Considerations Contra Cantorianism

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With the avoidance of Russell's paradox and its cognates as one paramount motivation, and the avoidance of ungrounded mathematical objects as another, the twentieth century from early on saw the initiation of various foundational theories which altogether avoided an invocation of infinite power sets. This is famously the case in the predicativist tradition going back to Herman Weyl's *Das Kontinuum*, and further investigated later principally by Solomon Feferman, but also by others. This was clearly also an important motivational aspect of the perhaps less rigorously formulated original intent of Luitzen Brouwer's intuitionist program, and it is presently manifest much more precisely within parts of the intuitionist tradition in that Per Martin Löf's constructive Type Theory lacks an analogue of the infinitary power-set operation, as does Peter Aczel's constructive set theory CZF.

The reverse mathematics program initiated by Harvey Friedman seemed to have established that only a very small fragment of Second order Arithmetic suffices for ordinary classical mathematics, i.e. mathematics which is not concerned with purely set-theoretic issues or those of its "higher reaches". The system we propose, *(minimalistic) librationism*[1], has mathematical strength beyond the fully impredicative system $\prod_1^1 -\mathrm{CA}_0$ plus the Bar-rule in a sense to be made more precise below. As librationism also sheds important light on other important phenomena, as the paradoxes, it seems that it may advantageously serve as a foundation for the mathematics needed for science, semantics and philosophical speculation.

I point out first in this separate paragraph that the last sentence in the previous paragraph may seem dated on account of other important very recent work by Friedman which shows that certain sentences of quite concrete mathematics in his Boolean Relation Theory needs something like ZFC plus the existence of all inaccessible strong Mahlo cardinals of finite order to be settled correctly. The interested reader should consult his home page for the book draft on Boolean Relation Theory. But Friedman has there also shown that what suffices to prove *the exotic cases* is ACA′ (which is often formulated as ACA_0 + the existence of the Turing jump for all subsets of ω) plus 1-Con(SMAH). The latter is an *arithmetical* schema which to the effect states the 1-consistency of Friedman's SMAH, i.e. that all $\sum_1^0$ sentences provable in ZF+ $\bigwedge_{n=0}^{n=\infty}$(there is a strong $\mathrm{n}-$Mahlo cardinal) are also true. As $\prod_1^1 -\mathrm{CA}_0$ plus the Bar-rule much exceeds ACA′, which is even weaker than ACA (=ACA(0) plus full induction), and our *semi formal* system librationism, for reasons that become clear below, settles all arithmetical sentences, we know *a priori* at a meta level that Friedman's exotic cases have librationist resolutions. How such resolutions may come about, and what they amount to, will remain to be seen. To find the resolutions will most likely not be a trivial exercise, perhaps quite on the contrary; it must, as is the standard procedure in isolating *partial* axiomatic and inferential principles of librationism, take its recourse to the

[1] I have coined the term "librationism" from the term "libration" which is used for certain oscillating phenomena e.g. in astronomy. This seems a useful name for our system as it is not taken, it reminds of the peculiar shift in perspectives which are involved in the theory's treatment of paradoxical phenomena and is also close in spelling and pronunciation to the term "liberalism" which I have used in some earlier lectures and publications. The latter term suggested itself because of the emancipatory feature that all set terms are allowed and dealt with in a comprehensive and, it is hoped, justified and edifying manner.

semi inductive semantics which we describe below. The point here is that we, from the librationist outlook, ultimately do not need to buy into ZFC-like points of view, or, more generally, opinions that include infinite power sets, in order to account even for Friedman's new exotic incompleteness phenomena.

There are two important traditions which we shall recall briefly in order to situate librationism. (1) Since the work of Saul Kripke and others there has been an explosive interest in self-referential truth. An important strand in that development was initiated independently by Hans Herzberger and Anil Gupta, and now serves as a background for semi inductive and revisionary style semantics. For us, the system LES introduced at the end of §69 in Andrea Cantini's important monograph has been influential. LES is an axiomatic theory of truth and abstraction justified by a semi inductive type semantics, and which respects classical logic. Librationism properly extends LES, and also respects classical logic. (2) Starting with Stanislaw Jaskowski's non-adjunctive system and the work of Newton da Costa, many formal calculi weaker than classical logic have been proposed in the paraconsistent tradition so as to allow naïve comprehension. But it cannot be seen that these views offer a satisfactory analysis of how mathematical objects are engendered by the accompanying naïve comprehension principle. Often e.g. ZFC is merely taken for granted as an inside paradox free theory, and paradoxical phenomena are just assumed to be inside an outside shell. One may challenge that these views thus ride piggy-back on classical views without offering the analysis we are in need of and which should be of our interest. Still, as the acute reader will see, librationism shares some important features with paraconsistent approaches.

Let us for this exposition simplify and take our language to have parentheses for punctuation, infinitely many variables v_0, v_1, $v_{2\ldots}$, connectives $\wedge$, $\vee$, $\equiv$, $\supset$, ~, quantifiers $\forall$, $\exists$, epsilon $\in$, the truth operator $\mathbf{T}$ and the set builder {:}. Formation rules and inter-definability relations are as would be expected by my audience, with the addition that A is a formula only if $\mathbf{T}$A is a formula. I point out that set brackets are not eliminable as in extensional set theories. Semantically, we rely on a semi-inductive process on ordinals. In our context we need no "boot-strapping policy". $\{v_i{:}A\}$ is a set-constant if A has at most v_i free. Fix an enumeration e(0), e(1), … of all set-constants. [A] is the Gödel-number of the formula A for a given coding. X is a function from ordinals to subsets of natural numbers and $\vDash$ a relation between such subsets and formulas as given by the double recursion: For any ordinal α,

(1) $X(\alpha)=\{[A]:\exists\beta(\beta\prec\alpha\&\forall\gamma(\beta\preceq\gamma\prec\alpha\Rightarrow X(\gamma)\vDash A))\}$
(2) $X(\alpha)\vDash\mathbf{T}A$ iff $[A]\in X(\alpha)$
(3) $X(\alpha)\vDash a\in\{v_i{:}A\}$ iff $X(\alpha)\vDash\mathbf{T}A(a/v_i)$
(4) $X(\alpha)\vDash A\wedge B$ iff $X(\alpha)\vDash A$ *and* $X(\alpha)\vDash B$
(5) $X(\alpha)\vDash{\sim}A$ iff *not* $X(\alpha)\vDash A$
(6) If a=e(i) then $X(\alpha)\vDash A(v_i)$ iff $X(\alpha)\vDash A(a)$
(7) $X(\alpha)\vDash\forall v_iA$ iff for all variables (names!) v_j, $X(\alpha)\vDash A(v_j/v_i)$.

By adapting results going back to Herzberger there will be a stabilization ordinal κ so that $X(\kappa)\vDash\mathbf{T}A$ iff $\forall\gamma(\gamma\succeq\kappa\Rightarrow X(\gamma)\vDash A)$. Notice that $X(\kappa)\vDash A$ or $X(\kappa)\vDash{\sim}A$, and also that by (7) the isolated *minimalist*[2] model, and thence the librationist system, is closed under the non-

[2] We think of the model and resulting librationist system as *minimalist* because the truth operator $\mathbf{T}$ has the empty extension at ordinal zero, i.e. $X(\emptyset)\vDash{\sim}\mathbf{T}A$ for all formulas A.

constructive Z-rule: If $X(\kappa)\models A(u)$ for all variables (or for all terms) u, then $X(\kappa)\models\forall xA(x)$. We at this point make a **crucial** shift in metalogical attention to $\{A: X(\kappa)\models\sim\mathbf{T}\sim A\}$ as our designated model (modulo the enumeration e invoked in the semantical set up), and define:

$$\Vdash A =_D X(\kappa)\models\sim\mathbf{T}\sim A.$$

By induced principles at $X(\kappa)$ we have $X(\kappa)\models\mathbf{T}A$ iff *not* $X(\kappa)\models\sim\mathbf{T}A$, so we define A to be a *maxim*, $\Vdash_M A$, iff $\Vdash A$ and *not* $\Vdash\sim A$. *Minors* are given by $\Vdash_m A$ iff $\Vdash A$ and $\Vdash\sim A$. $r\in r$ for $r=\{x:x\notin x\}$ is e.g. a minor. Theorems of classical logic are examples of maxims. It can be shown that identity can be introduced à la Leibniz. Induced inferential principles are unfamiliar, and one should not expect that these can be effectively circumscribed. We have isolated more than ten salient ones, but it is not of relevance to describe these or the isolated partial axiomatic principles in a short presentation as here; instead it is left to the reader to ponder on these matters, or else s/he is directed to other work pointed to or by the author. It is noteworthy that the traditional inferential schema modus ponens holds for $\Vdash_M$ (so we call this adjusted inferential schema *modus maximus* in the librationist framework) but not for $\Vdash_m$, as we do e.g. **not** have that the conjunction of $\Vdash_m A$ and $\Vdash_m A\supset\bot$ entails that $\Vdash_m\bot$. In consequence, librationism may be understood as a non-adjunctive system. All induced principles for truth and set-theoretic abstraction are, or so I argue, incontrovertible. One should, I again emphasize, appreciate that librationism is a semi-formal system, or framework, so that we will have that $\Vdash A$ or $\Vdash\sim A$ and that it is also closed under the Z-rule described above.

For the reader's benefit, I list some axiomatic and inferential principles of librationism in order to have a partial description (it is understood that all generalizations of instances of the following schemas are axioms, so that generalization is not a primitive inference rule):

L1M	$A\supset(B\supset A)$
L2M	$(A\supset(B\supset C))\supset((A\supset B)\supset(A\supset C))$
L3M	$(\sim B\supset\sim A)\supset(A\supset B)$
L4M	$A\supset\forall xA$, provided x is not free in A.
L5M	$\forall x(A\supset B)\supset(\forall xA\supset\forall xB)$
L6M	$\forall xA\supset A(t/x)$, if t is substitutable for x in A.

LO1M	$\mathbf{T}(A\supset B)\supset(\mathbf{T}A\supset\mathbf{T}B)$
LO2M	$\mathbf{T}A\supset\sim\mathbf{T}\sim A$
LO3M	$\mathbf{T}B\vee\mathbf{T}\sim B\vee(\mathbf{T}\sim\mathbf{T}\sim A\supset\mathbf{T}A)$
LO4M	$\mathbf{T}B\vee\mathbf{T}\sim B\vee(\mathbf{T}A\supset\mathbf{T}\mathbf{T}A)$
LO5M	$\mathbf{T}(\mathbf{T}A\supset A)\supset(\mathbf{T}A\vee\mathbf{T}\sim A)$
LO6M	$\exists x\mathbf{T}A\supset\mathbf{T}\exists xA$
LO7M	$\mathbf{T}\forall xA\supset\forall x\mathbf{T}A$
LO8m	$\mathbf{T}A\supset A$

LO9m $A\supset\mathbf{T}A$

LO10m $\forall x\mathbf{T}A\supset\mathbf{T}\forall xA$

LO11m $\mathbf{T}\exists xA\supset\exists x\mathbf{T}A$

Schemas that are maxims have a capital "M", while minor schemas, i.e. those that have minor instances, have a minuscule "m" at the end of its appellation as given here.

We next point out the librationist comprehension principle:

LCM $\forall x(x\in\{y:A\}\equiv\mathbf{T}A(x/y))$, if x is substitutable for y in A.

We proceed to provide some salient inferential principles that we can show hold in librationism:

IR1: If $\Vdash_M A$ and $\Vdash_M (A\supset B$ then $\Vdash_M B$ ("modus maximus")

IR2: If $\Vdash_m A$ and $\Vdash_M (A\supset B$ then $\Vdash B$ ("modus subiunctio")

IR3: If $\Vdash_M A$ and $\Vdash_m (A\supset B$ then $\Vdash_m B$ ("modus antecedentiae")

IR4: If $\Vdash_M A$ then $\Vdash_M \mathbf{T}A$ ("modus ascent maximus")

IR5: If $\Vdash_m A$ then $\Vdash_m \mathbf{T}A$ ("modus ascent minor")

IR6: If $\Vdash_M \mathbf{T}A$ then $\Vdash_M A$ ("modus descent maximus")

IR7: If $\Vdash_m \mathbf{T}A$ then $\Vdash_m A$ ("modus descent minor")

IR8: If $\Vdash_M {\sim}\mathbf{T}{\sim}A$ then $\Vdash_M \mathbf{T}A$ ("modus scandent maximus")

IR9: If $\Vdash_m {\sim}\mathbf{T}{\sim}A$ then $\Vdash_m \mathbf{T}A$ ("modus scandent minor")

IR10: If $\Vdash_M \forall x\mathbf{T}A$ then $\Vdash_M \mathbf{T}\forall xA$ ("modus Barcan")

IR11: If $\Vdash \mathbf{T}\exists xA$ then $\Vdash \exists x\mathbf{T}A$ ("modus attestor")

(Also: If $\Vdash_m \mathbf{T}\exists xA$ then $\Vdash_m \exists x\mathbf{T}A$)

IR12: If $\Vdash_m A$ and $\Vdash_m B$ then $\Vdash_m {\sim}\mathbf{T}{\sim}A\wedge{\sim}\mathbf{T}{\sim}B$ ("modus minor")

In this short presentation, the semantical verification of the axiomatic and inferential principles is left as an exercise to the reader. I also leave it as a problem to show that modus Barcan and modus attestor cannot be strengthened as one should intuitively expect.

We can by means of a fixed point construction going back to Andrea Cantini and Albert Visser isolate what we call *manifestation points*: If $A(x,y)$ is a formula with the free variables shown we can find a term h^A such that $\Vdash_M \forall z(z\in h^A\equiv\mathbf{TT}A(z,h^A))$. To prove this, take ordered pairs, e.g. à la Kuratowski, and suppose $d=\{\langle x,g\rangle:A(x,\{u:\langle u,g\rangle\in g\})\}$ and $h^A=\{x:\langle x,d\rangle\in d\}$. Define $KIND(x)=_D\forall y(\mathbf{T}y\in x\vee\mathbf{T}y\notin x)$. Taking H as the manifestation point of $KIND(x)\wedge x\subset y$, we have that $\Vdash_M a\in H$ iff $\Vdash_M KIND(a)\wedge a\subset H$. We can then show that we for $\mathbb{N}=\omega=\{x:\forall y(\emptyset\in y\wedge\forall z(z\in y\supset z'\in y)\supset x\in y)\}$, with $z'=\{w:w=z\vee w\in z\}$, have that $\Vdash_M \mathbb{N}\in H$. From what we pointed out in the second paragraph, we indeed have *fabulously* much more as regards maxims relative to H. We may intuitively think of H as the set of hereditarily non-paradoxical and, on account of our minimalist policy, wellfounded sets.

Let $\Vdash_M$KIND(f) and f a surjection from $\mathbb{N}$ to V={x:x=x}, and consider Cantor's s={x:x∈N∧x∉f(x)}, i.e. s={x:x∈$\mathbb{N}$∧∃y(<x,y>∈f∧x∉y)}. Since $\Vdash_M$KIND(f) and function(f) and $\Vdash_M 8\in\mathbb{N}$ this reduces to $\Vdash_M 8\in s\equiv\mathbf{T}(8\notin s)$. In the non-adjunctive framework of librationism it turns out that we only have the schemas $\Vdash_m A\supset\mathbf{T}A$ and $\Vdash_m \mathbf{T}A\supset A$ in full generality, and not always the conjunction of instances as minor schemas such as these have minor instances. This and further schemas and inferential principles only license the conclusion that $\Vdash 8\in s$ and $\Vdash 8\notin s$, i.e. $\Vdash_m 8\in s$. So s turns out to be paradoxical, just as Russell's set. We underline that <u>Cantor's arguments are perfectly valid</u>, but in librationism we find that the appropriate assumption to be discarded in the reductio is the camouflaged assumption that s is non-paradoxical. Since f was taken as a KIND surjection from $\mathbb{N}$ to V, it is a fortiori onto the power set of $\mathbb{N}$.

Other Cantorian arguments for higher infinities are dislodged for quite analogous or parallel reasons. I here leave residual matters for the reader's meditation. It turns out as a consequence that the set of real numbers, taken e.g. by Dedekind cuts, is paradoxical, and so not *listable*. By this we mean that there is no non-paradoxical function from $\mathbb{N}$ which has exactly non-paradoxical real numbers as values; there are, I mention *en passant*, paradoxical Dedekind cuts or real numbers as e.g. $\{x\in Q:(x<^Q 0\wedge r\in r)\vee(x<^Q 1\wedge r\notin r)\}$ where Q is the set of rational numbers, $<^Q$ its standard ordering and r is Russell's paradoxical set {x:x∉x}. Taking the real numbers by e.g. Cauchy-sequences does of course not alter the essential dialectics of the situation. Our observations are such that we are justified in thinking that librationism *allows*[3] a non-paradoxical function from $\mathbb{N}$ onto V, and so also onto the set of real numbers. There are thus according to our librationist point of view no more real numbers than there are natural numbers.

It is noteworthy in all of this to realize that virtually all power sets are paradoxical in librationism. I leave this as an exercise. Hint: Consider the manifestation point å such that $\Vdash_M \forall z(z\in å\equiv\mathbf{TT}z\notin å)$ and evaluate å's membership with respect to the power sets of any set m such that it is not a maxim that m is coextensional with V={x:x=x}.

We first pointed out that infinite power sets are not needed in order for the foundation of ordinary mathematics. Our results show that even if they are accommodated as in librationism, they do not here support Cantor's conclusion as paradoxical phenomena are accounted for in such a way as to forestall one of Cantor's essential though camouflaged assumptions. In this, we take librationism to *somehow* confirm the predicativist suspicions towards the use of power sets.

Cantorianism, as I understand it here, includes the point of view that there is at least one infinite cardinality larger than $\aleph_0$. The foregoing serves to challenge this tenet of Cantorianism and mathematical orthodoxy. There are further philosophical considerations which pull in the same direction that I in conclusion of this note just briefly mention: (1)

[3] We have of course not by this shown that librationism as so far developed ***has*** such a surjection from $\mathbb{N}$ to V. I have shown in work that goes beyond what is presented here that we may enlarge the librationist language with a new set constant Є and have its denotatum serve as a bijection from $\mathbb{N}$ to V by just slightly altering the semantical set up. Furthermore, and importantly, we may extend the librationist language with set constants Є *and* T and have the denotatum of the former serve as a bijection as pointed out while the denotatum of the latter serves as a truth *predicate* on Gödel numbers of formulas; the semantical alterations needed are minor. The accommodation of a truth predicate depends upon having the universe denumerable. It is of strong philosophical interest to include a truth predicate because it helps us provide what I think is a favorable librationist account of the Liar's paradox and related self referential semantical phenomena. Mathematically, a (non-paradoxical) bijection Є is very useful as it provides important choice principles in many settings where desirable.

Ontological economy suggests that Cantorianism is ontologically extravagant. (2) Cantorianism makes it impossible to presuppose an overarching philosophical meta-language. (3) If Cantorianism were true mathematical reality would need to exhibit disturbing essential traits of metaphysical incompleteness. (4) If we presuppose Cantorianism the question "How many objects are there?" becomes meaningless and cannot be answered; but it seems perfectly meaningful and in librationism has the answer that there are precisely countably infinitely many objects.

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