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Infinite horizon optimal control of forward-backward stochastic differential equations with delay

Nacira AGRAM* and Bernt ØKSENDAL†‡

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Abstract

We consider a problem of optimal control of an infinite horizon system governed by forward-backward stochastic differential equations with delay. Sufficient and necessary maximum principles for optimal control under partial information in infinite horizon are derived. We illustrate our results by an application to a problem of optimal consumption with respect to recursive utility from a cash flow with delay.

Keywords: Infinite horizon; Optimal control; Stochastic delay equation; Stochastic differentiel utility; Lévy processes; Maximum principle; Hamiltonian; Adjoint processes; Partial information.

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1 Introduction

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $B(\cdot)$ and an independent compensated Poisson random measure $\tilde{N}(dt, da) = N(dt, da) - \nu(da)dt$ are defined.

We study the following infinite horizon coupled forward-backward stochastic differential equations (FBSDEs, for short) control system with delay:

*Laboratory of Applied Mathematics, University Med Khider, Po. Box 145, Biskra (07000) Algeria. Email: agramnacira@yahoo.fr

†Center of Mathematics for Applications (CMA), University of Oslo, Box 1053 Blindern, N-0316 Oslo, Norway. Email: oksendal@math.uio.no

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(FORWARD EQUATION)

$$\begin{aligned} dX(t) &= b\left(t, X(t), X_1(t), X_2(t), u(t)\right) dt + \sigma\left(t, X(t), X_1(t), X_2(t), u(t)\right) dB(t) \\ &\quad + \int_{\mathbb{R}_0} \theta\left(t, X(t), X_1(t), X_2(t), u(t), a\right) \tilde{N}(dt, da); t \in [0, \infty) \\ X(t) &= X_0(t); \qquad\qquad t \in [-\delta, 0] \end{aligned} \tag{1.1}$$

where

$$X_1(t) = X(t-\delta) \ \text{ and } X_2(t) = \int_{t-\delta}^{t} e^{-\rho(t-r)} X(r) dr.$$

(BACKWARD EQUATION)

$$\begin{aligned} dY(t) &= -g\left(t, X(t), X_1(t), X_2(t), Y(t), Z(t), u(t)\right) dt + Z(t) dB(t) \\ &\quad + \int_{\mathbb{R}_0} K\left(t,\ a\right) \tilde{N}(dt, da); t \in [0, \infty) \cdot \end{aligned} \tag{1.2}$$

Throughout this paper, we introduce the following basic assumptions

$\delta > 0, \rho > 0$ are given constants,
$b : [0, \infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \Omega \to \mathbb{R}$,
$\sigma : [0, \infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \Omega \to \mathbb{R}$,
$g : [0, \infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \Omega \to \mathbb{R}$,
$\theta, K : [0, \infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \mathbb{R}_0 \times \Omega \to \mathbb{R}$,
$f : [0, \infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \times \mathcal{U} \times \Omega \to \mathbb{R}$,
$h : \mathbb{R} \to \mathbb{R}$,

where the coefficients $b, \sigma, \theta$and g are Fréchet differentiable (C^1) with respect to the variables (x, x_1, x_2, y, z, u).

We denote by $\mathcal{R}$, the set of all functions $k : \mathbb{R}_0 := \mathbb{R} \setminus \{0\} \to \mathbb{R}$.

We interprete the infinite horizon BSDE (1.2) in the sense of Pardoux [16], i.e. for all $T < \infty$, $(Y(t),\ Z(t), K(t, \cdot))$ solves the equation

$$\begin{aligned} Y(t) &= Y(T) + \int_t^T g\left(s, X(s), X_1(s), X_2(s), Y(s), Z(s)\right) ds - \int_t^T Z(s) dB(s) \\ &\quad - \int_t^T \int_{\mathbb{R}_0} K\left(s, a\right) \tilde{N}(ds, da);\ 0 \le t \le T, \end{aligned} \tag{1.3}$$

and moreover,

$$E[\sup_{t \ge 0} \ e^{\lambda t} Y^2(t) + \int_0^\infty e^{\lambda t} (Z^2(t) + \int_{\mathbb{R}_0} K^2\left(s, a\right) \nu(da)) dt] < \infty \tag{1.4}$$

for sufficiently large constant λ. See section 4 in [16] for more details.

Note that by the Itô representation theorem for Lévy processes (see [20]), equation (1.3) is equivalent to the equation

$$Y(t) = E[Y(T) + \int_t^T g\left(s, X(s), X_1(s), X_2(s), Y(s), Z(s)\right) ds \mid \mathcal{F}_t];\ t \leq T,$$
$$\text{for all } T < \infty. \tag{1.5}$$

Let $\mathcal{E}_t \subseteq \mathcal{F}_t$ be a given subfiltration, representing the information available to the controller at time t.

Let $\mathcal{U}$ be a non-empty convex subset of $\mathbb{R}$. We let $\mathcal{A}_{\mathcal{E}}$ denote a given locally convex family of admissible $\mathcal{E}_t$-predictable control processes with values in $\mathcal{U}$.

The corresponding performance functional is

$$J(u) = E[\int_0^\infty f\left(t, X(t), X_1(t), X_2(t), Y(t), Z(t), K\left(t, \cdot\right), u(t)\right)\ dt + h(Y(0))] \tag{1.6}$$

where we assume that the functions f and h are Fréchet differentiable (C^1) with respect to the variables $(x, x_1, x_2, y, z, k(\cdot), u)$ and $Y(0)$, respectively, and f satisfies

$$E[\int_0^\infty \left| f\left(t, X(t), X_1(t), X_2(t), Y(t), Z(t), K\left(t, \cdot\right), u(t)\right)\ \right| dt] < \infty. \tag{1.7}$$

The optimal control problem is to find an optimal control $u^* \in \mathcal{A}_{\mathcal{E}}$ and the value function $\Phi_{\varepsilon} \in \mathbb{R}$ such that

$$\Phi_{\varepsilon}(X_0) = \sup_{u \in \mathcal{A}_{\mathcal{E}}} J(u) = J(u^*) \tag{1.8}$$

We will study this problem by using a version of the maximum principle which is a combination of the infinite horizon maximum principle in [1] and the finite horizon maximum principle for FBSDEs in [12] and [15].

The Hamiltonian

$$H : [0, \infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times L^2(\nu) \times U \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times L^2(\nu) \to \mathbb{R}$$

is defined as

$$\begin{aligned} H(t, x, x_1, x_2, y, z, k(\cdot), u, \lambda, p, q, r(\cdot)) &= f(t, x, x_1, x_2, y, z, k, u) + g(t, x, x_1, x_2, y, z, u)\lambda \\ &+ b(t, x, x_1, x_2, u)p + \sigma(t, x, x_1, x_2, u)q + \int_{\mathbb{R}_0} \theta(t, x, x_1, x_2, u, a) r(t, a) \nu(da). \end{aligned} \tag{1.9}$$

We suppose that the Hamiltonian H is Fréchet differentiable (C^1) in the variables x, x_1, x_2, y, z, k.

We associate to the problem (1.8) the following pair of forward- backward SDEs in the adjoint processes $\lambda(t)$, ($p(t), q(t), r(t, \cdot)$):

(ADJOINT FORWARD EQUATION)

$$
\begin{aligned}
&d\lambda(t) = \tfrac{\partial H}{\partial y}(t, X(t), X_1(t), X_2(t), Y(t), Z(t), K(t,\cdot), u(t), \lambda(t), p(t), q(t), r(t,\cdot))\ dt \\
&+\tfrac{\partial H}{\partial z}(t, X(t), X_1(t), X_2(t), Y(t), Z(t), K(t,\cdot), u(t), \lambda(t), p(t), q(t), r(t,\cdot))\ dB(t) \\
&+\int_{\mathbb{R}_0} \nabla_k H(t, X(t), X_1(t), X_2(t), Y(t), Z(t), K(t,\cdot), u(t), \lambda(t), p(t), q(t), r(t,\cdot))\tilde{N}(dt, da) \\
&\lambda(0) = h^{'}(Y(0))
\end{aligned}
\tag{1.10}
$$

(ADJOINT BACKWARD EQUATION)

$$
dp(t) = E[\mu(t) \mid \mathcal{F}_t]dt + q(t)dB(t) + \int_{\mathbb{R}_0} r(t,a)\tilde{N}(dt, da); t \in [0, \infty) \tag{1.11}
$$

where

$$
\begin{aligned}
&\mu(t) = -\tfrac{\partial H}{\partial x}(t, X(t), X_1(t), X_2(t), Y(t), Z(t), K(t,\cdot), u(t), \lambda(t), p(t), q(t), r(t,\cdot)) \\
&-\tfrac{\partial H}{\partial x_1}\,(t+\delta, X(t+\delta), X_1(t+\delta), X_2(t+\delta), Y(t+\delta), Z(t+\delta), K(t+\delta,\cdot), \\
&u(t+\delta), \lambda(t+\delta), p(t+\delta), q(t+\delta), r(t+\delta,\cdot)) \\
&-e^{\rho t}(\int_t^{t+\delta} \tfrac{\partial H}{\partial x_2}(s, X(s), X_1(s), X_2(s), Y(s), Z(s), K(s,\cdot), u(s), \lambda(s), p(s), q(s), r(s,\cdot))e^{-\rho s} ds).
\end{aligned}
\tag{1.12}
$$

The unknown process $\lambda(t)$ is the adjoint process corresponding to the backward system $(Y(t), Z(t), K(t,a))$ and the triple unknown $(p(t), q(t), r(t,a))$ is the adjoint process corresponding to the forward system $X(t)$.

We show that in this infinite horizon setting the missing terminal conditions for the BSDEs for $(Y(t), Z(t), K(t,\cdot))$ and $(p(t), q(t), r(t,\cdot))$ should be replaced by asymptotic transversality conditions. See (H_1) and (H_5) below.

In this paper we obtain a sufficient and a necessary maximum principle for infinite horizon control of FBSDEs with delay. As an illustration we solve explicitly an infinite horizon optimal consumption problem with recursive utility. Related papers dealing with infinite horizon control, but either without FB systems or without delay, are [1], [8], [17] and [21]. Other related stochastic control publications dealing with finite horizon only are [2], [3], [4], [5], [6], [7], [9], [10], [11], [12], [13], [14], [15], [16], [18], [19],[20] and [22].

2 Sufficient maximum principle for partial information

We will prove in this section that under some assumptions the maximization of the Hamiltonian leads to an optimal control.

Theorem 2.1 *Let $\hat{u} \in \mathcal{A}_{\mathcal{E}}$ with corresponding solutions $\hat{X}(t), \hat{X}_1(t), \hat{X}_2(t), \hat{Y}(t), \hat{Z}(t)$, $\hat{K}(t,a), \hat{p}(t), \hat{q}(t), \hat{r}(t,a)$ and $\hat{\lambda}(t)$ of equations (1.1), (1.2), (1.10) and (1.11). Suppose that:*

(H_1): (Transversality conditions)

$$\underline{\lim}_{T\to\infty} E[\ \hat{p}(T)(\hat{X}(T) - X(T))] \leq 0$$

and

$$\overline{\lim}_{T\to\infty} E\ [\hat{\lambda}(T)(\hat{Y}(T) - Y(T))] \geq 0.$$

(H_2): (Concavity)

The functions $x \to h(x)$ and

$$(x, x_1, x_2, y, z, k, u) \to H(t, x, x_1, x_2, y, z, \hat{K}(\cdot), u, \hat{\lambda}, \hat{p}, \hat{q}, \hat{r}(\cdot))$$

are concave, for all $t \in [0, \infty)$.

(H_3): (The conditional maximum principle)

$$\max_{v\in U} E[H(t, \hat{X}(t), \hat{X}_1(t), \hat{X}_2(t), \hat{Y}(t), \hat{Z}(t), \hat{K}(t,\cdot), v, \hat{\lambda}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot)) \mid \mathcal{E}_t]$$
$$= E[H(t, \hat{X}(t), \hat{X}_1(t), \hat{X}_2(t), \hat{Y}(t), \hat{Z}(t), \hat{K}(t,\cdot), \hat{u}(t), \hat{\lambda}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot)) \mid \mathcal{E}_t].$$

(H_4): (Growth conditions) Suppose for all $u \in \mathcal{A}_{\mathcal{E}}$ that the following holds:

$$E[\int_0^\infty (\hat{Y}(t) - Y(t))^2 \{(\tfrac{\partial \hat{H}}{\partial y}(t))^2 + \int_{\mathbb{R}_0} \left\|\nabla_k \hat{H}(t,a)\right\|^2 \nu(da)\}dt] < \infty \tag{2.1}$$

$$E[\int_0^\infty \hat{\lambda}^2(t)\{(\hat{Z}(t) - Z(t))^2 + \int_{\mathbb{R}_0} (\hat{K}(t,a) - K(t,a))^2 \nu(da)\}dt] < \infty \tag{2.2}$$

$$E[\int_0^\infty (\hat{X}(t) - X(t))^2 \{\hat{q}^2(t) + \int_{\mathbb{R}_0} \hat{r}^2(t,a)\nu(da)\}dt] < \infty \tag{2.3}$$

$$E[\int_0^\infty \hat{p}^2(t)\{(\hat{\sigma}(t) - \sigma(t))^2 + \int_{\mathbb{R}_0} (\hat{\theta}(t,a) - \theta(t,a))^2 \nu(da)\}dt] < \infty \tag{2.4}$$

where $X(t)$, $X_1(t), X_2(t), Y(t), Z(t), K(t,a)$ are the solutions of (1.1), (1.2) corresponding to u, and we are using the notation

$$\frac{\partial \hat{H}}{\partial z}(t) = \frac{d}{dz} H(t, \hat{X}(t), \hat{X}_1(t), \hat{X}_2(t), \hat{Y}(t), \hat{Z}(t), \hat{K}(t,\cdot), z, \hat{\lambda}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot)) \mid_{z=\hat{Z}(t)}$$

and similarly with $\nabla_k \hat{H}(t,a)$.

Then $\hat{u}(t)$ is an optimal control for (1.8), i.e.

$$J(\hat{u}) = \sup_{u\in\mathcal{A}_{\mathcal{E}}} J(u).$$

Proof. Assume that $u \in \mathcal{A}_{\mathcal{E}}$. We want to prove that $J(\hat{u}) - J(u) \geq 0$, i.e.$\hat{u}$ is an optimal control.

We put

$$J(\hat{u}) - J(u) = I_1 + I_2 \tag{2.5}$$

where

$$I_1 = E[\int_0^\infty \{f(t,\hat{X}(t),\hat{X}_1(t),\hat{X}_2(t),\hat{Y}(t),\hat{Z}(t),\hat{K}(t,\cdot),\hat{u}(t))$$

$$-f\,(t,X(t),X_1(t),X_2(t),Y(t),Z(t),\ K\,(t,\cdot)\,,u(t))\}\ dt]$$

and

$$I_2 = E[h(\hat{Y}(0)) - h(Y(0))].$$

By the definition of H, we have

$$\begin{aligned} I_1 = E[\int_0^\infty \{(\hat{H}(t) - H(t)) - (\hat{g}(t) - g(t))\hat{\lambda}(t) - (\hat{b}(t) - b(t))\hat{p}(t) \\ -(\hat{\sigma}(t) - \sigma(t))\hat{q}(t) - \int_{\mathbb{R}_0} (\hat{\theta}(t,a) - \theta(t,a))\hat{r}(t,a)\nu(da)\}dt] \end{aligned} \tag{2.6}$$

where we have used the simplified notation

$$\hat{H}(t) = \hat{H}(t,\hat{X}(t),\hat{X}_1(t),\hat{X}_2(t),\hat{Y}(t),\hat{Z}(t),\hat{K}(t,\cdot),\hat{u}(t),\hat{\lambda}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))$$
$$H(t) = H(t,X(t),X_1(t),X_2(t),Y(t),Z(t),K(t,\cdot),u(t),\hat{\lambda}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot)) \text{ etc.}$$

Since h is concave, we have

$$h(\hat{Y}(0)) - h(Y(0)) \geq h'(\hat{Y}(0))(\hat{Y}(0) - Y(0)) = \hat{\lambda}(0)(\hat{Y}(0) - Y(0)) \ .$$

By Itô's formula, (H_4), (1.2) and (1.10), we have for all T

$$\begin{aligned} &E[\hat{\lambda}(0)(\hat{Y}(0) - Y(0))] = E[\hat{\lambda}(T)(\hat{Y}(T) - Y(T)) \\ &-\int_0^T \hat{\lambda}(t)d(\hat{Y}(t) - Y(t)) - \int_0^T (\hat{Y}(t) - Y(t))d\hat{\lambda}(t) \\ &-\int_0^T (\hat{Z}(t) - Z(t))\tfrac{\partial \hat{H}}{\partial z}(t)dt - \int_0^T \int_{\mathbb{R}_0} \nabla_k \hat{H}(t,a)(\hat{K}(t,a) - K(t,a))\nu(da)dt] \end{aligned}$$

Letting $T \to \infty$, we obtain

$$\begin{aligned} &E[\hat{\lambda}(0)(\hat{Y}(0) - Y(0))] = \overline{\lim_{T\to\infty}} E\ [\hat{\lambda}(T)(\hat{Y}(T) - Y(T))] \\ &-E[\int_0^\infty \{-\hat{\lambda}(t)(\hat{g}(t) - g(t)) + (\hat{Y}(t) - Y(t))\tfrac{\partial \hat{H}}{\partial y}(t) + (\hat{Z}(t) - Z(t))\tfrac{\partial \hat{H}}{\partial z}(t) \\ &+\int_{\mathbb{R}_0} \nabla_k \hat{H}(t,a)(\hat{K}(t,a) - K(t,a))\nu(da)\}dt]. \end{aligned} \tag{2.7}$$

Combining (2.6) – (2.7), we obtain

$$\begin{aligned} &J(\hat{u}) - J(u) \geq \overline{\lim_{T\to\infty}} E\ [\hat{\lambda}(T)(\hat{Y}(T) - Y(T))] \\ &+E[\int_0^\infty \{(\hat{H}(t) - H(t)) - (\hat{b}(t) - b(t))\hat{p}(t) - (\hat{\sigma}(t) - \sigma(t))\hat{q}(t) \\ &-\int_{\mathbb{R}_0} (\hat{\theta}(t,a) - \theta(t,a))\hat{r}(t,a)\nu(da) - (\hat{Y}(t) - Y(t))\tfrac{\partial \hat{H}}{\partial y}(t) - (\hat{Z}(t) - Z(t))\tfrac{\partial \hat{H}}{\partial z}(t) \\ &-\int_{\mathbb{R}_0} \nabla_k \hat{H}(t,a)(\hat{K}(t,a) - K(t,a))\nu(da)\}dt]. \end{aligned}$$

Since H is concave, we have

$$J(\hat{u}) - J(u) \geq \overline{\lim_{T\to\infty}} E\ [\hat{\lambda}(T)(\hat{Y}(T) - Y(T))] + E[\int_0^\infty \{(\hat{X}(t) - X(t))\tfrac{\partial \hat{H}}{\partial x}(t)$$
$$+(\hat{X}_1(t) - X_1(t))\tfrac{\partial \hat{H}}{\partial x_1}(t) + (\hat{X}_2(t) - X_2(t))\tfrac{\partial \hat{H}}{\partial x_2}(t) + (\hat{u}(t) - u(t))\tfrac{\partial \hat{H}}{\partial u}(t)$$
$$-(\hat{b}(t) - b(t))\hat{p}(t) - (\hat{\sigma}(t) - \sigma(t))\hat{q}(t) - \int_{\mathbb{R}_0} (\hat{\theta}(t,a) - \theta(t,a))\hat{r}(t,a)\nu(da)\}dt]. \tag{2.8}$$

Applying now (H_1), (H_4) together with the Itô formula to $\hat{p}(T)(\hat{X}(T) - X(T))$, we get

$$0 \geq \underline{\lim}_{T\to\infty} E\ [\ \hat{p}(T)(\hat{X}(T) - X(T))]$$
$$= E[\int_0^\infty \{(\hat{b}(t) - b(t))\hat{p}(t) - (\hat{X}(t) - X(t))E[\hat{\mu}(t) \mid \mathcal{F}_t]$$
$$+(\hat{\sigma}(t) - \sigma(t))\hat{q}(t) + \int_{\mathbb{R}_0} (\hat{\theta}(t,a) - \theta(t,a))\hat{r}(t,a)\nu(da)\}dt]$$
$$= E[\int_0^\infty \{(\hat{b}(t) - b(t))\hat{p}(t) - (\hat{X}(t) - X(t))\hat{\mu}(t)$$
$$+(\hat{\sigma}(t) - \sigma(t))\hat{q}(t) + \int_{\mathbb{R}_0} (\hat{\theta}(t,a) - \theta(t,a))\hat{r}(t,a)\nu(da)\}dt]. \tag{2.9}$$

By the definition (1.12) of μ , we have

$$E[\ \int_0^\infty (\hat{X}(t) - X(t))\hat{\mu}(t)dt]$$
$$= \overline{\lim_{T\to\infty}}\ E[\ \int_\delta^{T+\delta} ((\hat{X}(t-\delta) - X(t-\delta))\hat{\mu}(t-\delta)dt)]$$
$$= \overline{\lim_{T\to\infty}} E[\ (-\ \int_\delta^{T+\delta} \tfrac{\partial \hat{H}}{\partial x}(t-\delta)(\hat{X}(t-\delta) - X(t-\delta))dt$$
$$-\int_\delta^{T+\delta} \tfrac{\partial \hat{H}}{\partial x_1}\,(t)\,(\hat{X}_1(t) - X_1(t))dt - \int_\delta^{T+\delta} (\int_{t-\delta}^{t} \tfrac{\partial \hat{H}}{\partial x_2}\,(s)\,e^{-\rho s}ds)\cdot$$
$$\cdot e^{\rho(t-\delta)}(\hat{X}(t-\delta) - X(t-\delta)))dt]\ . \tag{2.10}$$

Substituting $r = t - \delta$, we obtain

$$\int_0^T \tfrac{\partial \hat{H}}{\partial x_2}(s)(\hat{X}_2(s) - X_2(s))ds$$
$$= \int_0^T \tfrac{\partial \hat{H}}{\partial x_2}(s) \int_{s-\delta}^{s} e^{-\rho(s-r)}(\hat{X}(r) - X(r))dr\ ds$$
$$= \int_0^T (\int_r^{r+\delta} \tfrac{\partial \hat{H}}{\partial x_2}(s)e^{-\rho s}ds)e^{\rho r}(\hat{X}(r) - X(r))\ dr$$
$$= \int_\delta^{T+\delta} (\int_{t-\delta}^{t} \tfrac{\partial \hat{H}}{\partial x_2}\,(s)\,e^{-\rho s}ds)e^{\rho(t-\delta)}(\hat{X}(t-\delta) - X(t-\delta))dt\ . \tag{2.11}$$

Combining (2.8) with (2.9) – (2.11), we deduce that

$$\begin{aligned}
J(\hat{u}) - J(u) &\geq \overline{\lim_{T\to\infty}} E\ [\hat{\lambda}(T)(\hat{Y}(T) - Y(T))] - \underline{\lim}_{T\to\infty} E[\hat{p}(T)(\hat{X}(T) - X(T))] \\
&\quad + E[\int_0^\infty (\hat{u}(t) - u(t)) \tfrac{\partial \hat{H}}{\partial u}(t) dt] \\
&= \overline{\lim_{T\to\infty}} E[\ \hat{\lambda}(T)(\hat{Y}(T) - Y(T))] - \underline{\lim}_{T\to\infty} E\ [\hat{p}(T)(\hat{X}(T) - X(T))] \\
&\quad + E[\int_0^\infty E\{(\hat{u}(t) - u(t)) \tfrac{\partial \hat{H}}{\partial u}(t) \mid \mathcal{E}_t\} dt].
\end{aligned}$$

Then

$$\begin{aligned}
J(\hat{u}) - J(u) &\geq \overline{\lim_{T\to\infty}}\ E[\hat{\lambda}(T)(\hat{Y}(T) - Y(T))] - \underline{\lim}_{T\to\infty} EE\ [\hat{p}(T)(\hat{X}(T) - X(T))] \\
&\quad + E[\int_0^\infty E\{\tfrac{\partial \hat{H}}{\partial u}(t) \mid \mathcal{E}_t\}(\hat{u}(t) - u(t)) dt].
\end{aligned}$$

By assumptions (H_1) and (H_3), we conclude

$$J(\hat{u}) - J(u) \geq 0$$

i.e.$\hat{u}$ is an optimal control. ■

3 Necessary conditions of optimality for partial information

A drawback of the previous section is that the concavity condition is not always satisfied in applications. In view of this, it is of interest to obtain conditions for an optimal control with partial information where concavity is not needed. We assume the following:

(A_1) For all $u \in \mathcal{A}_{\mathcal{E}}$ and all $\beta \in \mathcal{A}_{\mathcal{E}}$ bounded, there exists ϵ>0 such that

$$u + s\beta \in \mathcal{A}_{\mathcal{E}} \qquad \text{for all } s \in (-\epsilon, \epsilon).$$

(A_2) For all t_0, h and all bounded $\mathcal{E}_{t_0}$-mesurable random variables α, the control process $\beta(t)$ defined by

$$\beta(t) = \alpha 1_{[s,s+h)}(t) \tag{3.1}$$

belongs to $\mathcal{A}_{\mathcal{E}}$.

(A_3) For all bounded $\beta \in \mathcal{A}_{\mathcal{E}}$, the derivative processes

$$\xi(t) := \tfrac{d}{ds} X^{u+s\beta}(t) \mid_{s=0} \tag{3.2}$$

$$\phi(t) := \tfrac{d}{ds} Y^{u+s\beta}(t) \mid_{s=0} \tag{3.3}$$

$$\eta(t) := \tfrac{d}{ds} Z^{u+s\beta}(t) \mid_{s=0} \tag{3.4}$$

$$\psi(t,a) := \tfrac{d}{ds} K^{u+s\beta}(t,a) \mid_{s=0} \tag{3.5}$$

exist and

$$E[\int_0^\infty \{ \left| \tfrac{\partial f}{\partial x}(t)\xi(t) \right| + \left| \tfrac{\partial f}{\partial x_1}(t)\xi(t-\delta) \right| + \left| \tfrac{\partial f}{\partial x_2}(t) \int_{t-\delta}^t e^{-\rho(t-r)}\xi(r)dr \right| + \left| \tfrac{\partial f}{\partial y}(t)\phi(t) \right|$$
$$+ \left| \tfrac{\partial f}{\partial z}(t)\eta(t) \right| + \left| \tfrac{\partial f}{\partial u}(t)\beta(t) \right| + \int_{\mathbb{R}_0} |\nabla_k f(t,a)\psi(t,a)| \, \nu(da) \} dt < \infty. \tag{3.6}$$

We can see that

$$\tfrac{d}{ds} X_1^{u+s\beta}(t) \mid_{s=0} = \tfrac{d}{ds} X^{u+s\beta}(t) \mid_{s=0} = \xi(t-\delta)$$

and

$$\frac{d}{ds} X_2^{u+s\beta}(t) \mid_{s=0} = \int_{t-\delta}^t e^{-\rho(t-r)}\xi(t)dr.$$

Note that

$$\xi(t) = 0 \text{ for } t \in [-\delta, 0].$$

Theorem 3.1 *Suppose that $\hat{u} \in \mathcal{A}_{\mathcal{E}}$ with corresponding solutions $\hat{X}(t), \hat{X}_1(t), \hat{X}_2(t), \hat{Y}(t), \hat{Z}(t), \hat{K}(t,a)$, $\hat{\lambda}(t), \hat{p}(t), \hat{q}(t)$ and $\hat{r}(t,a)$ of equations (1.1), (1.2), (1.10) and (1.11).*
Assume that (2.1) – (2.4) and the following transversality conditions hold:

(H_5)

$$\lim_{T\to\infty} E[\ \hat{p}(T)\xi(T)] = 0,$$
$$\lim_{T\to\infty} E[\hat{\lambda}(T)\phi(T)] = 0.$$

(H_6) *Moreover, assume that the following growth condition holds*

$$E[\int_0^T \{\hat{\lambda}^2(t)(\eta^2(t) + \int_{\mathbb{R}_0} \psi^2(t,a)\nu(da)) + \phi^2(t)((\tfrac{\partial \hat{H}}{\partial z})^2(t) + \int_{\mathbb{R}_0} \nabla_k \hat{H}^2(t,a)\nu(da))$$
$$+\hat{p}^2(t)(\tfrac{\partial \sigma}{\partial x}(t)\xi(t) + \tfrac{\partial \sigma}{\partial x_1}(t)\xi(t-\delta) + \tfrac{\partial \sigma}{\partial x_2}(t) \int_{t-\delta}^t e^{-\rho(t-r)}\xi(r)dr + \tfrac{\partial \sigma}{\partial u}(t)\beta(t))^2$$
$$+\hat{p}^2(t)(\int_{\mathbb{R}_0} \{\tfrac{\partial \theta}{\partial x}(t,a)\xi(t) + \tfrac{\partial \theta}{\partial x_1}(t,a)\xi(t-\delta) + \tfrac{\partial \theta}{\partial x_2}(t,a) \int_{t-\delta}^t e^{-\rho(t-r)}\xi(r)dr + \tfrac{\partial \theta}{\partial u}(t,a)\beta(t)\}^2\nu(da))\}dt] < \infty$$

for all $T < \infty$.

Then the following assertions are equivalent.
(i) For all bounded $\beta \in \mathcal{A}_{\mathcal{E}}$,

$$\frac{d}{ds} J(\hat{u} + s\beta) \mid_{s=0} = 0.$$

(*ii*) *For all* $t \in [0,\infty)$,

$$E[\frac{\partial}{\partial u}H(t,\hat{X}(t),\hat{X}_1(t),\hat{X}_2(t),\hat{Y}(t),\hat{Z}(t),\hat{K}(t,\cdot),u,\hat{\lambda}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot)) \mid \mathcal{E}_t]_{u=\hat{u}(t)} = 0.$$

Proof. $(i) \Longrightarrow (ii)$:

It follows from (1.1) that

$$\begin{aligned}
d\xi(t) &= \{\tfrac{\partial b}{\partial x}(t)\xi(t) + \tfrac{\partial b}{\partial x_1}(t)\xi(t-\delta) + \tfrac{\partial b}{\partial x_2}(t)\int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr + \tfrac{\partial b}{\partial u}(t)\beta(t)\}dt \\
&+\{\tfrac{\partial \sigma}{\partial x}(t)\xi(t) + \tfrac{\partial \sigma}{\partial x_1}(t)\xi(t-\delta) + \tfrac{\partial \sigma}{\partial x_2}(t)\int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr + \tfrac{\partial \sigma}{\partial u}(t)\beta(t)\}dB(t) \\
&+\int_{\mathbb{R}_0}\{\tfrac{\partial \theta}{\partial x}(t,a)\xi(t) + \tfrac{\partial \theta}{\partial x_1}(t,a)\xi(t-\delta) + \tfrac{\partial \theta}{\partial x_2}(t,a)\int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr + \tfrac{\partial \theta}{\partial u}(t,a)\beta(t)\}\tilde{N}(dt,da),
\end{aligned}$$

and

$$\begin{aligned}
d\phi(t) = \{&-\tfrac{\partial g}{\partial x}(t)\xi(t) - \tfrac{\partial g}{\partial x_1}(t)\xi(t-\delta) - \tfrac{\partial g}{\partial x_2}(t)\int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr - \tfrac{\partial g}{\partial y}(t)\phi(t) \\
&-\tfrac{\partial g}{\partial u}(t)\beta(t) - \tfrac{\partial g}{\partial z}(t)\eta(t)\}dt + \eta(t)dB(t) + \int_{\mathbb{R}_0}\psi(t,a)\tilde{N}(dt,da),
\end{aligned}$$

where for simplicity of notation, we have set

$$\frac{\partial}{\partial x}b(t) = \frac{\partial}{\partial x}b(t,X(t),X_1(t),X_2(t),u(t)) \text{ etc.}$$

Suppose that assertion (i) holds. Then

$$\begin{aligned}
0 &= \tfrac{d}{ds}J(\hat{u}+s\beta)\mid_{s=0} \\
&= E[\int_0^{\infty}\{\tfrac{\partial f}{\partial x}(t)\xi(t) + \tfrac{\partial f}{\partial x_1}(t)\xi(t-\delta) + \tfrac{\partial f}{\partial x_2}(t)\int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr + \tfrac{\partial f}{\partial y}(t)\phi(t) \\
&+\tfrac{\partial f}{\partial z}(t)\eta(t) + \tfrac{\partial f}{\partial u}(t)\beta(t) + \int_{\mathbb{R}_0}\nabla_k f(t,a)\psi(t,a)\nu(da)\}dt + h'(\hat{Y}(0))\phi(0)].
\end{aligned} \tag{3.7}$$

We know by the definition of H that

$$\frac{\partial f}{\partial x}(t) = \frac{\partial H}{\partial x}(t) - \frac{\partial g}{\partial x}(t)\lambda(t) - \frac{\partial b}{\partial x}(t)p(t) - \frac{\partial \sigma}{\partial x}(t)q(t) - \int_{\mathbb{R}_0}\frac{\partial \theta}{\partial x}(t,a)r(t,a)\nu(da)$$

and similarly for $\frac{\partial f}{\partial x_1}(t)$, $\frac{\partial f}{\partial x_2}(t)$, $\frac{\partial f}{\partial u}(t)$, $\frac{\partial f}{\partial y}(t)$, $\frac{\partial f}{\partial z}(t)$ and $\nabla_k f(t,a)$.

By the Itô formula and (H_6), we get

$$\begin{aligned}
&E[h'(\hat{Y}(0)\phi(0))] = E[\hat{\lambda}(0)\phi(0)] \\
&= \lim_{T\to\infty}E[\hat{\lambda}(T)\phi(T)] \\
&-\lim_{T\to\infty}E[\int_0^{T}\{\hat{\lambda}(t)(-\tfrac{\partial g}{\partial x}(t)\xi(t) - \tfrac{\partial g}{\partial x_1}(t)\xi(t-\delta) - \tfrac{\partial g}{\partial x_2}(t)\int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr - \tfrac{\partial g}{\partial y}(t)\phi(t) \\
&-\tfrac{\partial g}{\partial z}(t)\eta(t) - \tfrac{\partial g}{\partial u}(t)\beta(t)) + \phi(t)\tfrac{\partial H}{\partial y}(t) + \eta(t)\tfrac{\partial H}{\partial z}(t) + \int_{\mathbb{R}_0}\nabla_k H(t,a)\psi(t,a)\nu(da)\}dt].
\end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.7) we get

$$
\begin{aligned}
0 &= \tfrac{d}{ds}J(\hat{u}+s\beta)\mid_{s=0} \\
&= E[\int_0^\infty \{\tfrac{\partial f}{\partial x}(t)\xi(t) + \tfrac{\partial f}{\partial x_1}(t)\xi(t-\delta) + \tfrac{\partial f}{\partial x_2}(t)\int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr + \tfrac{\partial f}{\partial y}(t)\phi(t) \\
&+ \tfrac{\partial f}{\partial z}(t)\eta(t) + \tfrac{\partial f}{\partial u}(t)\beta(t) + \int_{\mathbb{R}_0} \nabla_k f(t,a)\psi(t,a)\nu(da) \\
&- \hat{\lambda}(t)(-\tfrac{\partial g}{\partial x}(t)\xi(t) - \tfrac{\partial g}{\partial x_1}(t)\xi(t-\delta) - \tfrac{\partial g}{\partial x_2}(t)\int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr - \tfrac{\partial g}{\partial y}(t)\phi(t) \\
&- \tfrac{\partial g}{\partial z}(t)\eta(t) - \tfrac{\partial g}{\partial u}(t)\beta(t)) + \phi(t)\tfrac{\partial H}{\partial y}(t) + \eta(t)\tfrac{\partial H}{\partial z}(t) + \int_{\mathbb{R}_0} \nabla_k H(t,a)\psi(t,a)\nu(da)\}dt].
\end{aligned}
\tag{3.9}
$$

Applying the Itô formula to $\hat{p}(T)\xi(T)$ and using (H_6), we get

$$
\begin{aligned}
0 &= \lim_{T\to\infty} E[\hat{p}(T)\ \xi(T)] \\
&= E[\int_0^\infty \hat{p}(t)\ \{\tfrac{\partial b}{\partial x}(t)\xi(t) + \tfrac{\partial b}{\partial x_1}(t)\xi(t-\delta) + \tfrac{\partial b}{\partial x_2}(t)\int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr + \tfrac{\partial b}{\partial u}(t)\beta(t)\}dt \\
&+ \int_0^\infty \xi(t)E[\mu(t)\mid\mathcal{F}_t]dt + \int_0^\infty \hat{q}(t)\{\tfrac{\partial \sigma}{\partial x}(t)\xi(t) + \tfrac{\partial \sigma}{\partial x_1}(t)\xi(t-\delta) + \tfrac{\partial \sigma}{\partial x_2}(t)\int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr + \tfrac{\partial \sigma}{\partial u}(t)\beta(t)\}dt \\
&+ \int_0^\infty \int_{\mathbb{R}_0} \hat{r}(t,a)\{\tfrac{\partial \theta}{\partial x}(t,a)\xi(t) + \tfrac{\partial \theta}{\partial x_1}(t,a)\xi(t-\delta) + \tfrac{\partial \theta}{\partial x_2}(t,a)\int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr + \tfrac{\partial \theta}{\partial u}(t,a)\beta(t)\}\nu(da)dt] \\
&= -\tfrac{d}{ds}J(\hat{u}+s\beta)\mid_{s=0} + E[\int_0^\infty \tfrac{\partial H}{\partial u}(t)\beta(t)dt].
\end{aligned}
\tag{3.10}
$$

Adding (3.9) and (3.10) we obtain

$$E[\int_0^\infty \frac{\partial H}{\partial u}(t)\beta(t)dt] = 0.$$

Now apply this to

$$\beta(t) = \alpha 1_{[s,s+h)}(t)$$

where $\alpha(\omega)$ is bounded and $\mathcal{E}_{t_0}$-mesurable, $s \geq t_0$. Then we get

$$E[\int_s^{s+h} \frac{\partial H}{\partial u}(s)ds\ \alpha] = 0$$

Differentiating with respect to h at $h = 0$ we obtain

$$E[\tfrac{\partial H}{\partial u}(s)\ \alpha] = 0$$

Since this holds for all $s \geq t_0$ and all α, we conclude

$$E[\tfrac{\partial H}{\partial u}(t_0)\mid\mathcal{E}_{t_0}] = 0\ .$$

This proves that (i) implies (ii).

$(ii) \Longrightarrow (i)$:

The argument above shows that

$$\frac{d}{ds}J(u+s\beta)\mid_{s=0}=E[\int_0^\infty \frac{\partial H}{\partial u}(t)\beta(t)dt]$$

for all u, $\beta \in \mathcal{A}_\mathcal{E}$ with β bounded. So to complete the proof we use that every bounded $\beta \in \mathcal{A}_\mathcal{E}$ can be approximated by linear combinations of controls β of the form (3.1). We omit the details. ■

4 Application to optimal consumption with respect to recursive utility

4.1 A general optimal recursive utility problem

Let $X(t) = X^{(c)}(t)$ be a cash flow modelled by

$$\begin{cases} dX(t) = X(t-\delta)[b_0(t)dt + \sigma_0(t)dB(t) + \int_{\mathbb{R}_0} \gamma(t,a)\tilde{N}(dt,da)] - c(t)dt; t \geq 0 \\ X(0) = x > 0 \end{cases} \tag{4.1}$$

where $b_0(t)$, $\sigma_0(t)$ and $\gamma(t,a)$ are given bounded $\mathcal{F}_t$-predictable processes, $\delta \geq 0$ is a fixed delay and $\gamma(t,a) > -1$ for all $(t,a) \in [0,\infty) \times \mathbb{R}$.

The process $u(t) = c(t) \geq 0$ is our control process, interpreted as our relative consumption rate such that $X^{(c)}(t) > 0$ for all $t \geq 0$. We let $\mathcal{A}$ denote the family of all $\mathcal{F}_t$-predictable relative consumption rates. To every $c \in \mathcal{A}$ we associate a recursive utility process $Y^{(c)}(t) = Y(t)$ defined as the solution of the infinite horizon BSDE

$$Y(t) = E[Y(T) + \int_t^T g\left(s, Y(s), c(s)\right) ds \mid \mathcal{F}_t] \text{ for all } t \leq T, \tag{4.2}$$

valid for all deterministic $T < \infty$. We assume that $Y(t)$ satisfies (1.4) (See Duffie & Epstein (1992)).

Suppose the solution $Y(t)$ of the infinite horizon BSDE (4.2) satisfies the condition (1.4) and let $c(s); s \geq 0$ be the consumption rate.

We assume that the function $g(t,y,c)$ satisfies the following conditions:

- $g(t,y,c)$ is concave with respect to y and c
- $$\int_0^\infty E\left[|g(s,Y(s),c(s))|\right] ds < \infty \text{ for all } c \in \mathcal{A} \tag{4.3}$$
- $\frac{\partial}{\partial c}g(t,y,c)$ has an inverse:

$$I(t,v,w) = \begin{cases} 0 & \text{if } v \geq v_0(t,w) \\ (\frac{\partial}{\partial c} g(t,y,c))^{-1}(v) & \text{if } 0 \leq v \leq v_0(t,w) \end{cases}$$

where v_0 is $\frac{\partial}{\partial c} g(t,y,0)$.
We study the problem to find $c^* \in \mathcal{A}$ such that

$$\sup_{c \in \mathcal{A}} Y^{(c)}(0) = Y^{(c^*)}(0). \tag{4.4}$$

We call such a process c^* a recursive utility optimal consumption rate.
We see that the problem (4.5) is a special case of problem (1.8) with

$$J(u) = Y(0)$$

$f = 0$, $h(y) = y$, $u = c$ and

$$\begin{aligned} b(t,x,x_1,x_2,u) &= x_1 b_0(t) - c \\ \sigma(t,x,x_1,x_2,u) &= x_1 \sigma_0(t) \\ \theta(t,x,x_1,x_2,u,a) &= x_1 \gamma(t,a) \end{aligned}$$

In this case the Hamiltonian defined in (1.9) takes the form

$$\begin{aligned} H(t,x,x_1,x_2,y,z,k(\cdot),u,\lambda,p,q,r(\cdot)) &= \lambda g(t,y,c) + (x_1 b_0(t) - c)\, p \\ &\quad + x_1 \sigma_0(t) q + x_1 \int_{\mathbb{R}_0} \gamma(t,a) r(a) \nu(da) \end{aligned} \tag{4.5}$$

Maximizing H as a function of c gives the first order condition

$$\lambda(t) \frac{\partial g}{\partial c}(t, Y(t), c(t)) = E[p(t) \mid \mathcal{E}_t] \tag{4.6}$$

for an optimal $c(t)$.
The pair of adjoint processes (1.10)-(1.11) is given by

$$\begin{cases} d\lambda(t) = \lambda(t) \frac{\partial g}{\partial y}(t, Y(t), c(t)) dt \\ \lambda(0) = 1 \end{cases} \tag{4.7}$$

and

$$dp(t) = E[\mu(t) \mid \mathcal{F}_t] dt + q(t) dB(t) + \int_{\mathbb{R}_0} r(t,a) \tilde{N}(dt,da); t \in [0,\infty) \tag{4.8}$$

where

$$\begin{aligned} \mu(t) = -[&b_0(t) p(t+\delta) + \sigma_0(t) q(t+\delta) \\ &+ \int_{\mathbb{R}_0} \gamma(t,a) r(t+\delta,a) \nu(da)] \end{aligned} \tag{4.9}$$

Equation (4.7) has the solution

$$\lambda(t) = \exp(\int_0^t \frac{\partial g}{\partial y}(s, Y(s), c(s))ds); t \geq 0 \tag{4.10}$$

which substituted into (4.6) gives

$$\frac{\partial g}{\partial c}(t, Y(t), c(t)) \exp(\int_0^t \frac{\partial g}{\partial y}(s, Y(s), c(s))ds) = E[p(t) \mid \mathcal{E}_t] \tag{4.11}$$

Hence, to find the optimal consumption rate c it suffices to find

$$E[p(t) \mid \mathcal{E}_t]; t \geq 0.$$

We refer to Theorem 5.1 in [1] for a proof of the existence of the solution of the ABSDE (4.8).

4.2 A solvable special case

In order to get a solvable case we choose the driver g in (4.2) to be of the form

$$g(t, y, c) = -\alpha(t)y + \ln c \tag{4.12}$$

where $\alpha(t) \geq \alpha > 0$ is an $\mathcal{F}_t$-adapted process.

We also choose

$$\delta = 0 \text{ and } \mathcal{E}_t = \mathcal{F}_t; t \geq 0 \tag{4.13}$$

and we represent the consumption rate $c(t)$ as

$$c(t) = \rho(t)X(t), \tag{4.14}$$

where $\rho(t) \geq 0$ is the relative consumption rate.

We assume that ρ is bounded away from 0. This set of controls is denoted by $\mathcal{A}$.

The FBSDE system now has the form

$$\begin{cases} dX(t) = X(t^-)[(b_0(t) - \rho(t))dt + \sigma_0(t)dB(t) + \int_{\mathbb{R}_0} \gamma(t, a)\tilde{N}(dt, da)]; t \geq 0 \\ X(0) = x > 0 \end{cases} \tag{4.15}$$

and

$$Y(t) = Y^{(\rho)}(t) = E[Y(T) + \int_t^T (-\alpha(s)Y(s) + \ln c(s))\, ds \mid \mathcal{F}_t] \tag{4.16}$$

i.e.

$$dY(t) = -\left(-\alpha(t)Y(t) + \ln c(t)\right) dt + Z(t)dB(t); t \geq 0 \tag{4.17}$$

We want to find $\rho^* \in \mathcal{A}$ such that

$$\sup_{\rho \in \mathcal{A}} Y^{(\rho)}(0) = Y^{(\rho^*)}(0) \tag{4.18}$$

In this case the Hamiltonian (1.9) gets the form

$$H(t,x,y,\rho,\lambda,p,q,r) = \lambda(-\alpha(t)y + \ln(\rho x)) + x\,(b_0(t)-\rho)\,p \\ + x\sigma_0(t)q + x\int_{\mathbb{R}_0} \gamma(t,a)r(a)\nu(da) \tag{4.19}$$

Maximizing H with respect to ρ gives the first order equation

$$\lambda(t)\frac{1}{\rho(t)} = p(t)X(t) \tag{4.20}$$

where, by (1.10) – (1.11) $\lambda(t)$ and $(p(t),q(t),r(t,a))$ satisfy the FBSDEs

$$\begin{cases} d\lambda(t) = -\alpha(t)\lambda(t)dt \\ \lambda(0) = 1 \end{cases} \tag{4.21}$$

and

$$\begin{aligned} dp(t) &= -[\lambda(t)\tfrac{1}{X(t)} + (b_0(t)-\rho(t))\,p(t) + \sigma_0(t)q(t) \\ &+ \int_{\mathbb{R}_0} \gamma(t,a)r(a)\nu(da)]dt + q(t)dB(t) + \int_{\mathbb{R}_0} r(t,a)\tilde{N}(dt,da) \end{aligned} \tag{4.22}$$

The infinite horizon BSDE (4.22) has a unique solution, (see Theorem 3.1 in [8]).

Then, the solutions of (4.21) – (4.22) are respectively,

$$\lambda(t) = \exp(-\int_0^t \alpha(s)ds) \tag{4.23}$$

and, for all $0 \le t \le T$ and all $T < \infty$,

$$p(t)\Gamma(t) = E[p(T)\Gamma(T) + \int_t^T \lambda(s)\frac{\Gamma(s)}{X(s)}ds \mid \mathcal{F}_t], \tag{4.24}$$

where $\Gamma(t)$ is given by

$$\begin{cases} d\Gamma(t) = \Gamma(t^-)[(b_0(t)-\rho(t))\,dt + \sigma_0(t)dB(t) \\ \qquad\qquad + \int_{\mathbb{R}_0} \gamma(t,a)\tilde{N}(dt,da)]; t \ge 0 \\ \Gamma(0) = 1 \end{cases} \tag{4.25}$$

(See e.g.[14, 18]).

This gives

$$\begin{aligned} \Gamma(t) = \exp(&-\int_0^t \sigma_0(s)dB(s) + \int_0^t \{b_0(s)-\rho(s)-\tfrac{1}{2}\sigma_0^2(s)\}ds \\ &+\int_0^t\int_{\mathbb{R}_0} \{\ln(1+\gamma(s,a))-\gamma(s,a)\}\nu(da)ds \\ &+\int_0^t\int_{\mathbb{R}_0} \ln(1+\gamma(s,a))\tilde{N}(ds,da); t \ge 0 \end{aligned} \tag{4.26}$$

Comparing with (4.15) we see that

$$X(t) = x\Gamma(t); t \geq 0 \tag{4.27}$$

Substituting this into (4.24) we obtain

$$p(t)X(t) = E[p(T)X(T) + \int_t^T \exp(-\int_0^s \alpha(r)dr)ds \mid \mathcal{F}_t] \tag{4.28}$$

Since ρ is bounded away from 0 we deduce from (4.20) that

$$p(T)X(T) = \frac{\lambda(T)}{\rho(T)} = \frac{1}{\rho(T)} \exp(-\int_0^T \alpha(r)dr) \to 0 \text{ dominatedly as } T \to \infty. \tag{4.29}$$

Hence, by letting $T \to \infty$ in (4.28) we get

$$p(t)X(t) = E[\int_t^\infty \exp(-\int_0^s \alpha(r)dr)ds \mid \mathcal{F}_t] \tag{4.30}$$

By (4.20) we therefore get the optimal relative consumption rate

$$\rho(t) = \rho^*(t) = \frac{\exp(-\int_0^s \alpha(r)dr)}{E[\int_t^\infty \exp(-\int_0^s \alpha(r)dr)ds \mid \mathcal{F}_t]}; t \geq 0 \tag{4.31}$$

In particular, if $\alpha(r) = \alpha > 0$ (constant) for all r, then

$$\rho^*(t) = \alpha; t \geq 0. \tag{4.32}$$

With this choice of ρ^* the transversality conditions (H_1) and (H_5) hold and we have proved:

Theorem 4.1 *The optimal relative consumption rate* $\rho^*(t)$ *for problem* (4.12)–(4.18) *is given by* (4.31).

In particular, if $\alpha(r) = \alpha > 0$ *(constante) for all* r*, then*$\rho^*(t) = \alpha$; *for all* t.

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