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## LIBRATIONIST CLOSURES OF THE PARADOXES

**Abstract:** We present a semi-formal foundational theory of *sorts*, akin to sets, named *librationism* because of its way of dealing with paradoxes. Its semantics is related to Herzberger’s semi inductive approach, it is negation complete and free variables (*noemata*) name sorts. Librationism deals with paradoxes in a novel way related to paraconsistent dialethic approaches, but we think of it as *bialethic* and *parasistent*. Classical logical theorems are retained, and none contradicted. Novel inferential principles make recourse to theoremhood and failure of theoremhood. Identity is introduced à la Leibniz-Russell, and librationism is highly non-extensional.  $\Pi_1^1$ -comprehension with ordinary Bar-Induction is accounted for (to be lifted). Power sorts are generally paradoxical, and Cantor’s Theorem is blocked as a camouflaged premise is naturally discarded.

**Keywords:** Bialethicism, Burali-Forti Paradox, Cantor’s Theorem, Curry’s Paradox, Dialetheism, Foundations of Mathematics, Liar’s Paradox, Paraconsistency, Parasistency, Paradoxes, Reverse Mathematics, Russell’s Paradox, Second Order Arithmetic, Semantical paradoxes, Set Theoretic Paradoxes, Set Theory, Theory of Truth.

### §0 Introduction

In the following we present some of the main features of the librationist foundational system, with emphasis upon its dealing with paradoxes and its provision of an alternative foundation for mathematics. Librationism has its new-coined name from the word “libration”, which the reader is asked to look up if unfamiliar. This replaces the term “liberalism” which was used in some superseded publications and lectures on account of the theory’s emancipatory feature that all abstraction terms are allowed. The new name, which was available, is meant to also remind of the oscillating manner of its dealing with paradoxical phenomena. According to recent nomenclature, librationism could be considered a theory of properties. Confer the influential opening remark of [15]: “Gödel said to me more than once “There never were any set-theoretic paradoxes, but the property-theoretic paradoxes are still unresolved”; and he may well have said the same thing in print.” This author agrees with Gödel’s attitude that the word “set” should best be reserved for those in the more iterative, extensional and non-paradoxical sense, and the term “property-theoretic paradox” is very appropriate and useful in the sense that it points out that there are other objects than iterative and extensional sets which succumb to triviality when naïve abstraction is brought into play to understand them. However, it does not follow that the term “property theory” is an appropriate term for such theories that endeavor to understand objects which are appropriately analyzed by those means which suggest themselves in approximating naïve abstraction. This is

because there are many properties, like the property of having pain, which for obvious reasons cannot conceivably be dealt with in such frameworks. On the basis of such grounds as these we steer a middle way and understand the theory to be developed in this essay as a theory of sorts. All sorts may be regarded as properties, but not vice versa. All sets in a more classical sense, as far as their existence is supported by librationism, are sorts, but not vice versa. It is not known to what extent librationism supports the existence of sets. The librationist theory of sorts supports the existence of non-well founded sorts, and also contains universal sorts; librationism is a highly non-extensional theory, and it e.g. turns out that there are infinitely many distinct non-paradoxical empty, and hence also universal sorts. One should keep in mind that in librationism, all conditions give rise to a corresponding sort. We are able to isolate a sort  $H$  of hereditarily non-paradoxical and iterative sorts. With respect to  $H$  we may in some contexts use bi-simulation to recapture extensionality and other desirable principles, e.g. concerning ordinals. Results so far have established that we by using manifestation-points (see §6) can establish that librationism gives an interpretation of finitely iterated inductive definitions  $ID_{<\omega}$  plus the Bar-rule; librationism is therefore stronger than the impredicative subsystem of second order arithmetic  $\Pi_1^1 - CA_0 +$  ordinary Bar-Induction in a sense to be made more precise.<sup>1</sup> This will be lifted.

The language of librationism may succinctly but just approximately be described as that of ordinary set theory extended with a truth operator  $\mathbf{T}$ . There are five caveats concerning this which we draw attention to here: Firstly, the terms of our language are taken to denote sorts. Secondly, the truth operator is eliminable as concerns the purely sort theoretic aspects of librationism, but it facilitates presentation and is of importance e.g. if and when we extend the theory with a truth predicate. Thirdly, sort brackets are included in what we here took as the language of ordinary set theory, and these are not eliminable in librationist sort theory as in extensional set theories. Fourthly, the identity sign “=” is not taken as a primitive sign in the librationist language, as a Leibnizian definition of identity with Russell’s simplification can be shown to be adequate. Fifthly, in the presentation we give below we define the primitive language more austere in a Polish manner for metalogical and, as we shall see, philosophical reasons.

Librationism may be understood as an heir both to the semi-inductive type of approach to semantic paradoxes originating with [11, 12] ([9] independently suggested the very much related revision theory at the same time) as well as to some tenets of various paraconsistent points of view. In our semantics, it is of vital importance that we fix our focus on one designated model, and in our setup what is usually regarded as free variables serve as names of sorts via an enumeration of these in the metalanguage. But then the expression *variable* in such contexts is very much misleading indeed, and so we instead opt for using the expressions *noema* (singular) and *noemata* (plural). This is inter alia

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<sup>1</sup>Readers unfamiliar with the invoked notions related to reverse mathematics are advised to consult the now classical [16]. The Wikipedia entry on *Reverse Mathematics* can also be a good place to start gaining some familiarity with central notions.

justified by the fact that one meaning of the word *noema* as listed in the Oxford English Dictionary is: *A figure of speech whereby something stated obscurely is nevertheless intended to be understood or worked out.* Also, the Greek letter  $\nu$  in the original Greek word  $\nu\acute{o}\eta\mu\alpha$  typographically very much resembles lower case *v*.

In the Herzberger process we invoke, all sorts have the empty extension at the ordinal zero; this is not an essential assumption from a formal point of view (other consistent beginnings could, so seen, serve as well), and one may think of the version we develop as *minimalist* librationism. The author thinks that minimalistic librationism is preferable to other variants of librationism for philosophical and, if that is different here, esthetical reasons. The minimalist policy also has the advantage of justifying the regularity rule for hereditarily kind and iterative sorts (cfr. §9).

Herzberger's semi inductive semantics was geared towards dealing with the semantic Liar's paradoxes with a self referential truth predicate. But he was aware of the possibility of using the same type of semantics for what we call a sort theory. In footnote 11 of [12] this is stated very clearly: "Similar questions might be raised in set theory by applying semi-inductive methods to the construction of the membership relation." Our approach may be seen as following this line of research, but as well extending it e.g. with including infinitary inferential principles (see below).

We take as librationistically valid all those formulas that hold unboundedly below the closure ordinal reached in the designated Herzberger process described. This contrasts with what would in this context have been the standard semi inductive approach, as it would, in this way of expressing things, have taken as valid all and only those formulas whose negations are not unbounded under the closure ordinal. If we assume the ordinals used are von Neumann ordinals, we may state this alternatively by saying that a formula is librationistically valid iff the union of the set of those ordinals below the closure ordinal where it holds is the closure ordinal. With the same assumption, a formula is valid according to the standard semi inductive approach iff the union of the set of ordinals where the negation of the formula holds is smaller than the closure ordinal. We may, as is usual, take a formula to be stably in (stably out) iff there is an ordinal  $\gamma$  below the closure ordinal such that it (its negation) holds at all ordinals  $\delta$  larger than  $\gamma$  and below the closure ordinal. A formula is unstable iff it is neither stably in nor stably out. According to a standard semi inductive approach, a formula will be counted as valid iff stably in. From the librationist viewpoint put forward here, a formula is counted as valid iff stably in or unstable; only formulas stably out are discounted in librationism.

We will at this point illustrate the difference with a couple of examples, and we first consider predicate logical tautologies. Classical logical theorems hold at all ordinals in the Herzberger process, and negations of such theorems fail at all ordinals. So theorems of classical logic are librationistically valid, and negations of theorems of classical logic are not. In the standard semi inductive approach, classical logical theorems are valid because the set of ordinals where the negation of any such theorem holds in the Herzberger process is just the empty set, and

negations of classical theorems are not valid because the set of ordinals in the Herzberger process where their negations hold is unbounded under the closure ordinal. We next consider  $r \in r$ , with  $r = \{x : x \notin x\}$ . Given our semantic setup with our alethic comprehension principle, we here have that if  $r \in r$  holds at an ordinal then  $r \notin r$  holds at its successor, and  $r \notin r$  holds at an ordinal only if  $r \in r$  holds at the next ordinal. So both the set of ordinals where  $r \in r$  holds below the closure ordinal as well as the set of ordinals where  $r \notin r$  holds below the closure ordinal are unbounded under the closure ordinal, and so both  $r \in r$  and  $r \notin r$  are librationistically valid. According to the standard semi inductive approach, neither  $r \in r$  nor  $r \notin r$  is valid.

We write  $\Vdash A$  for the statement that  $A$  is librationistically valid, and call the symbol “ $\Vdash$ ” the *roadstyle* when referred to. As pointed out, if  $A$  is a theorem of classical logic then  $\Vdash A$  and not  $\Vdash \sim A$ , whereas, for  $r = \{x : x \notin x\}$ , we have that  $\Vdash r \in r$  as well as  $\Vdash r \notin r$ . On account of this, we distinguish between maxims that are theorems whose negations are not theorems, and minors which are theorems that are not maxims. Theorems of classical logic are examples of maxims, and  $r \in r$  for  $r = \{x : x \notin x\}$  an example of a minor. We say that a schema is minor if all its instances are theorems and it has minor instances, and a schema is maximal if all its instances are maxims.  $\Vdash_M A$  signifies that  $A$  is a maxim, and  $\Vdash_m A$  means that  $A$  is a minor. We use the roadstyle without subscript if it is left undecided whether the theorem is a maxim or a minor.

We here stress that the induced inference rules for librationism are novel, and that e.g. modus ponens for  $\Vdash$  is not a valid inference rule. This will be covered precisely below, but needs mentioning here in order to forestall hasty dismissals.

We have seen that both  $r \in r$  and  $r \notin r$  are librationist theorems. This does not mean, however, that  $r \in r \wedge r \notin r$  is a librationist theorem, as, quite on the contrary,  $r \in r \vee r \notin r$  is a maxim. To forestall irrelevant objections appealing to something like what is thought of as the one and only true meaning of connectives, we suggest that the skeptical reader e.g. presupposes the following slightly alternative names to the most common connectives used in the main bulk of this presentation of librationism: negjunction ( $\sim$ ), adjunction ( $\wedge$ ), veljunction ( $\vee$ ), subjunction ( $\supset$ ) and equijunction ( $\equiv$ ). The first of these names is a seemingly etymologically justified neologism whose pronunciation is not too far off from “negation”. The last name returned some very few occurrences with the intended meaning of material equivalence on an internet search. The name “veljunction” is sometimes, but very rarely used for disjunction, and “adjunction” perhaps even more rarely for conjunction. It seems that “subjunction” has come to be used sometimes in grammar literature for material conditionals and their cognates. (In some logic literature “subjunctive conditional” is taken to refer to a conditional akin to the counterfactual conditional.)

The reader may associate with these different names for connectives in the librationist setting in part to avoid the prejudice that these are to be thought of as gaining their meanings from purely truth functional considerations. We will not adhere strictly to this in our own presentation. As in standard theories, the meaning of connectives in librationism must be understood syncategore-

matically, as the schoolmen would have expressed it; this is to say that they do not have a meaning in virtue of a denotatum, but rather obtain one from their appropriate use in conjunction with other formulas. But, importantly, some pretheoretically expected usages fail in librationism. We cannot, e.g., infer from  $\Vdash \sim A$  to  $\not\Vdash A$ , nor vice versa, as in a standard semantical framework. Nor do we always have adjunctivity for theoremhood, i.e. that  $\Vdash A$  and  $\Vdash B$  only if  $\Vdash A \wedge B$ , and so in this single respect there is a resemblance between librationism and Stanislaw Jaskowski's non-adjunctive paraconsistent system. And yet the standard interdefinability connections between connectives hold maximally.

Librationism is a semi-formal system. An important difference e.g. between Peano arithmetic and omega logic, i.e. Peano arithmetic with the omega rule, is that the latter is quite categorical with respect to content. For this reason we use the expression *contentual system* as synonymous with, or as a replacement for, *semi-formal system*. This neologism seems to fit the analogous distinction between form and content appropriately in our context. Also, the term *semi-formal* does not seem to carry important information. However, on many occasions *semi-formal* is used parenthetically as a reminder.

The contentual (semi-formal) system librationism is not recursively axiomatizable, but it serves to isolate many partial formal systems. It is important in this connection to point out the validity of what we, in analogy with the  $\omega$ -rule, by picking the last letter of the Latin alphabet, call the Z-rule: from  $\Vdash_M A(v)$  for all noemata (free variables)  $v$ , infer  $\Vdash_M \forall x A(x)$ . The intuitive and prima facie weaker infinitary rule that  $\Vdash_M \forall x A(x)$  holds if  $\Vdash_M A(t)$  holds for all terms  $t$  entails the Z-rule given the facts that noemata (free variables) in librationism are names of sorts and all sorts are named. The corresponding rules with no subscript or minor subscript are not validated.

As noemata (free variables) serve as names of sorts we may e.g. have that  $\Vdash v_{37} = \{v_3 : v_3 \in v_3\}$ . But generalizing this would of course be absurd. In stating partial axiomatic schemas which do allow generalization, the precaution is taken that all generalizations of the stated schemas are also axioms. A simple inductive argument going back to Tarski shows that generalization will hold as a derived inference rule for the partial systems consisting of such schemas as here described.

The validity of the Z-rule makes it the case that librationism verifies the consistency not only of a wide range of first order axiomatizable theories such as Peano arithmetic, but also much stronger theories. In this there is nothing whatsoever which detracts from Gödel's seminal insights, as librationism itself is not recursively axiomatizable. Indeed, it is important to stress that we in librationism always see things from a semantical point of view. Thence traditional soundness and completeness considerations are inappropriate in the librationist setting. The axiomatic and inferential principles of librationism which we are able to isolate are therefore always partial.

Librationism may be regarded as a paraconsistent system given contemporary terminology, but the reader is asked to pay attention to the very significant differences between it and such frameworks. The author also has some impor-

tant issues with the nomenclature in the area as concerns librationism (and not only for etymological reasons) and thinks *parasistent*, which etymologically signifies the property of standing up beyond, is a much more suitable term than *paraconsistent*, which etymologically rather seems to signify the property of being beyond a safe place to stand.

Provided a theory is regarded as inconsistent iff it has theorems of the form  $A \wedge \sim A$ , then librationism is a consistent theory. We will conform to this usage, and consider librationism consistent.

There is then the question of whether librationism should be considered a *contradictory* theory on account of the fact that for some sentences  $A$  both  $\Vdash A$  and  $\Vdash \sim A$ . Pragmatic considerations here strongly suggest that we should avoid the term *contradictory* if at all possible, for it seems not to be in accordance with commonly adopted standards for language and rationality that a theory contradicts itself. It is important in this to pay attention to the fact that standard usage has it that if two sentences are contradictory then it is impossible for both to be true.

And closer inspection indeed suggests that a contradiction need not be thought to be involved here. We do not, in librationism, commit ourselves to the idea that the Aristotelian principle of non-contradiction fails in paradoxical contexts. Let the *significance* of a formula be the set of ordinals below the closure ordinal where it holds in the Herzberger process. A formula is then librationistically valid just in case the union of its significance is the closure ordinal (assuming von Neumann ordinals). When we have both  $\Vdash A$  and  $\Vdash \sim A$ , what we have is that the two sentences  $A$  and  $\sim A$  have what we take to be *complementary* significances in the sense that the union of these significances is the closure ordinal itself, their intersection is empty and both significances are unbounded under the closure ordinal. We think of sentences as contradictory just in case the union of their significance is the closure ordinal, the intersection of their significances is empty and it is not the case that both significances are unbound under the closure ordinal. Contradictory and complementary sentences as  $A$  and  $\sim A$  are always *incompatible* in the sense that their conjunction (adjunction) must fail to be a librationist theorem.

With this as background we can offer a librationist diagnosis of why it is wrong to assert  $A$  as well as to assert  $\sim A$  when  $A$  and  $\sim A$  are contradictory; this is because exactly one of  $A$  and  $\sim A$  is false. Similarly, we see that if  $A$  and  $\sim A$  are complementary, then they are both true from the librationist point of view; so we *can* in this case truthfully assert  $A$  as well as truthfully assert  $\sim A$ .

It is worthwhile to point out and emphasize that our connectives behave quite classically when regarded as operating upon the significances of formulas. Given the significances of formulas  $A$  and  $B$  as the sets of ordinals below the closure ordinal where they hold, the significance of  $\sim A$  is the complement of the significance of  $A$  relative to the closure ordinal, the significance of  $A \vee B$  is the union of the significance of  $A$  with the significance of  $B$  and the significance of  $A \wedge B$  is the intersection of the significance of  $A$  with the significance of  $B$ . The significance of subjunctives and equijunctives are defined similarly according to their standard definitions in terms of other connectives. The significance of e.g.

$\mathbf{T}A$  given that of  $A$  is more complicated to express, and these two will always differ in our minimalist approach.

The author sides with those who, like recently [6], are dissatisfied with the formation of the terms “dialetheism” and “dialethic” on etymological grounds, to the preference of “dialethism” and “dialethic”, and prefers the terms “bialethism” and “bialethic” for usage in characterizing librationism in order to distinguish the point of view from common expositions of dialetheism which have it that the latter view is characterized by accepting the truth of some contradictions.

In summing up, librationism may be understood as a parasistent, consistent, complementary and bialethic theory. Librationism is related to paraconsistent theories, though it has some very special features which sets it apart from such approaches. This warrants special terminology.

Instead of having restrictions on syntax, as e.g. in type theory, or, alternatively, weakening classical logic and keeping the naive comprehension schema, as e.g. in certain traditional or hypothetical paraconsistent approaches to set theory (or “property theory”...), we may instead opt for syntactic freedom, keep classical logic and weaken the naive comprehension schema. In librationism this is, as in the ZF-tradition, a central trait of the strategy which is followed. In articulating the librationist strategy, we conveniently make use of a truth-operator in what we may think of as an alethic comprehension schema which we may for now state as follows:

$$\forall x(x \in \{w : A\} \equiv \mathbf{T}A(x/w))$$

Here  $A$  is a formula where  $w$  may occur free and  $A(x/w)$  is the result of substituting  $x$  for  $w$  in  $A$ . It is presupposed that  $x$  is substitutable for  $w$  in  $A$ .  $\mathbf{T}$  is a monadic formula-forming formula operator. Intuitively, we may think of  $\mathbf{T}$  as our truth-operator. The sort brackets are used as one should expect. If we were to conjoin alethic comprehension with the naive truth principle  $A \equiv \mathbf{T}A$ , we would of course recover naive comprehension and triviality, i.e. that everything follows, in the context of classical logic. In librationism we instead have a series of axiom schemas and inferential principles which in sum approximate the naive truth principle very strongly while avoiding triviality.

The system we isolate is, as pointed out, importantly, contentual (semi-formal), i.e. infinitary proof principles hold, and in that sense it goes beyond standard formal systems. We focus upon one designated model, and this is instrumental in isolating the *provability verb*. We hope that we will give occasion to appreciate the adequacy of such a move in connection with our discussion of Curry-paradoxicalities in §11. By adequacy is here meant that our discussion of the Curry-paradoxicalities reveals that a contentual (semi-formal) approach is indeed needed in order to deal with paradoxes in a general setting. It turns out that the Curry-paradoxicality in the librationist framework is transformed into a metalogical reminder that librationism is negation (negjunction) complete and so only serves to reiterate that librationism is a contentual system and that what we present of it must only be understood as a partial axiomatization.

It is a surprising fact that Cantor's reductio argument for the uncountability of power-sorts of infinite sorts does not go through in librationism. Instead, Cantor's reductio argument, which of course is entirely valid, serves to discard the assumption that there is a non-paradoxical sort  $s = \{x : x \in \mathbb{N} \wedge x \notin f(x)\}$  given a function  $f$  from the sort  $\mathbb{N}$  of natural numbers onto the power sort of  $\mathbb{N}$ . Indeed, we may even postulate that there is such a function from  $\mathbb{N}$  onto the full universe  $V$  of all sorts, and this does not fall prey to Cantor's argument. Also, generally power-sorts are paradoxical in librationism.

As the reader comes to study more details, she or he is encouraged to appreciate that there are, in a certain sense, very few intuitively or pretheoretically plausible principles of truth which fail. Librationism does not generally have the naïve truth principle  $A \equiv \mathbf{T}A$ , but it always has both halves, i.e. both  $A \supset \mathbf{T}A$  and  $\mathbf{T}A \supset A$  are (at least minor) theorems. Also, if  $A$  is a theorem, then so is  $\mathbf{T}A$ , and vice versa. In consequence of the foregoing, transparency, as it has been recently called, in the sense of having full substitutivity of the sentences  $A$  and  $\mathbf{T}A$  in all contexts, will of course fail in the general case. But such transparency will hold whenever the sentence  $A$  is not paradoxical. Further, and more subtle, deviations from the naive picture of truth and abstraction are not pointed out here, but accounted for below.

We have stressed that librationism is a contentual (semi-formal) system several times. It at this point seems appropriate to quote from a post by Martin Davis on the Foundations of Mathematics mailing list on Friday the 16th of March 1998: "For me, it has been clear since I was a boy (a very long time ago) that an acceptable account of Gödel's incompleteness theorem would necessarily take the natural numbers as given in their totality with objective properties beyond what could be derived in any particular formal system. As my teacher Emil Post put it (even longer ago): "this ... must result in at least a partial reversal of the entire axiomatic trend of the late nineteenth and early twentieth centuries, with a return to meaning and truth as being of the essence of mathematics." See in such a way, the fact that librationism is a contentual system is not something which one should too easily hold against it. We suggest on the contrary.

## §1 The formal language

In order to avoid certain complexities in some of our metalogical reasoning we shall at the outset presuppose a rather austere language in a Polish fashion. Another important reason for this austerity is that the Polish formulation brings to the fore the point that sorts may be regarded as properties. As our primitive alphabet we take the 6 signs in the list ' $v, ., \mathbf{T}, \forall, |, \wedge$ '. The noemata (free variables) are generated by the clauses: (1)  $v$  is a noema; (2) If something is a noema then that noema concatenated with  $.$  is also a noema; (3) Nothing else is a noema. Instead of using the austere expressions " $v$ ", " $v.$ ", " $v..$ ", etc. we will in our exposition on occasions make use of numerals and write " $v_0$ ", " $v_1$ ", ..., and also we use "i", "k" ... etc. to stand for arbitrary numerals. These are numerals used for metamathematical convenience and not objects which in

themselves are terms which can be acted upon by quantifiers, and we therefore use boldface fonts to distinguish. Usually, we will for convenience be using noemata like  $x, y, z$  in the metalinguistic exposition.

The primitive alphabet also contains the monadic formula forming formula operator  $\mathbf{T}$ , the dyadic connective or formula forming formula operator  $|$ , signifying the truth function neither-nor, the dyadic quantifier or formula forming noema-cum-formula operator  $\forall$  and the dyadic sortifier or term forming noema-cum-formula operator  $\hat{\cdot}$ .

We use upper case  $A$  and  $B$  etc. for arbitrary formulas and lower case  $a$  and  $b$  etc. for arbitrary terms, though in some exceptional cases we will use upper case letters for terms (sorts) which are of special interest (e.g.  $\emptyset, V, \mathbb{N}, H$ ). The formation rules can be stated by the double recursion:

FR1: All noemata are terms.

FR2: If  $a$  and  $b$  are terms then  $ba$  is a formula.

FR3: If  $A$  is a formula then  $\mathbf{T}A$  is a formula.

FR4: If  $A$  and  $B$  are formulas then  $|AB$  is a formula.

FR5: If  $A$  is a formula and  $v_i$  is a noema, then  $\forall v_i A$  is a formula.

FR6: If  $A$  is a formula and  $v_i$  is a noema, then  $\hat{v}_i A$  is a term.

FR7: Nothing else is a term or a formula.

All and only terms and formulas are expressions.

Notice that although e.g.  $v_i$  and  $v_j$  are considered noemata as taken in isolation, the austere expression  $\forall v_i v_j v_i$  contains only  $v_j$  as a noema. The two occurrences of  $v_i$  in  $\forall v_i v_j v_i$  are variables and not noemata; the one occurring nearest the quantifier is the *binding* variable and the other is a *bound* variable.

With these notions we define the set of noemata of expressions as follows (we use square brackets for sets as used in the metalanguage):

$$\begin{aligned}\nu(v_i) &= [v_i]; \\ \nu(ba) &= \nu(a) \cup \nu(b); \\ \nu(\mathbf{T}A) &= \nu(A); \\ \nu(|AB) &= \nu(A) \cup \nu(B); \\ \nu(\forall v_i A) &= \nu(A) \setminus [v_i]; \\ \nu(\hat{v}_i A) &= \nu(A) \setminus [v_i].\end{aligned}$$

We say that a noema  $v_i$  is *present* in a formula  $A$  iff  $v_i \in \nu(A)$ , and present in a term  $a$  iff  $v_i \in \nu(a)$ . A noema  $v_i$  occurring in a formula  $A$  (term  $a$ ) is a variable in  $A$  ( $a$ ) iff  $v_i$  is not present in  $A$  ( $a$ ). A formula  $A$  is a *proposition* iff no noema is present in  $A$ . A term  $a$  is a *nomen* iff no noema is present in  $a$ . A formula  $A$  is atomic iff  $A$  is of the form  $ba$  with terms  $a$  and  $b$ . For a formula  $A$  and noema  $v_i$  we write  $A(v_i)$  to signify that  $v_i$  is present in  $A$ .

With this terminology, all propositions are sentences and all *nomina* (pl) are sort constants. (We do not presuppose that propositions are extralinguistic entities in the context of our framework.) However, as it turns out, in librationism all formulas are sentences and all terms are sort constants. But not all sentences

are propositions and not all sort constants are nomina. No nomen is a noema and no noema is a nomen, but both nomina and noemata are sort constants. All and only terms are sort constants, but some terms, as  $\hat{v}_i v_j v_i$ , are neither nomina nor noemata.

The substitution function  $(\_/\_)$  from expressions to expressions has the following definition:

$$\begin{aligned}
(a/v_{\mathbf{k}})v_i &= a \text{ if } \mathbf{i} = \mathbf{k}, \text{ otherwise } (a/v_{\mathbf{k}})v_i = v_i; \\
(a/v_{\mathbf{k}})cb &= (a/v_{\mathbf{k}})c(a/v_{\mathbf{k}})b; \\
(a/v_{\mathbf{k}})\mathbf{T}A &= \mathbf{T}(a/v_{\mathbf{k}})A; \\
(a/v_{\mathbf{k}})|AB &= |(a/v_{\mathbf{k}})A(a/v_{\mathbf{k}})B; \\
(a/v_{\mathbf{k}})\forall v_i A &= \forall v_i (a/v_{\mathbf{k}})A \text{ if } \mathbf{i} \neq \mathbf{k}, \text{ else } (a/v_{\mathbf{k}})\forall v_i A = \forall v_i A; \\
(a/v_{\mathbf{k}})\hat{v}_i A &= \hat{v}_i (a/v_{\mathbf{k}})A \text{ if } \mathbf{i} \neq \mathbf{k}, \text{ else } (a/v_{\mathbf{k}})\hat{v}_i A = \hat{v}_i A.
\end{aligned}$$

We will make use of a suffix notation and write  $A(a/v_{\mathbf{k}})$  for  $(a/v_{\mathbf{k}})A$ . Iterated uses of the substitution function like  $(a_0/v_0)(a_1/v_1)\dots(a_n/v_n)$  should be written as  $(a_0/v_0, \dots, a_n/v_n)$ .

We define the notion ‘ $a$  is substitutable for  $v_{\mathbf{k}}$  in...’ by the recursion:  $a$  is substitutable for  $v_{\mathbf{k}}$  in  $v_j$ ;  $a$  is substitutable for  $v_{\mathbf{k}}$  in  $cb$  iff  $a$  is substitutable for  $v_{\mathbf{k}}$  in  $b$  and in  $c$ ;  $a$  is substitutable for  $v_{\mathbf{k}}$  in  $\mathbf{T}A$  iff  $a$  is substitutable for  $v_{\mathbf{k}}$  in  $A$ ;  $a$  is substitutable for  $v_{\mathbf{k}}$  in  $|AB$  iff  $a$  is substitutable for  $v_{\mathbf{k}}$  in  $A$  and in  $B$ ;  $a$  is substitutable for  $v_{\mathbf{k}}$  in  $\forall v_i A$  iff  $v_i$  does not occur in  $a$  or  $v_{\mathbf{k}}$  is not present in  $A$ , and  $a$  is substitutable for  $v_{\mathbf{k}}$  in  $A$ ;  $a$  is substitutable for  $v_{\mathbf{k}}$  in  $\hat{v}_i A$  iff  $v_i$  does not occur in  $a$  or  $v_{\mathbf{k}}$  is not present in  $A$ , and  $a$  is substitutable for  $v_{\mathbf{k}}$  in  $A$ .

We usually write  $A(v_{\mathbf{k}})$  instead of  $A(v_i)(v_{\mathbf{k}}/v_i)$  when  $v_{\mathbf{k}}$  is substitutable for  $v_i$  in  $A$ , and on occasions simply write e.g.  $A(a)$  and  $A(b)$ , where it is then understood that they are given by  $A(v_i)(a/v_i)$  and  $A(v_i)(b/v_i)$  for some noema  $v_i$  such that  $a$  and  $b$  are substitutable for  $v_i$  in  $A(v_i)$ .

We will, as mentioned, later make use of noema signs “ $x$ ”, “ $y$ ”, “ $z$ ”.... to stand for arbitrary noemata, and also introduce definitions as follows in order to more conveniently work in the metalanguage as we provide partial axiomatic and inferential principles and work in the contentual system of librationism. Parentheses are invoked for punctuation. We use the definitions:

$$\begin{aligned}
\{x : A\} &=_D \hat{v}_x A; \\
\sim A &=_D |AA; \\
(A \wedge B) &=_D \sim |AB; \\
(A \vee B) &=_D \sim (\sim A \wedge \sim B); \\
(A \supset B) &=_D \sim (A \wedge \sim B); \\
(A \equiv B) &=_D ((A \supset B) \wedge (B \supset A)); \\
(\exists x)A &=_D \sim (\forall x) \sim A.
\end{aligned}$$

Instead of the applicative Polish expression  $ba$  we will in general be using the standard infix epsilon notation  $a \in b$ . Our reasons for having presupposed the austere Polish notions lie in the facts that this simplifies some of the following metalogical reasoning and that it brings to the fore that sorts may fundamentally be regarded as a special kind of properties.

## §2 The model

We now describe the semi-inductive type of Herzberger process which provides a model that validates our librationist principles. For related descriptions of this kind of semantics, see [7,11,12]. Our modelling of librationism will, as announced, contain some additional twists. Let there be a Gödel-coding of our language so that we have the set of natural numbers which, under this coding, are codes of formulas as seen at the metalevel. As it turns out that librationism accommodates more than arithmetic it is strong enough to provide its own Gödel coding, and so  $Fm(x)$ , for  $x$  is the Gödel number of a formula of the librationist language, can in the following be regarded both as a statement in the object language and as a meta statement. We use square brackets to denote sets presupposed metalogically for the semantic setup, as in  $[x : Fm(x)]$  for the set of Gödel numbers of formulas. We let  $\ulcorner A \urcorner$  stand for the Gödel-number of the formula  $A$ . We define a semi inductive style model  $(X, \models, e)$  by a semi inductive process  $(X, \models)$  built upon a given enumeration  $e (e(0), e(1), \dots)$  of all nomina (i.e. terms not containing noemata but only bound or binding variables) by a double transfinite recursion on (e.g. von Neumann) ordinals which are taken as given.

For  $\alpha$  any ordinal, we require:

- $P(0)$   $X(\alpha) = [\ulcorner A \urcorner : Fm(\ulcorner A \urcorner) \wedge \exists \beta (\beta < \alpha \ \& \ \forall \gamma (\beta \leq \gamma < \alpha \rightarrow X(\gamma) \models A))]$
- $P(1)$   $X(\alpha) \models \mathbf{TA}$  iff  $\ulcorner A \urcorner \in X(\alpha)$
- $P(2)$   $X(\alpha) \models \mathbf{AB}$  iff neither  $X(\alpha) \models A$  nor  $X(\alpha) \models B$
- $P(3)$   $X(\alpha) \models \hat{v}_i A$  iff  $a$  is substitutable for  $v_i$  in  $A$  and  $X(\alpha) \models \mathbf{TA}(a/v_i)$
- $P(4)$   $X(\alpha) \models \forall v_i A$  iff for all  $a$  substitutable for  $v_i$  in  $A$ ,  $X(\alpha) \models A(a/v_i)$
- $P(5)$  If  $a = e(i)$  then  $X(\alpha) \models A(a)$  iff  $X(\alpha) \models A(v_i)$

Define:

$$\begin{aligned} IN(X, \models) &= [\ulcorner A \urcorner : Fm(\ulcorner A \urcorner) \ \& \ \exists \beta \forall \gamma (\beta \leq \gamma \rightarrow \ulcorner A \urcorner \in X(\gamma))] \\ OUT(X, \models) &= [\ulcorner A \urcorner : Fm(\ulcorner A \urcorner) \ \& \ \exists \beta (\forall \gamma (\beta \leq \gamma \rightarrow \ulcorner A \urcorner \notin X(\gamma))] \\ STAB(X, \models) &= IN(X, \models) \cup OUT(X, \models) \\ UNSTAB(X, \models) &= [\ulcorner A \urcorner : Fm(\ulcorner A \urcorner)] \setminus STAB(X, \models) \end{aligned}$$

Definitions:

- (i) Limit  $\kappa$  covers  $(X, \models)$  iff for every  $\gamma \geq \kappa$ ,  $IN(X, \models) \subset X(\gamma)$  and  $X(\gamma) \subset IN(X, \models) \cup UNSTAB(X, \models)$ .
- (ii) Limit  $\sigma$  stabilizes  $(X, \models)$  iff  $\sigma$  covers  $(X, \models)$  and  $X(\sigma) \subset IN(X, \models)$ .

*Theorem 0:* (i) There is an ordinal  $\kappa$  which covers  $(X, \models)$ . (ii) There is an ordinal  $\sigma$  which stabilizes  $(X, \models)$ .

Proof (i): Any member  $\ulcorner A \urcorner$  of  $STAB(X, \models)$  will stabilize at  $HT(\ulcorner A \urcorner) =$  the least ordinal  $\gamma$  such that for all  $\delta \geq \gamma$ ,  $\ulcorner A \urcorner \in X(\delta)$  ( $\ulcorner A \urcorner \notin X(\delta)$ ). By Löwenheim-Skolem-style arguments (see [11,19,20]) members of  $STAB(X, \models)$  will stabilize at a countable ordinal. Any limit ordinal  $\kappa$  larger than the supremum of  $[HT(\ulcorner B \urcorner) : \ulcorner B \urcorner \in STAB(X, \models)]$  will cover  $(X, \models)$ .

Proof (ii): Let  $\delta$  be the least ordinal which covers  $(X, \vDash)$ . Let  $[f(n) : n \in \omega]$  by a trick of Cantor be an enumeration of all elements of  $UNSTAB(X, \vDash)$  where each element  $\ulcorner B \urcorner$  of  $UNSTAB(X, \vDash)$  recurs infinitely often in the sense that if  $\ulcorner B \urcorner = f(m)$  and  $m < n \in \omega$ , then there is a natural number  $n'$ ,  $n < n' \in \omega$ , such that  $f(n') = \ulcorner B \urcorner$ . Define recursively:  $F(0) = \delta$  and  $F(n+1) =$  the least  $\nu > F(n)$  such that  $f(n) \in X(\nu)$  iff  $f(n) \notin X(F(n))$ . We define  $\Phi = [\gamma : \exists m \exists \nu (m \in \omega \ \& \ \nu = F(m) \ \& \ \gamma \in \nu)]$ . It is obvious that  $\Phi$  is a limit ordinal which covers  $(X, \vDash)$ . It is also clear that if  $m < n \in \omega$  then  $F(m) < F(n)$ . Since  $\Phi$  covers  $(X, \vDash)$ , it suffices to show that  $\ulcorner B \urcorner \in X(\Phi)$  entails that  $\ulcorner B \urcorner \in STAB(X, \vDash)$  in order to establish that  $\Phi$  stabilizes  $(X, \vDash)$ . Suppose  $\ulcorner B \urcorner \in X(\Phi)$ . Since  $\Phi$  is a limit ordinal, this entails by  $P(0)$  that we for some ordinal  $\nu$  have that

$$\text{a) } \quad \forall \mu (\nu \leq \mu < \Phi \Rightarrow \ulcorner B \urcorner \in X(\mu))$$

Since  $F$  is increasing with  $\Phi$  as its range, we will then for some natural number  $m \in \omega$  have that  $\nu \leq F(m) < \Phi$ , so that

$$\text{b) } \quad \forall \mu (F(m) \leq \mu < \Phi \Rightarrow \ulcorner B \urcorner \in X(\mu))$$

Suppose  $\ulcorner B \urcorner \notin STAB(X, \vDash)$ . By our enumeration of unstable elements where each term recurs infinitely often, we will have that  $\ulcorner B \urcorner = f(n)$  for some natural number  $n$ ,  $m < n \in \omega$ . It follows that  $F(m) < F(n) < \Phi$ . From a) and b) we can then infer that  $\ulcorner B \urcorner \in X(F(n))$ , since we have supposed that  $\ulcorner B \urcorner \in X(\Phi)$ . But from the construction of the function  $F$  it would then follow that  $\ulcorner B \urcorner \notin X(F(n+1))$ , contradicting b). It follows that  $\ulcorner B \urcorner \in X(\Phi)$  only if  $\ulcorner B \urcorner \in STAB(X, \vDash)$ , so that  $\Phi$  stabilizes  $(X, \vDash)$ .

The least stabilizing ordinal for  $(X, \vDash)$  is called the closure ordinal for the process  $(X, \vDash)$ . We henceforth let “ $\Phi$ ” denote the closure ordinal. Notice that it will hold that  $\ulcorner A \urcorner \in X(\Phi)$  iff for all  $\gamma \geq \Phi$ ,  $X(\gamma) \vDash A$ . Since all members of  $STAB(X, \vDash)$  stabilize at a countable ordinal, the closure ordinal is countable.

In the proof of Theorem 0 we have mainly adapted [7], pp. 391-2. The construction goes back to [11,12]. Notice that we need no “boot-strapping policy” in our framework.

We now make the crucial librationist twist in order to isolate the intended model of librationism. We shift our attention to those formulas (as noemata serve as names, sentences)  $A$  which are such that  $X(\Phi) \vDash \sim \mathbf{T} \smile A$ . So our official definition of the roadstyle sign is given by  $\vDash A =_D X(\Phi) \vDash \sim \mathbf{T} \smile A$ . It is a fact that  $X(\Phi) \vDash$  is maximal consistent in the sense that  $X(\Phi) \vDash B$  iff not  $X(\Phi) \vDash \sim B$ . Suppose not  $\vDash A$ . It follows that  $X(\Phi) \vDash \mathbf{T} \sim A$ . But we can show that  $X(\Phi) \vDash \mathbf{T}B \supset \sim \mathbf{T} \sim B$  (see  $LO2_M$  in the next paragraph) and that modus ponens holds for  $X(\Phi) \vDash$ , so it follows that  $X(\Phi) \vDash \sim \mathbf{T}A$ , i.e.  $\vDash \sim A$ . So:  $\vDash A$  or  $\vDash \sim A$ , as announced.

Notice from this that our definition of the roadstyle supports the following more precise definitions of maxims and minors:  $\vDash_M A =_D X(\Phi) \vDash \mathbf{T}A$  and  $\vDash_m A =_D \vDash A \ \& \ \vDash \sim A$ .

We again stress that on account of P(5) all noemata name nomina (i.e terms which contain no noemata but only bound or binding variables), and as a consequence all formulas of librationism are in reality sentences. We will in the main bulk of what is to follow let that be reflected in our terminology.

### §3 Axiomschemas and inference rules of librationism

We first give a partial list of axiomatic principles, presupposing the definitions introduced at the end of §1. Maximal schemas are indicated with subscript  $M$ , and minor schemas, i.e. schemas which have minor instances, are indicated with subscript  $m$ . We remind that all axiom schemas that follow hold with all generalizations, so that generalization is not a primitive inference rule. We can show, however, by an inductive argument going back to Tarski, that generalization holds as a derived inference rule relative to theorems which follow from the axiom schemas presupposed with all generalizations.

$L1_M$	$A \supset (B \supset A)$
$L2_M$	$(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
$L3_M$	$(\sim B \supset \sim A) \supset (A \supset B)$
$L4_M$	$A \supset \forall xA$ , provided $x$ is not present in $A$ .
$L5_M$	$\forall x(A \supset B) \supset (\forall xA \supset \forall xB)$
$L6_M$	$\forall xA \supset A(t/x)$ , if $t$ is substitutable for $x$ in $A$ .
$LO1_M$	$\mathbf{T}(A \supset B) \supset (\mathbf{T}A \supset \mathbf{T}B)$
$LO2_M$	$\mathbf{T}A \supset \sim \mathbf{T} \sim A$
$LO3_M$	$\mathbf{T}B \vee \mathbf{T} \sim B \vee (\mathbf{T} \sim \mathbf{T} \sim A \supset \mathbf{T}A)$
$LO4_M$	$\mathbf{T}B \vee \mathbf{T} \sim B \vee (\mathbf{T}A \supset \mathbf{T}\mathbf{T}A)$
$LO5_M$	$\mathbf{T}(\mathbf{T}A \supset A) \supset (\mathbf{T}A \vee \mathbf{T} \sim A)$
$LO6_M$	$\exists x\mathbf{T}A \supset \mathbf{T}\exists xA$
$LO7_M$	$\mathbf{T}\forall xA \supset \forall x\mathbf{T}A$
$LO8_m$	$\mathbf{T}A \supset A$
$LO9_m$	$A \supset \mathbf{T}A$
$LO10_m$	$\forall x\mathbf{T}A \supset \mathbf{T}\forall xA$
$LO11_m$	$\mathbf{T}\exists xA \supset \exists x\mathbf{T}A$

The alethic comprehension principle is as follows:

$$AC_M \quad \forall x(x \in \{y : A\} \equiv \mathbf{T}A(x/y)), \text{ if } x \text{ is substitutable for } y \text{ in } A.$$

We next point out some salient inference rules for librationism:

$R_1$	$\Vdash_M A \ \& \ \Vdash_M (A \supset B) \Rightarrow \Vdash_M B$	<i>modus maximus</i>
$R_2$	$\Vdash_m A \ \& \ \Vdash_M (A \supset B) \Rightarrow \Vdash B$	<i>modus subiunctionis</i>
$R_3$	$\Vdash_M A \ \& \ \Vdash_m (A \supset B) \Rightarrow \Vdash_m B$	<i>modus antecedentiae</i>
$R_4$	$\Vdash_M A \Rightarrow \Vdash_M \mathbf{T}A$	<i>modus ascendens maximus</i>
$R_5$	$\Vdash_m A \Rightarrow \Vdash_m \mathbf{T}A$	<i>modus ascendens minor</i>
$R_6$	$\Vdash_M \mathbf{T}A \Rightarrow \Vdash_M A$	<i>modus descendens maximus</i>
$R_7$	$\Vdash_m \mathbf{T}A \Rightarrow \Vdash_m A$	<i>modus descendens minor</i>
$R_8$	$\Vdash_M \mathbf{T} \smile A \Rightarrow \Vdash_M \mathbf{T}A$	<i>modus scandens maximus</i>
$R_9$	$\Vdash_m \mathbf{T} \smile A \Rightarrow \Vdash_m \mathbf{T}A$	<i>modus scandens minor</i>
$R_{10}$	$\Vdash_M \forall x \mathbf{T}A \Rightarrow \Vdash_M \mathbf{T}\forall x A$	<i>modus Barcanicus</i>
$R_{11}$	$\Vdash \mathbf{T}\exists x A \Rightarrow \Vdash \exists x \mathbf{T}A$	<i>modus attestans generalis</i>
$R_{12}$	$\Vdash_m \mathbf{T}\exists x A \Rightarrow \Vdash_m \exists x \mathbf{T}A$	<i>modus attestans minor</i>
$R_{13}$	$\Vdash_m A \ \& \ \Vdash_m B \Rightarrow \Vdash_m \mathbf{T} \smile A \wedge \smile \mathbf{T} \smile B$	<i>modus minor</i>
$R_Z$	$\Vdash_M A(v) \text{ for all noemata } v \Rightarrow \Vdash_M \forall x A(x)$	<i>The Z-rule</i>

This list of axiom schemas and inference principles is, we again stress, not complete, as librationism is not recursively axiomatizable and no such list can be safeguarded as complete. Moreover, we have aimed at providing a fairly comprehensive list instead of circumscribing an independent list of schemas and inference rules.

We have no explicit *librationist* comprehension principle. However, one may think of librationist comprehension as being implicitly defined by the sum total of such principles as librationism provides.

We show how some of the axiom schemas and inference rules are validated and leave the rest as exercises. §69 in [7] will be helpful on some, but not all, issues as regards other axiom schemas; the inferential principles are all novel with librationism. Notice well the subscripts in  $R_{10}$ - $R_{12}$  above, as pretheoretically plausible strengthenings have counterexamples.

$LO1_M$ : Suppose  $\beta = \gamma + 1$  is a successor ordinal and  $X(\beta) \models \mathbf{T}(A \supset B)$  and  $X(\beta) \models \mathbf{T}A$ . Then  $X(\gamma) \models (A \supset B)$  and  $X(\gamma) \models A$ , hence by modus ponens  $X(\gamma) \models B$ , thence  $X(\beta) \models \mathbf{T}B$ . For  $\beta$  a limit,  $X(\beta) \models \mathbf{T}(A \supset B)$  and  $X(\beta) \models \mathbf{T}A$  entails that  $\forall \delta(\gamma \leq \delta < \beta \Rightarrow X(\delta) \models (A \supset B))$  and  $\forall \delta(\varepsilon \leq \delta < \beta \Rightarrow X(\delta) \models A)$  as from some ordinals  $\gamma$  and  $\varepsilon$  smaller than  $\beta$ . Let  $\kappa = \max(\gamma, \varepsilon)$ . Again by modus ponens,  $\forall \delta(\kappa \leq \delta < \beta \Rightarrow X(\delta) \models B)$ , so  $X(\beta) \models \mathbf{T}B$ . It follows that  $X(\beta) \models \mathbf{T}(A \supset B) \supset (\mathbf{T}A \supset \mathbf{T}B)$  for any ordinal  $\beta$  below  $\Phi$ . Consequently  $X(\Phi) \models \mathbf{T}(\mathbf{T}(A \supset B) \supset (\mathbf{T}A \supset \mathbf{T}B))$ , and so  $\Vdash_M \mathbf{T}(A \supset B) \supset (\mathbf{T}A \supset \mathbf{T}B)$ .

$LO3_M$ : We notice that for  $\beta = \gamma + 1$  a successor ordinal  $X(\beta) \models (\mathbf{T}B \vee \mathbf{T} \smile B)$ . This follows from the definition of  $X$  and  $\models$  as it entails that  $X(\gamma) \models B$  or  $X(\gamma) \models \smile B$ . We show that for limit  $\beta$ ,  $X(\beta) \models (\mathbf{T} \smile \mathbf{T} \smile A \supset \mathbf{T}A)$ . Suppose  $\beta$  is a limit and that  $X(\beta) \models \mathbf{T} \smile \mathbf{T} \smile A$ . Then for some ordinal  $\gamma < \beta$ ,  $\forall \delta(\gamma \leq \delta < \beta \Rightarrow X(\delta) \models \smile \mathbf{T} \smile A)$ . As for all  $\gamma$ ,  $X(\gamma + 1) \models \smile \mathbf{T} \smile A$  only if  $X(\gamma) \models A$ , it will hold that  $\forall \delta(\gamma \leq \delta < \beta \Rightarrow X(\delta) \models A)$ , hence  $X(\beta) \models \mathbf{T}A$ . We have shown that  $X(\beta) \models (\mathbf{T} \smile \mathbf{T} \smile A \supset \mathbf{T}A)$  for all limit ordinals  $\beta$ . As we have that  $X(\beta) \models (\mathbf{T}B \vee \mathbf{T} \smile B)$  for all successor ordinals  $\beta$ , this justifies that  $X(\Phi) \models \mathbf{T}(\mathbf{T}B \vee \mathbf{T} \smile B \vee (\mathbf{T} \smile \mathbf{T} \smile A \supset \mathbf{T}A))$ , from which it follows that  $\Vdash_M \mathbf{T}B \vee \mathbf{T} \smile B \vee (\mathbf{T} \smile \mathbf{T} \smile A \supset \mathbf{T}A)$ .

*LO5<sub>M</sub>*: For  $\beta$  a successor ordinal this holds trivially as the consequent holds. Let  $\beta$  be a limit ordinal and suppose  $X(\beta) \models \mathbf{T}(\mathbf{T}A \supset A)$ . Then for some ordinal  $\gamma < \beta$ ,  $\forall \delta(\gamma \leq \delta < \beta \Rightarrow X(\delta) \models \mathbf{T}A \supset A)$ . Suppose there is some ordinal  $\kappa$  such that  $\gamma \leq \kappa < \beta$  and  $X(\kappa) \models A$ ; then  $\forall \delta(\kappa \leq \delta < \beta \Rightarrow X(\delta) \models A)$ , so  $X(\beta) \models \mathbf{T}A$ . In case there is no such ordinal  $\kappa$ , we have that  $X(\beta) \models \mathbf{T} \smile A$ . In either case,  $X(\beta) \models (\mathbf{T}A \vee \mathbf{T} \smile A)$ . So  $X(\beta) \models \mathbf{T}(\mathbf{T}A \supset A) \supset (\mathbf{T}A \vee \mathbf{T} \smile A)$ . As  $\beta$  can be taken as arbitrary below  $\Phi$ , we have that  $\Vdash_M \mathbf{T}(\mathbf{T}A \supset A) \supset (\mathbf{T}A \vee \mathbf{T} \smile A)$ .

*R<sub>2</sub>*: Suppose  $\Vdash_m A$  and  $\Vdash_M A \supset B$ . It follows that  $X(\Phi) \models \smile \mathbf{T}A$  and  $X(\Phi) \models \smile \mathbf{T} \smile A$  as well as  $X(\Phi) \models \mathbf{T}(A \supset B)$ . It is straightforward to observe that  $\Vdash B$  as  $A$  is unbounded under  $\Phi$  and  $(A \supset B)$  holds below  $\Phi$  as from some ordinal below it. But we do not have enough information to know whether  $B$  is a maxim or a minor.

*R<sub>3</sub>*: Suppose  $\Vdash_M A$  and  $\Vdash_m A \supset B$ . We then have that  $X(\Phi) \models \mathbf{T}A$ ,  $X(\Phi) \models \smile \mathbf{T} \smile (A \supset B)$  and  $X(\Phi) \models \smile \mathbf{T}(A \supset B)$ . That  $X(\Phi) \models \smile \mathbf{T} \smile (A \supset B)$  means that  $A \supset B$  is unbounded under  $\Phi$ . That  $X(\Phi) \models \mathbf{T}A$  means that  $A$  holds as from some ordinal below  $\Phi$ . As modus ponens holds at all ordinals, this means that  $B$  is unbounded under  $\Phi$ , i.e.  $X(\Phi) \models \smile \mathbf{T} \smile B$ . That  $X(\Phi) \models \smile \mathbf{T}(A \supset B)$ , i.e.  $X(\Phi) \models \smile \mathbf{T} \smile (A \wedge \smile B)$ , means that  $A \wedge \smile B$  is unbounded under  $\Phi$ . But so a fortiori also  $\smile B$  is unbounded under  $\Phi$ , i.e.  $X(\Phi) \models \smile \mathbf{T}B$ . So  $\Vdash B$  and  $\Vdash \smile B$ , i.e.  $\Vdash_m B$ .

*R<sub>10</sub>*: Suppose  $\Vdash_M \forall x \mathbf{T}A(x)$ . Then  $X(\Phi) \models \mathbf{T}\forall x \mathbf{T}A(x)$ . But then it obviously follows that  $X(\Phi) \models \mathbf{T}\mathbf{T}\forall x A(x)$ , and so  $\Vdash_M \mathbf{T}\forall x A(x)$ . Cfr. §11 as to why *R<sub>10</sub>* cannot be strengthened as we would intuitively expect.

*R<sub>11</sub>*: Suppose  $\Vdash \mathbf{T}\exists x A$ . We then have that  $X(\Phi) \models \smile \mathbf{T} \smile \mathbf{T}\exists x A$ , so that for all  $\gamma < \Phi$  there is a  $\beta$ ,  $\gamma < \beta < \Phi$ , such that  $X(\beta) \models \mathbf{T}\exists x A$ . But then, whether  $\beta$  is a successor or not, there is a  $\delta$  such that  $\beta > \delta \geq \gamma$  and such that  $X(\delta) \models \exists x A$ . By *P*(2) and *P*(4) it follows that  $X(\delta) \models A(a/x)$  for some term  $a$  substitutable for  $x$  in  $A$ . So  $X(\delta + 1) \models \mathbf{T}A(a/x)$ , and so by existential generalization we have that  $X(\delta + 1) \models \exists x \mathbf{T}A$ . So that for all  $\gamma < \Phi$  there is a  $\beta$ ,  $\gamma < \beta < \Phi$ , such that  $X(\beta) \models \exists x \mathbf{T}A$ . It follows that  $X(\Phi) \models \smile \mathbf{T} \smile \mathbf{T}\exists x A$  implies  $X(\Phi) \models \smile \mathbf{T} \smile \exists x \mathbf{T}A$ . This means that if  $\Vdash \mathbf{T}\exists x A$  then also  $\Vdash \exists x \mathbf{T}A$ . As to why we in addition to *R<sub>11</sub>* and *R<sub>12</sub>* cannot have the rule that  $\Vdash_M \mathbf{T}\exists x A$  only if  $\Vdash_M \exists x \mathbf{T}A$ , cfr. §11.

## §4 Identity

We are able to justify the following:

*Theorem 1:*

- (i)  $X(\Phi) \models \mathbf{T}A \equiv \mathbf{T}\mathbf{T}A$
- (ii)  $X(\Phi) \models \mathbf{T} \smile \mathbf{T} \smile A \equiv \mathbf{T}A$
- (iii)  $X(\Phi) \models \mathbf{T}(A \supset B) \equiv \mathbf{T}(\mathbf{T}A \supset \mathbf{T}B)$
- (iv)  $X(\Phi) \models \mathbf{T}(A \supset \mathbf{T}A) \equiv \mathbf{T}(\mathbf{T}A \supset A)$
- (v)  $X(\Phi) \models \forall x \mathbf{T}A \supset \mathbf{T}\forall x A$
- (vi)  $X(\Phi) \models \mathbf{T}A \supset A$

Proof: We do (i) and (iv) and leave the rest as exercises. (Notice that all the isolated principles of Theorem 1 will hold as minor schemas.) (i): Let  $r$  be  $\{x : x \notin x\}$ . From alethic comprehension and universal instantiation we have that  $X(\Phi) \models r \in r \equiv \mathbf{T}r \notin r$ . As by Theorem 1 (vi)  $X(\Phi) \models \mathbf{T}A \supset A$  this gives us  $X(\Phi) \models r \in r \supset r \notin r$ , i.e.  $X(\Phi) \models r \notin r$ . This gives us  $X(\Phi) \models \sim \mathbf{T}r \notin r$ , and we also get  $X(\Phi) \models \sim \mathbf{T}r \in r$  on account of  $X(\Phi) \models \mathbf{T}r \in r \supset r \in r$  and that modus tollens is respected by  $X(\Phi) \models$ .  $X(\Phi) \models \mathbf{T}r \in r \vee \mathbf{T}r \notin r \vee (\mathbf{T}A \supset \mathbf{T}A)$  is an instance of  $LO4_M$ . Since  $X(\Phi) \models \sim \mathbf{T}r \in r$  and  $X(\Phi) \models \sim \mathbf{T}r \notin r$  we have that  $X(\Phi) \models \mathbf{T}A \supset \mathbf{T}A$ . The reverse direction comes from Theorem 1 (vi). (iv): Assume  $X(\Phi) \models \mathbf{T}(A \supset \mathbf{T}A)$ . Because of  $LO2_M$  we then have that  $X(\Phi) \models \mathbf{T}(A \supset \sim \mathbf{T} \sim A)$ , so by contraposition  $X(\Phi) \models \mathbf{T}(\mathbf{T} \sim A \supset \sim A)$ . On account of  $LO5_M$  it therefore follows that  $X(\Phi) \models \mathbf{T}A \vee \mathbf{T} \sim A$ . If  $X(\Phi) \models \mathbf{T}A$  it follows, using  $LO1_M$ , that  $X(\Phi) \models \mathbf{T}(\mathbf{T}A \supset A)$ . If  $X(\Phi) \models \mathbf{T} \sim A$ , we have  $X(\Phi) \models \mathbf{T} \sim \mathbf{T}A$  by Theorem 1 (ii), and so by  $LO1_M$  we again get that  $X(\Phi) \models \mathbf{T}(\mathbf{T}A \supset A)$ . The reverse direction is similar.

We next justify

$$\text{Lemma 1:} \quad \Vdash_M \forall x, y (\forall u (x \in u \supset y \in u) \supset \mathbf{T} \forall u (x \in u \supset y \in u))$$

Proof: By logic

$$\Vdash_M \forall x, y (\forall u (x \in u \supset y \in u) \supset (x \in \{z : \forall u (x \in u \supset z \in u)\} \supset y \in \{z : \forall u (x \in u \supset z \in u)\})).$$

But clearly  $\Vdash_M \forall x (x \in \{z : \forall u (x \in u \supset z \in u)\})$ , so

$$\Vdash_M \forall x, y (\forall u (x \in u \supset y \in u) \supset y \in \{z : \forall u (x \in u \supset z \in u)\}).$$

Lemma 1 follows by alethic comprehension.

We justify the *Substitution Axiom Schema*:

$$\Vdash_M \forall u (a \in u \supset b \in u) \supset (A(a) \supset A(b))$$

Proof: Suppose  $X(\Phi) \models \sim \mathbf{T}(\forall u (a \in u \supset b \in u) \supset (A(a) \supset A(b)))$ . By Theorem 1 (ii),  $X(\Phi) \models \sim \mathbf{T} \sim \mathbf{T}(\forall u (a \in u \supset b \in u) \wedge A(a) \wedge \sim A(b))$ . From  $LO1_M$  we get that  $X(\Phi) \models \sim \mathbf{T} \sim (\mathbf{T} \forall u (a \in u \supset b \in u) \wedge \mathbf{T}A(a) \wedge \mathbf{T} \sim A(b))$ . By using Theorem 1 (iv) and Lemma 1, on the other hand, we establish that  $X(\Phi) \models \mathbf{T} \forall x, y (\mathbf{T} \forall u (x \in u \supset y \in u) \supset \forall u (x \in u \supset y \in u))$ . From these it follows that  $X(\Phi) \models \sim \mathbf{T} \sim (\forall u (a \in u \supset b \in u) \wedge \mathbf{T}A(a) \wedge \mathbf{T} \sim A(b))$ . Using  $LO2_M$  on the third conjunct,  $X(\Phi) \models \sim \mathbf{T} \sim (\forall u (a \in u \supset b \in u) \wedge \mathbf{T}A(a) \wedge \sim \mathbf{T}A(b))$ . By  $AC_M$ ,  $X(\Phi) \models \sim \mathbf{T} \sim (\forall u (a \in u \supset b \in u) \wedge a \in \{x : A(x)\} \wedge b \notin \{x : A(x)\})$ . But then also  $X(\Phi) \models \sim \mathbf{T} \sim (\forall u (a \in u \supset b \in u) \wedge \exists u (a \in u \wedge b \notin u))$ , which is impossible. So  $X(\Phi) \models \mathbf{T}(\forall u (a \in u \supset b \in u) \supset (A(a) \supset A(b)))$ , and the Substitution Axiom Schema holds.

We justify the *Symmetry Theorem* (given our substitution function it does not follow directly from the Substitution Axiom Schema but needs separate consideration):

$$\Vdash_M \forall x, y (\forall u (x \in u \supset y \in u) \supset \forall u (y \in u \supset x \in u))$$

Proof: We have that

$$\begin{aligned} & \Vdash_M \forall u(a \in u \supset b \in u) \supset \\ & (a \in \{z : \forall u(z \in u \supset a \in u)\} \supset b \in \{z : \forall u(z \in u \supset a \in u)\}). \end{aligned}$$

By rearrangement

$$\begin{aligned} & \Vdash_M a \in \{z : \forall u(z \in u \supset a \in u)\} \supset \\ & (\forall u(a \in u \supset b \in u) \supset b \in \{z : \forall u(z \in u \supset a \in u)\}). \end{aligned}$$

As  $\Vdash_M a \in \{z : \forall u(z \in u \supset a \in u)\}$  we use modus maximus and alethic comprehension to get  $\Vdash_M \forall u(a \in u \supset b \in u) \supset \mathbf{T}\forall u(b \in u \supset a \in u)$ . From Lemma 1 and Theorem 1 (iv),  $\Vdash_M \mathbf{T}\forall u(b \in u \supset a \in u) \supset \forall u(b \in u \supset a \in u)$ , so by a hypothetical syllogism  $\Vdash_M \forall u(a \in u \supset b \in u) \supset \forall u(b \in u \supset a \in u)$ .

As the relation  $\forall u(a \in u \supset b \in u)$  is also reflexive and transitive, we presuppose the Leibnizian-Russellian definition

$$\text{Definition (=):} \quad a = b =_D \forall u(a \in u \supset b \in u)$$

## §5 Arithmetic

Definitions:

$$\begin{aligned} \text{KIND}(a) & =_D \forall x(\mathbf{T}x \in a \vee \mathbf{T}x \notin a) \\ \emptyset & =_D \{x : x \neq x\} \\ a' & =_D \{x : x \in a \vee x = a\} \\ \mathbb{N} & =_D \{x : \forall y(\emptyset \in y \wedge \forall z(z \in y \supset z' \in y) \supset x \in y)\} \end{aligned}$$

We call a sort  $a$  a kind, or kind, if  $\Vdash_M \text{KIND}(a)$ . Following standard notation, we also set  $\omega =_D \mathbb{N}$ .

*Theorem 2:* (i)  $\Vdash_M \emptyset \in \mathbb{N}$ , (ii)  $\Vdash_M \forall z(z \in \mathbb{N} \supset z' \in \mathbb{N})$ , (iii)  $\Vdash_M \text{KIND}(\mathbb{N})$ , (iv) sort-induction:  $\Vdash_M \forall y(\emptyset \in y \wedge \forall z(z \in y \supset z' \in y) \supset \forall w(w \in \mathbb{N} \supset w \in y))$  and (v) full induction:  $\Vdash_M A(\emptyset) \wedge \forall z(A(z) \supset A(z')) \supset \forall w(w \in \mathbb{N} \supset A(w))$ .

Proof: (i): This follows from  $\Vdash_M \forall y(\emptyset \in y \wedge \forall z(z \in y \supset z \in y) \supset \emptyset \in y)$  and alethic comprehension. (ii): By predicate logic  $\forall y(\emptyset \in y \wedge \forall z(z \in y \supset z' \in y) \supset x \in y) \supset \forall y(\emptyset \in y \wedge \forall z(z \in y \supset z' \in y) \supset x' \in y)$  so  $X(\Phi) \Vdash \mathbf{T}(\forall y(\emptyset \in y \wedge \forall z(z \in y \supset z' \in y) \supset x \in y) \supset \forall y(\emptyset \in y \wedge \forall z(z \in y \supset z' \in y) \supset x' \in y))$ . So by Theorem 1 (iii)  $X(\Phi) \Vdash \mathbf{T}(\mathbf{T}\forall y(\emptyset \in y \wedge \forall z(z \in y \supset z' \in y) \supset x \in y) \supset \mathbf{T}\forall y(\emptyset \in y \wedge \forall z(z \in y \supset z' \in y) \supset x' \in y))$ . Using alethic comprehension and the definition of  $\mathbb{N}$  we have that  $X(\Phi) \Vdash \mathbf{T}(x \in \mathbb{N} \supset x' \in \mathbb{N})$ . As  $x$  was arbitrary, it follows that  $X(\Phi) \Vdash \forall x \mathbf{T}(x \in \mathbb{N} \supset x' \in \mathbb{N})$ . But by Theorem 1 (v) we then have that  $X(\Phi) \Vdash \mathbf{T}\forall x(x \in \mathbb{N} \supset x' \in \mathbb{N})$ , so  $\Vdash_M \forall x(x \in \mathbb{N} \supset x' \in \mathbb{N})$ . (iii): From predicate logic we get  $X(\Phi) \Vdash \mathbf{T}(\emptyset \in \mathbb{N} \wedge (\forall x)(x \in \mathbb{N} \supset x' \in \mathbb{N}) \supset ((\forall y)(\emptyset \in y \wedge \forall z(z \in y \supset z' \in y) \supset a \in y) \supset a \in \mathbb{N}))$ . Using Theorem 2 (i) and (ii) and the fact that  $X(\Phi) \Vdash \mathbf{T}(A \supset B) \supset (\mathbf{T}A \supset \mathbf{T}B)$ , it follows that  $X(\Phi) \Vdash \mathbf{T}(\forall y)((\emptyset \in y \wedge \forall z(z \in y \supset z' \in y) \supset a \in y) \supset a \in \mathbb{N})$ . Using  $LO7_M$ , Theorem 1 (iii), alethic comprehension and the definition of  $\mathbb{N}$  it follows

that  $X(\Phi) \models \mathbf{T}(a \in \mathbb{N} \supset \mathbf{T}a \in \mathbb{N})$ . Using Theorem 1 (iv) we get that  $X(\Phi) \models \mathbf{T}(\mathbf{T}a \in \mathbb{N} \supset a \in \mathbb{N})$ , and so by  $LO5_M$  we have that  $X(\Phi) \models \mathbf{T}a \in \mathbb{N} \vee \mathbf{T}a \notin \mathbb{N}$ . But  $a$  was arbitrary, hence  $X(\Phi) \models \mathbf{T}KIND(\mathbb{N})$  and  $\Vdash_M KIND(\mathbb{N})$ . (iv): Immediate. v): We strengthen an idea of [7] (p. 356). Let  $A(x)$  be an arbitrary sentence and define  $A'(x) =_D A(\emptyset) \wedge \forall y(A(y) \supset A(y')) \supset A(x)$ . We will have by logic that  $X(\Phi) \models \mathbf{T}(A'(\emptyset) \wedge \forall x(A'(x) \supset A'(x')))$ , so from established principles also  $X(\Phi) \models \mathbf{T}(\emptyset \in \{u : A'(u)\} \wedge \forall x(x \in \{u : A'(u)\} \supset x' \in \{u : A'(u)\}))$ . Using sort-induction we therefore have that  $X(\Phi) \models \mathbf{T}\forall w(w \in \mathbb{N} \supset w \in \{u : A'(u)\})$ . As  $\mathbb{N}$  is kind we obtain that  $X(\Phi) \models \mathbf{T}\forall w(\mathbf{T}w \in \mathbb{N} \supset \mathbf{T}A'(w))$ . But this entails, by  $LO7_M$ , that  $X(\Phi) \models \forall w\mathbf{T}(w \in \mathbb{N} \supset \mathbf{T}A'(w))$ , and so, by Theorem 1 (iii), we have that  $X(\Phi) \models \forall w\mathbf{T}(w \in \mathbb{N} \supset A'(w))$ . Given Theorem 1 (v) it follows that  $X(\Phi) \models \mathbf{T}\forall w(w \in \mathbb{N} \supset A'(w))$ . By the definition of  $A'(x)$  it follows that  $X(\Phi) \models \mathbf{T}\forall w(w \in \mathbb{N} \supset (A(\emptyset) \wedge \forall y(A(y) \supset A(y')) \supset A(w)))$ . The unrestricted induction schema follows by rearrangement.

Theorem 2, with its obvious elaborations, establishes Peano-arithmetic. The proof that  $\Vdash_M \forall x, y(x, y \in \mathbb{N} \supset (x' = y' \supset x = y))$  is facilitated by the regularity rule for the sort  $H$  of hereditarily iterative non-paradoxical sorts, pointed out in §9.

## §6 Manifestation-points and non-extensionality

The following construction goes back to [7], p 78, and, in a related context, [18]. We can isolate a fixed-point construction, which we call *manifestation-points*, as follows. If we let  $A(x, y)$  be any sentence with the noemata shown, we can find a term  $h^A$  such that  $\Vdash_M \forall z(z \in h^A \equiv \mathbf{T}\mathbf{T}A(z, h^A))$ . Proof: Let  $\langle a, b \rangle$  be the ordered pair e.g. à la Kuratowski,  $d = \{\langle x, g \rangle : A(x, \{u : \langle u, g \rangle \in g\})\}$  and  $h^A = \{x : \langle x, d \rangle \in d\}$ .

The next theorem shows that librationism is a highly non-extensional:

*Theorem 3:* Let  $a =_E b$  abbreviate  $\forall x(x \in a \equiv x \in b)$  and  $KIND(x)$  be as defined in §5. (i)  $\Vdash_M \exists x(KIND(x) \wedge x =_E \emptyset \wedge x \neq \emptyset)$  (ii) If  $a$  is any kind then there is a kind  $b$  such that  $\Vdash_M a =_E b \wedge a \neq b$

A proof of (i) is by letting  $A(x, y)$  be  $x = y \wedge x = \emptyset$  and considering its manifestation-point  $k$  such that  $\Vdash_M \forall x(x \in k \equiv \mathbf{T}\mathbf{T}(x = k \wedge x = \emptyset))$ . Suppose some  $b \in k$ . Then  $b = k \wedge b = \emptyset$  and the empty sort  $\emptyset$  has a member. So  $k$  is empty, and due to the maximality of identity statements, it is a maxim that  $k$  is empty. Suppose that  $k = \emptyset$ . But then clearly  $\emptyset \in k$ , which is impossible. So  $k$  is distinct from  $\emptyset$  and maximally coextensional with  $\emptyset$ . This is called “Gordeev’s paradox” by [7], p. 73.<sup>2</sup> Notice that  $k$  is kind because of the logic of identity. The following type of proof of (ii) is credited to Pierluigi Minari by [7], p. 74. Let  $a$  be any kind and consider the manifestation point  $b$  such that

<sup>2</sup>Lev Gordeev has related to me that he had discovered and communicated the same kind of result based upon combinatoric logic in the context of Explicit Mathematics to Solomon Feferman and to Michael Beeson around 1981. The result was published with acknowledgement to Gordeev in [1].

$\Vdash_M \forall x(x \in b \equiv \mathbf{TT}((a = b \wedge a \notin a) \vee (a \neq b \wedge x \in a)))$ . As  $a$  is kind also  $b$  will be kind, so  $\Vdash_M \forall x(x \in b \equiv ((a = b \wedge a \notin a) \vee (a \neq b \wedge x \in a)))$ . It is now an easy exercise to verify that  $\Vdash_M a =_E b \wedge a \neq b$ .  $\square$

*Theorem 4:* (i) *There are infinitely many mutually distinct kinds coextensional with  $\emptyset = \{x : x \neq x\}$ .* (ii) *If  $a$  is any kind, then there are infinitely many mutually distinct kinds coextensional with  $a$ .*

Proof: (i) We extend the idea in the proof of Theorem 3 (i). Write  $\underline{0}$  for  $\emptyset$  (and  $\underline{1}$  for  $k$  as in that proof). Write  $\bigvee_{i=0}^{i=n} (x = \underline{i})$  for the disjunction (veljunction) of  $n$  identities. Our definitions of the kinds are now given by  $\underline{0} = \emptyset$  and  $\underline{n+1}$  as provided by the manifestation point of  $x = y \wedge \bigvee_{i=0}^{i=n} (x = \underline{i})$ . It follows by identity theory that  $\Vdash_M \forall x(x \in \underline{n+1} \equiv (x = \underline{n+1} \wedge \bigvee_{i=0}^{i=n} (x = \underline{i})))$ . We show by an induction that  $\underline{n+1}$  is kind and distinct from all of  $\underline{0}, \dots, \underline{n}$ . Suppose  $b \in \underline{n+1}$ . Then  $b = \underline{n+1}$  and ( $b = \underline{0}$  or ... or  $b = \underline{n}$ ). By identity theory  $b \in \underline{0}$  or ... or  $b \in \underline{n}$ . But  $\underline{0}$  to  $\underline{n}$  are empty kinds by the induction hypothesis. So  $\underline{n+1}$  is empty, and kind by identity theory. If  $\underline{n+1}$  were to be identical with one of  $\underline{0}$  to  $\underline{n}$  we would have  $\underline{n+1} \in \underline{n+1}$ , contradicting its emptiness. (ii) Exercise. Hint: Generalize Minari's strategy used in Theorem 3 (ii) in a similar way as the proof of Theorem 3 (i) was generalized in Theorem 4 (i).<sup>3</sup>  $\square$

## §7 $\hat{a}$ and the paradoxicality and infinitude of power-sorts

We show the existence of an exotic sort  $\hat{a}$ <sup>4</sup>, that virtually all power sorts are paradoxical and that all power sorts have infinitely many members.

*Theorem 5:* *There is a sort  $\hat{a}$  such that  $\Vdash_m \forall x(x \in \hat{a})$  and  $\Vdash_m \forall x(x \notin \hat{a})$ .*

Hint: Let  $A(x, y)$  be  $x \notin y$ , and let  $\hat{a}$  be its manifestation-point.

*Theorem 6:* *If  $\Vdash_M \exists x(x \notin a)$ ,  $\wp(a) = \{x : x \subset a\} = \{x : \forall y(y \in x \supset y \in a)\}$  is paradoxical.*

Hint: Employ  $\hat{a}$  and reason semantically

For  $\hat{a}$  the construction needed to prove Theorem 6 fails, but in that case we will e.g. for  $V = \{x : x = x\}$  have that  $\Vdash_m V \in \{x : x \subset \hat{a}\}$ . The author does not know of any sort not maximally coextensional with kind universal sorts which does not have a paradoxical power sort.

*Theorem 7:* *All power-sorts have infinitely many members.*

<sup>3</sup>During the revision process of this paper the author was communicated a different but related construction in Theorem 4 of [10] (forthcoming), which gives a result similar to our Theorem 4 (ii) in the context of fuzzy set theory. This inspired the insight that also Minari's construction can be generalized.

<sup>4</sup> $\hat{a}$  is the minuscule of the Scandinavian letter  $\text{\AA}$ .

Proof: Let  $a$  be any sort. Let  $b$  be any of the infinitely many kinds coextensional with  $\emptyset$  as provided by Theorem 4 (i). For any such  $b$ ,  $\Vdash_M b \subset a$  and hence  $\Vdash_M b \in \{x : x \subset a\}$ .

For the notion of infinitude invoked here cfr. the following paragraph. Notice that Theorem 7, counter intuitive as it may seem, even holds for finite sorts including empty sorts. Our librationist results on power sorts confirm, as it were, predicativist and related scruples about and suspicions concerning power sets. But in librationism this is made more precise and more general. Power sorts are accommodated in librationism, but in a sense of the word “sense”, power sorts do not make sense; they are virtually always paradoxical. This does not exclude that there can be inner models where a restricted power sort behaves non-paradoxically.

### §8 Resisting Cantor’s conclusion

There is no doubt whatsoever that Cantor’s arguments for the conclusion that there are uncountable cardinalities are perfectly valid reductio arguments. However, we have learned from Duhem and Quine and others that in the face of contrary evidence a theory might many times be changed in various ways. In the light of librationism the assumption that there may be a function from the natural numbers onto its power-sort, or indeed, onto the universe itself, does not need to be discarded in the face of Cantor’s evidence. Instead, as we shall, see, a hidden assumption concerning the non-paradoxicality of certain sorts is discarded in the librationist framework.

We define some central concepts as they are cashed out in the librationist setting. A sort  $f$  is a relation iff  $\Vdash_M \forall x(x \in f \supset \exists y, z(x = \langle y, z \rangle))$ .  $f$  is a function iff it is a relation and  $\Vdash_M \forall x, y, y'(\langle x, y \rangle \in f \wedge \langle x, y' \rangle \in f \supset y = y')$ .  $a$  is a preimage (domain) of a function  $f$  iff  $\Vdash_M \forall x(x \in a \equiv \exists y(\langle x, y \rangle \in f))$ . A sort  $a$  is an image (sometimes imprecisely called range) of a function  $f$  iff  $\Vdash_M \forall y(y \in a \equiv \exists x(\langle x, y \rangle \in f))$ . We used the indefinite article for preimage and image in the two previous sentences on account of librationism’s highly non-extensional character as borne out by Theorem 4. A function  $f$  is a bijection from preimage to image if  $\Vdash_M \forall x, y, y'(\langle x, z \rangle \in f \wedge \langle y, z \rangle \in f \supset x = y)$ . A function  $f$  is onto a sort  $a$  (a surjection) iff  $\Vdash_M \forall y(y \in a \supset \exists x(\langle x, y \rangle \in f))$ . Notice that all functions are surjections to their images, given these notions. It follows that if  $\Vdash_M a \subset b$  and  $f$  is a surjection to  $b$  then  $f$  is also a surjection to  $a$ . A sort  $a$  has cardinality  $n$ , for  $n \in \mathbb{N}$ , iff there is a kind bijection from  $n$  to  $a$ . A sort  $a$  has cardinality  $\omega$  iff there is a kind bijection from  $\omega = \mathbb{N}$  to  $a$ . A sort has cardinality iff it has cardinality  $\omega$  or it has cardinality  $n$  for some  $n \in \mathbb{N}$ . A sort  $a$  is finite iff there is a sort  $b$  such that  $\Vdash_M a \subset b$  and  $b$  has cardinality  $n$  for some  $n \in \mathbb{N}$ . A sort  $a$  is infinite iff for all  $n \in \mathbb{N}$  there is a  $b$  with cardinality  $n$  such that  $\Vdash_M b \subset a$ . A sort  $a$  is *unfinite* iff it is not finite. All infinite sorts are unfinite, but not vice versa; a sort is properly unfinite if it is unfinite and not infinite. There are both finite, properly unfinite and infinite sorts which have no cardinality: An example of the first type is

$\{x : (x = \emptyset \wedge r \in r) \vee ((x = \emptyset \vee x = \{\emptyset\}) \wedge r \notin r)\}$  for  $r = \{x : x \notin x\}$ .  $\hat{a}$  is an example of the second type. Examples of the third type are  $r = \{x : x \notin x\}$  and  $\{x : x \subset \emptyset\}$ . A sort  $a$  is countable iff there is a surjection from  $\omega$  to  $a$ . As it turns out, in librationism all sorts are countable, i.e. none are uncountable. A sort is *listable* iff it has a cardinality.

Assume there is a function  $f$  from  $\mathbb{N}$  onto the full universe  $V = \{x : x = x\}$ . We also assume that  $f$  is a kind, i.e. that  $\Vdash_M \forall x(\mathbf{T}x \in f \vee \mathbf{T}x \notin f)$ . We now consider Cantor's sort  $s = \{x : x \in \mathbb{N} \wedge x \notin f(x)\}$ . Clearly  $s$  exists according to librationism, as all expressible conditions correspond to a sort according to the librationist point of view. We will write  $m \approx f(n)$  as shorthand for  $\langle n, m \rangle \in f$ , avoiding the identity sign as is commonly used as there in librationism are paradoxical functions. The use of the identity sign for the purpose of abbreviating functional mapping would at best be misleading in librationism, and sometimes turns out to be just wrong as identity logic holds maximally in librationism (see below in this paragraph for more on this).

Let numerals stand for finite von Neumann ordinals as usual. Suppose now e.g. that  $s \approx f(8)$ . Since  $f$  is maximal we will have that this is a maxim. We then consider whether  $8 \in s$ . What we obtain from all this and our comprehension principle  $AC_M$  is that  $\Vdash_M 8 \in s \equiv \mathbf{T}(8 \in \mathbb{N} \wedge 8 \notin s)$ . But  $\Vdash_M 8 \in \mathbb{N}$ , so this reduces to  $\Vdash_M 8 \in s \equiv \mathbf{T}(8 \notin s)$ . But the available axiomatic principles and inference rules only license the conclusion that  $s$  is a paradoxical sort, and that we thus have both  $\Vdash 8 \in s$  and  $\Vdash 8 \notin s$ . The assumption that  $s$  must be non-paradoxical is an essential requirement in this Cantorian argument for the existence of higher cardinalities. In the librationist framework the assumption is naturally discarded, and the Cantorian argument does not support the conclusion that  $f$  cannot exist.

We assumed  $f$  to be a kind (non-paradoxical) function from  $\mathbb{N}$  onto the full universe of sorts, and noticed that such an assumption can be made without falling prey to Cantor's considerations. It holds, a fortiori, that we may assume that there is such a function from  $\mathbb{N}$  onto its power-sort.

Other Cantorian type arguments, including Cantor's first argument, for higher types of infinities fall prey to similar considerations. For example, if (as is indeed suggested by the present framework) the sort of real numbers (e.g. taken in a Dedekindian way) is a paradoxical sort, there is no way to collect exactly all the real numbers by means of a non-paradoxical function from the natural numbers. The sort of real numbers so taken is paradoxical in the librationist framework, just as is the power sort of the natural numbers and, indeed, as we saw, power-sorts more generally. There even are paradoxical real numbers with such a Dedekindian setup (e.g.  $\{x : (x <^Q 0^Q \wedge r \in r) \vee (x <^Q 1^Q \wedge r \notin r)\}$  with  $<^Q$  the standard order of rational numbers,  $0^Q$  ( $1^Q$ ) rational zero (one) and  $r = \{x : x \notin x\}$ ), and there is no non-paradoxical sort which maximally collects exactly the non-paradoxical real numbers. The situation is as follows: If there were a non-paradoxical function from the natural numbers having exactly the sort of (non-paradoxical) real numbers as its range, then the sort of (non-paradoxical) real numbers would be non-paradoxical. But we can show that the sort of (non-paradoxical) real numbers so taken, for independent rea-

sons, is paradoxical. So there is no such function. The sort of real numbers is not *listable*. Still, there is nothing which licences the conclusion that there is no non-paradoxical function from the natural numbers *onto* the sort of (non-paradoxical) real numbers, and so in this fundamental and perfectly adequate sense the sort of real numbers remains countable, i.e. it does not have a cardinality larger than  $\omega$ . No sort has a cardinality larger than  $\omega$  in librationism, though some, as the sort of real numbers do not have a cardinality. There are no more real numbers than there are natural numbers.

We have of course not by the foregoing shown that librationism as so far developed has such a surjection as assumed from  $\mathbb{N}$  to  $V$ . To achieve such a strengthened countable framework, we enlarge the librationist language with a new nomen  $\epsilon$  and have its denotatum serve as a bijection from  $\mathbb{N}$  to the full universe by just slightly altering the semantical setup.

We first change into an enumeration where  $\epsilon$  is reckoned amongst the nomina. The semantical principle  $P(5)$  of §2 is now changed into  $P_\epsilon(5)$ : (1) If  $a = e(0)$  then  $X(\alpha) \models A(a)$  iff  $X(\alpha) \models A(v_0)$ . (2) For successor numerals  $\mathbf{i} + \mathbf{1}$ , if there is a natural number  $n$  such that for all numerals  $\mathbf{k}$  smaller than  $\mathbf{i} + \mathbf{1}$ ,  $X(\alpha) \models \sim (\forall u)(e(n) \in u \supset v_{\mathbf{k}} \in u)$ , then  $X(\alpha) \models (\forall u)(e(m) \in u \supset v_{\mathbf{i}+1} \in u)$  iff  $m$  is the smallest number such that for all numerals  $\mathbf{k}$  smaller than  $\mathbf{i} + \mathbf{1}$ ,  $X(\alpha) \models \sim (\forall u)(e(m) \in u \supset v_{\mathbf{k}} \in u)$ . Otherwise  $X(\alpha) \models (\forall u)(v_{\mathbf{i}} \in u \supset v_{\mathbf{i}+1} \in u)$ .

Notice that we in defining  $P_\epsilon(5)$  have presupposed the Leibnizian-Russellian definition of identity in §4. We also point out that if we stay with the notation of  $\text{Def}(=)$  in §4, one must keep in mind that it is only at very large ordinals of the semantical process that identity is adequately captured. We e.g. have that  $X(0) \models \forall x \forall y (x = y)$ . But already  $X(2) \models \{x : x \in x\} \neq \{x : x \notin x\}$ . The generation of non-identities is monotonous in the external semantical process, so that if  $\alpha < \beta$  and  $X(\alpha) \models a \neq b$  then  $X(\beta) \models a \neq b$ .

Presupposing e.g. a Kuratowskian definition of ordered pairs and  $\mathbb{N}$  as defined above, we further assume a new semantical principle  $P(6)$ :  $X(\alpha) \models u \in \epsilon$  iff for some sort  $a$  and some natural number  $n$  and noema with corresponding numeral  $\mathbf{n}$ ,  $X(\alpha) \models u = \langle n, a \rangle \wedge n \in \mathbb{N} \wedge a = v_{\mathbf{n}}$ . Our semantical setup is now such that  $\Vdash_M \text{KIND}(\epsilon)$ . This follows from the logic of identity and the fact that  $\mathbb{N}$  is kind. It holds that  $\epsilon$  is a bijection from  $\mathbb{N}$  to the full universe, as distinct noemata are now unique standard names for distinct sorts, i.e. all sorts will have a unique noema as its standards name and all noemata denote a unique sort. Given this we may also accommodate an appropriate substitution function and by slight alterations in the semantical setup include also a truth predicate; we then justify an *Axiom of Truth* which expresses the appropriate correspondence between the truth operator  $\mathbf{T}$  and the truth predicate. The truth predicate is then best thought of as a sort of natural numbers, and it is a paradoxical sort. The Liar's Paradox and related paradoxes are now accounted for librationistically in a way which at this point will be understood in its outlines by my audience; we invoke the Carnap-Gödel Diagonal Lemma. We mention that Yablo's alleged non-circular paradox can be accounted for in our framework.

Given that  $\epsilon$  is a kind bijection from  $\mathbb{N}$  to the full universe, an appropriate

partial function from the natural numbers  $\mathbb{N}$  onto its power sort  $\wp(\mathbb{N})$  is provided by  $f = \{ \langle x, y \rangle : \langle x, y \rangle \in \mathbb{E} \wedge y \subset \mathbb{N} \}$ . An image of  $f$  will indeed be  $\{x : x \subset \mathbb{N}\}$ . For any  $b$ , and so also if  $\Vdash b \in \{x : x \subset \mathbb{N}\}$ , there will be some unique sort  $a$  such that  $\Vdash_M \langle a, b \rangle \in \mathbb{E}$ . Here  $\Vdash_M a \in \mathbb{N}$ . So suppose  $\Vdash b \in \{x : x \subset \mathbb{N}\}$  and  $a \in \mathbb{N}$  such that  $\Vdash_M \langle a, b \rangle \in \mathbb{E}$ . By alethic comprehension and modus descendens we get  $\Vdash b \subset \mathbb{N}$ . By classical logic and modus subiunctionis then  $\Vdash (\langle a, b \rangle \in \mathbb{E} \wedge b \subset \mathbb{N})$ . By modus ascendens,  $\Vdash \mathbf{T}(\langle a, b \rangle \in \mathbb{E} \wedge b \subset \mathbb{N})$ , and so by alethic comprehension and the definition of  $f$ ,  $\Vdash \langle a, b \rangle \in f$ . The function  $f$ , partial on  $\mathbb{N}$ , can be seen to be a bijection from e.g. the proper domain  $\{x : \exists y(\langle x, y \rangle \in \mathbb{E} \wedge y \subset \mathbb{N})\}$  to its images.

We show that the function  $f = \{ \langle x, y \rangle : \langle x, y \rangle \in \mathbb{E} \wedge y \subset \mathbb{N} \}$  itself is paradoxical. To see this consider our sort  $\hat{a}$  defined in §7 which is such that  $\Vdash \forall x(x \in \hat{a})$  and  $\Vdash \forall x(x \notin \hat{a})$ . Given that  $\mathbb{E}$  is a kind bijection from  $\mathbb{N}$  to the full universe, there will be a unique  $n \in \mathbb{N}$  so that  $\Vdash_M \langle n, \hat{a} \rangle \in \mathbb{E}$ . Since  $\Vdash \hat{a} \subset \mathbb{N}$  it follows that  $\Vdash \langle n, \hat{a} \rangle \in \mathbb{E} \wedge \hat{a} \subset \mathbb{N}$  by classical logic and modus subiunctionis. By modus ascendens,  $\Vdash \mathbf{T}(\langle n, \hat{a} \rangle \in \mathbb{E} \wedge \hat{a} \subset \mathbb{N})$ , and so by alethic comprehension and the definition of  $f$ ,  $\Vdash \langle n, \hat{a} \rangle \in f$ . Now, since also  $\Vdash \hat{a} \not\subset \mathbb{N}$ , it will as well hold that  $\Vdash \langle n, \hat{a} \rangle \notin \mathbb{E} \vee \hat{a} \not\subset \mathbb{N}$ . By modus ascendens it follows that  $\Vdash \mathbf{T}(\langle n, \hat{a} \rangle \notin \mathbb{E} \vee \hat{a} \not\subset \mathbb{N})$ . By  $LO2_M$  and modus subiunctionis it follows that  $\Vdash \neg \mathbf{T}(\langle n, \hat{a} \rangle \in \mathbb{E} \wedge \hat{a} \subset \mathbb{N})$ . By alethic comprehension and the definition of  $f$ ,  $\Vdash \langle n, \hat{a} \rangle \notin f$ . So  $\Vdash \langle n, \hat{a} \rangle \in f$  and  $\Vdash \langle n, \hat{a} \rangle \notin f$ .  $f$  is a paradoxical function. Still, it *is* a function in that it is maximally a relation and  $\Vdash_M \forall x, y, z(\langle x, y \rangle \in f \wedge \langle x, z \rangle \in f \supset y = z)$ .

The fact that there are paradoxical functions conjoined with the fact that identity statements are always maximally true or maximally false, justify the symbolical innovation introduced above for the librationist setting.<sup>5</sup> As we know it has been common to write  $g(a) = b$  for  $\langle a, b \rangle \in g$  when  $g$  is a function. But this notation is in the librationist framework not advisable, since it, in conjunction with the librationist theory of identity, would imply that functions cannot be paradoxical. Instead we suggest to write  $g(a) \approx b$  for  $\langle a, b \rangle \in g$  when  $g$  is a function. If e.g.  $\Vdash_M \langle 13, \hat{a} \rangle \in \mathbb{E}$  and  $f$  is as in the two previous paragraphs, we conclude that  $\Vdash_m \langle 13, \hat{a} \rangle \in f$  and write  $\Vdash_m \hat{a} \approx f(13)$ .

It is conceivable that one *could* presuppose a librationist framework for dealing with the paradoxes and at the same time retain or cling to the idea that there are uncountable infinities. The author would regard such an approach, if possible, as quite disingenuous. It is a virtue to postulate as few entities as possible in order to account for a phenomenon. Given this attitude, we should not postulate uncountable entities unless we are compelled to. Moreover, the author does not believe there are uncountable infinities. But we are in the librationist framework not compelled to postulate uncountable infinities, and we ought to regard this as a strong advantage which counts in its favour. Here also the Löwenheim-Skolem theorem is on our side, as it, as stressed by Skolem, shows that the notion of uncountability is one that we can have only in a very theory relative sense.

<sup>5</sup>The author first suggested this symbolic innovation in [3].

## §9 Introducing hereditarily kind and iterative sorts

We first report some results without proof. For related results in a different formal setting, see [7] §12 and §23. As before, we let  $KIND(y)$  abbreviate  $\forall x(\mathbf{T}x \in y \vee \mathbf{T}x \notin y)$ .

Given the manifestation-point  $\Vdash_M \forall x(x \in H \equiv \mathbf{TT}(KIND(x) \wedge x \subset H))$  it follows that  $\Vdash_M x \in H$  iff  $\Vdash_M KIND(x) \wedge x \subset H$ . If  $\Vdash_M a \in H$  then  $a$  is hereditarily kind (non-paradoxical) and iterative. ( $a$  is hereditarily kind iff  $a$  is kind and all members of  $a$  are hereditarily kind. We explain our notion of iterativity below.) From Theorem 2 (iii),  $\Vdash_M \mathbb{N} \in H$ . Further,  $H$  is closed under pairing and union, in that if  $\Vdash_M a, b \in H$  then  $\Vdash_M \{a, b\} \in H$  and  $\Vdash_M \cup b \in H$ .  $H$  is in fact closed under all the remaining Jensen rudimentary functions.<sup>6</sup>

$$\begin{aligned} F_1(x, y) &= x \setminus y \\ F_2(x, y) &= x \times y \\ F_3(x, y) &= \{ \langle u, z, v \rangle : z \in x \wedge \langle u, v \rangle \in y \} \\ F_4(x, y) &= \{ \langle u, v, z \rangle : z \in x \wedge \langle u, v \rangle \in y \} \\ F_6(x, y) &= Dom(x) \\ F_7(x, y) &= \in \cap x^2 \\ F_8(x, y) &= \{ x''\{z\} : z \in y \} \end{aligned}$$

in the same sense, and so also under  $\Delta_0$ -separation. From §5 we know that the full induction schema for  $\mathbb{N}$  holds. Indeed, from the following paragraph we will see that much more is true.

One should pay attention to the fact that  $H$  itself is not kind. For example,  $\Vdash_m \hat{a} \in H$ . This is left as an exercise.

We wrote that  $H$  is a sort of iterative sorts. This holds in the following sense of a regularity rule:

$$\text{If } \Vdash_M b \in H \text{ then } \Vdash_M \exists x(x \in b) \supset \exists x(x \in b \wedge \forall y(y \in b \supset y \notin x))$$

We can justify the regularity rule briefly as follows: Suppose instead that  $\Vdash_M b \in H$  and  $\Vdash \exists x(x \in b) \wedge \forall x(x \in b \supset \exists y(y \in b \wedge y \in x))$ . As  $b$  is hereditarily kind it follows that  $\Vdash_M \exists x(x \in b) \wedge \forall x(x \in b \supset \exists y(y \in b \wedge y \in x))$ . But the latter can only be satisfied if  $b$  is circular, a cycle or has an infinitely descending chain. Given the nature of  $H$ , it would follow that  $X(0) \models \exists x(x \in H)$ , which is contrary to our minimalist stipulations. Hence,  $H$  only contains well-founded sorts as maximal members.

If  $\Vdash_M b \in H$  we will say that  $b$  is a good. Here the word “good” is used as a noun, but we also on occasion use it adjectivally. All goods are kinds, but not vice versa. In as far as librationism supports the existence of sets in a more classical sense (e.g. as those sorts which belong to a good defined as the least sort built up iteratively from  $\mathbb{N}=\omega$  by closing off with the Jensen rudimentary functions) such sets will be goods. But not all goods are sets. E.g. extensionality fails badly for  $H$ . For this, see Theorems 3 and 4.

<sup>6</sup>See [13], especially Lemma 1.8 on p. 239.

## §10 Finitely Iterated Inductive Definitions and Transfinite Induction

The notion of a  $y$ -positive sentence will be central here, so we define  $y$ -positive and  $y$ -negative sentences relative to goods  $e$  of  $H$  as follows: If  $y$  is not among the noemata of a sentence  $A$  then  $A$  is both  $y$ -positive and  $y$ -negative. The sentence  $t \in y$  is  $y$ -positive. If  $A$  and  $B$  are both  $y$ -positive ( $y$ -negative) then  $A \wedge B$ ,  $A \vee B$ ,  $\exists x(x \in e \wedge A)$  and  $\forall x(x \in e \supset A)$  are  $y$ -positive ( $y$ -negative). If  $A$  is  $y$ -positive ( $y$ -negative) then  $\neg A$  is  $y$ -negative ( $y$ -positive).

The theory of finitely iterated inductive definitions  $ID_{<\omega}$  extends  $ACA_0$  by least fixed-point principles for successively newly introduced set terms. If  $A(x, y)$  has  $x$  and  $y$  present and is  $y$ -positive, let the expression  $[xyA(x, y)]$  temporarily stand for such a new term with  $x$  and  $y$  bound so that:

$$\begin{aligned} FP & \quad \forall z(z \in [xyA(x, y)] \equiv A(z, [xyA(x, y)])) \\ LEAST & \quad \forall z(\forall x(A(x, z) \supset x \in z) \supset ([xyA(x, y)] \subset z)) \end{aligned}$$

It follows from what we pointed out in §9 that  $H$  suffices to interpret  $ACA$ . We will now show that analogues of  $FP$  and  $LEAST$  can be captured, and we use manifestation points. Since we are now first of all interested in the generation of subsorts of  $\mathbb{N}$ , we let  $A(x, y)$  be of the form  $x \in \mathbb{N} \wedge A'(x, y)$ , where  $A'(x, y)$  is  $y$ -positive relative to (goods of)  $H$ . This means that all quantifiers in the build-up of  $A(x, y)$  are to be bound by goods in  $H$ . All and only good parameters from  $H$  are allowed. We now consider the manifestation-point such that  $\Vdash_M \forall x(x \in h^A \equiv \mathbf{T}A(x, h^A))$  and will show that  $FP$  and  $LEAST$  hold for  $h^A$ . We first consider  $FP$ .

It is sufficient to show that  $h^A$  is non-paradoxical, which in its turn is necessary in order to show that  $\Vdash_M h^A \in H$ . OBSERVATION: Since  $h^A$  is an operator with the indicated restrictions to  $H$ , we will for all ordinals  $\alpha$  have that  $X(\alpha) \models \forall x(x \in h^A \supset \mathbf{T}x \in h^A) \supset \forall x(A(x, h^A) \supset \mathbf{T}A(x, h^A))$ . This follows from the build up of  $A(x, y)$ , it being positive in  $y$ . We will show first that  $X(\Phi) \models \mathbf{T}\forall x(x \in h^A \supset \mathbf{T}x \in h^A)$ . If  $\delta$  is a limit below  $\Phi$  or  $\delta = \emptyset$  then clearly also  $X(\delta) \models \forall x(x \in h^A \supset \mathbf{T}x \in h^A)$  and  $X(\delta + 1) \models \forall x(x \in h^A \supset \mathbf{T}x \in h^A)$ . Suppose  $\beta = \gamma + 2$  and that the hypothesis that  $\forall x(x \in h^A \supset \mathbf{T}x \in h^A)$  holds below  $\beta$ . Suppose so that  $X(\beta) \models a \in h^A$ . Then  $X(\gamma) \models A(a, h^A)$ . But by the induction hypothesis,  $X(\gamma) \models \forall x(x \in h^A \supset \mathbf{T}x \in h^A)$ . From our OBSERVATION, it then follows that  $X(\gamma) \models \mathbf{T}A(a, h^A)$ , which in its turn entails that  $X(\beta) \models \mathbf{T}a \in h^A$ . Hence  $X(\beta) \models (a \in h^A \supset \mathbf{T}a \in h^A)$ . By a transfinite induction it follows that we at the closure ordinal have that  $X(\Phi) \models \mathbf{T}(a \in h^A \supset \mathbf{T}a \in h^A)$ , and hence  $\Vdash_M KIND(h^A)$ . Now clearly also  $\Vdash_M h^A \subset H$ , so that  $\Vdash_M h^A \in H$ . It follows as a matter of course that we also have  $\models_M \forall x(x \in h^A \equiv A(x, h^A))$ , so that we have  $FP$ .

We show that  $h^A$  satisfies  $LEAST$ . Let  $\beta$  be an ordinal equal to or below the closure ordinal and assume  $X(\beta) \models \forall x(A(x, z) \supset x \subset z)$ . Also, let  $\beta$  be larger than  $HT(\ulcorner KIND(h^A) \urcorner)$  (cfr. §2), so that  $X(\beta) \models \forall x(x \in h^A \supset \mathbf{T}x \in h^A)$ . We first show by a transfinite induction on ordinals  $\gamma$  below  $\beta$  that if  $X(\gamma) \models a \in h^A$  then  $X(\beta) \models a \in z$ . (1) Let  $\gamma = 0$  or a limit. Then in the first case not

$X(\gamma) \models a \in h^A$ , so the conditional holds trivially. In the second case, supposing that  $X(\gamma) \models a \in h^A$ , the conclusion follows by the induction hypothesis because  $a \in h^A$  holds at yet smaller ordinals. (2) Let  $\gamma = 1$  or  $\delta + 1$  where  $\delta$  is a limit ordinal. In the first case again not  $X(\gamma) \models a \in h^A$ , so the conditional holds trivially. The second case is similar to the second case under (1), as it, due to  $h^A$  being a manifestation point, entails that  $X(\delta) \models a \in h^A$ . (3) Let  $\gamma = \delta + 2$  and assume that  $X(\gamma) \models a \in h^A$ . It follows that  $X(\delta) \models A(a, h^A)$ . Now, reasoning at the meta-level, the set  $[y : X(\delta) \models y \in h^A]$  is, by the induction hypothesis, a subset of  $[y : X(\beta) \models y \in z]$ . Since  $A(x, y)$  is a kind operator positive in  $y$ , and  $X(\delta) \models A(a, h^A)$ , it follows that  $X(\beta) \models A(a, z)$ . But  $X(\beta) \models (A(a, z) \supset a \in z)$ , so  $X(\beta) \models a \in z$ .  $\beta$  was taken to be an arbitrary ordinal equal to or below the closure ordinal and above  $HT(\ulcorner KIND(h^A) \urcorner)$ . If we assume that  $X(\beta) \models a \in h^A$  it therefore follows that  $X(\beta) \models \mathbf{T}a \in h^A$ . But if so,  $X(\gamma) \models a \in h^A$  for some ordinal  $\gamma$  below  $\beta$ , so that by our induction,  $X(\beta) \models a \in z$ . Putting things together, it follows that  $X(\beta) \models \forall z(\forall x(A(x, z) \supset x \subset z) \supset (h^A \subset z))$ . But  $\beta$  was arbitrary at or below the closure ordinal and above  $HT(\ulcorner KIND(h^A) \urcorner)$ , so it follows that  $X(\Phi) \models \mathbf{T}\forall z(\forall x(A(x, z) \supset x \in z) \supset (h^A \subset z))$  so that  $\Vdash_M \forall z(\forall x(A(x, z) \supset x \in z) \supset (h^A \subset z))$ .

We now first show that a transfinite induction rule, the Bar rule, holds along good well-founded relations in  $H$ . Let  $\prec = \{ \langle x, y \rangle : A(x, y) \}$ , where we shall assume  $A(x, y)$  to be such that  $\Vdash_M \prec \in H$ . Instead of  $\langle x, y \rangle \in \prec$  we write  $x \prec y$ . Define:

$$\begin{aligned} \text{Progr}(\prec, B) &=_D \forall x(\forall y(y \prec x \supset B(y)) \supset B(x)) \\ \text{Progr}(\prec, z) &=_D \forall x(\forall y(y \prec x \supset y \in z) \supset x \in z) \\ W(\prec, u) &=_D \forall z(\text{Progr}(\prec, z) \supset u \in z) \\ WF(\prec) &=_D \{u : W(\prec, u)\} \end{aligned}$$

The transfinite induction rule, or Bar rule, we show is:  $\Vdash_M a \in WF(\prec)$  only if  $\Vdash_M (\text{Progr}(\prec, B) \supset B(a))$ . Suppose the contrary  $X(\Phi) \models \mathbf{T}a \in WF(\prec)$  and  $X(\Phi) \models \sim \mathbf{T}(\text{Progr}(\prec, B) \supset B(a))$ . The first means that  $X(\Phi) \models \mathbf{T}\forall z(\text{Progr}(\prec, z) \supset a \in z)$ , i.e. that  $X(\Phi) \models \mathbf{T}\forall z((\forall x(\forall y(y \prec x \supset y \in z) \supset x \in z) \supset a \in z)$ . The second means that there is an unbounded set of ordinals  $\delta$  below  $\Phi$  such that  $X(\delta) \models \text{Progr}(\prec, B) \wedge \sim B(a)$ . Spelled out this means that  $X(\delta) \models \forall x(\forall y(y \prec x \supset B(y)) \supset B(x)) \wedge \sim B(a)$ . But then it follows that  $X(\delta + 1) \models \mathbf{T}(\forall x(\forall y(y \prec x \supset B(y)) \supset B(x)) \wedge \sim B(a))$ . From  $LO1_M$  and  $LO7_M$  we get that  $X(\delta + 1) \models \forall x(\mathbf{T}\forall y(y \prec x \supset B(y)) \supset \mathbf{T}B(x)) \wedge \mathbf{T}\sim B(a)$ . Since  $\delta + 1$  is a successor ordinal, the Barcan-principle ( $LO10_m$ ) holds, so that  $X(\delta + 1) \models \forall x(\forall y\mathbf{T}(y \prec x \supset B(y)) \supset \mathbf{T}\forall y(y \prec x \supset B(y)))$ . Combined with  $LO2_M$  it follows that  $X(\delta + 1) \models \forall x(\forall y\mathbf{T}(y \prec x \supset B(y)) \supset \mathbf{T}B(x)) \wedge \sim \mathbf{T}B(a)$ . As  $\delta + 1$  is a successor ordinal and  $\Vdash_M \prec \in H$  we have that  $X(\delta + 1) \models (y \prec x \supset \mathbf{T}B(y)) \equiv \mathbf{T}(y \prec x \supset B(y))$ . From this we obtain  $X(\delta + 1) \models \forall x(\forall y(y \prec x \supset \mathbf{T}B(y)) \supset \mathbf{T}B(x)) \wedge \sim \mathbf{T}B(a)$ . Using comprehension, this means that  $X(\delta + 1) \models \forall x(\forall y(y \prec x \supset y \in \{u : B(u)\}) \supset x \in \{u : B(u)\}) \wedge a \notin \{u : B(u)\}$ . But, as we have supposed  $\Vdash_M a \in WF(\prec)$ ,  $X(\Phi) \models \mathbf{T}\forall z(\forall x(\forall y(y \prec x \supset y \in z) \supset x \in z) \supset a \in z)$ , so we also have that  $X(\delta + 1) \models \forall z(\forall x(\forall y(y \prec x \supset y \in z) \supset x \in z) \supset a \in z)$ . Instantiating with  $\{u : B(u)\}$  for  $z$  we have a contradiction. So the Bar-rule is

valid.

If also  $\vDash_M WF(\prec) \in H$  we can moreover show that full Bar-Induction  $\Vdash_M \forall x(x \in WF(\prec) \supset (Progr(\prec, B) \supset B(x)))$  holds. It is worth pointing out that  $\vDash_M \prec \in H$  suffices for  $\Vdash \forall x(x \in WF(\prec) \supset (Progr(\prec, B) \supset B(x)))$ , and so in combination with the Bar-rule there is little that goes amiss compared with the strength of full Bar-induction. Nonetheless, it would be of interest to know whether  $\vDash_M \prec \in H$  entails  $\vDash_M WF(\prec) \in H$  or not. If the answer is no, it would be of interest to know counterexamples.

Since  $\Vdash_M h^A \in H$  iterations of such fixed-points are allowed as in the theory of finitely iterated inductive definitions  $ID_{<\omega}$ . By established results of proof theory this shows that librationism accommodates  $\Pi_1^1 - CA_0$  in the sense of what can be proved maximally in H. Possibly, the precise gauge here is  $\Pi_1^1 - CA$  as the condition  $A(x, y)$  allows goods from  $H$  as parameters.

### §11 Closures of Paradox

We have seen how librationism deals with Russell's paradox, in that we both have that  $\Vdash r \in r$  and  $\Vdash r \notin r$ , i.e.  $\Vdash_m r \notin r$  for Russell's sort  $r = \{x : x \notin x\}$ . We will now discuss how librationism deals with a selection of other and some more complicated paradoxes.

The Liar's paradox can be treated in a way very much like Russell's paradox if we extend our language with a bijection from  $\mathbb{N}$  to  $V$  and with a truth predicate, as well as with an Axiom of Truth which links the truth predicate with the truth operator  $\mathbf{T}$  in the appropriate way.

The Curry-paradox has deservedly captured much attention since its inception. In librationism it has a somewhat surprising resolution. Let  $F$  be any sentence, and define  $c = \{x : x \in x \supset F\}$ .  $LOS_m$  and alethic comprehension gives us  $\Vdash c \in c \supset (c \in c \supset F)$ . But  $\Vdash_M (c \in c \supset (c \in c \supset F)) \supset (c \in c \supset F)$ , so by modus subiunctionis it follows that  $\Vdash (c \in c \supset F)$ . By modus ascendens we get  $\Vdash \mathbf{T}(c \in c \supset F)$ , and so next  $\Vdash c \in c$  follows from alethic comprehension by modus subiunctionis. So we have that both  $\Vdash c \in c$  and  $\Vdash c \in c \supset F$  for any arbitrary sentence  $F$ . Now,  $\Vdash c \in c$  being a theorem, it must either be a minor or a maxim. If  $\Vdash_M c \in c$ , we easily derive that  $F$  is a maxim, i.e.  $\Vdash_M F$ . If  $c \in c$  is a minor ( $\Vdash_m c \in c$ ), it follows that  $\smile F$  is a theorem. For then also  $\Vdash_m c \notin c$  so by alethic comprehension and modus subiunctionis  $\Vdash \smile \mathbf{T}(c \in c \supset F)$  and thence by modus scandens and modus descendens,  $\Vdash c \in c \wedge \smile F$ . By tautologies and modus subiunctionis,  $\Vdash \smile F$ . It follows, by parity of reasoning, that for any sentence  $F$ , either  $F$  is a maxim,  $F$  (and hence also  $\smile F$ ) is a minor, or  $\smile F$  is a maxim. But in our contentual framework this is perfectly as it should be; remember our observation that librationism is negation (negjunction) complete at the end of §2.

We will now discuss a paradoxicality related to an observation in [14]. Let  $\mathbb{N}$  as usual be the sort of natural numbers as defined earlier, and let the use of ' as superscript signify ordinal succession. Let, for any sort  $s$ ,  $s_N$  be given by:

$$\{x : \forall y(< \emptyset, s > \in y \wedge \forall z, w(< z, w > \in y \supset < z', \{u : u \in w\} > \in y) \supset x \in y)\}$$

(Remember that extensionality fails in librationism, and  $w$  is generally distinct from  $\{u : u \in w\}$ .) Let  $r = \{x : x \notin x\}$ ,  $t = \{x : x = r \wedge x \notin x \wedge \mathbf{T}x \in x\}$  and  $>$  be the usual order on the natural numbers. Let  $B(x)$  be the sentence  $(x \in \mathbb{N} \supset \exists y(y > x \wedge \exists w(< y, w > \in t_{\mathbb{N}} \wedge r \in w))$ ). If we now for any limit-ordinal  $\alpha$  (under the closure ordinal) consider the limit-ordinal  $\beta$  such that  $\beta = \alpha + \omega$ , we will (and we leave this as an exercise) realize that  $X(\beta) \models \forall x \mathbf{T}B(x)$  while also  $X(\beta) \models \sim \mathbf{T}\forall x B(x)$ . So not  $X(\Phi) \models \mathbf{T}(\forall x \mathbf{T}B(x) \supset \mathbf{T}\forall x B(x))$ . In consequence this also clarifies why only a minor schema of the Barcan-formula can be assumed in librationism. Now, our construction also reveals that  $X(\Phi) \models \sim \mathbf{T} \sim \forall x \mathbf{T}B(x)$  and  $X(\Phi) \models \sim \mathbf{T}\forall x \mathbf{T}B(x)$ . So  $\Vdash_m \forall x \mathbf{T}B(x)$ . But also  $X(\Phi) \models \mathbf{T}\exists x \sim B(x)$  (exercise), so  $\Vdash_M \exists x \sim B(x)$ . By  $R_4$  we now have that  $\Vdash_M \mathbf{T}\exists x \sim B(x)$ . Using modus maximus with  $LO2_M$  we obtain  $\Vdash_M \sim \mathbf{T}\forall x B(x)$ . But then, it is not the case that  $\Vdash \mathbf{T}\forall x B(x)$ , and a fortiori not the case that  $\Vdash_m \mathbf{T}\forall x B(x)$ . So we have that  $\Vdash_m \forall x \mathbf{T}B(x)$  and not  $\Vdash \mathbf{T}\forall x B(x)$  (and hence also not  $\Vdash_m \mathbf{T}\forall x B(x)$ ). Consequently  $R_{10}$  cannot be strengthened in such ways as one would intuitively suspect. It is with intricacies such as in this paragraph that librationism evades omega inconsistencies. Importantly, librationism is omega complete, and avoids the type of omega inconsistency encountered e.g. in the Friedman-Sheard logic of truth.

Another curious phenomenon arises in connection with the fact that the inference rule  $\Vdash_M \mathbf{T}\exists x A$  only if  $\Vdash_M \exists x \mathbf{T}A$  is not generally valid. To see this, the reader is left to realize that  $X(\Phi) \models \mathbf{T}\exists x(x = \emptyset \equiv r \in r)$ , where  $r$  is again Russell's sort. But also  $X(\Phi) \models \sim \exists x \mathbf{T}(x = \emptyset \equiv r \in r)$ , and as a consequence  $X(\Phi) \models \sim \mathbf{T}\exists x \mathbf{T}(x = \emptyset \equiv r \in r)$  by Theorem 1 (vi). So  $\Vdash_M \mathbf{T}\exists x(x = \emptyset \equiv r \in r)$  and not  $\Vdash_M \exists x \mathbf{T}(x = \emptyset \equiv r \in r)$  in this exotic case.

The first paradox to receive attention in the modern mathematical literature on these was that of Burali-Forti which concerned itself with well-orderings. We will point out some distinctive features in the way librationism tackles this challenging paradox.

In order to emulate von Neumann ordinals, we utilize the fact that goods in  $H$  are well-founded, and define the sort of ordinals by

$$\Vdash_M \forall x(x \in Ord \equiv x \in H \wedge Tr(x) \wedge \forall y(y \in x \supset Tr(y))).$$

(Here  $Tr(x)$  is short for  $\forall y(y \in x \supset \forall z(z \in y \supset z \in x))$ .)

Since  $H$  is non-extensional, we need to make use of bi-simulation in order to recapture standard information on good ordinals. Let (global) bi-similarity  $\cong$  be given by the manifestation point

$$\begin{aligned} \models_M \forall x(x \cong \equiv \mathbf{T}\mathbf{T}\exists u, v(x = \langle u, v \rangle \wedge \forall w(w \in u \supset \\ \exists z(z \in v \wedge \langle w, z \rangle \in \cong))) \wedge \forall w(w \in v \supset \exists z(z \in u \wedge \langle w, z \rangle \in \cong))). \end{aligned}$$

Instead of  $\langle u, v \rangle \in \cong$  we write  $u \cong v$ . Define  $a \eta b =_D \exists c(a \cong c \wedge c \in b)$ . For relations between good ordinals one writes  $\alpha \prec \beta$  instead of  $a \eta \beta$ . We can now establish e.g. that if  $\Vdash_M \alpha, \beta \in Ord$  then  $\Vdash_M \alpha \prec \beta \vee \alpha \cong \beta \vee \alpha \succ \beta$ . Further principles of ordinal arithmetic similarly depend upon the condition that the ordinals are good.

The Burali-Forti paradox is resolved in the context of librationism because  $Ord$  itself is paradoxical and so not good; we e.g. have that  $\Vdash_m \hat{a} \in Ord$ .

## §12 Closures of Mathematical Phenomena

Since librationism is a contentual system which is negation complete, it may seem somewhat malapropos to compare its strength to formal systems such as Peano arithmetic and  $\Pi_1^1 - CA$ . E.g., as librationism accommodates the omega-rule, as a special instance of the Z-rule, it has no proof theoretic ordinal. Let the *information strength* of a system  $S$  be the set of sentences  $\Sigma$  such that  $\sigma \in \Sigma$  iff  $S$  entails or contains that  $\sigma$  and does not entail or contain that  $\sim \sigma$ . A trivial system which entails everything will then have no information strength, and also a system  $S$  which has no theorems will have zero information strength. Some systems will have incomparable information strengths. Maximal consistent sets of sentences have maximal information strength; however, such sets in the language of librationism cannot preserve sufficiently many of our pretheoretic intuitions concerning abstraction and truth. Librationism is geared to have as much information strength as possible while also providing a considered account of theoremhood and paradoxicality which preserves as many of our pretheoretical intuitions as possible. Its tremendous information strength follows from its being arithmetically complete and extending Weak König Lemma which is equivalent over  $RCA_0$  to Gödel's Completeness Theorem. It follows from this that any consistent first order formal theory has a countable model in librationism. So librationism has greater information value than any consistent first order formal theory. But such information strength is immensely obscured, and will also predominately depend upon delicate issues of interpretation. It is therefore of interest to consider *revealed* information strength as various formal theories may gain a librationist justification in that manner. Our discussion in §10 has established that the revealed information strength of librationism is greater than that of  $\Pi_1^1 - CA_0$  plus ordinary Bar-Induction. Recent work of the author suggests, as is to be expected, that the revealed information strength of librationism will be significantly increased. Notice that according to librationism no mathematical problem is absolutely unsolvable.

## §13 Concluding Words

We come to closure on something, e.g. grief, when we have come to accept the reason for our grief and manage to live with the grief without being paralyzed or made powerless by it. In this way, it seems to the author that librationism offers a closure in our dealing with paradoxes. We are in the librationist framework able to accommodate paradoxes and accept their existence without giving up on highly important inventory of our intellectual heritage, such as classical logical theorems and means to sustain advanced reasoning. As pointed out in our motivation of librationism's name, the theory on offer here deals with paradoxical phenomena in a way which offers justice to the shifts in perspectives which are

involved in our reasoning in such contexts. Importantly, librationism achieves this without falling prey to revenge paradoxicalities.

Closures in mathematics abound in another sense. Many sets, or sorts in our context, may be regarded as e.g. the least sort containing this and closed under that. Our manifestation-points offer another way of obtaining similar, or related, closure. Other constructions available may be regarded in similar ways. From what we have pointed out, librationism offers an alternative foundation of mathematics with great potential for closure.

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