin{aligned} Y_t &= \mathbb{E}\Big[-(X_T - k)\exp\Big\{\int_t^T \rho_s\,ds\Big\}\Big|\mathcal{G}_t\Big] \\ &= \big(k - X_t\big)\mathbb{E}\Big[\exp\Big\{\int_t^T \rho_s\,ds\Big\}\Big|\mathcal{G}_t\Big] + \mathbb{E}\Big[(X_t - X_T)\exp\Big\{\int_t^T \rho_s\,ds\Big\}\Big|\mathcal{G}_t\Big]. \end{aligned}$$

The study of the above representation hints that the process Y takes the form $Y_t = A_t X_t + C_t$ where A and C are some $\mathbb{G}$-adapted processes of finite variation and $A_T = -1$, $C_T = k$. Together with some results in the Lévy case, see e.g. [FØS04], we can take a guess of the structure of Y for an optimal control candidate and use our verification theorem to actually determine it's optimality.

Proof of Theorem 7.1. Denote a and c as of the derivatives of A and C with respect to t, so that $dA_t = a_t\,dt$ and $dC_t = c_t\,dt$. Set

$$\hat{u}_t := -\frac{\rho_t\big(A_t\hat{X}_t + C_t\big) + A_t\rho_t\hat{X}_t + \hat{X}_t a_t + c_t}{A_t(\alpha_t - \rho_t)}, \tag{7.12}$$

$$\hat{Y}_t := A_t\hat{X}_t + C_t, \tag{7.13}$$

where A and C are given by (7.10)-(7.11). We need to show

i) that (7.13) indeed is the process Y in the solution of (7.7) corresponding to the control $\hat{u}$.
ii) that $\hat{u}$ (7.12) satisfies (6.17) in Theorem 6.2.

To prove i), we combine (7.3) and (7.13) to get

$$\begin{aligned} d\hat{Y}_t &= A_t\big(\rho_t\hat{X}_t + (\alpha_t - \rho_t)\hat{u}_t\big)\,dt + A_t\hat{u}_t\int_{\mathbb{R}} \psi_t(z)\,\mu(dt,dz) \\ &\quad + \hat{X}_t a_t\,dt + c_t\,dt; \qquad \hat{Y}_T = k - \hat{X}_T. \end{aligned} \tag{7.14}$$

Inserting (7.12) in (7.14) and by the fact that $A_T = -1$ and $C_T = k$, we see that

$$\hat{Y}_t = k - \hat{X}_T + \int_t^T \hat{Y}_{s\text{-}}\rho_s\,ds - \int_t^T\int_{\mathbb{R}} A_s\hat{u}_s\psi_s(z)\,\mu(ds,dz). \tag{7.15}$$

By the uniqueness of the BSDE-solution (Theorem 4.5) we see that $(\hat{Y},\hat{\phi})$ with

$$\hat{\phi}_t(z) = A_t\hat{u}_t\psi_t(z), \tag{7.16}$$

solves (7.7) with $\hat{u}$. Hence i) is proved.

Next we study ii). The Hamiltonian (7.6) is a linear function in $u \in \mathcal{U} = \mathbb{R}$ and, composed with $(\hat{Y}, \hat{\phi})$, it is:

$$\mathcal{H}_t(\lambda, u, x, \hat{Y}_t, \hat{\phi}_t) = \rho_t x \hat{Y}_t + u\Big[(\alpha_t - \rho_t)\hat{Y}_t + \psi_t(0)\hat{\phi}_t(0)\lambda_t^B + \int_{\mathbb{R}_0} \big[\psi_t(z)\hat{\phi}_t(z)\big]\, \lambda_t^H \nu(dz)\Big].$$

To have (6.16) well defined and (6.17) satisfied for the control $\hat{u}$ we see that

$$(\alpha_t - \rho_t)\hat{Y}_t + \psi_t(0)\hat{\phi}_t(0)\lambda_t^B + \int_{\mathbb{R}_0} \big[\psi_t(z)\hat{\phi}_t(z)\big]\, \lambda_t^H \nu(dz) = 0 \tag{7.17}$$

is necessary and sufficient. Indeed we can see that equation (7.17) is satisfied. Substituting (7.13) and (7.16) into (7.17), we obtain the equation:

$$\hat{u}_t = \frac{-(\alpha_t - \rho_t)\big(A_t\hat{X}_t + C_t\big)}{A_t\big(|\psi_t(0)|^2\lambda_t^B + \int_{\mathbb{R}_0} |\psi_t(z)|^2\, \lambda_t^H\, \nu(dz)\big)}. \tag{7.18}$$

Substituting the definition of $\hat{u}$ (7.12), we have

$$\begin{aligned}&\big(\alpha_t - \rho_t\big)^2\big(A_t\hat{X}_t + C_t\big)\\ &= \Big(2\rho_t A_t\hat{X}_t + \rho_t C_t + \hat{X}_t a_t + c_t\Big)\Big(|\psi_t(0)|^2\lambda_t^B + \int_{\mathbb{R}_0} |\psi_t(z)|^2\, \lambda_t^H\, \nu(dz)\Big),\end{aligned}$$

which is verified once the definitions of A and C (7.10)-(7.11) are inserted. Hence, in the setting of the theorem, we conclude that $\hat{u}^{\mathbb{G}} = \hat{u}$ (7.12)-(7.18) is optimal. □

We can now solve problem (7.9) using similar arguments.

Theorem 7.2. *Consider the feedback control $\hat{u}^{\mathbb{F}} \in \mathcal{A}^{\mathbb{F}}$ given by*

$$\hat{u}_t^{\mathbb{F}} = -\frac{(\alpha_t - \rho_t)\Big(\mathbb{E}\big[A_t\big|\mathcal{F}_t\big]\hat{X}_t + \mathbb{E}\big[C_t\big|\mathcal{F}_t\big]\Big)}{\mathbb{E}\big[A_t\big|\mathcal{F}_t\big]\big(|\psi_t(0)|^2\lambda_t^B + \int_{\mathbb{R}_0} |\psi_t(z)|^2\, \lambda_t^H\, \nu(dz)\big)}, \tag{7.19}$$

where $\hat{X}$ refers to (7.4) *with $\hat{u}^{\mathbb{F}}$ and the processes A and C are given by* (7.10)-(7.11)*. If* (6.15) *holds then $\hat{u}^{\mathbb{F}}$ solves* (7.9)*.*

Apart from a technical point involving conditional expectations, the proof is similar to Theorem 7.1.

Proof. Let $\hat{u}$ be given by (7.19), i.e.

$$\hat{u}_t = -\frac{(\alpha_t - \rho_t)\Big(\mathbb{E}\big[A_t\big|\mathcal{F}_t\big]\hat{X}_t + \mathbb{E}\big[C_t\big|\mathcal{F}_t\big]\Big)}{\mathbb{E}\big[A_t\big|\mathcal{F}_t\big]\big(|\psi_t(0)|^2\lambda_t^B + \int_{\mathbb{R}_0} |\psi_t(z)|^2\, \lambda_t^H\, \nu(dz)\big)}.$$

The adjoint equation is given by

$$\begin{aligned}\hat{Y}_t &= \mathbb{E}\big[A_t\big|\mathcal{F}_t\big]\hat{X}_t + \mathbb{E}\big[C_t\big|\mathcal{F}_t\big],\\ \hat{\phi}_t(z) &= \mathbb{E}\big[A_t\big|\mathcal{F}_t\big]\hat{u}_t\psi_t(z),\end{aligned}$$

with A and C given by (7.10) and (7.11). We can verify that $\hat{Y}$, $\hat{\phi}$ is indeed the solution of (7.7) using the same arguments as in the proof of Theorem 7.1. First we show that $d\mathbb{E}[A_t|\mathcal{F}_t] = \mathbb{E}[a_t|\mathcal{F}_t]\,dt$ and $d\mathbb{E}[C_t|\mathcal{F}_t] = \mathbb{E}[c_t|\mathcal{F}_t]\,dt$, where

$$\mathbb{E}[a_t|\mathcal{F}_t] = \Big(\frac{(\alpha_t-\rho_t)^2}{|\psi_t(0)|^2\lambda_t^B + \int_{\mathbb{R}_0}|\psi_t(z)|^2\,\lambda_t^H\,\nu(dz)} - 2\rho_t\Big)\mathbb{E}\big[A_t\big|\mathcal{F}_t\big],$$
$$\mathbb{E}[c_t|\mathcal{F}_t] = \Big(\frac{(\alpha_t-\rho_t)^2}{|\psi_t(0)|^2\lambda_t^B + \int_{\mathbb{R}_0}|\psi_t(z)|^2\,\lambda_t^H\,\nu(dz)} - \rho_t\Big)\mathbb{E}\big[C_t\big|\mathcal{F}_t\big].$$

The argument is described for A, the case of C is identical. Define

$$J_t := \frac{(\alpha_t-\rho_t)^2}{|\psi_t(0)|^2\lambda_t^B + \int_{\mathbb{R}_0}|\psi_t(z)|^2\,\lambda_t^H\,\nu(dz)} - 2\rho_t.$$

We have

$$\begin{aligned}
&\lim_{\Delta t\to 0^+}\frac{\mathbb{E}\big[A_{t+\Delta t}\big|\mathcal{F}_{t+\Delta t}\big] - \mathbb{E}\big[A_t\big|\mathcal{F}_t\big]}{\Delta t}\\
&\quad= \lim_{\Delta t\to 0^+}\frac{1}{\Delta t}\Big(\mathbb{E}\Big[\exp\Big\{\int_{t+\Delta t}^{T} J_s\,ds\Big\}\Big|\mathcal{F}_{t+\Delta t}\Big] - \mathbb{E}\Big[\exp\Big\{\int_{t}^{T} J_s\,ds\Big\}\Big|\mathcal{F}_t\Big]\Big)\\
&\qquad - \mathbb{E}\Big[\exp\Big\{\int_0^T J_s\,ds - \int_0^t J_s\,ds\Big\}\Big|\mathcal{F}_t\Big]\Big)\\
&\quad= \lim_{\Delta t\to 0^+}\frac{1}{\Delta t}\Big(\exp\big\{-\int_0^{t+\Delta t} J_s\,ds\big\}\mathbb{E}\Big[\exp\Big\{\int_0^T J_s\,ds\Big\}\Big|\mathcal{F}_{t+\Delta t}\Big]\\
&\qquad - \exp\big\{-\int_0^t J_s\,ds\big\}\mathbb{E}\Big[\exp\Big\{\int_0^T J_s\,ds\Big\}\Big|\mathcal{F}_t\Big]\Big).
\end{aligned}$$

By the existence of a càdlàg modification, $\mathbb{E}\big[\exp\{\int_0^T J_s\,ds\}\big|\mathcal{F}_{t+\Delta t}\big]$ $\to \mathbb{E}\big[\exp\{\int_0^T J_s\,ds\}|\mathcal{F}_t\big]$ as Δt goes to 0 $dt\times d\mathbb{P}$ a.e. Hence

$$\begin{aligned}
&\lim_{\Delta t\to 0^+}\frac{\mathbb{E}\big[A_{t+\Delta t}\big|\mathcal{F}_{t+\Delta t}\big] - \mathbb{E}\big[A_t\big|\mathcal{F}_t\big]}{\Delta t}\\
&\quad= \lim_{\Delta t\to 0^+}\frac{\exp\big\{-\int_t^{t+\Delta t} J_s\,ds\big\} - \exp\{0\}}{\Delta t}\mathbb{E}\Big[\exp\Big\{\int_t^T J_s\,ds\Big\}\Big|\mathcal{F}_t\Big]\\
&\quad= \mathbb{E}\big[a_t\big|\mathcal{F}_t\big].
\end{aligned}$$

Following Theorem 6.3 we define

$$\mathcal{H}_t^{\mathbb{F}}(\lambda, u, x, \hat{Y}_t, \hat{\phi}_t) = \rho_t x \mathbb{E}\big[\hat{Y}_t \big| \mathcal{F}_t\big] + u\Big\{\big(\alpha_t - \rho_t\big)\mathbb{E}\big[\hat{Y}_t \big| \mathcal{F}_t\big] + \psi_t(0)\mathbb{E}\big[\hat{\phi}_t(0)\big|\mathcal{F}_t\big]\lambda^B + \int_{\mathbb{R}_0} \psi_t(z)\mathbb{E}\big[\hat{\phi}_t(z)\big|\mathcal{F}_t\big]\,\lambda^H \nu(dz)\Big\}.$$

For $\hat{u}$ to be optimal, from (6.22)-(6.23), it is sufficient to show that

(7.20)

$$\big(\alpha_t - \rho_t\big)\mathbb{E}\big[\hat{Y}_t \big| \mathcal{F}_t\big] + \psi_t(0)\mathbb{E}\big[\hat{\phi}_t(0)\big|\mathcal{F}_t\big]\lambda_t^B + \int_{\mathbb{R}_0} \psi_t(z)\mathbb{E}\big[\hat{\phi}_t(z)\big|\mathcal{F}_t\big]\,\lambda_t^H \nu(dz) = 0.$$

Replacing $\hat{\phi}_t(z) = A_t \hat{u}_t \psi_t(z)$ and inserting (7.19) in (7.20), we obtain the desired result. □

7.2. **Utility maximization of final wealth.** For the utility maximization problem of the final wealth we set

$$J(u) = E\big[U(X_T)\big]$$

where $U : \mathbb{R} \to \mathbb{R}$ is a differentiable utility function, increasing and concave. The BSDE is given by

$$Y_t = U'(X_T) + \int_t^T Y_{s\text{-}} \rho_s \, ds - \int_t^T \int_{\mathbb{R}} \phi_s(z)\, \mu(ds, dz)$$

(where $U'(x) = \frac{d}{dx}U(x)$). By arguments as in the mean-variance portfolio problem, the sufficient condition from the maximum principle for optimal $\hat{u} \in \mathcal{A}^{\mathbb{G}}$ can be reduced to

$$(\alpha_t - \rho_t)\hat{Y}_t = \psi_t(0)\hat{\phi}_t(0)\lambda_t^B + \int_{\mathbb{R}_0} \big[\psi_t(z)\hat{\phi}_t(z)\big]\,\lambda_t^H \nu(dz),$$

or

$$(\alpha_t - \rho_t)\Big(U'(\hat{X}_T) + \int_t^T \hat{Y}_{s\text{-}} \rho_s \, ds - \int_t^T \int_{\mathbb{R}_0} \hat{\phi}_s(z)\, \mu(ds, dz)\Big) = \psi_t(0)\hat{\phi}_t(0)\lambda_t^B + \int_{\mathbb{R}_0} \big[\psi_t(z)\hat{\phi}_t(z)\big]\,\lambda_t^H \nu(dz),$$

where $\hat{Y}$, $\hat{\phi}$ depends on $\hat{u}$.

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