Accounting for the income of non-renewable resources
Four essays on new theoretical approaches

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Introduction

Non-renewable resources\(^1\) are widely extracted and used in today’s global economy. On the one hand, the use of these resources generates huge cash flows that can be used for current consumption. On the other hand, the extraction of these resources implies less wealth of these natural resources in the future. To compensate for this wealth decrease, we have to accumulate other kinds of real wealth, like man-made capital, human capital, and renewable resources. The wealth accumulation can be achieved by activities of investments. Reasonable financial support for the investments should come from cash flows related to non-renewable resources. Hence, cash flows related to non-renewable resources should be divided for two basic purposes: current consumption and savings to support activities of investments. If the cash flows are used too much for current consumption, then we may suffer by being impoverished in the future. If we save too much at current time, we may sacrifice our current well-being. The trade-off between current consumption and savings invokes a fundamental question: how much of the cash flows can we use for current consumption without impoverishing the future?

Following the literature line from Hicks (1946, Chapter 14) via Samuelson (1961) to Sefton and Weale (2006)\(^2\), in the first essay of this dissertation, Geir Asheim and I develop a theory of sectoral income. The theory can be applied

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\(^1\)A non-renewable resource has a finite stock. It can not be regenerated naturally (at least in the foreseeable future). Fossil fuels like crude oil and natural gas are typical examples.

\(^2\)Hicks (1946, p. 172) defines “...a man’s income as the maximum value which he can consume during a week, and still expect to be as well off at the end of the week as he was at the beginning.” By such a definition, Hicks (1946) intends to make income an indicator to give people an amount "which they can consume without impoverishing themselves." This definition of income has been discussed extensively in the literature at an aggregate level. A recent contribution to the literature is provided by Sefton and Weale (2006). A survey of relevant literature can be found in Section 2 of the first essay titled "Sectoral income."
to estimate income at a *sectoral* level. By this notion of sectoral income, we can answer the fundamental question concerning the division of the cash flows generated by non-renewable resources into consumption and savings. In particular, by applying the new theory in this dissertation, I provide a new method for estimating *real income* generated by non-renewable resources.

The new method has at least two advantages when compared with the wealth-based method, which is a commonly applied method in practical resource accounting. By the wealth-based method, sectoral income is associated with the interest on the sector’s wealth, where the wealth is estimated as the present value of cash flows generated from the sector (see, e.g., Aslaksen et al., 1990, and Brekke, 1997, Sections II.C and IV). However, as pointed out by Hicks (1946, p. 174), income as interest on wealth can not serve as an indicator of prudent behavior if the real interest rate is expected to change\(^3\). An advantage of the new method is that it is still plausible even if real interest rate is expected to change over time. Another advantage is that the new method provides an alternative way to estimate net investments of a *sector*. This can be applied when information on sectoral capital stocks is not available as it may be difficult to observe, measure, or evaluate the change of capital stocks.

This dissertation consists of four essays. In the first essay, Geir Asheim and I propose the theory of sectoral income. In the second essay, I apply this theory to identify and classify the income generated by a non-renewable resource. In particular, sectoral net investments is shown to coincide with the shadow value of the change in capital stocks of the sector in a general framework. Then I

\(^3\)In Section 7 of the first essay, this observation is nicely illustrated by the Dasgupta-Heal-Solow model (Dasgupta and Heal, 1974, 1979; Solow, 1974) of capital accumulation and resource depletion where the real interest rate is decreasing along a path where capital is accumulated and resource flow diminishes.
illustrate the new theory by estimating sectoral income based on real data of the Norwegian petroleum sector in the third essay. As a first step to take into account real world uncertainty, I study the concept of national income in a stochastic setting in the last essay.

1 Sectoral income

Following Hicks (1946), current income is associated with the maximum value we can consume now without being impoverished in the future. A recent contribution is Sefton and Weale (2006). They associate real income with the present value of real interest on future consumption. By their definition of real income, Sefton and Weale (2006) also show that national income can be decomposed to individual income by using information on individual consumption. Hence, the real income can be estimated to guide individuals’ prudent behavior as long as their consumption information is available.

In the essay, we go one step further by studying sectoral income, i.e. income of a sector in an economy. Consider a sector like extraction activities of crude oil and natural gas. The final purpose of the sector is to supply goods and services for individual consumption. This implies the sector contributes to individual income since individual income is related to individual consumption. By summing up all the individual income that originally generated by the sector, we can obtain a definition of sectoral income. In this way, we develop a new theory of sectoral income that provides an alternative method to estimate income of a sector.

In the theory, sectoral income is associated with the present value of real

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4The first essay has appeared in the journal of Environmental and Resource Economics (2009) 42: 65-87.
interest on future cash flows to the sector. Hence, we can estimate sectoral income directly based on future cash flows to the sector. In practical non-renewable resource accounting (eg. Brekke, 1997; SEEA, 2003), the same information on future cash flows is necessary to estimate resource wealth, which is associated with the present value of future cash flows generated by the resource.

In the essay, we decompose sectoral income to three elements. One element is current cash flow and the other two are called sectoral net investments and price change effects. Sectoral net investments are defined as the present value of the change of all future commodity flows (including both inflows and outflows) to the sector\(^5\). Price change effects represent the effects of future price change alone. By the new theory, sectoral income represents the maximum value to consume now without impoverishing us in the future for the given path of cash flows to the sector. If a share of current cash flow equal to sectoral income is used for current consumption, then the minimum value to save without impoverishing us in the future is the rest of current cash flow, which is the sum of the sectoral net investments and the price change effects.

We also illustrate the theory by two examples in the essay. One example considers functional income shares in a Cobb-Douglas Dasgupta-Heal-Solow (DHS) model (Dasgupta and Heal, 1974, 1979; Solow, 1974), where three sectors are considered: capital owners, workers, and resource owners. This example shows the application of the new theory in the setting of a general model.

Another example considers a reservoir of a resource like crude oil. This shows the application of the new theory in the setting of a partial model. In

\(^5\)The commodity flows include flows of all the goods and services used to calculate the cash flows to the sector.
the setting, cash flow at each point in time equals Hotelling rent plus Ricardian rent. By definition, income of the reservoir consists of two parts: interest on the present value of future Hotelling rent and interest on the present value of future Ricardian rent. On the other hand, by the decomposition of sectoral income, income of the reservoir equals exactly current Ricardian rent since net investments of the reservoir cancel out current Hotelling rent. Ricardian rent can be calculated solely on the basis of information available today, and no future information is required in this special example.

2 Income generated by a non-renewable resource

According to the new definition described above, sectoral income can be divided into three elements: current cash flow, sectoral net investments, and price change effects. Of the three elements, current cash flow and price change effects have clear meanings as shown by their names. On the contrary, it is not obvious why the remaining element, the present value of the change of future commodity flows to the sector, can be called sectoral net investments. Hence, in the second essay, I show that sectoral net investments coincide with the shadow value of the change in all the sectoral capital stocks for a competitive sector. This justifies the terminology we have adopted in this component.

I then apply the results to a non-renewable resource to deepen our understanding on income and net investments arising from the resource. By applying a general form of production function, I show that sectoral net investments in extraction of the resource coincide with the shadow value of the resource stock if man-made capital keeps constant and technology does not change over time.
In particular, the net investments coincide with the Hotelling rent of the resource under certain assumptions.

The essay also compares the sectoral income method with other practical accounting methods: net price method, user cost method, and net present value method. If the real interest rate is constant over time, the resource income by our sectoral income method coincides exactly with other methods. In addition, the value of the resource depletion defined in SNA (1993) and SEEA (2003) can be divided in three elements: the interest rate change effects, net investments, and the price change effects. Hence, only if the real interest rate is constant over time, the SNA/SEEA definition of the value of resource depletion provides a plausible indicator of required savings, i.e. the sum of sectoral net investments and price change effects.

3 Net investments and price change effects on income arising from Norwegian petroleum sector

In this essay, I use real data of the Norwegian petroleum sector to illustrate the theory of sectoral income. In the beginning of the essay, the theory of sectoral income is expressed in discrete time to facilitate the application of real data. Then, I adopt a reference scenario for the Norwegian petroleum sector. The scenario provides a plausible development path of the petroleum

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6 A short version of the essay was presented at the 28th USAEE/IAEE North American Conference.

7 This reference scenario is designed and adopted by Statistics Norway in a computable general equilibrium model (Heide et al., 2004).
extraction in Norway from 2006 to 2050. Based on the scenario, real income and its components are estimated for the sector for each year between 2006 and 2050.

Furthermore, I analyze the sensitivity of the sectoral income and its components with respect to real interest rate and the change of price forecast. The focus is on the beginning year 2006. If real interest rate is chosen in a reasonable range, price change effects are quite stable for the Norwegian petroleum sector. Both sectoral income and net investments become larger with increasing real interest rates. On the other hand, the change of price forecast may influence dramatically the price change effects, which indicates the importance of the price change effects. In all the cases, net investments are always negative and considerable. In the extreme case when the petroleum income is higher than current cash flow\(^8\), the negative net investments are still there due to high positive price change effects. These results show that net investments of the sector differ from price change effects. It can be misleading if any one of them is ignored.

4 Stochastic national income

Uncertainty is an unavoidable issue for the estimation of real income generated by non-renewable resources. As the first step to explore the issue, I develop a stochastic version of the definition of real national income in the last essay. This will, hopefully, serve as a basis for developing a stochastic version of the theory of sectoral income.

As shown above, national income is defined as the present value of real

\(^8\)This case should be common at the beginning stage of the extraction activities since cash flows to the sector can be quite small or even negative.
interest on future national consumption in the deterministic setting. However, at current time, future national consumption is always unknown and far from deterministic. Then it is necessary to define national income in an uncertainty setting.

The essay studies the concept of real national income in a stochastic setting. If prices of consumption and net investment are properly defined, then stochastic national income defined here has plausible interpretation from both a welfare and productive respective. Furthermore, stochastic real national income can be interpreted as the expectation of present value of real interests on future national consumption if a real consumption interest rate is properly defined.

Then I use a stochastic one-good model to illustrate the theory. The model shows that whether real national income equals current production or not depends on the future uncertainty and the curvature of utility function. The curvature of utility function has dramatic effects on real national income under uncertainty. Higher production does not necessarily imply higher real income if it is associated with more uncertainty.

References


Sectoral income*

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Abstract

What is the income of each sector of an economy? E.g., in the case of a country endowed with petroleum resources, what is the income of its petroleum sector? Here we present a definition of sectoral income, which is compatible with an important line of theoretical literature on comprehensive national accounting. We do so first by splitting national income into individual income and then defining sectoral income by considering the contributions to individual income that the sectors give rise to.

Keywords and Phrases: Sectoral income, comprehensive national accounting.

JEL Classification Numbers: C43, D60, O47.

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1 Introduction

What is the income of each sector of an economy? E.g., in the case of a country endowed with petroleum resources, what is the income of its petroleum sector?

In practical applications, sectoral income has often been measured by wealth-based measures, whereby the present value of the cash flow from a sector is estimated, and its income is equated with the interest on the sector’s wealth determined in this manner (see, e.g., Aslaksen et al., 1990, and Brekke, 1997, Sections II.C and IV).

In contrast, a line of theoretical literature—from Hicks (1946, Chapter 14) via Samuelson (1961) to Sefton and Weale (2006)—has taken a quite different route by associating income with the present value of real interest on future consumption and savings with the present value of future consumption changes.

In this paper we extend this line of theoretical literature on comprehensive national accounting by presenting a definition of sectoral income which is consistent with such consumption-based income definitions. To be precise, we define sectoral income by keeping track of how each sector contributes to individual income (in the sense of Sefton and Weale, 2006, Definition 2).

We start in Section 2 by presenting a short survey of relevant literature and the income concepts presented in these contributions. Then, in Sections 3—5 we present definitions of real income at the national and individual level in line with Sefton and Weale (2006), while generalizing their welfare results slightly. Our definition of sectoral income is presented and analyzed in Section 6, before illustrating this concept both in the setting of a general model (in Sections 7) and in the setting of a partial model (in Sections 8). Throughout we seek to derive expressions for sectoral income that can be useful in practical applications.

There are two appendices, one contains welfare results referred to in the main part of our paper, while the other analyzes alternative wealth-based concepts of sectoral income.
2 What is income?

At a national level, in particularly in the context of a closed economy with a stationary technology, income can be derived from net national product, measuring the value of the flows of goods and services that are produced by the productive assets of society. National income derived in this way has also welfare significance, as established by Weitzman (1976) and later references (e.g., Aronsson et al., 1997, 2004; Asheim and Weitzman, 2001). At a sectoral level, it is however hard to determine what a sector’s “net product” is, since much of the return on the sector’s assets may derive from expected capital gains.\(^1\) In particular, the remaining deposits of a non-renewable resource is not productive as a stock, but yields its owners positive returns by being moved closer to the time of depletion. This motivates a brief survey of relevant literature on income concepts.

Income in the tradition of Fisher (1906) and Lindahl (1933, Section II) is associated with interest on wealth, where wealth is the present value of future consumption. If, at each point in time, national consumption equals the sum of the cash flows from the different sectors of the economy, this definition allows national income to be split into sectoral income so that sectoral income summed over all sectors adds up to national income.

Hicks (1946), in Chapter 14 of Value and Capital, suggests that “the practical purpose of income is to serve as a guide for prudent conduct” by giving “people an indication of the amount which they can consume without impoverish themselves” (both quotes from Hicks, 1946, p. 172).

Hicks (1946, p. 174) points out that income as interest on wealth is not an indicator of prudent behavior if the real interest rate is expected to change. This observation is nicely illustrated by the Dasgupta-Heal-Solow model (Dasgupta and Heal, 1974, 1979; Solow, 1974) of capital accumulation and resource depletion—

\(^1\)See, however, an interesting attempt to do so in Sefton and Weale (2006, Section 6.2).
which we will return to in Section 7—where the real interest rate is decreasing along a path where capital is accumulated and resource flow diminishes. In this model, income as interest on wealth exceeds both net national product and consumption along an efficient path with constant consumption (see Appendix B for details). Hence, in this setting, the consumers of the economy would impoverish themselves if they were consuming the interest on their wealth.

If we instead use Hicks’s (1946) suggestion to obtain alternative income concepts, then we must operationalize what is meant by “the amount which they can consume without impoverish themselves”. Hicks (1946) himself offers the following operationalization, referring to the corresponding concept as “Income No. 3”:

“Income No. 3 must be defined as the maximum amount of money which the individual can spend this week, and still be able to spend the same amount in real terms in each ensuing week” (Hicks, 1946, p. 174). “The standard stream corresponding to Income No. 3 is constant in real terms . . . . We ask . . . . how much he would be receiving if he were getting a standard stream of the same present value as his actual expected receipts. This amount is his income. (Hicks, 1946, p. 184)”

Hence, income is associated with the “stationary equivalent of future consumption” (Weitzman, 1976, p. 160).

In an economy where wellbeing depends on a single consumption good, this concept of income can be defined as the constant level of consumption with the same present value as the actual future stream of consumption. Such wealth equivalent income can be determined at both a national and sectoral level in such way that sectoral income summed over all sectors adds up to national income (see Appendix B). Moreover, the concept is designed to be an indicator of prudent behavior (although wealth equivalent income is only hypothetically sustainable if interest rates are changed when consumption is transformed into a constant and efficient path).
Unfortunately, as pointed out by Asheim (1997) and Sefton and Weale (2006, Section 3.1.2) (and discussed in Appendix B), such wealth equivalent income at the national level does not equal net national product, even in a closed economy with a stationary technology, unless the interest rate is constant (which is the case analyzed by Weitzman, 1976) or the consumption level is constant (in which case Hartwick’s rule\(^2\) implies that net national product equals this constant level and, thus, the result follows). Moreover, this concept is hard to generalize to the empirically relevant case of multiple consumption goods, since determining an amount constant in real terms leads to an indexing problem if relative consumption prices are changing.

However, Hicks’s “amount which they can consume without impoverish themselves” can be interpreted in an alternative manner. Hicks (1946, p. 172) writes that “it seems that we ought to define a man’s income as the maximum value which he can consume during a week, and still expect to be as well off at the end of the week as he was at the beginning.” One attractive possibility, suggested by Pemberton and Ulph (2001) and Sefton and Weale (2006), is to associate “as well off” with the level of dynamic welfare.

It is an insight first pointed out by Samuelson (1961, pp. 51–52) that the present value of future consumption changes measures welfare improvement in a market economy following an optimal path. This gives a welfare foundation for interpreting the present value of future consumption changes as national savings. Adding current consumption to this notion of savings (measured in the same numeraire) leads to a concept of national income with nice properties:

(1) It follows from Samuelson’s insight that such a concept of national income is an indicator of prudent behavior, since the present value of future consumption changes is positive—and thus, dynamic welfare improves—if and only if consumption is smaller than national income.

\(^2\)Cf. Hartwick (1977) and Dixit, Hammond and Hoel (1980).
(2) It follows through integration by parts that such a concept of national income can be expressed as the present value of real interest on future national consumption.

(3) It follows from the analysis of Sefton and Weale (1996) and Weitzman (2003, Chapter 6) that such a concept of national income equals net national product in a closed economy with a stationary technology.

In Sections 3 and 4, and backed up by the results of Appendix A, we establish formally properties (1)–(3) under assumptions more general than those imposed by Sefton and Weale (2006); in particular, we do not assume that a discounted utilitarian welfare function is maximized, and we do not require that the technology satisfies constant-returns-to-scale. In Sections 5 and 6 we then turn to the up-to-now unresolved question of how to split this concept of national income into sectoral income in such way that sectoral income summed over all sectors adds up to national income. We do so first, in Section 5, by splitting national income into individual income, building on analysis presented by Sefton and Weale (2006), and then, in Section 6, by defining sectoral income by considering the contributions to individual income that the sectors give rise to. Throughout (and in line with the analysis of Sefton and Weale, 2006), consumer price indices play a central and natural role when turning nominal into real prices.

3 Defining national income

Consider a national economy, where \( \mathbf{c} \) is a comprehensive vector of consumption flows, implying that all determinants of current well-being are included in \( \mathbf{c} \). Let \( \{c(t)\}_{t=0}^{\infty} \) be the path of consumption flows in this economy, and let \( \{p_{c(t)}\}_{t=0}^{\infty} \) be the corresponding path of market (or calculated) present value prices of consumption. The term “present value” reflects that discounting is taken care of by the prices. In particular, if relative consumption prices are constant throughout and there is
constant real interest rate $R$, then it holds that $p_c(t) = e^{-Rt}p_c(0)$. However, we will allow for non-constant relative consumption prices and will return to the question of how to determine real interest rates from $\{p_c(t)\}_{t=0}^{\infty}$ in this more general case.

Differentiation of $p_c(t)c(t)$ yields

$$\frac{d}{dt}(p_c(t)c(t)) = \dot{p}_c(t)c(t) + p_c(t)\dot{c}(t).$$

Integrating on both sides under the assumption that $p_c(\tau)c(\tau) \to 0$ as $\tau \to \infty$, leads to the following equation:

$$-p_c(t)c(t) = \int_{t}^{\infty} \dot{p}_c(\tau)c(\tau)d\tau + \int_{t}^{\infty} p_c(\tau)\dot{c}(\tau)d\tau.$$  

By rearranging this equality we obtain

$$\int_{t}^{\infty} \left( -p_c(\tau) \right)c(\tau)d\tau = p_c(t)c(t) + \int_{t}^{\infty} p_c(\tau)\dot{c}(\tau)d\tau. \quad (1)$$

Here, we will interpret the l.h.s. as national income at time $t$ and the second term on the r.h.s. as national savings at time $t$. As we will argue next, these interpretations can be supported in both a welfare and a productive perspective.

In line with Samuelson (1961, pp. 51–52), one can argue that $\int_{t}^{\infty} p_c(\tau)\dot{c}(\tau)d\tau$ measures welfare improvement in a market economy following an optimal path. A precise and more general statement of this result is proven in Appendix A through Proposition 4. In particular, we need not assume that the dynamic welfare is discounted utilitarian. Moreover, by allowing for the possibility that the prices are calculated, we need not assume that the economy implements a welfare maximizing path of consumption flows through an intertemporal market equilibrium.

Thus, Proposition 4 gives a welfare foundation for interpreting $\int_{t}^{\infty} p_c(\tau)\dot{c}(\tau)d\tau$ as national savings. Then, if national income is to serve as a guide for prudent conduct in the sense that dynamic welfare improves if and only if national consumption is smaller than national income, we obtain that national income equals $p_c(t)c(t) + \int_{t}^{\infty} p_c(\tau)\dot{c}(\tau)d\tau$, which by (1) can be transformed to $\int_{t}^{\infty} \left( -\dot{p}_c(\tau) \right)c(\tau)d\tau$. 

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If an economy implements a path with constant instantaneous well-being and the vector of consumption prices $\mathbf{p}_c(t)$ is at any time proportional to the contributions that the various consumption flows make to instantaneous well-being, then it follows that $\mathbf{p}_c(t) \hat{c}(t) = 0$ at all times. Hence, national income equals the value of consumption and shows that this concept of income serves as a guide for prudent conduct also in this special case.

Under the assumptions of the technology being stationary and the economy realizing a competitive equilibrium, then it follows from Dixit, Hammond and Hoel (1980, proof of Theorem 1) that

$$\mathbf{p}_c(t) \hat{c}(t) + \frac{d}{dt}(\mathbf{p}_k(t) \hat{k}(t)) = 0,$$

where $\{k(t)\}_{t=0}^{\infty}$ is the path of the vector of capital stocks in this economy and $\{p_k(t)\}_{t=0}^{\infty}$ is the corresponding path of market (or calculated) present value prices of net investment flows.

Integrating on both sides under the assumption that $p_k(\tau) \hat{k}(\tau) \to 0$ as $\tau \to \infty$, entails that the following equation holds for all $t$:

$$\int_t^{\infty} \mathbf{p}_c(\tau) \hat{c}(\tau)d\tau = \mathbf{p}_k(t) \hat{k}(t).$$

Combined with (1) we obtain:

$$\int_t^{\infty} \left(-\mathbf{p}_c(\tau)\right) c(\tau)d\tau = \mathbf{p}_c(t)c(t) + \mathbf{p}_k(t) \hat{k}(t). \quad (2)$$

Hence, national income as defined through (1) equals net national product under the assumptions of the technology being stationary and the economy realizing a competitive equilibrium.

A precise and more general statement of the result that the value of the net investment flows equals the present value of future consumption changes is proven in Appendix A through Proposition 5. In particular, we need not assume that the economy implements a competitive equilibrium. By allowing for the possibility that
the prices are calculated, it is sufficient that the path of consumption flows and capital stocks is implemented by a stationary resource allocation mechanism (as introduced by Dasgupta and Mäler, 2000; Dasgupta, 2001; Arrow et al., 2003).

Example: Cake-eating economy. It is instructive to illustrate this definition of national income in the setting of a cake-eating economy, faced with the problem

$$\max_{\{c(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} u(c(t))dt \quad \text{s.t.} \quad \int_0^{\infty} c(t)dt \leq S(0) \text{ and } c(t) \geq 0 \text{ for all } t \geq 0$$

for some twice differentiable and strictly concave utility function $u : [0, \infty) \to \mathbb{R}$ satisfying $\lim_{c \to 0} u'(c) = \infty$, utility discount rate $\rho > 0$ and initial cake $S(0) > 0$. The optimal path, $\{c(t)\}_{t=0}^{\infty}$, is differentiable and satisfies

$$p_c(t) := e^{-\rho t} u'(c(t)) = u'(c(0)) \text{ for all } t \geq 0.$$

Since $\dot{p}_c(t) = 0$ for all $t \geq 0$, national income at each time $t$ equals zero:

$$\int_0^{\infty} (-\dot{p}_c(\tau)) c(\tau) d\tau = 0.$$

Moreover, it follows from (1) that the positive value of consumption at each time $t$ exactly cancels the negative present value of the future consumption changes, the latter term measuring the change in dynamic welfare as the remaining cake vanishes:

$$p_c(t)c(t) + \int_t^{\infty} p_c(\tau) \dot{c}(\tau) d\tau = p_c(t)c(t) + p_c(t) \int_t^{\infty} \dot{c}(\tau) d\tau = p(t) (c(t) - c(t)) = 0$$

since $p_c(\tau) = p_c(t)$ for all $\tau \geq t$ and $\lim_{\tau \to -\infty} c(\tau) = 0$.

4 Expressions for real national income

To find real (rather than present value) prices, consider the Divisia consumer price index $\{\pi(t)\}_{t=0}^{\infty}$ defined by $\pi(0) = 1$ and

$$\frac{\dot{\pi}(t)}{\pi(t)} = \frac{\dot{p}_c(t)c(t)}{p_c(t)c(t)}.$$ 

(3)
for all $t \geq 0$. Define the path of market (or calculated) real prices of consumption $\{P_e(t)\}_{t=0}^{\infty}$ by

$$P_e(t) = p_e(t)/\pi(t)$$ (4)

for all $t \geq 0$. Define the path of market (or calculated) real consumption interest rates $\{R(t)\}_{t=0}^{\infty}$ by

$$R(t) = -\hat{\pi}(t)/\pi(t)$$ (5)

for all $t \geq 0$. Then, by applying (3)–(5),

$$(-\dot{p}_e(t))c(t) = -\hat{\pi}(t)/\pi(t)p_e(t)c(t) = \pi(t)R(t)P_e(t)c(t).$$ (6)

Hence, it follows from (1) that real national income, $\int_t^\infty (-\dot{p}_e(\tau))c(\tau)d\tau/\pi(t)$, is equal to the present value of real interest on future national consumption, as stated in the following definition.

**Definition 1** Real national income at time $t$ is determined as

$$Y(t) := \int_t^\infty \frac{\pi(\tau)}{\pi(t)}R(\tau)P_e(\tau)c(\tau)d\tau.$$

By using (4) in (1), we can express real national income as the sum of current real national consumption and the real national savings, as stated in Proposition 1 below. Furthermore, by differentiating $Y(t)$ w.r.t. $t$, we obtain as the second part of the proposition that $\dot{Y}(t) \geq 0$ is equivalent to $P_e(t)c(t) \leq Y(t)$ and $\int_t^\infty (\pi(\tau)/\pi(t))P_e(\tau)c(\tau)d\tau \geq 0$ if the real interest rate $R(t)$ is positive; hence, $\dot{Y}(t) \geq 0$ can serve as an alternative guide for prudent behavior.

**Proposition 1** Real national income at time $t$ can be expressed as

$$Y(t) = P_e(t)c(t) + \int_t^\infty \frac{\pi(\tau)}{\pi(t)}P_e(\tau)\dot{c}(\tau)d\tau.$$

Furthermore,

$$\dot{Y}(t) = R(t)(Y(t) - P_e(t)c(t)) = R(t)\left(\int_t^\infty \frac{\pi(\tau)}{\pi(t)}P_e(\tau)\dot{c}(\tau)d\tau\right).$$
Example: Cake-eating economy (continued). In the case of the cake-eating economy introduced in Section 3, \( p_c(t) = p_c(0) \) for all \( t \geq 0 \). It follows from \( \pi(0) = 1 \) and (3) that \( \pi(t) = 1 \) for all \( t \geq 0 \). Furthermore, by (5), \( R(t) = 0 \) for all \( t \geq 0 \). Hence, by applying Definition 1, we obtain that real national income, \( Y(t) \), in a cake-eating economy equals zero for all \( t \). Furthermore, since the real interest rate equals zero for all \( t \), \( \dot{Y}(t) \geq 0 \) cannot serve as a guide for prudent behavior. This is caused by the fact that a cake-eating economy has only one asset, the “cake”, which is unproductive as a stock. In the respect, a cake-eating economy represents an extreme case, which corresponds neither to the models that economists usually analyze nor to real economies. In the Dasgupta-Heal-Solow model analyzed in Section 7, we combine a non-renewable resource, being unproductive as a stock, with a productive asset. This leads to a real interest rate which is positive throughout.

Definition 1 and Proposition 1 yield expressions for income that can be used at a national level also if the technology is not stationary, and it also facilitates the definition and expression of income for individuals and in different sectors of a national economy. We turn to such definitions next.

5 Defining individual income

Divide the national economy into \( m \) infinitely lived individuals (or dynasties), so that each individual \( i \) is in \( I := \{1, \ldots, m\} \). For each \( i \in I \), denote by \( \{c_i(t)\}_{t=0}^{\infty} \) the path of the vector of individual \( i \)’s consumption flows, where

\[
c(t) = \sum_{i \in I} c_i(t) \quad \text{for all} \quad t \geq 0.
\]  

(7)

In line with Sefton and Weale (2006, Definition 2) and the analysis of Section 3, define individual income by \( \int_t^{\infty} (- \dot{p}_c(\tau))c_i(\tau)d\tau \), which can be transformed to

\[
\int_t^{\infty} (- \dot{p}_c(\tau))c_i(\tau)d\tau = p_c(t)c_i(t) + \int_t^{\infty} p_c(\tau)\dot{c}_i(\tau)d\tau
\]  

(8)

individual income

individual savings
through integration by parts, provided that $\dot{p}_c(\tau)c_i(\tau) \to 0$ as $\tau \to \infty$. It follows from Proposition 6 of Appendix A that $\int_t^\infty p_c(\tau)c_i(\tau)d\tau$ measures individual welfare improvement if the individual faces $\{p_c(t)\}_{t=0}^{\infty}$ as market prices and maximizes dynamic welfare subject to a budget constraint. This result does not rely on individual welfare being discounted utilitarian. Hence, individual income as defined above serves as a guide for prudent conduct under general conditions.

To define real individual income at time $t$ in the same numeraire as real national income at time $t$, and to obtain alternative expressions, consider individual Divisia consumer price indices $\{\pi_i(\tau)\}_{\tau=t}^{\infty}$ defined by $\pi_i(t) = \pi(t)$ and

$$\hat{\pi}_i(\tau) = \frac{\dot{p}_c(\tau)c_i(\tau)}{p_c(\tau)c_i(\tau)},$$

(9)

for all $\tau \geq t$. Define the path of market (or calculated) real prices of consumption $\{P_{ci}(\tau)\}_{\tau=t}^{\infty}$ for individual $i$ by

$$P_{ci}(\tau) = \frac{p_c(\tau)}{\pi_i(\tau)}$$

(10)

for all $\tau \geq t$. Define the path of market (or calculated) real consumption interest rates $\{R_i(\tau)\}_{\tau=t}^{\infty}$ for individual $i$ by

$$R_i(\tau) = -\hat{\pi}_i(\tau)/\pi_i(\tau)$$

(11)

for all $\tau \geq t$. Then, by applying (9)–(11),

$$\left(-\dot{p}_c(\tau)\right)c_i(\tau) = \frac{\hat{\pi}_i(\tau)}{\pi_i(\tau)}p_c(\tau)c_i(\tau) = \pi_i(\tau)R_i(\tau)P_{ci}(\tau)c_i(\tau).$$

(12)

Hence, by $\pi_i(t) = \pi(t)$ and (12), real individual income, $\int_t^\infty \left(-\dot{p}_c(\tau)\right)c_i(\tau)d\tau/\pi(t)$, is equal to the present value of real interest on future individual consumption, as stated in the following definition.

**Definition 2** Real individual income at time $t$ is determined as

$$Y_i(t) := \int_t^\infty \frac{\pi_i(\tau)}{\pi_i(t)}R_i(\tau)P_{ci}(\tau)c_i(\tau)d\tau.$$
By using (10) in (8), we can also express real individual income as the sum of current real individual consumption and the real individual savings. Furthermore, real individual incomes add exactly up to real national income since

$$\sum_{i \in I} Y_i(t) = \sum_{i \in I} \left( \frac{1}{\pi_i(t)} \int_t^\infty (-\dot{p}_e(\tau)) c_i(\tau) d\tau \right)$$

$$= \frac{1}{\pi(t)} \int_t^\infty (-\dot{p}_e(\tau)) \sum_{i \in I} c_i(\tau) d\tau$$

$$= \frac{1}{\pi(t)} \int_t^\infty (-\dot{p}_e(\tau)) c(\tau) d\tau = Y(t),$$

where the first equality follows from (12), the second from $\pi_i(t) = \pi(t)$ for all $i \in I$, the third from (7), and the fourth from (6). We state these results as follows.

**Proposition 2** Real individual income at time $t$ can be expressed as

$$Y_i(t) = P_e(t) c_i(t) + \int_t^\infty \frac{\pi_i(\tau)}{\pi_i(t)} P_e(\tau) c_i(\tau) d\tau.$$

Furthermore,

$$\sum_{i \in I} Y_i(t) = Y(t).$$

If all individuals consume goods in the same proportions, i.e., for each $i \in I$ there exists $\{\gamma_i(t)\}_{t=0}^\infty$ such that

$$c_i(t) = \gamma_i(t) c(t)$$

for all $t \geq 0$, then it follows from (3) and (9) that, for each $i \in I$, $\pi_i(\tau) = \pi(\tau)$ for all $\tau \geq t$. Hence, in this case the individual prices and interest rates as determined by (10) and (11) coincide with the real prices and real interest rate at a national level, as determined by (4) and (5).
6 Defining sectoral income

Divide the national economy into \( n \) sectors, so that each sector \( j \) is in \( J := \{1, \ldots, n\} \). For each \( j \in J \), denote by \( \{x^j(t)\}_{t=0}^{\infty} \) the path of sector \( j \)'s vector of commodity flows excluding consumption flows acquired for end use, and let \( \{p_x(t)\}_{t=0}^{\infty} \) be the corresponding path of market (or calculated) present value prices of these commodity flows. Assume that aggregate cash flows are zero. Then the value of national consumption \( c \) equals the cash flow generated by \( \sum_{j \in J} x^j \) at each point in time:

\[
 p_c(t)c(t) = \sum_{j \in J} p_x(t)x^j(t)
\]

for all \( t \geq 0 \). Furthermore, for each \( i \in I \), let \( \{\sigma_i^j(t)\}_{t=0}^{\infty} \) be the path of individual \( i \)'s share of sector \( j \), where for each \( j \in J \),

\[
 \sum_{i \in I} \sigma_i^j(t) = 1 \quad \text{for all} \quad t \geq 0,
\]

so that

\[
 p_c(t)c_i(t) = \sum_{j \in J} \sigma_i^j(t)p_x(t)x^j(t),
\]

and consumer \( i \)'s income from sector \( j \) equals \( \int_{t}^{\infty} R_i(\tau)\sigma_i^j(\tau)p_x(\tau)x^j(\tau)d\tau \). By summing over individuals, we can now define each sector \( j \)'s income by

\[
 \int_{t}^{\infty} \left( \sum_{i \in I} R_i(\tau)\sigma_i^j(\tau) \right) p_x(\tau)x^j(\tau)d\tau.
\]

To define real sectoral income at time \( t \) in the same numeraire as real national income at time \( t \), and obtain alternative expressions for sectoral income, define for each \( j \in J \) the path of market (or calculated) real consumption interest rates \( \{R^j(\tau)\}_{\tau=t}^{\infty} \) for sector \( j \) as a weighted average of the individual interest rates,

\[
 R^j(\tau) := \sum_{i \in I} \sigma_i^j(\tau)R_i(\tau)
\]

for all \( \tau \geq t \), and derive a Divisia consumer price index for sector \( j \) from its path of real consumption interest rates,

\[
 \pi^j(\tau) := \pi(t)e^{-\int_{t}^{\tau} R^j(s)ds}
\]

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for all $\tau \geq t$. Clearly we have that $\pi^j(t) = \pi(t)$ and
\[
R^j(\tau) = -\dot{\pi}^j(\tau)/\pi^j(\tau)
\] (14)
for all $\tau \geq t$. Define the path of market (or calculated) real commodity prices
$\{P^j_x(\tau)\}_{\tau=t}^\infty$ for sector $j$ by
\[
P^j_x(\tau) = p_x(\tau)/\pi^j(\tau)
\]
for all $\tau \geq t$. By these definitions we obtain
\[
\left( \sum_{i \in I} R_i(\tau)\sigma^j_i(\tau) \right) p_x(\tau) x^j(\tau) = \pi^j(\tau) R^j(\tau) P^j_x(\tau) x^j(\tau) = \left( -\dot{\pi}^j(\tau) \right) P^j_x(\tau) x^j(\tau). \tag{15}
\]
Hence, by $\pi^j(t) = \pi(t)$ and (15), \textit{real sectoral income}, $\int_t^\infty \left( \sum_{i \in I} R_i(\tau)\sigma^j_i(\tau) \right) p_x(\tau) x^j(\tau) d\tau/\pi(t)$, is equal to the present value of real interest on future sectoral cash flow, as stated in the following definition.

**Definition 3** \textit{Real sectoral income} at time $t$ is determined as
\[
Y^j(t) := \int_t^\infty \frac{\pi^j(\tau)}{\pi^j(t)} R^j(\tau) P^j_x(\tau) x^j(\tau) d\tau.
\]

Differentiation of $\pi^j(\tau) P^j_x(\tau) x(\tau)$ yields
\[
\frac{d}{dt} \left( \pi^j(\tau) P^j_x(\tau) x(\tau) \right) = \dot{\pi}^j(\tau) P^j_x(\tau) x(\tau) + \pi^j(\tau) \dot{P}^j_x(\tau) x(\tau) + \pi^j(\tau) P^j_x(\tau) \dot{x}(\tau).
\]
Integrating on both sides under the assumption that $\pi^j(\tau) P^j_x(\tau) x(\tau) \to 0$ as $t \to \infty$, leads to the following equation:
\[
-\pi^j(t) P^j_x(t) x(t) = \int_t^\infty \dot{\pi}^j(\tau) P^j_x(\tau) x(\tau) d\tau + \int_t^\infty \pi^j(\tau) \dot{P}^j_x(\tau) x(\tau) d\tau + \int_t^\infty \pi^j(\tau) P^j_x(\tau) \dot{x}(\tau) d\tau.
\]
By rearranging this equality, invoking (14) and applying Definition 3, we obtain the first part of Proposition 3 below. The second part of the proposition follows since
\[
\sum_{j \in J} \left( \sum_{i \in I} R_i(\tau)\sigma^j_i(\tau) \right) p_x(\tau) x^j(\tau) = \sum_{i \in I} R_i(\tau) \left( \sum_{j \in J} \sigma^j_i(\tau) p_x(t) x^j(\tau) \right)
\]
\[
= \sum_{i \in I} \left( -\frac{\dot{\pi}^j_i(\tau)}{\pi^j_i(\tau)} \right) p_x(\tau) c^i(\tau) = \sum_{i \in I} \left( -\dot{p}_c(\tau) \right) c^i(\tau) = \left( -\dot{p}_c(\tau) \right) c(\tau),
\]
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using (9), (11) and (13); i.e., real sectoral incomes add exactly up to real national income:

\[
\sum_{j \in J} Y^j(0) = \sum_{j \in J} \left( \frac{1}{\pi(t)} \int_t^\infty \left( \sum_{i \in I} R_i(\tau) \sigma_i^j(\tau) p_i(\tau) x_i^j(\tau) d\tau \right) \right) \\
= \frac{1}{\pi(t)} \int_t^\infty \left( - \dot{p}_c(\tau) \right) c(\tau) d\tau = Y(0) .
\]

**Proposition 3** Real sectoral income at time \( t \) can be expressed as

\[
Y^j(t) = \left[ \mathbf{P}^j(t) \mathbf{x}(t) \right]_{\text{current cash flow}} + \int_t^\infty \frac{\pi^j(t)}{\pi^j(\tau)} \mathbf{P}^j(\tau) \mathbf{x}(\tau) d\tau \bigg|_{\text{sectoral net investments}} + \int_t^\infty \frac{\pi^j(t)}{\pi^j(\tau)} \dot{\mathbf{P}}^j(\tau) \mathbf{x}(\tau) d\tau \bigg|_{\text{price change effects}} .
\]

Furthermore,

\[
\sum_{j \in J} Y^j(t) = Y(t) .
\]

The first part of Proposition 3 means that we are able to split the sector’s income into its current cash flow, its net investments, and its price change effects, by using a consumer price index associated with the use of the cash flow from sector \( j \).

For each sector \( j \), \( R^j(\tau) = \sum_{i \in I} R_i(\tau) \sigma_i^j(\tau) \) is an average real consumption interest rate to be used for the calculation of sector \( j \)'s income. If the individual interest rates as determined by (11) coincide with the real interest rate at a national level as determined by (5), then the average consumption interest for each sector \( j \) also coincide with the real interest at a national level. Also if, for each \( i \in I \), \( \sigma_i^j(\tau) = \sigma_i(\tau) \) for all \( j \in N \), then it follows that the average consumption interest for each sector \( j \) coincide with the real interest rate at a national level.

In this section, “sector” has been used as an abstract term. The examples of the two next sections illustrate ways in which an economy can be divided into different “sectors”. 

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7 Functional income shares in a DHS model

Consider the Cobb-Douglas Dasgupta-Heal-Solow (DHS) model (Dasgupta and Heal, 1974, 1979; Solow, 1974). Hence, production, $q(t)$ at time $t$ is given by

$$q(t) = k(t)^\alpha r(t)^\beta$$

where $k$ is the capital stock, $r$ is resource input being extracted at no cost from a finite stock, and the available labor $\ell$ is constant and normalized to one (i.e, $\ell(t) = 1$ for all $t$), and where we assume that

$$1 > \alpha + \beta > \alpha > \beta.$$ 

Production can be split into consumption $c(t)$ and accumulation of capital $\dot{k}(t)$:

$$q(t) = c(t) + \dot{k}(t).$$

Since this is a one-consumption good model, price indices need not be invoked. Consequently, the real price of consumption can be set to 1 for all $t \geq 0$, and the real wage $\bar{P}_t(t)$, the real price of resource input $P_r(t)$, and the real interest rate $R(t)$ equals the marginal productivities of inputs:

$$\bar{P}_t(t) = (1 - \alpha - \beta)q(t),$$
$$P_r(t) = \beta q(t)/r(t),$$
$$R(t) = \alpha q(t)/k(t).$$  \hspace{1cm} (16)

Furthermore, along an efficient path, the Hotelling rule,

$$\pi(t)P_r(t) = P_r(0),$$

is satisfied, where $\{\pi(t)\}_{t=0}^\infty$ is the path of present value prices of consumption:

$$\pi(t) = e^{-\int_0^t R(\tau) d\tau}$$  \hspace{1cm} (17)

for all $t \geq 0$. 

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Assume that the economy follows the efficient constant consumption path, which exists under these assumption. This path is characterized by a constant production \( q \), with the constant consumption being a fixed share of production: \( c = (1 - \beta)q \), with the reminder being used for capital accumulation:

\[
\dot{k}(t) = \beta q. \tag{18}
\]

Consider three sectors, corresponding to the supply of labor, the supply of resource input, and the production sector. We assume that the production sector owns the capital stock and is responsible for capital accumulation. The cash flow to each of these sectors at each point in time is as follows:

Labor: \[ P_L(t) = (1 - \alpha - \beta)q \]

Resource: \[ P_r(t)r(t) = \beta q \tag{19} \]

Production/Capital: \[ R(t)k(t) - \dot{k}(t) = (\alpha - \beta)q \]

It is easy to check that the cash flow from these sectors add up to national consumption at each point in time. In order to find sectoral income, note that

\[
\int_t^\infty \frac{\pi(\tau)}{\pi(t)} R(\tau) d\tau = \frac{1}{\pi(t)} \int_t^\infty \left( -\pi(\tau) \right) d\tau = \frac{\pi(t)}{\pi(t)} = 1, \quad (20)
\]

provided that \( \pi(\tau) \to 0 \) as \( \tau \to \infty \). This implies that sectoral income at time \( t \) is given by

Labor: \[ Y^L(t) = \int_t^\infty \frac{\pi(\tau)}{\pi(t)} R(\tau)(1 - \alpha - \beta)q d\tau = (1 - \alpha - \beta)q \]

Resource: \[ Y^R(t) = \int_t^\infty \frac{\pi(\tau)}{\pi(t)} R(\tau)\beta q d\tau = \beta q \]

Production/Capital: \[ Y^C(t) = \int_t^\infty \frac{\pi(\tau)}{\pi(t)} R(\tau)(\alpha - \beta)q d\tau = (\alpha - \beta)q \]

Since the real interest rate is decreasing over time, resource income is lower than the interest on the resource wealth and production/capital income is lower than the interest on capital, as demonstrated in Appendix B.
Alternatively, we can use Proposition 3 to derive expressions for sectoral income.

\[ Y^\ell(t) = P_\ell(t)\ell(t) + \int_t^\infty \frac{\pi(\tau)}{\pi(t)} P_\ell(\tau)\ell(\tau)d\tau + \int_t^\infty \frac{\pi(\tau)}{\pi(t)} P_\ell(\tau)\ell(\tau)d\tau \]

\[ = P_\ell(t) = (1 - \alpha - \beta)q \quad \text{since } P_\ell(\tau) = (1 - \alpha - \beta)q \text{ and } \ell(\tau) = 1 \text{ for all } \tau. \]

Hence, for labor, there is no net investments or price change effects.

\[ Y^\tau(t) = P_r(t)r(t) + \int_t^\infty \frac{\pi(\tau)}{\pi(t)} P_r(\tau)r(\tau)d\tau + \int_t^\infty \frac{\pi(\tau)}{\pi(t)} \dot{P}_r(\tau)r(\tau)d\tau \]

\[ = P_r(t)r(t) + P_r(t) \int_t^\infty \dot{r}(\tau)d\tau + \int_t^\infty \frac{\pi(\tau)}{\pi(t)} \dot{P}_r(\tau)r(\tau)d\tau \quad \text{by Hotelling’s rule,} \]

\[ = \int_t^\infty \frac{\pi(\tau)}{\pi(t)} \dot{P}_r(\tau)r(\tau)d\tau \quad \text{since } \lim_{\tau \to \infty} r(\tau) = 0 \implies \int_t^\infty \dot{r}(\tau)d\tau = -r(t), \]

\[ = \frac{1}{\pi(t)} \int_t^\infty (-\dot{\pi}(\tau)) P_r(\tau)r(\tau)d\tau \quad \text{since } \dot{\pi}P_r + \pi \dot{P}_r = 0 \text{ by Hotelling’s rule,} \]

\[ = \beta q \quad \text{by (20) since } P_r(\tau)r(\tau) = \beta q \text{ for all } \tau. \]

This means that resource income can be split like this:

\[ Y^\tau(t) := \beta q - \beta q + \beta q, \]

where the negative net investments equal the Hotelling rents and cancel out the value of production.

\[ Y^k(t) = R(t)k(t) - \dot{k}(t) + \int_t^\infty \frac{\pi(\tau)}{\pi(t)} (R(\tau)\dot{k}(\tau) - \dot{k}(\tau))d\tau + \int_t^\infty \frac{\pi(\tau)}{\pi(t)} \dot{R}(\tau)k(\tau)d\tau \]

\[ = R(t)k(t) - \dot{k}(t) + \int_t^\infty \frac{\pi(\tau)}{\pi(t)} \dot{R}(\tau)k(\tau)d\tau \text{ by (20) as } \dot{k}(\tau) = \beta q \text{ for all } \tau, \]

\[ = R(t)k(t) - \int_t^\infty \frac{\pi(\tau)}{\pi(t)} R(\tau)\dot{k}(\tau)d\tau \quad \text{since } \dot{R}(\tau)/R(\tau) = -\dot{k}(\tau)/k(\tau) \text{ for all } \tau, \]

\[ = R(t)k(t) - \dot{k}(t) = (\alpha - \beta)q \quad \text{by (20) as } \dot{k}(\tau) = \beta q \text{ for all } \tau, \]

This means that production/capital income can be split like this:

\[ Y^k(t) := (\alpha - \beta)q + \beta q - \beta q, \]

so that the negative price change effects due to the decreasing interest rate cancel out the positive net investments.

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8 Income of a reservoir

Consider a reservoir of a resource, say petroleum. The reservoir has a fixed size $S(0)$, and non-negative resource extraction $r$ depends on non-negative extractive effort $e$ through an increasing, strictly concave, and continuously differentiable production function $F$,

$$r = F(e),$$

satisfying $F(0) = 0$. The real prices of the extracted resource and extractive effort are constant in real terms and equal $P_r$ and $P_e$ respectively. Hence the real cash-flow at time $t$ is given by

$$P_r r(t) - P_e e(t),$$

where $r(t) = F(e(t))$ for all $t \geq 0$. By defining the cost function $C$ by

$$C(r) = P_e F^{-1}(r)$$

(21)

for all $r$ in the range of $F$, the real cash-flow at time $t$ can be rewritten as follows:

$$P_r r(t) - C(r(t)) = (P_r - C'(r(t))) r(t) + (C'(r(t))) r(t) - C(r(t)).$$

Under the assumption that there is a constant interest real rate $R$, so that $\pi(t) = e^{-Rt}$, and the reservoir is extracted in a profit-maximizing manner, we have that

$$e^{-Rt}(P_r - C'(r(t))) = \text{constant}$$

(22)

for all $t \in [0, T]$, where $T$ is the time at which the reservoir is exhausted:

$$\int_0^T r(t) dt = S(0).$$

Hence, by (22) the income at time $t$ of the reservoir can be written as

$$Y^r(t) = \int_t^T e^{-R(\tau-t)} R(P_r r(\tau) - C(r(\tau))) d\tau$$

$$= R(P_r - C'(r(t))) S(t) + R \int_t^T e^{-R(\tau-t)} (C'(r(\tau)) r(\tau) - C(r(\tau))) d\tau.$$

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The first term is interest on the present value of future Hotelling rent, while the second term is interest on the present value of future Ricardian rent.

Alternatively, we can use Proposition 3 to derive expressions for the income of a reservoir. Since $P_r$ and $P_e$ are assumed to be constant, we obtain

$$Y^*(t) = P_r(t)r(t) - P_e(t)e(t) + \int_t^\infty e^{-R(t-\tau)}(P_r(\tau)\dot{r}(\tau) - P_e(\tau)\dot{e}(\tau))d\tau$$

$$= P_r(t)r(t) - C(t) + \int_t^\infty e^{-R(t-\tau)}(P_r(\tau) - C'(r(\tau)))\dot{r}(\tau)d\tau \text{ by (21)},$$

$$= P_r(t)r(t) - C(t) + (P_r(t) - C'(r(t)))\int_t^\infty \dot{r}(\tau)d\tau \text{ by (22)},$$

$$= C'(r(t))r(t) - C(r(t)) \text{ since } \lim_{\tau \to \infty} r(\tau) = 0 \text{ implies } \int_t^\infty \dot{r}(\tau)d\tau = -r(t).$$

Hence, we arrive at the result that the income of a reservoir – given the assumptions that we have made – equals current Ricardian rent.

By interpreting $F$ to be derived from a constant returns to scale production function $\tilde{F}$ that depends on both effort $e$ and the ground $g$ from which the resource is extracted, by setting $g = 1$, i.e.,

$$F(e) = \tilde{F}(e, 1),$$

we obtain the interpretation that the income of a reservoir is equal to the marginal productivity of the ground evaluated at the resource price net of the Hotelling rent:

$$C'(r) \frac{\partial \tilde{F}(e, 1)}{\partial g} = C'(r)(F(e) - F'(e)e) = C'(r)r - C(r),$$

since $C'(r)F'(e) = P_e$ and $P_e e = C(r)$ by the definition of $C$.

9 Concluding remarks

In this paper we have presented a definition of sectoral income which is compatible with an important line of welfare-based theory of comprehensive national accounting in the tradition of Hicks (1946), Samuelson (1961), Weitzman (1976) and Sefton and Weale (2006). The definition has the desirable properties that sectoral income summed over all sectors adds up to a concept of national income which
• is a guide to prudent behavior in the sense that dynamic welfare improves if and only if consumption is less than national income, and

• equals net national product in a closed economy with a stationary technology.

We have decomposed sectoral income into current cash flow, sectoral net investments and price change effects. Our definition of sectoral income and its decomposition have been illustrated though application to a general model of capital resource accumulation and resource depletion as well as a partial model of a single resource reservoir.

As noted in Section 2, we do not require that a discounted utilitarian welfare function is maximized. Rather, as explained in Appendix A, the formal analysis builds on the assumption that the economy’s actual decisions are taken according to a resource allocation mechanism (as introduced by Dasgupta and Mäler, 2000; Dasgupta, 2001; Arrow et al., 2003). The resource allocation mechanism is allowed to be inefficient, due to, e.g., externalities, monopolistic competition, or distortionary taxation.³ This is particular relevant in a world facing serious environmental problems caused by uninternalized externalities. It is not trivial to calculate the relevant accounting prices under such conditions. Guidelines for practical calculation of accounting prices are outside the scope of the present paper. Arrow et al. (2003) discuss problems that arise within such a framework and is a useful reference.

³The general results of Appendix A (Propositions 4 and 5) that we use to justify the national income definition do not require that dynamic welfare is maximized. However, the result of Appendix A (Proposition 6) that we use to justify the individual income definition requires that individuals make decisions according to the relevant accounting prices. The propositions of the main text remain valid even if the premise of Proposition 6 does not hold.
Appendix A: Welfare results

If dynamic welfare is forward-looking and numerically representable, then dynamic welfare, denoted \( V \), is a functional \( G \) of the path of consumption flows:

\[
V(t) = G(\{c(\tau)\}_{\tau=t}^{\infty}, t) .
\]  

(A1)

The functional \( G \) is **time-invariant** if

\[
G(\{c(\tau)\}_{\tau=t}^{\infty}, t) = G(\{\hat{c}(\tau)\}_{\tau=0}^{\infty}, 0) .
\]  

(A2)

where \( \{c(\tau)\}_{\tau=0}^{\infty} \) is determined by \( \hat{c}(\tau) = c(t + \tau) \) for all \( \tau \geq 0 \). Furthermore, \( G \) satisfies a condition of independent future if

\[
G(\{c'(\tau)\}_{\tau=0}^{\infty}, 0) < G(\{c''(\tau)\}_{\tau=0}^{\infty}, 0) \quad \Leftrightarrow \quad G(\{c'(\tau)\}_{\tau=t}^{\infty}, t) < G(\{c''(\tau)\}_{\tau=t}^{\infty}, t)
\]

whenever \( \{c'(\tau)\}_{\tau=0}^{\infty} \) and \( \{c''(\tau)\}_{\tau=0}^{\infty} \) coincides during the interval \([0, t]\).

**Proposition 4** Let dynamic welfare be numerically representable by a forward-looking and time-invariant functional \( G \) of the path of consumption flows, satisfying a condition of independent future. Then, if \( G \) is smooth, there exist a path of present value consumer prices \( \{p_c(t)\}_{t=0}^{\infty} \) such that welfare improves along the implemented path \( \{c(t)\}_{t=0}^{\infty} \) at time \( t \) if and only if

\[
\int_t^\infty p_c(\tau) \hat{c}(\tau) d\tau > 0 .
\]

**Proof.** Since \( G \) is smooth and satisfies independent future, there exists a path of present value consumer prices \( \{p_c(t)\}_{t=0}^{\infty} \), unique up to a choice of numeraire, supporting the implemented path of consumption flows \( \{c(t)\}_{t=0}^{\infty} \) in the sense that, for all \( t \),

\[
\lambda(t)dV(t) = \int_t^\infty p_c(\tau) dc(\tau) .
\]  

(A3)

for some \( \lambda(t) > 0 \). Furthermore, since \( G \) is time-invariant,

\[
\lambda(t)\hat{V}(t) = \lambda(t) \frac{d}{dt} G(\{c(\tau)\}_{\tau=t}^{\infty}, t) = \lambda(t) \frac{d}{dt} G(\{c(t + \tau)\}_{\tau=0}^{\infty}, 0) = \int_t^\infty p_c(\tau) \hat{c}(\tau) d\tau ,
\]  

(A4)

This appendix is partly based on analysis and results in Asheim (2007)
where the first equality follows from (A1), the second from (A2), and the third from (A3), thus establishing the result that local welfare comparisons across time for a given path of consumption flows \( \{c(t)\}^{\infty}_{t=0} \) depends on the present value of future consumption growth.

\[ \text{Example: Discounted utilitarianism.} \] Let the welfare functional be given as
\[
G(\{c(\tau)\}_{\tau=t}^{\infty}, t) = \int_{t}^{\infty} e^{-\rho(\tau-t)}u(c(\tau))d\tau.
\]

Then it follows that
\[
\frac{d}{dt}\left(\int_{t}^{\infty} e^{-\rho(\tau-t)}u(c(\tau))d\tau\right) = -u(c(t)) + \rho\int_{t}^{\infty} e^{-\rho(\tau-t)}u(c(\tau))d\tau \tag{A5}
\]
\[
= e^{\rho t}\int_{t}^{\infty} e^{-\rho(\tau-t)}\nabla u(c(\tau))\dot{c}(\tau)d\tau,
\]

where the second equality follows by integrating by parts. This verifies (A4) in the case of discounted utilitarianism by setting, for all \( t, \lambda(t) = e^{-\rho t} \) and \( p_c(t) = e^{-\rho t} \nabla u(c(t)) \). Note also that time-invariance corresponds to
\[
\int_{t}^{\infty} e^{-\rho(\tau-t)}u(c(\tau))d\tau = \int_{0}^{\infty} e^{-\rho(\tau-0)}u(c(t+\tau))d\tau,
\]
which clearly holds in the case of discounted utilitarianism.

Consider that the economy’s actual decisions are taken according to a resource allocation mechanism (RAM) that assigns some attainable consumption-net investment pair \((c, \dot{k})\) to any vector of capital stocks \( k \). Hence, for any vector of capital stocks \( k \), the RAM determines the consumption and net investment flows. The net investment flows in turn map out the development of the capital stocks. The RAM thereby implements a feasible path of consumption flows, net investment flows, and capital stocks, for any initial vector of capital stocks. If the set of all attainable consumption-net investment pairs to any vector of capital stocks \( k \) is time-invariant (i.e., in closed economy with a stationary technology), one can assume that the RAM in the economy is Markovian (i.e., the chosen consumption-net investment pair depends only on \( k \)) and time-invariant. If the welfare functional \( G \) is time-invariant and a stationary (i.e., Markovian and time-invariant) RAM implements a unique path, then the dynamic welfare at \( t \) of the implemented path,
\[
V(t) = V(k(t)),
\]
depends solely on the current vector of capital stocks \( \mathbf{k}(t) \).

**Proposition 5** Let (1) dynamic welfare be numerically representable by a forward-looking and time-invariant functional \( \mathcal{G} \) of the path of consumption flows, satisfying a condition of independent future, and (2) the RAM be stationary. Then, if \( \mathcal{G} \) is smooth, the RAM implements a unique path, and the state valuation function \( \mathcal{V} \) is differentiable, there exist a path of present value consumer prices \( \{ \mathbf{p}_c(t) \}_{t=0}^{\infty} \) and present value net investment prices \( \{ \mathbf{p}_k(t) \}_{t=0}^{\infty} \) such that welfare improves along the implemented path \( \{ c(t), \dot{\mathbf{k}}(t), \mathbf{k}(t) \}_{t=0}^{\infty} \) at time \( t \) if and only if

\[
\int_t^\infty p_c(\tau) \dot{c}(\tau) d\tau = p_k(t) \dot{\mathbf{k}}(t) > 0.
\]

**Proof.** By the proof of Proposition 4, there exists a path of supporting present value consumer prices \( \{ \mathbf{p}_c(t) \}_{t=0}^{\infty} \), unique up to a choice of numeraire, such that, for all \( t \), (A3) is satisfied for some \( \lambda(t) > 0 \). Since \( \mathcal{V} \) exists and is differentiable, there exists, at any \( t \), a supporting vector of capital prices \( \mathbf{p}_k(t) \) satisfying

\[
\lambda(t) \nabla \mathcal{V}(\mathbf{k}(t)) = \mathbf{p}_k(t).
\]

Hence, since \( \mathcal{V} \) is time-invariant, local welfare comparisons across time for a given implemented path \( \{ c(t), \dot{\mathbf{k}}(t), \mathbf{k}(t) \}_{t=0}^{\infty} \) depends on the value of net investments:

\[
\lambda(t) \dot{\mathcal{V}}(t) = \lambda(t) \nabla \mathcal{V}(\mathbf{k}(t)) \dot{\mathbf{k}}(t) = \mathbf{p}_k(t) \dot{\mathbf{k}}(t).
\]

The result follows by combining (A4) and (A6). \( \blacksquare \)

**Example: Discounted utilitarianism (continued).** Equation (A5) can be rewritten as

\[
\nabla \mathcal{V}(\mathbf{k}(t)) \dot{\mathbf{k}}(t) = -u(c(t)) + \rho \mathcal{V}(\mathbf{k}(t))
\]

or

\[
u(c(t)) + \nabla \mathcal{V}(\mathbf{k}(t)) \dot{\mathbf{k}}(t) = \rho \mathcal{V}(\mathbf{k}(t)).
\]

Differentiating once more w.r.t. time yields:

\[
\nabla u(c(t)) \dot{c}(t) + \frac{d[\nabla \mathcal{V}(\mathbf{k}(t)) \dot{\mathbf{k}}(t)]}{dt} = \rho \nabla \mathcal{V}(\mathbf{k}(t)) \dot{\mathbf{k}}(t),
\]

or equivalently, since \( \mathbf{p}_c(t) = e^{-\rho t} \nabla u(c(t)) \) and \( \mathbf{p}_k(t) = e^{-\rho t} \nabla \mathcal{V}(\mathbf{k}(t)) \),

\[
\mathbf{p}_c(t) \dot{c}(t) = -\frac{d(\mathbf{p}_k(t) \dot{\mathbf{k}}(t))}{dt}
\]
as $d(V(k(t)) \dot{k})/dt = d(e^{\rho t} p_k \dot{k})/dt = e^{\rho t} d(p_k \dot{k})/dt + \rho p_k \dot{k}$. This means that the equality in Proposition 5 follows through integration, provided that the following net investment value transversality condition holds:

$$\lim_{t \to \infty} p_c(t) \dot{k}(t) = 0.$$ 

If individual welfare is forward-looking and numerically representable, then individual welfare, denoted $V_i$, is a functional $G_i$ of the path of individual consumption flows:

$$V_i(t) = G_i(\{c_i(\tau)\}_{\tau=t}^{\infty}, t).$$

**Proposition 6** Let individual welfare be numerically representable by a forward-looking and time-invariant functional $G_i$ of the path of individual consumption flows, satisfying a condition of independent future. Then, if $G_i$ is smooth and the individual path $\{c_i(t)\}_{t=0}^{\infty}$ maximizes individual welfare subject to the budget constraint $\int_0^\infty p_c(t)c_i(t)dt = m(0)$, individual welfare improves along the implemented path at time $t$ if and only if

$$\int_t^\infty p_c(\tau)c_i(\tau)d\tau > 0.$$ 

**Proof.** Since $G_i$ is smooth and satisfies independent future, and $\{c_i(t)\}_{t=0}^{\infty}$ maximizes individual welfare subject to the budget constraint $\int_0^\infty p_c(t)c_i(t)dt = m(0)$, it follows that, for all $t$,

$$\lambda_i(t)dV_i(t) = \int_t^\infty p_c(\tau)d\lambda_i(\tau).$$

for some $\lambda_i(t) > 0$. The result follows by applying the argument of (A4) to $G_i$. 

**Appendix B: Other income concepts**

In this appendix we define formally the concepts of income as interest on wealth and wealth equivalent income, as alternative concepts to the one analyzed in the main part of this paper. Both these wealth-based concepts of income depend on the way in which nominal
prices are turned into real prices, thus raising an indexing issue. To sidestep this issue we choose to define these alternative income concepts in a setting where consumption is a scalar.

With only one consumption good, the basic setting of the main text can be restated as follows: Let \( \{c(t)\}_{t=0}^{\infty} \) be the path of consumption flows in the economy, and let \( \{\pi(t)\}_{t=0}^{\infty} \) be the corresponding path of positive market (or calculated) present value prices of consumption, with \( \int_0^{\infty} \pi(\tau) d\tau < \infty \). Define the path of market (or calculated) real consumption interest rates \( \{R(t)\}_{t=0}^{\infty} \) by \( R(t) = -\dot{\pi}(t)/\pi(t) \) for all \( t \geq 0 \). For each \( j \in J := \{1, \ldots, n\} \), denote by \( \{x^j(t)\}_{t=0}^{\infty} \) the path of sector \( j \)'s commodity flows excluding consumption flow acquired for end use, and let \( \{p_x(t)\}_{t=0}^{\infty} \) be the corresponding path of market (or calculated) present value prices of these commodity flows. Assume that aggregate cash holdings are zero. Then the value of national consumption equals total cash flow at each point in time: \( \pi(t)c(t) = \sum_{j \in J} p_x(t)x^j(t) \) for all \( t \geq 0 \). In this setting, real income as the present value of real interest on future consumption (at the aggregate level) and cash flow (at the sectoral level) at time \( t \) is determined as follows:

At the national level:
\[
Y(t) := \int_t^{\infty} R(\tau) \frac{\pi(\tau)}{\pi(t)} c(\tau) d\tau
\]

At the sectoral level:
\[
Y^j(t) := \int_t^{\infty} R(\tau) \frac{p_x^j(\tau)}{\pi(t)} x^j(\tau) d\tau.
\]

For the wealth-based income concepts below, we must define real wealth at time \( t \):

At the national level:
\[
W(t) := \int_t^{\infty} \frac{\pi(\tau)}{\pi(t)} c(\tau) d\tau
\]

At the sectoral level:
\[
W^j(t) := \int_t^{\infty} \frac{p_x^j(\tau)}{\pi(t)} x^j(\tau) d\tau.
\]

**Income as interest on wealth**

**Definition 4** Real income as interest on wealth at time \( t \) is determined as follows:

At the national level:
\[
Z(t) := R(t)W(t) = c(t) + \dot{W}(t)
\]

At the sectoral level:
\[
Z^j(t) := R(t)W^j(t) = \frac{p_x^j(t)}{\pi(t)} x^j(t) + \dot{W}^j(t).
\]
Since the current value of sectoral cash flow sum up to national consumption for all \( t \geq 0 \), it follows that real income as interest on wealth has the property that sectoral income summed over all sectors add up to national income, \( Z(t) = \sum_{j \in J} Z^j(t) \).

In view of Proposition 5 of Appendix A we can check whether real income as interest on wealth at the national level equals net national product in a closed economy with a stationary technology by comparing \( Z(t) \) with \( Y(t) \). By Sefton and Weale (2006, eq. (19)),
\[
Y(t) - Z(t) = \int_t^\infty R(\tau) \frac{\pi(\tau)}{\pi(t)} W(\tau) d\tau .
\]
Hence, the definition of real income as interest on wealth is unproblematic as long as the economy is in a steady state with a constant real interest rate. In this case, \( Z(t) \) can also serve as indicator for prudent behavior, since we obtain
\[
Z(t) - c(t) = \bar{W}(t) = \int_t^\infty \frac{\pi(\tau)}{\pi(t)} \bar{c}(\tau) d\tau + Z(t) - Y(t)
\]
by writing \( W(t) = \int_0^\infty (\pi(t + \tau)/\pi(t)) c(t + \tau) d\tau \) and differentiating with respect to \( t \).

However, there are important models where there exists no steady state and where the real interest rate is not constant. In particular, in the DHS model—which represents a closed economy with a stationary technology and which we have presented in its Cobb-Douglas version in Section 7—the real interest rate is decreasing along a path where capital is accumulated and resource flow diminishes. Indeed, (16) and (18) implies that the real interest is given by
\[
R(t) = \frac{\alpha}{k(0)/q + \beta t}
\]
along an efficient path with constant consumption in the DHS model, where \( \alpha \) and \( \beta \) are the functional shares of capital and resource input, \( k(0) \) is the initial capital stock, and \( q \) is the constant production along the egalitarian path. Since (17) and (A7) imply that
\[
R(t) \int_t^\infty \frac{\pi(\tau)}{\pi(t)} d\tau = \frac{\alpha}{\alpha - \beta} ,
\]
for all \( t \geq 0 \), it follows from (19) that sectoral income as defined in Definition 4 is given by

Labor: \( Z^L(t) = R(t) \int_t^\infty \frac{\pi(\tau)}{\pi(t)} (1 - \alpha - \beta) q d\tau = \frac{\alpha}{\alpha - \beta} (1 - \alpha - \beta) q \)

Resource: \( Z^R(t) = R(t) \int_t^\infty \frac{\pi(\tau)}{\pi(t)} \beta q d\tau = \frac{\alpha}{\alpha - \beta} \beta q \)

Production/Capital: \( Z^K(t) = R(t) \int_t^\infty \frac{\pi(\tau)}{\pi(t)} (\alpha - \beta) q d\tau = \frac{\alpha}{\alpha - \beta} (\alpha - \beta) q \).
Moreover, since consumption equals \((1 - \beta)q\), income at the national level is given by
\[
Z(t) = R(t) \int_t^\infty \frac{\pi(\tau)}{\pi(t)} (1 - \beta)q d\tau = \frac{1}{1 - \beta} (1 - \beta)q.
\]

Since net national product equals \((1 - \beta)q\) along the egalitarian path, it now follows that national income as defined in Definition 4 will exceed net national product and exceed consumption along an efficient path with constant consumption. This contradicts that national income as interest on wealth equals net national product and also means that real income as interest on wealth can not serve as an indicator for prudent behavior.

**Wealth equivalent income**

Real wealth equivalent income is the stream, constant in real terms, yielding the same wealth as the implemented path. Writing \(H\) for such income, with reference to Hicks (1946), this leads to the following defining equations.

At the national level:
\[
\int_t^\infty \pi(\tau) H(t) d\tau = \int_t^\infty \pi(\tau) c(\tau) d\tau
\]

At the sectoral level:
\[
\int_t^\infty \pi(\tau) H^j(t) d\tau = \int_t^\infty \mathbf{p}_\pi(\tau) \mathbf{x}^j(\tau) d\tau.
\]

Solving with respect to wealth equivalent income and defining the infinitely long-term interest rate, \(R_\infty\) by
\[
R_\infty(t) := \frac{\pi(t)}{\int_t^\infty \pi(\tau) d\tau} = \frac{\int_t^\infty \pi(\tau) R(\tau) d\tau}{\int_t^\infty \pi(\tau) d\tau}.
\]  

(A8)

we obtain the expressions of Definition 5.

**Definition 5** Real wealth equivalent income at time \(t\) is determined as follows:

At the national level:
\[
H(t) := \frac{\int_t^\infty \pi(\tau) c(\tau) d\tau}{\int_t^\infty \pi(\tau) d\tau} = R_\infty(t) W(t)
\]

At the sectoral level:
\[
H^j(t) := \frac{\int_t^\infty \mathbf{p}_\pi(\tau) \mathbf{x}^j(\tau) d\tau}{\int_t^\infty \pi(\tau) d\tau} = R_\infty(t) W^j(t).
\]

Note that (A8) entails that \(R_\infty(t)\) is a discount-weighted average of \(\{R(\tau)\}_{t=\tau}^\infty\); hence, \(R_\infty(t') < R_\infty(t) < R(t)\) for \(t' > t\) if \(R(\tau)\) is a decreasing function of \(\tau\) for \(\tau \geq t\).

Since the current value of sectoral cash flow sum up to national consumption for all \(t \geq 0\), it follows that real wealth equivalent income has the property that sectoral income
summed over all sectors add up to national income, \( H(t) = \sum_{j \in J} h^j(t) \). Moreover, it also follows that
\[
\dot{H}(t) = R_\infty(t) \left( H(t) - c(t) \right),
\]
entailing that \( c(t) \leq H(t) \) (and equivalently, \( \dot{H}(t) \geq 0 \)) is an indicator of prudent behavior.\(^5\) In particular, real wealth equivalent income at the national level is equal to consumption if consumption is constant.

Unfortunately, as discussed by Asheim (1997) and Sefton and Weale (2006), this approach is not compatible with the property that real income at the national level equals net national product in a closed economy with a stationary technology. To see this, note that by Sefton and Weale (2006, eq. (23)),
\[
Y(t) - H(t) = \int_t^\infty \left( 1 - \frac{R_\infty(t)}{R_\infty(\tau)} \right) \frac{\pi(\tau)}{\pi(t)} \dot{c}(\tau) d\tau.
\]
Hence, if consumption is always increasing and the real interest rate (and therefore the real infinitely long-term interest rate, too) is always decreasing, then \( \dot{c}(\tau) > 0 \) and \( 1 - R_\infty(t)/R_\infty(\tau) < 0 \), so that \( Y(t) < H(t) \).

References


\(^5\)Note that \( H(t) \) is the maximal consumption that can be sustained indefinitely only if intertemporal redistribution of consumption can be made at relative prices \( \{ \pi(t) \}_{t=0}^\infty \) without changing these prices. Due to general equilibrium effects, this will not be the case at the national level.


Income generated by a non-renewable resource

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Abstract

This paper applies the theory of sectoral income developed by Asheim and Wei (2009) for the purpose of determining and classifying the components of income generated by a non-renewable resource. It is shown that "sectoral net investments" in the theory of sectoral income equals the value of the change of all capital stocks in the sector. Furthermore, it compares the sectoral income method with other practical methods and concludes that the value of resource depletion defined in SEEA (2003) provides correct information only if the real interest rate is constant.

Key words: Income, sectoral net investments, non-renewable resource, comprehensive national accounting

JEL classification: C43, C82, D60, O47, Q32

1 Introduction

Every year considerable revenue is generated worldwide from non-renewable resources like oil and gas. How much of the revenue can be used for consumption if consumption is bounded by a guideline of prudent behavior? Recently, following the line of theoretical literature from Hicks (1946, Chapter 14) via Samuelson (1961) and Weitzman (1976) to

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Sefton and Weale (2006), a concept of sectoral income has been developed by Ashen and Wei (2009). This concept associates income with the present value of real interest on future consumption and savings with the present value of future consumption changes. A main result is that sectoral income can be split into three terms: current cash flow, net investments and price change effects. In the theory, net investments are associated with the present value of the change of future commodity flows in the sector. However, Ashem and Wei (2009) do not justify why they call this term net investments. The present paper will show that under certain assumptions, the concept of net investments coincide with the present value of the change of all the capital stocks in the sector. In this way, the terminology is justified.

In the theory of comprehensive national accounting, if a stationary technology is assumed, i.e. any variable determinant of current productive capacity is included in capital stocks, then net investments are associated with the change of capital stocks, where capital stocks includes not only the usual reproductive capital stocks, but also stocks of natural resources, environmental assets, human capital, and other productive assets. The value of national net investments are accordingly calculated as the sum of net investments evaluated at suitable accounting prices. In the same way, the value of net investments of a sector in an economy under stationary assumptions can be calculated as the sum of net investments in the sector evaluated at suitable accounting prices.

In practical accounting, however, information required by the method may not be available. First, it may be difficult to observe and measure the change of capital stocks at a given time. Since the depreciation of an in-use truck is not easy to estimate, it becomes difficult to determine the quantity of the truck. The development of an oil well is another example. Along with the oil well being deeper, the state of the well is changing over time. How can we properly measure the state of oil well and the change of the oil well? If we can not measure it properly, then we can not estimate net investments as the value of the change of capital stocks. Human capital is another typical example. How can we know properly how much human capital we have and how large its change is? Second, the prices of capital stocks may not be available even if the change of capital stocks can be measured properly. What is the proper price for the oil well mentioned above? Even though the historical costs of the well is known, it does not help us to evaluate the real
price of the well. As the well becomes deeper, its real price must be adjusted over time since the expected returns to the oil well are changing. This may also happen for un-marketed capital stocks like environmental assets and human capital stocks. Hence, we may prefer an alternative method to calculate the value of net investments at a sectoral level. The theory of sectoral income proposed by Asheim and Wei (2009) offers such an alternative. By the theory, the value of net investments can be estimated based on the projection of future cash flows to the resource sector.

In the present paper, after the notion of net investments in the theory is justified as the value of the change of all sectoral capital stocks, the theory is further applied to a non-renewable resource. Under certain assumptions, the net investments of the resource can be shown to equal exactly the shadow value of the resource depletion. Furthermore, several controversial questions are clarified. For example, how much is the net investments of a non-renewable resource? Is Hotelling rent a part of the income? How about Ricardian rent? How does the technical progress contribute to the income?

There are some practical methods used to estimate the income generated by non-renewable resources. Three of them are: net price method provided by Repetto et al. (1989), the user cost method developed by El Serafy (1981), and the net present value (NPV) method recommended by SNA (1993) and SEEA (2003). The paper compares these methods to the sectoral income method. An interesting result is that the value of resource depletion defined in SEEA (2003) can be divided to three parts: interest rate change effects, net investments, and price change effects. If interest rates are not constant over time, then these practical methods provide biased information on resource income and savings for tomorrow.

The paper is organized as follows. After a brief introduction to the theory of sectoral income in Section 2, Section 3 shows that sectoral net investments defined in the sectoral income coincide with the value of the change of capital stocks for any given sector, which provides a solid theoretical basis for the notion. Section 4 applies the theory to a non-renewable resource by assuming a competitive market economy. After practical methods are compared with the sectoral income method in Section 5, the final section offers concluding remarks. In the Appendices, I provide a technical explaining of national and sectoral income, a proof that sectoral net investments sum up to national net invest-
ments, and two applications of the theory respectively for a monopolist and exogenously determined extraction paths of a resource.

2 Sectoral income

Samuelson (1961, pp. 51–52) was the first to point out that welfare improvement can be indicated by the present value of future consumption changes in a market economy following an optimal path. This gives a welfare foundation for interpreting the present value of future consumption change as national savings. By using this notion of savings, Sefton and Weale (2006) prove that in an idealized world with given consumption flows, the concept of national income can be defined as a weighted average of future national consumption. Following the literature, Asheim and Wei (2009) define sectoral income as the present value of real interest on future sectoral cash flows. Appendix A briefly explains these concepts. The remainder of this section gives the main expressions for sectoral income.

Define the path of market (or calculated) real interest rates \( \{R(t)\}_{t=0}^{\infty} \) by

\[
R(t) = -\frac{\dot{\pi}(t)}{\pi(t)}
\]

for all time \( t \geq 0 \), where the the dot (\( \cdot \)) over a variable represents the first order derivative w.r.t. time and \( \pi(t) \) is a discount factor at time \( t \geq 0 \). By definition, \( \pi(0) = 1 \) and

\[
\pi(t) = e^{-\int_{0}^{t} R(\tau) d\tau}
\]

for all \( t \geq 0 \). Then real income from a sector \( j \) at any point in time \( t \geq 0 \) can be defined by

\[
Y^j(t) = \int_{t}^{\infty} \frac{\pi(\tau)}{\pi(t)} R(\tau) P^j(\tau) x^j(\tau) d\tau,
\]

where \( \{x^j(t)\}_{t=0}^{\infty} \) is the path of sector \( j \)’s vector of commodity flows excluding consumption flows acquired for end use, and \( \{P^j(t)\}_{t=0}^{\infty} \) is the corresponding path of market (or calculated) real prices of these commodity flows. \( P^j(\tau) x^j(\tau) \) represents the cash flow to sector \( j \) at any time \( \tau \geq 0 \). An explaining of the definition is given in Appendix A.
Differentiation of $\pi(\tau)P^j_x(\tau)x(\tau)$ yields

$$\frac{d}{d\tau}(\pi(\tau)P^j_x(\tau)x(\tau)) = \pi(\tau)P^j_x(\tau)x(\tau) + \pi(\tau)\dot{P}^j_x(\tau)x(\tau) + \pi(\tau)P^j_x(\tau)\dot{x}(\tau).$$

Integrating on both sides under the assumption that $\pi^j(\tau)P^j_x(\tau)x(\tau) \to 0$ as $t \to \infty$, leads to the following equation:

$$-\pi^j(t)P^j_x(t)x(t) = \int_t^\infty \pi^j(\tau)P^j_x(\tau)x(\tau)d\tau + \int_t^\infty \pi^j(\tau)\dot{P}^j_x(\tau)x(\tau)d\tau + \int_t^\infty \pi^j(\tau)P^j_x(\tau)\dot{x}(\tau)d\tau.$$

By rearranging this equation, invoking (1) and applying (3), we decompose the real sectoral income as three terms: current cash flow, net investments, and price change effects.

$$Y^j(t) = P^j_x(t)x^j(t) + \int_t^\infty \frac{\pi(\tau)}{\pi(t)}P^j_x(\tau)x^j(\tau)d\tau + \int_t^\infty \frac{\pi(\tau)}{\pi(t)}\dot{P}^j_x(\tau)x^j(\tau)d\tau.$$

In the expression, net investments reflect the effects on sectoral income due to change of future sectoral commodity flows alone. Price change effects shows the share of sectoral income caused by change of future prices of all the sectoral commodity flows alone. Then net investments can be taken as a volume indicator and price change effects a price indicator.

3 Sectoral net investments

This section will show the consistence between the concept of sectoral net investments and the value of the change of capital stocks for any given sector.

Consider any given sector in a competitive economy. At each point in time, corresponding to commodities entering (or exiting) the sector, cash flows in the form of payment (or revenue) are exiting (or entering) the sector. Assume the commodity flows to the sector at each time $t$ is $n$-dimensional and denoted by a vector $x(t)^1$. A positive

1In the section, I ignore the superscript $j$, which indicates the sector $j$, for notational simplification.
element of the vector \( \mathbf{x} \) implies its corresponding commodity goes out of the sector as output and the sector receives revenue. On the contrary, if an element is negative, its corresponding commodity enters the sector as input and the sector pays out. Let \( \mathbf{P}_x(t) \) denote the real market prices of \( \mathbf{x}(t) \) at each time \( t \). All the prices are taken as given by producers in the sector since the market is competitive.

At each point in time, there exist certain productive capital stocks that the sector can utilize for income generation. The equipment of a firm is an example. As mentioned at the beginning of the paper, it may be difficult to measure the quantities of these capital stocks even though their historical costs are known. The real prices of these capital stocks may not be available. Even though these necessary information may not be available in practice, they are presumed to be known in the section. Hence, the net investments in the theory of sectoral income can be shown theoretically to equal the value of the change of all the sectoral capital stocks.

The capital stocks of the sector may change over time on the condition of certain constraints due to the scarcity of capital stocks. At each point in time, the constraints are related to four types of variables: the commodity flows to the sector, available capital stocks, the change of capital stocks, and pure time-related variables like exogenous technical progress. The constraints may be one or many. Let \( \mathbf{K} \) denote the \( m \)-dimensional vector of available capital stocks in the sector, and \( \dot{\mathbf{K}} \) the \( m \)-dimensional vector of instantaneous change of \( \mathbf{K} \). We may apply the general form for all the possible constraints on capital stocks at time \( \tau \) as follows:

\[
\mathbf{f} \left( \mathbf{x}(\tau), \mathbf{K}(\tau), \dot{\mathbf{K}}(\tau), \tau \right) \geq 0,
\]

where \( \mathbf{f} \) is an \( r \)-dimensional vector of functions \( (1 \leq r < \infty) \), so that the inequality \( \mathbf{f}(\cdot) \geq 0 \) represents the \( r \) inequalities

\[
f_k \left( \mathbf{x}(\tau), \mathbf{K}(\tau), \dot{\mathbf{K}}(\tau), \tau \right) \geq 0, \quad k = 1, \ldots, r.
\]

Suppose \( \mathbf{f} \) is a convex function and the boundary \( \mathbf{f}(\cdot) = 0 \) is smooth and differentiable w.r.t. all the arguments.
Given current available capital stocks, the competitive sector aims to maximize the present value of future cash flows such that the function (5) is satisfied:

\[
\max_x \int_t^\infty \frac{\pi(\tau)}{\pi(t)} P_x(\tau) x(\tau) \, d\tau
\]

s.t. \( f(x(\tau), K(\tau), \dot{K}(\tau), \tau) \geq 0 \), \( K(t) \) is given, \hfill (6)

where \( \pi \) is a discount factor chosen by producers with initial value \( \pi(0) = 1 \) for the sector and two neighboring vectors in expressions mean the inner-product between them. By solving the problem, we will show that the definition of sectoral net investments given by Asheim and Wei (2009) coincides with the shadow value of the change of sectoral capital stocks.

The Hamiltonian of the problem is

\[
H = \frac{\pi(\tau)}{\pi(t)} P_x(\tau) x(\tau) + \lambda(\tau) \dot{K}(\tau),
\]

and the Lagrangian is

\[
L = H + \sum_{i=1}^{r} \mu_i(\tau) f_i(x(\tau), K(\tau), \dot{K}(\tau), \tau),
\]

where \( \lambda = (\lambda_1, \lambda_2, ..., \lambda_m) \) is the vector of shadow prices of the state variable \( K \) and \( \mu = (\mu_1, \mu_2, ..., \mu_r) \) the shadow prices of the constraints \( f \). The necessary conditions for an optimum, among others, include

\[
\frac{\partial L}{\partial x_j} : \quad \frac{\pi(\tau)}{\pi(t)} P_{xj}(\tau) = -\sum_{i=1}^{r} \mu_i(\tau) \frac{\partial f_i}{\partial x_j} \quad j = 1, ..., n \tag{7}
\]

\[
\frac{\partial L}{\partial K_j} : \quad \lambda_j(\tau) = -\sum_{i=1}^{r} \mu_i(\tau) \frac{\partial f_i}{\partial K_j} \quad j = 1, ..., m \tag{8}
\]

\[
\frac{\partial L}{\partial \dot{K}_j} : \quad \dot{\lambda}_j(\tau) = -\sum_{i=1}^{r} \mu_i(\tau) \frac{\partial f_i}{\partial \dot{K}_j} \quad j = 1, ..., m \tag{9}
\]

\[
\mu_i(\tau) \geq 0, \quad \mu_i(\tau) f_i(\tau) = 0 \quad i = 1, ..., r. \tag{10}
\]

First condition (7) shows that total shadow value of marginal production w.r.t. each commodity is equated to the present value price of the commodity. Condition (8) implies the
shadow price of each capital is determined by total shadow value of marginal production w.r.t. the capital change. The arbitrage condition (9) means that total marginal value of each capital is equated to the instantaneous change of shadow price of the capital over time. The last one (10) is the complementary slackness conditions.

Since the market prices $P_x$ are non-negative for normal commodities and shadow prices $\lambda, \mu \geq 0$, then by the first two conditions (7) and (8), the derivatives of the function $f$ exhibit properties as follows,

$$\sum_{i=1}^{r} \mu_i(\tau) \frac{\partial f_i}{\partial x_j} \leq 0, \quad (11)$$

$$\text{and} \quad \sum_{i=1}^{r} \mu_i(\tau) \frac{\partial f_i}{\partial K_j} \leq 0. \quad (12)$$

Given other things being equal, inequality (11) shows if more commodities are available to go out of the sector, i.e. more output from the sector, then the production approaches much closer to the boundary of the constraints $f(\cdot) = 0$ as a whole; If one more unit of input enters the sector, i.e. its corresponding element of $x$ is one unit smaller, then the constraints are relaxed to allow more output to produce. Hence, if the input is denoted as positive in the vector $x$, then the sign of (11) should be reversed to be non-negative, which is the case of a non-renewable resource described in the next section. By inequality (12), the change of capital stocks can be thought of as a kind of output of the sector, which is not sold at current time. The value of the change of capital stocks will be realized in the future production and contribute to future cash flow to the sector.

By condition (9), if the shadow price of a capital is decreasing all the time, i.e. $\dot{\lambda} \leq 0$, then as a whole, the constraints are relaxed with more available capital stocks: $\sum_{i=1}^{r} \mu_i(\tau) \frac{\partial f_i}{\partial K_j} \geq 0$.

By the condition (10), we can always have

$$\sum_{i=1}^{r} \mu_i(\tau) \frac{df_i(\tau)}{d\tau} = 0 \quad (13)$$

Since if $\mu_i(\tau) > 0$, then $f_i(\tau) = 0$, which implies $df_i(\tau) / d\tau = 0$. By the definition of
\[ f(\cdot) \text{ in (5), we have} \]
\[
\frac{df_i}{d\tau} = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} \dot{x}_j(\tau) + \sum_{j=1}^{m} \frac{\partial f_i}{\partial K_j} \dot{K}_j(\tau) + \sum_{j=1}^{m} \frac{\partial f_i}{\partial \dot{K}_j} \frac{d\dot{K}_j}{d\tau}(\tau) + \frac{\partial f_i}{\partial \tau} \quad i = 1, \ldots, r. \tag{14}
\]

Then we substitute (14) to (13) to obtain:
\[
\sum_{i=1}^{r} \sum_{j=1}^{n} \mu_i(\tau) \frac{\partial f_i}{\partial x_j} \dot{x}_j(\tau) + \sum_{i=1}^{r} \sum_{j=1}^{m} \mu_i(\tau) \frac{\partial f_i}{\partial K_j} \dot{K}_j(\tau) + \sum_{i=1}^{r} \sum_{j=1}^{m} \mu_i(\tau) \frac{\partial f_i}{\partial \dot{K}_j} \frac{d\dot{K}_j}{d\tau}(\tau) + \sum_{i=1}^{r} \mu_i(\tau) \frac{\partial f_i}{\partial \tau} = 0. \tag{15}
\]

By applying conditions (7), (8), (9) and the equation (15), we can express the present value of the change of commodity flows at each point in time by
\[
\frac{\pi(\tau)}{\pi(t)} \sum_{j=1}^{n} P_{x_j}(\tau) \dot{x}_j(\tau) \tag{16}
\]
\[
\begin{align*}
&= -\sum_{i=1}^{r} \sum_{j=1}^{n} \mu_i(\tau) \frac{\partial f_i}{\partial x_j} \dot{x}_j(\tau) \quad \text{by (7)} \\
&= \sum_{i=1}^{r} \sum_{j=1}^{m} \mu_i(\tau) \frac{\partial f_i}{\partial K_j} \dot{K}_j(\tau) + \sum_{i=1}^{r} \sum_{j=1}^{m} \mu_i(\tau) \frac{\partial f_i}{\partial \dot{K}_j} \frac{d\dot{K}_j}{d\tau}(\tau) + \sum_{i=1}^{r} \mu_i(\tau) \frac{\partial f_i}{\partial \tau} \quad \text{by (15)} \\
&= -\sum_{j=1}^{m} \lambda_j(\tau) \dot{K}_j(\tau) - \sum_{j=1}^{m} \lambda_j(\tau) \frac{d\dot{K}_j}{d\tau}(\tau) + \sum_{i=1}^{r} \mu_i(\tau) \frac{\partial f_i}{\partial \tau} \quad \text{by (8) and (9)} \\
&= -\sum_{i=1}^{m} \frac{d}{d\tau} \left[ \lambda_i(\tau) \dot{K}_i(\tau) \right] + \sum_{i=1}^{r} \mu_i(\tau) \frac{\partial f_i}{\partial \tau} \quad \text{by integration by parts.}
\end{align*}
\]

Then integration on both sides of (16) yields the sectoral net investments as the present
value of the change of future sectoral commodity flows

\[
\int_t^\infty \frac{\pi(\tau)}{\pi(t)} P_x(\tau) \dot{x}(\tau) \, d\tau = \int_t^\infty \frac{\pi(\tau)}{\pi(t)} \sum_{j=1}^n P_{x_j}(\tau) \dot{x}_j(\tau) \, d\tau
\]

\[
= - \int_t^\infty \sum_{i=1}^m \frac{d}{d\tau} \left[ \lambda_i(\tau) \dot{K}_i(\tau) \right] \, d\tau + \int_t^\infty \sum_{i=1}^r \mu_i(\tau) \frac{\partial f_i}{\partial \tau} \, d\tau
\]

\[
= \sum_{i=1}^m \lambda_i(t) \dot{K}_i(t) + \int_t^\infty \sum_{i=1}^r \mu_i(\tau) \frac{\partial f_i}{\partial \tau} \, d\tau
\]

\[
= \underline{\lambda(t) \dot{\mathbf{K}}(t)} + \int_t^\infty \underline{\mu(\tau) \frac{\partial f}{\partial \tau}} \, d\tau
\]

\[
\text{shadow value of the change of capital stocks} \quad \text{value of passage of time}
\]

given the transversality conditions \( \lambda(\tau) \dot{\mathbf{K}}(\tau) \to 0 \) and \( \mu(\tau) \frac{\partial f}{\partial \tau} \to 0 \) as \( \tau \to \infty \). The expression shows \textit{sectoral net investments} in the theory of sectoral income can be expressed by two terms: the sum of shadow value of the change of capital stocks, and the shadow value of pure time due to factors related to time other than commodities \( \mathbf{x} \) and capital \( \mathbf{K} \). The second term on the r.h.s. of (17) measures the so-called "value of passage of time", which is taken as one part of the net investments in the literature (for a discussion, see Asheim, 2003, p. 124-5). The result in (17) justifies the notion a solid theoretical basis for the notion of sectoral net investments in the theory of sectoral income. Since the information of both commodity flows and their prices in the future are required by the practical wealth-based income methods, the l.h.s. of (17) can be used as an alternative method to estimate the value of sectoral net investments without more information demand than wealth-based income methods.

\textit{Stationary technology.} If at the beginning, we assume the constraints \( f(\mathbf{x}(\tau), \mathbf{K}(\tau), \dot{\mathbf{K}}(\tau)) \geq 0 \),

then the second term on the r.h.s. of (17) disappears and the sectoral net investments can be simplified as one term

\[
\int_t^\infty \frac{\pi(\tau)}{\pi(t)} P_x(\tau) \dot{x}(\tau) \, d\tau = \lambda(t) \dot{\mathbf{K}}(t),
\]

(18)
which shows sectoral net investments defined as present value of the change of future commodity flows are equivalent to the shadow value of the change of all the capital stocks evaluated by producers in the sector.

Notice that the shadow prices of capital stocks $\lambda$ may be different for producers in an imperfect capital market. If the capital market is not perfect, the market prices of capitals may not represent the real value of capital stocks for a producer. Then it is not appropriate to estimate sectoral net investments by the market prices. Further, the value of passage of time is not easy to estimate in practice. In these cases, the l.h.s. of (17) really provides a better alternative to estimate the value of sectoral net investments. The approach would be useful for some un-marketed capital like natural resources, environmental amenities, and human capital.

*Competitive capital market and stationary technology.* If further the capital market is competitive and perfect, then the shadow prices of the capital stocks coincide with their market prices, i.e. $P_K(t) = \lambda(t)$, and the sectoral net investments defined by (4) also coincide with the market value of the change of all the capital stocks $^2$,

$$
\int_t^\infty \frac{\pi(\tau)}{\pi(t)} P_x(\tau) \dot{x}(\tau) d\tau = P_K(t) \dot{K}(t).
$$

(19)

Furthermore, the notion exhibits the property that sectoral net investments sum up to equal national net investments as shown in Appendix B. This implies that price change effects by sectors defined by (4) exactly cancel out each other and price change effects disappear at the national level. If the technology is not stationary, the sum of sectoral price change effects may differ from zero as the value of passage of time appears. One special case studied by Sefton and Weale (1996) is the effects of changing terms-of-trade faced by a resource exporter.

It might also be interesting to consider income of a monopolist. An example could be the OPEC in the world oil market. Since a monopolist can control prices by choosing output level, price change over time may result from the market power owned by the monopolist. The value of the change of capital stocks are influenced by such kinds of

---

$^2$In this case, we can apply a simple method following the proof of Theorem 1 in Dixit, Hammond and Hoel (1980), where they proved a similar equation for an economy.
price change. As shown in Appendix C, the price change effects caused by the market power contribute to the value of the change of all the sectoral capital stocks. Hence, the identity (17) does not hold in this special case. However, if there is no other kinds of price change over time, the net investments and price change effects estimated by the sectoral income method sum up to equal the value of the change of all the sectoral capital stocks. This provides a correct indicator of savings for tomorrow.

4 A non-renewable resource in a competitive market

In this section, the notions of sectoral income and sectoral net investments are applied to a sector of a non-renewable resource extraction, e.g. the crude oil and natural gas extraction. The resource extracted from the underground may be heterogenous since a resource reservoir generally consists of stocks of various characteristics and even several types of resource. The initial level of the resource stocks are assumed to be given as a vector $S(0)$ and no new discoveries in the future. At any time $t \geq 0$, the resource extraction rate is denoted by a vector $r(t) \geq 0$ and the remaining resource stock by $S(t)$. Since the initial resource stocks are given, then the resource will be used up at a future point in time, which is denoted by $T$. Then the remaining resource stocks equal the sum of all the future extraction rates, i.e.

$$S(t) = \int_{t}^{T} r(s) ds$$

(20)

for all $t \geq 0$. Differentiation on both sides of (20) w.r.t. time $t$ yields

$$\dot{S}(t) = -r(t),$$

(21)

which shows the resource stocks at any time is diminishing if there is any positive amount of the resource extracted from the underground. This simplifies the path of change of the resource stock.

The extraction cost tends to rise as the remaining stocks decline ceteris paribus. For example, a petroleum resource may occur near the surface of the earth's crust and at
various depths and in various degrees of contiguous abundance. Oil pressure is commonly augmented with water and sometimes natural gas, injected into a declining stock during the extraction. A usual way to capture such effects is to include the remaining stocks $S$ in the extraction function.

Moreover, the technical progress always happens over time. It tends to improve the total factor productivity (TFP) and alleviate the increasing marginal cost. Here the technical progress is thought of as exogenously determined and an individual producer adopts the popular technology. The extraction function in the section is designed to depend directly on the time variable $t$ to capture the effects of technical progress.

The extractive efforts at each time denoted by a vector $a(t) \geq 0$ are used to develop the extraction project for two reasons$^3$. One reason is that some efforts is used to form man-made capital represented by a vector $k(t) \geq 0$, such as equipment, which is utilized to produce in the future. such kind of efforts may happen any time. Another reason is that some efforts, like labor and energy, is directly used to current extraction activities by combination with existing man-made capital, remaining resource stocks, and the popular technology. It is a fact that at the beginning of the project, no resource is extracted from the underground and all efforts are used to form man-made capital for future production. Then at each point in time, there are not one-to-one relations between extraction rate $r$ and extractive efforts $a$, and neither are the relations between the extractive efforts $a$ and the change of man-made capital $\dot{k}$. A differentiable function can be used to represent the extraction activities at any time $t \geq 0$,

$$g \left( r(t), a(t), S(t), k(t), \dot{k}(t), t \right) = 0$$  \hfill (22)

where $S$ is the remaining resource stock, $k$ is man-made capital stocks accumulated from the very beginning, and the argument of time $t$ represents the exogenously known technology.

Given other things being equal, more extractive efforts are required to extract more resource from the underground; more resource stock or man-made capital implies easier

$^3$In practice, some efforts are used to explore more resource stock. In the paper, the exploration activities are abstracted and so are the efforts for exploration.
extraction; technical progress means less extractive efforts required; and more accumu-
lation of man-made capital implies less extraction rate. These facts enable us to assume
the properties of first order derivatives as follows: $g_{r_i}' \leq 0$, $g_{a_i}' \geq 0^4$, $g_{s_i}' \geq 0$, $g_{k_i}' \geq 0$, $g_{z}' \geq 0$, and $g_{k_i}' \leq 0$, where the subscripts $i$ represent the element $i$ of the vectors.

The extracted resource at any time $t \geq 0$ is sold immediately at a real market price
denoted by a vector $\mathbf{P}_r(t)$. Suppose the real market price of the extractive efforts is
known for sure at any time $t$ as a vector $\mathbf{P}_a(t)$. Then the real cash flow to the resource
sector at any time $t$ is given by

$$ Q(t) = \mathbf{P}_r(t) \mathbf{r}(t) - \mathbf{P}_a(t) \mathbf{a}(t). $$

(23)

According to the definition of sectoral income (3), by the expression for cash flows in
(23), the income generated by the resource can be written as

$$ Y^r(t) = \int_t^\infty \frac{\pi(\tau)}{\pi(t)} R(\tau) \left[ \mathbf{P}_r(\tau) \mathbf{r}(\tau) - \mathbf{P}_a(\tau) \mathbf{a}(\tau) \right] d\tau. $$

(24)

The resource income is the present value of real interest on all the future cash flows
generated by the resource extraction.

Moreover, by expression (4), resource income is decomposed to be three elements:
current cash flow, net investments, and price change effects, i.e.

$$ Y^r(t) = \underbrace{\mathbf{P}_r(t) \mathbf{r}(t) - \mathbf{P}_a(t) \mathbf{a}(t)}_{\text{current cash flow}} + \int_t^\infty \frac{\pi(\tau)}{\pi(t)} \left[ \mathbf{P}_r(\tau) \dot{\mathbf{r}}(\tau) - \mathbf{P}_a(\tau) \dot{\mathbf{a}}(\tau) \right] d\tau $$

$$ + \int_t^\infty \frac{\pi(\tau)}{\pi(t)} \left[ \dot{\mathbf{P}}_r(\tau) \mathbf{r}(\tau) - \dot{\mathbf{P}}_a(\tau) \mathbf{a}(\tau) \right] d\tau, $$

(25)

which shows that changes of both the extraction rates and the extractive efforts contribute
to the net investments and similarly, the changes of both prices contributes to price change
effects.

\footnote{Here the efforts $a$ is input to the resource sector and denoted as positive flows. If it is denoted
by negative flows to the sector, then the partial derivative is also negative, the same as that wrt. the
extraction rate $r$.}
The method expressed by (25) can be applied to any planned path of extraction rates. Appendix D gives some expressions for several specific paths of extraction rates and exogenous prices.

As shown in the previous section, the net investments should correspond to the value of the change of all the capital stocks, which are resource stock, man-made capital stocks and technical progress for the resource sector. Next subsection shows it is true for the resource sector.

4.1 Net investments generated by the resource

Assume the resource market is competitive. The resource producers maximize their profits by taking the real market prices as given, i.e. an individual producer has no effects on the market prices or the effects can be ignored from the producer’s viewpoint. Hence, the path of real market prices \( \{ P_r (\tau) \}_{\tau=0}^{\infty} \) is taken as exogenously given even though it may not be a constant path over time.

Suppose at any given time, a producer is able to adjust her extraction rates and extractive efforts freely to obtain the maximum of her profits. Hence, what the producer can do is to find out an optimal path of future extraction rates and extractive efforts such that the present value of future cash flows is maximized at any time \( t \geq 0 \),

\[
\text{max}_{r(\tau), a(\tau)} \int_t^{\infty} \frac{\pi(\tau)}{\tau(t)} [P_r (\tau) r (\tau) - P_a (\tau) a (\tau)] d\tau
\]

s.t. \( \dot{S}(\tau) = -r(\tau) \)

\[
g \left( r(\tau), a(\tau), S(\tau), k(\tau), \dot{k}(\tau), \tau \right) = 0
\]

\( S(t), k(t) \) is given. \hfill (26)

The maximization problem can be re-organized as a special example of the general framework described in the previous section. Let vectors \( P_x (\tau) = [P_r (\tau), P_a (\tau)] \), \( x(\tau) = [r(\tau), -a(\tau)] \), \( K(\tau) = [S(t), k(t)] \), and the vector of functions \( f = [f_1, f_2] \) in the constraints (5), where

\[
f_1 \left( x(\tau), K(\tau), \dot{K}(\tau), \tau \right) = -\dot{S}(\tau) - r(\tau) \quad \text{and}
\]

\[
f_2 \left( x(\tau), K(\tau), \dot{K}(\tau), \tau \right) = g \left( r(\tau), a(\tau), S(\tau), k(\tau), \dot{k}(\tau), \tau \right).
\]
Then the maximization problem (26) can be rewritten as

\[
\max_x \int_t^\infty \frac{\pi(\tau)}{\pi(t)} P_x(\tau) x(\tau) \, d\tau \\
\text{s.t. } f_1 = -\dot{S}(\tau) - r(\tau) \geq 0 \\
f_2 = g \left( r(\tau), a(\tau), S(\tau), k(\tau), \dot{k}(\tau), \tau \right) \geq 0
\]

where the inequality \( f_1 \geq 0 \) implies the resource could be discarded after extracted from the underground and another inequality \( f_2 \geq 0 \) represents the production probabilities set. A profit-maximizer will produce efficiently until the equations hold, which are exactly the same constraints as in the problem (26).

Hence, we are able to apply directly the results by solving the problem (6) in the previous section\(^5\). By (17), net investments generated by the resource sector are

\[
\begin{align*}
\int_t^\infty & \frac{\pi(\tau)}{\pi(t)} \left| P_r(\tau) \dot{r}(\tau) - P_a(\tau) \dot{a}(\tau) \right| \\
= & \int_t^\infty \frac{\pi(\tau)}{\pi(t)} P_x(\tau) \dot{x}(\tau) \, d\tau \\
= & \frac{\lambda(t) \dot{K}(t)}{\text{shadow value of the change of capital stocks}} + \int_t^\infty \frac{\mu(\tau)}{\text{value of passage of time}} \frac{\partial f_2}{\partial \tau} \, d\tau \\
= & \lambda_s(t) \dot{S} + \lambda_k(t) \dot{k} + \int_t^\infty \mu(\tau) \frac{\partial f_2}{\partial \tau} \, d\tau
\end{align*}
\]

given the transversality conditions \( \lambda_s(\tau) \dot{S} \to 0, \lambda_k(\tau) \dot{k} \to 0 \) as \( \tau \to \infty \), where \( \lambda_s, \lambda_k, \) and \( \mu \) are the shadow prices of the remaining resource stock \( S \), man-made capital \( k \), and the constraint \( f_2(\cdot) \geq 0 \) respectively.

The expression (28) shows that the net investments as present value of the change of commodity flows alone coincide with the shadow value of the change of all the capital stocks including resource stock, man-made stocks, and the technology. Since the shadow prices are neither observable nor easy to estimate properly in practical accounting, the alternative method to estimate sectoral net investments provided by (28) is more plausible, which, in addition, demands no more information than the popular wealth-based

---

\(^5\)The same results can be found by directly solving the problem (26), which is found in Appendix E.
methods in practical accounting.

Since the technical progress influences the production/cost directly by reducing the extractive efforts $a$, its effects enters the net investments term. In some cases, however, it may be plausible to take it as kind of supply of extraction service, which reduces virtually the real prices of the efforts and increases the net revenue earned by the resource. In this sense, the effects of technical progress can be taken as part of price change effects instead of net investments (Wei, 2006, Sections 5.5, 5.6).

Next subsection provides a more specific example. By the example, Hotelling rent of the resource is shown to cancel out the net investments of the resource and only Ricardian rent is a part of the resource income.

### 4.2 Income of a reservoir

Consider a reservoir of a resource like oil described in Section 8 of Asheim and Wei (2009). The size of the reservoir is known as $S(0)$ at initial time, and producers may choose to extract or not, i.e. the resource extraction rates $r(t) \geq 0$ for all time $t \geq 0$. The extraction is conducted though a well-defined production function $h$,

$$r = h(e),$$

satisfying $h(0) = 0$, where $e$ is non-negative extractive efforts. Assume constant real prices of the extracted resource and extractive efforts as $P_r$ and $P_e$ respectively. Hence the cash-flow at time $t$ is given by

$$P_r r(t) - P_e e(t),$$

where $r(t) = h(e(t))$ for all $t \geq 0$. The extractive efforts $e$ can be freely obtained in the market at the given price $P_e$ so that producers do not worry about the scarcity of the efforts $e$. However, the extracted resource $r$ is restricted by the size of the reservoir. We always have

$$\dot{S}(t) = -r(t).$$

(29)
Hence, the producers extract the resource from the reservoir in a profit-maximizing manner,

$$\max_{r,e} \int_0^\infty e^{-Rt} \left[ P_r r(t) - P_e e(t) \right] dt$$

s.t. \( \dot{S}(t) = -r(t) \)

\( r(t) = h(e(t)) \),

(30)

where \( R \) is a constant real interest rate. The Hamiltonian of the problem is

$$H = e^{-Rt} \left[ P_r r(t) - P_e e(t) \right] - \lambda(t) r(t)$$

$$= e^{-Rt} \left[ P_r h(e(t)) - P_e e(t) \right] - \lambda(t) h(e(t)),$$

where \( \lambda \geq 0 \) is the shadow price of the resource. First order conditions w.r.t. \( e \) and \( S \) can be expressed by

$$e^{-Rt} [P_r h' - P_e] = \lambda(t) h'.$$

(31)

$$\dot{\lambda}(t) = 0 \Rightarrow \lambda(t) = \lambda = \text{constant}$$

(32)

Then the net investments of the reservoir can be calculated by

$$\int_0^\infty e^{-Rt} \left[ P_r \dot{r}(t) - P_e \dot{e}(t) \right] dt$$

(33)

$$= \int_0^\infty e^{-Rt} \left[ P_r h' - P_e \right] \dot{e}(t) dt \quad \text{by (30)}$$

$$= \int_0^\infty \lambda(t) h' \dot{e}(t) dt \quad \text{by (31)}$$

$$= \int_0^\infty \lambda(t) \dot{r}(t) dt \quad \text{by (30)}$$

$$= \lambda r(t)|_0^\infty \quad \text{by (32)}$$

$$= \lambda \dot{S}(0) \quad \text{by (29)}.$$

Hence, the reservoir’s net investments equals the shadow value of the change of the reservoir’s size, which can be thought of as the depletion value of the reservoir at current time. by (31) and (29), we know the constant shadow price \( \lambda = e^{-Rt} [P_r - P_e h'] \), which
is the unit Hotelling rent since $P_e/\ell' = P_e d\ell / dr$ can be interpreted as marginal extraction costs. Together with (29), the net investments can be interpreted as the negative Hotelling rent of the resource.

By (25), the income of the reservoir can be decomposed as

$$Y^r (t) = \underbrace{P_e r(t) - P_e c(t)}_{\text{current cash flow}} + \underbrace{\lambda \dot{S}(0)}_{\text{net investments}} + \underbrace{0}_{\text{price change effects}}.$$ 

In this case, current cash flow is the sum of Hotelling rent and Ricardian rent. Then the resource income coincides with Ricardian rent since the net investments cancel out Hotelling rent.

## 5 Comparison with other practical methods

To simplify the notation, this section assumes one dimensional stock and extraction rate of a non-renewable resource.

### 5.1 Net price method

The net price method is a practical method, which is provided by Repetto et al. (1989). Roughly speaking, the method is based on the Hotelling rent model (Hotelling, 1931). The core assumption of the method is that the rent per unit of the non-renewable resource will keep increasing at a rate equal to the interest rate as the resource becomes scare. The assumption then implies the present value of unit resource rent is constant all the time and there is no need to discount future resource rent. Hence, the value of the resource stock is calculated as the current rent per unit of resource times the size of the resource stock. The resource income is the interest on the value of the resource stock. Moreover, the difference between current resource rent and the income by the method is thought of as the value of resource depletion.

The method assumes that the entire per unit resource rent at the current time is Hotelling rent, which implicitly assumes the marginal cost of the extraction coincides the average unit cost at any time. Then the method is not plausible if there exists Ricardian rent, which is generated due to marginal cost varying with the extraction rate.
By using the notation in this paper, the net price method can be described as follows.

First calculate the average unit resource rent at current time \( t \), \( P_r(t) - P_a(t)a(t)/r(t) \). Then the assumption of constant present value of the average unit resource rent implies

\[
\pi(\tau) \left[ P_r(\tau) - P_a(\tau)a(\tau)/r(\tau) \right] = \pi(t) \left[ P_r(t) - P_a(t)a(t)/r(t) \right]
\]

(34)

for all \( \tau \geq t \). By using (34), the resource income by the method equals real interest on the present value of the resource stock,

\[
Z(t) = R(t) \left[ P_r(t) - P_a(t)a(t)/r(t) \right] S(t),
\]

(35)

and the value of resource depletion defined by the method equals

\[
\left[ P_r(t)r(t) - P_a(t)a(t) \right] \left[ 1 - R(t) \frac{S(t)}{r(t)} \right].
\]

(36)

which is determined by the current cash flow, current real interest rate, and the life expectancy of the resource.

Under the assumption (34), by the expression of sectoral income (24), the real resource income is calculated by

\[
Y^r(t) = \left[ P_r(t) - P_a(t)a(t)/r(t) \right] \int_t^\infty R(\tau)r(\tau) d\tau
\]

(37)

\[
= R(t) \left[ P_r(t) - P_a(t)a(t)/r(t) \right] S(t) + \left[ P_r(t) - P_a(t)a(t)/r(t) \right] \int_t^\infty \hat{R}(\tau) S(\tau) d\tau,
\]

where the second equation is obtained by integration by parts using (20). The first term on the r.h.s. is the interest on the value of the resource stock evaluated at the current average unit resource rent and the second term is the effects of interest rate change. Obviously the first term coincides with the income by the net price method in (35). If the interest rate is constant over time, then the income estimated by the two methods coincides with each other.

By (25) and (37), the value of resource depletion defined by the net price method in (36) can be decomposed as three parts: interest rate change effects, the net investments
and price change effects.

\[
\left[ P_r(t) - P_a(t)a(t)/r(t) \right] \int_t^\infty \hat{R}(\tau) S(\tau) d\tau \tag{38}
\]

\[
- \int_t^\infty \frac{\pi(\tau)}{\pi(t)} \left[ P_r(\tau) \hat{r}(\tau) - P_a(\tau) \dot{a}(\tau) \right] d\tau - \int_t^\infty \frac{\pi(\tau)}{\pi(t)} \left[ \dot{P}_r(\tau) r(\tau) - \dot{P}_a(\tau) a(\tau) \right] d\tau,
\]

If the interest rate is constant over time, then the value of the depletion is simplified as
the sum of net investments and price change effects.

Hence, constant real interest rate over time is crucial for the net price method to
achieve the correct results for the presumed setting. If real interest rates vary over time,
then the income by net price method differs from the one by sectoral income method as
shown in (37). And as shown in (38), the so-called value of resource depletion differs from
the minimum savings for tomorrow suggested by the sectoral income method, which is
indicated by the sum of net investments and price change effects by the sectoral income
method.

5.2 User cost method

The user cost method initially put forward by El Serafy (1981) converts the future resource
rent to a so-called permanent income stream by making strict assumptions on future
extraction rates and cash flows. The future extraction rates are assumed to be exactly the
same as the current one until the resource is exhausted. In addition, the method assumes
constant real cash flows generated by the resource until the resource is exhausted. The
method also assumes constant interest rate \( R \) over time. One example satisfying these
assumptions is the case 1 in Appendix D.

The user cost method can be stated as follows. Before the time \( T \) when the resource
is exhausted, the extraction rate \( r \) is the same as the current extraction rate; After the
time \( T \), it becomes zero. i.e. \( r(s) = r(t) \) for all \( t \leq s \leq T \) and \( r(s) = 0 \) for all \( s > T \).

The assumption of constant real cash flows then implies

\[
P_r(\tau) r(\tau) - P_a(\tau) a(\tau) = P_r(t) r(t) - P_a(t) a(t) \tag{39}
\]
for all time $t \leq \tau \leq T$. It is not necessary to keep all prices and the efforts constant.

The user cost method calculates resource income by assuming the present value of the future resource rent equals the present value of a constant cash flow until infinity. The constant cash flow until infinity is called the resource income by the user cost method, $Z^u(t)$, which satisfies

$$
\int_t^{\infty} Z^u(t) \frac{\pi(\tau)}{\pi(t)} d\tau = \int_t^{T} [P_r(\tau)r(t) - P_a(\tau)a(\tau)] \frac{\pi(\tau)}{\pi(t)} d\tau.
$$

By using (39) to solve the equation, the resource income by the user cost method is expressed by

$$
Z^u(t) = \underbrace{P_r(t)r(t) - P_a(t)a(t)}_{\text{current cash flow}} - \frac{\pi(T)}{\pi(t)} \left[ P_r(t)r(t) - P_a(t)a(t) \right] - \underbrace{\int_t^{T} \frac{\pi(\tau)}{\pi(t)} P_a(\tau) \dot{a}(\tau) d\tau}_{\text{user cost}}.
$$

The first term on the r.h.s. is the current cash flow and the second term is the so-called "user cost", which can be interpreted as the value of the resource depletion by the method.

In light of the expression (25), the real resource income can be written as\(^6\)

$$
Y^r(t) = \underbrace{P_r(t)r(t) - P_a(t)a(t)}_{\text{current cash flow}} - \frac{\pi(T)}{\pi(t)} \left[ P_r(t)r(t) - P_a(t)a(t) \right] - \underbrace{\int_t^{T} \frac{\pi(\tau)}{\pi(t)} P_a(\tau) \dot{a}(\tau) d\tau}_{\text{net investments}}
$$

$$
+ \underbrace{\int_t^{T} \frac{\pi(\tau)}{\pi(t)} \left[ \dot{P}_r(\tau)r(t) - \dot{P}_a(\tau)a(\tau) \right] d\tau}_{\text{price change effects}}.
$$

Then the user cost in (40) is one part of the net investments in (41), which corresponds to the effects of the sudden exhaustion of the resource at the time $T$.

---

\(^{6}\)To justify the argument, the path of extraction rate could be approximated near the time $T$. e.g. let

$$
\tau_s = \begin{cases} 
\tau_t & \text{if } t \leq s \leq T \\
\tau_t (1 - (s - T) / h) & \text{if } T \leq s < T + h \\
0 & \text{if } s \geq T + h
\end{cases}
$$

where $h > 0$ and $h \to 0$. The efforts $a$ near the time $T$ is also approximated in the same way. In fact, the result can be got directly by the application of Riemann-Stieltjes integral (or more generally Lebesgue integral).
The differentiation on both sides of (39) w.r.t. time \( \tau \) gives

\[
\dot{P}_r (\tau) r(t) - \dot{P}_a (\tau) a(\tau) = P_a (\tau) \dot{a} (\tau) .
\]

By substituting (42) into (41), we observe that the net investments caused by the change of the efforts cancels off the price change effects and the real resource income expressed by (41) coincides with the income by the user cost method in (40). The user cost in (40) is the sum of the net investments and price change effects in (41). If and only if the efforts \( a \) is also constant until the resource is exhausted, the user cost in (40) coincides with the net investments in (41).

As a result, under the specific assumptions, the income by user cost method coincides with the income by sectoral income method. Furthermore, the user cost coincides with the sum of the net investments and price change effects by sectoral income method. Then the user cost method provides correct information even though it does not distinguish net investments from price change effects.

### 5.3 Net present value method

Net present value (NPV) method is recommended by the SNA (1993) and SEEA (2003) for valuing mineral resource stocks. Domingo and Lopez-Dee (2007, p 7) mentions that many countries adopted the NPV method in the mineral resource accounting. By the NPV method stated in Chapter 10 of SEEA (2003), the value of the resource stock is estimated as the present value of the resource rent of future extractions. Then the resource income is estimated as the interest on the value of the resource stock. The change of the value of the resource stock is interpreted as the value of resource depletion, which is the difference between the current resource rent and the resource income by the NPV method. The method does not make any special assumptions on the future paths of commodity flows and their prices. Then the method just states an accounting principle for the practice.

By the notation in the present paper, the value of the resource stock can be calculated
\[ V(t) = \int_t^\infty \frac{\pi(\tau)}{\pi(t)} [P_r(\tau)r(\tau) - P_a(\tau)a(\tau)] \, d\tau, \]

which is the present value of all the future cash flows to the resource. Then the resource income is estimated as the interest on the value of the resource stock, \( R(t)V(t) \). By integration by parts, the income is further expressed by two terms,

\[ R(t)V(t) = [P_r(t)r(t) - P_a(t)a(t)] - \left[ -\dot{V}(t) \right], \tag{43} \]

where the second term on the r.h.s. is the change of value of the resource stock, which, in SEEA (2003), is interpreted as the value of the resource depletion.

On the other hand, in light of (24), the real resource income is evaluated as

\[ Y^r(t) = \int_t^\infty \frac{\pi(\tau)}{\pi(t)} R(\tau) [P_r(\tau)r(\tau) - P_a(\tau)a(\tau)] \, d\tau, \]

which can be re-expressed by \( \dot{Y} \) (ref. Sefton and Weale, 2006, eq. 19)

\[ Y^r(t) = R(t)V(t) + \int_t^\infty \dot{R}(\tau) \frac{\pi(\tau)}{\pi(t)} V(\tau) \, d\tau. \tag{45} \]

As long as the interest rate \( R \) is approximately constant over time, then the estimation

\[ \frac{d}{d\tau} \left[ \frac{\pi(\tau)}{\pi(t)} V(\tau) \right] = \frac{\pi(\tau)}{\pi(t)} \dot{V}(\tau) + V(\tau) \left[ -\frac{R(\tau)\pi(\tau)}{\pi(t)} \right] \]

\[ = \frac{\pi(\tau)}{\pi(t)} \left( \dot{V}(\tau) - R(\tau) V(\tau) \right) \]

\[ = -\frac{\pi(\tau)}{\pi(t)} [P_r(\tau)r(\tau) - P_a(t)a(t)] \quad \text{By (43)} \tag{44} \]

Then integrating both sides after multiplying with \( R(\tau) \),

\[ \int_t^\infty R(\tau) \frac{\pi(\tau)}{\pi(t)} P_r(\tau)r(\tau) \, d\tau \]

\[ = -\int_t^\infty R(\tau) \frac{d}{d\tau} \left[ \frac{\pi(\tau)}{\pi(t)} V(\tau) \right] \, d\tau \]

\[ = -R(\tau) \frac{\pi(\tau)}{\pi(t)} V(\tau)|_t^\infty + \int_t^\infty \dot{R}(\tau) \left[ \frac{\pi(\tau)}{\pi(t)} V(\tau) \right] \, d\tau \]

\[ = R(t)V(t) + \int_t^\infty \dot{R}(\tau) \left[ \frac{\pi(\tau)}{\pi(t)} V(\tau) \right] \, d\tau. \]
by NPV method is approximately equal to the real resource income $Y^r$.

Moreover, by (25), the real resource income by the sectoral income method can be decomposed to be

$$Y^r (t) = \frac{P_r(t) r(t) - \bar{P}_a(t) a(t)}{\pi(t)} + \int_t^\infty \frac{\pi(\tau)}{\pi(t)} \left[ P_r(\tau) \dot{r}(\tau) - \bar{P}_a(\tau) \dot{a}(\tau) \right] d\tau$$

(46)

+ \int_t^\infty \frac{\pi(\tau)}{\pi(t)} \left[ \dot{P}_r(\tau) r(\tau) - \dot{\bar{P}}_a(\tau) a(\tau) \right] d\tau.

price change effects

Then by (43), (45), and (46), the change of the value of the resource stock can be expressed by

$$-\dot{V}(t) = \int_t^\infty \dot{R}(\tau) \frac{\pi(\tau)}{\pi(t)} V(\tau) d\tau$$

(47)

$$-\int_t^\infty \frac{\pi(\tau)}{\pi(t)} \left[ P_r(\tau) \dot{r}(\tau) - \bar{P}_a(\tau) \dot{a}(\tau) \right] d\tau - \int_t^\infty \frac{\pi(\tau)}{\pi(t)} \left[ \dot{P}_r(\tau) r(\tau) - \dot{\bar{P}}_a(\tau) a(\tau) \right] d\tau.$$

interest rate change effects

net investments

price change effects

Since $-\dot{V}(t)$ is thought of as the value of the resource depletion, then the concept of the value of the resource depletion defined by SEEA (2003) includes three terms: the interest rate change effects, net investments, and the price change effects. If both the interest rate and real prices are constant over time, then the value of the resource depletion is the same as the net investments in absolute terms. In practice, it is common to assume constant interest rate over time, then the value of the resource depletion always includes the net investments and the price change effects.

Therefore, the present value method can provide correct indicators of income and savings for tomorrow as long as the real interest rate is constant over time. Otherwise, it might be misleading.

6 Concluding remarks

In the paper, the theory of sectoral income provided by Asheim and Wei (2009) has been applied to a non-renewable resource. In particular, sectoral net investments has been
shown to equal the shadow value of the change of all the sectoral capital stocks. By the analysis of a reservoir, it is clarified that net investments of the reservoir equal the negative current Hotelling rent. Since there are no price change effects in the case, the income of the reservoir coincides with current Ricardian rent as current cash flow consists of Hotelling rent and Ricardian rent.

The new method provides an alternative way to estimate sectoral net investments other than the value of the change of all the sectoral capital stocks. Thus, the net investments generated by a non-renewable resource can be estimated on the basis of the future cash flows arising from the resource. This is useful if information on the change of capital stocks is not available.

The paper also compared the sectoral income method with other practical accounting methods. Only if the real interest rate is constant over time, the resource income by the sectoral income method coincides with other methods and the SNA/SEEA definition of the value of resource depletion provides a plausible indicator of required savings, i.e. the sum of sectoral net investments and price change effects.

Appendix A: Explaining of national and sectoral income

Let $u (c)$ denote the unidimensional utility at the national level, which is a time-invariant, concave and non-decreasing function with continuous second derivatives w.r.t. the vector of national consumption $c$. For a given smooth path of national consumption $\{c (s)\}_{s=0}^{\infty}$ over time, dynamic welfare at any time $t \geq 0$ is defined by the discounted utilitarian,

$$W (t) = \int_t^\infty e^{-p(s-t)} u (c (s)) \, ds,$$

where $p$ is the constant utility discount rate.

Assume the transversality condition holds: $\lim_{s \to 0} e^{-p(s-t)} u (c (s)) = 0$, which implies the infinite future utility means nothing for current time, then by integrating by parts, we obtain

---

8 This appendix is a special case of analysis and results in Asheim and Wei (2009).
the instantaneous welfare change at time $t$,

$$
\dot{W}(t) = \int_t^\infty e^{-\rho(s-t)} \nabla u(c(s)) \dot{c}(s) \, ds,
$$

(48)

where the dot $(\cdot)$ over a variable represents the derivative w.r.t. time and $\nabla u(c(s))$ is a vector of derivatives with respect to consumption $c$ evaluated at time $s$. Define the vector of the present value consumer prices, $\{p_c(s)\}_{s=0}^\infty$, for all $s \geq 0$, satisfying,

$$
p_c(s) = e^{-\rho(s-t)} \nabla u(c(s)).
$$

Substituting it into (48), we achieve that

$$
\dot{W}(t) = \int_t^\infty p_c(s) \dot{c}(s) \, ds,
$$

which shows that instantaneous change of welfare is represented by the present value of future consumption change. As shown in Proposition 4 of Asheim and Wei (2009), $\int_t^\infty p_c(s) \dot{c}(s) \, ds$ can be interpreted as national savings. Then, if national income is to serve as a guide for prudent behavior such that the dynamic welfare improves if and only if national consumption is smaller than national income, we obtain a definition of national income as the sum of current consumption value and the present value of future consumption change,

$$
y(t) = p_c(t) c(t) + \int_t^\infty p_c(s) \dot{c}(s) \, ds.
$$

(49)

Just by integrating by parts, we directly derive another expression of national income from (49),

$$
y(t) = \int_t^\infty (-\dot{p}_c(s)) c(s) \, ds,
$$

(50)

since $u(c)$ has continuous second derivatives w.r.t. consumption.

A Divisia consumer price index $\{\pi(s)\}_{s=0}^\infty$ is defined to satisfy $\pi(0) = 1$ and

$$
\frac{\dot{\pi}(s)}{\pi(s)} = \frac{\dot{p}_c(s) c(s)}{p_c(s) c(s)}
$$

(51)
for all $s \geq 0$. Then define the path of real consumption interest rates $\{R(s)\}_{s=0}^{\infty}$ by
\[
R(s) = -\frac{\dot{\pi}(s)}{\pi(s)}
\]
(52)
for all $s \geq 0$. The real consumption price flow $\{P_c(s)\}_{s=0}^{\infty}$ can be defined by
\[
P_c(s) = \frac{P_c(s)}{\pi(s)}
\]
(53)
for all $s \geq 0$. Since for all $s \geq 0$,
\[
-\dot{p}_c(s)c(s) = -\frac{\dot{\pi}(s)}{\pi(s)}p_c(s)c(s) \quad \text{by (51)}
\]
\[
= R(s)p_c(s)c(s) \quad \text{by (52)}
\]
\[
= R(s)\pi(s)P_c(s)c(s) \quad \text{by (53)},
\]
Then by (50), real national income can be defined as follows.

**Definition 1** Real national income is the sum of present value of real interest on future national consumption:
\[
Y(t) = \int_t^{\infty} \frac{\pi(s)}{\pi(t)} R(s) P_c(s) c(s) ds.
\]
(54)
Assume real interest rates of all individuals and sectors in the society coincide with the national one. Suppose the society can be divided to be $n$ sectors. The real cash flow to each sector $j = 1, 2, ..., n$ is denoted by $P_x(s)x^j(s)$ at any time $s$, where $x^j(s)$ is the sector $j$’s vector of commodity flows excluding consumption flows acquired for end use, and $P_x(t)$ the corresponding market (or calculated) real prices of these commodity flows. Since aggregate cash flows are zero, then the value of national consumption $c$ equals the cash flow generated by all sectors at each point in time: $P_c(s)c(s) = \sum_{j=1}^{n} P_x(s)x^j(s)$ for all $s \geq 0$. Hence, we have the following definition for real sectoral income:

**Definition 2** Real sectoral income at time $t$ is defined as
\[
Y^j(t) := \int_t^{\infty} \frac{\pi(\tau)}{\pi(t)} R(\tau) P_x(\tau)x^j(\tau)d\tau.
\]
(55)
Appendix B: Sectoral and national net investments

To illustrate the relations between sectoral and national net investments, we assume competitive capital market and stationary technology here. Hence, (19) holds for any sector in an economy. Assume the economy consists of \( N \) sectors. by (19), we denote the net investments in any sector \( j \) of the economy by

\[
\int_{t}^{\infty} \frac{\pi^j(\tau)}{\pi^j(t)} P_x(\tau) \dot{x}^j(\tau) d\tau = P_K(t) \dot{K}^j(t), \quad j = 1, 2, \ldots N. \tag{56}
\]

Then the sum of net investments across sectors exactly equals the national net investments:

\[
\sum_{j=1}^{N} \int_{t}^{\infty} \frac{\pi^j(\tau)}{\pi^j(t)} P_x(\tau) \dot{x}^j(\tau) d\tau = \sum_{j=1}^{N} P_K(t) \dot{K}^j(t), \tag{57}
\]

where the sectoral discount factor \( \pi^j \) may differ from each other. The result in (57) holds for both a closed and an open economy.

By Proposition 3 in Asheim and Wei (2009), sectoral income can be expressed by three terms

\[
Y^j(t) = \underbrace{P_x(t) x^j(t)}_{\text{current cash flow}} + \underbrace{\int_{t}^{\infty} \frac{\pi^j(\tau)}{\pi^j(t)} P_x(\tau) \dot{x}^j(\tau) d\tau}_{\text{sectoral net investments}} + \underbrace{\int_{t}^{\infty} \frac{\pi^j(\tau)}{\pi^j(t)} \dot{P}_x(\tau) \dot{x}^j(\tau) d\tau}_{\text{price change effects}}. \tag{58}
\]

and national income is the sum of sectoral income

\[
Y(t) = \sum_{j=1}^{N} Y^j(t). \tag{59}
\]

Furthermore, the value of national consumption equals total sectoral cash flow at each point in time:

\[
P_x(t)c(t) = \sum_{j=1}^{N} P_x(t) x^j(t) \tag{60}
\]

for all \( t \geq 0 \). Hence, by applying (58) to (59), national income can be expressed by

\[
Y(t) = P_x(t)c(t) + \sum_{j=1}^{N} P_K(t) \dot{K}^j(t) + \sum_{j=1}^{N} \int_{t}^{\infty} \frac{\pi^j(\tau)}{\pi^j(t)} P_x(\tau) \dot{x}^j(\tau) d\tau \quad \text{by (60) and (57).} \tag{61}
\]
On the other hand, under the assumptions of the technology being stationary and the economc realizing a competitive equilibrium, national income equals net national product (Asheim and Wei, 2009, eq. (2)), i.e.

\[ Y(t) = P_x(t)c(t) + \sum_{j=1}^{N} P_K(t) \dot{K}^j(t). \]

Comparison with expression (61) yields:

\[ \sum_{j=1}^{N} \int_t^\infty \frac{\pi^j(\tau)}{\pi^j(t)} P^j_x(\tau) x(\tau) d\tau = 0, \]

which shows the price change effects across sectors cancel out each other. Hence, the future price change effects at the sectoral level may be ignored from estimates of income at the national level. However, the price change effects should be part of the sectoral income.

If the technology is not stationary, the sum of sectoral price change effects may differ from zero as the value of passage of time appears. One special case studied by Sefton and Weale (1996) is the effects of changing terms-of-trade faced by a resource exporter.

**Appendix C: A monopolist**

We consider a monopolist sector in this appendix. The two notions of sectoral net investments coincide each other for a competitive sector as shown in (17) of Section 3. Will the consistence still keep if the market is not competitive? For example, a monopolist sector? The question is interesting for empirical accounting. One example is that OPEC countries have considerable control power over the oil prices in the world market.

The monopolist sector aims to maximize the present value of future cash flows such that the function (5) is satisfied:

\[
\max_x \int_t^\infty \frac{\pi(\tau)}{\pi(t)} P_x(\tau) x(\tau) d\tau \\
\text{s.t. } h(x(\tau), K(\tau), \dot{K}(\tau), \tau) \geq 0, \\
K(t) \text{ is given,}
\]

where the prices of commodity flows \( P_x \) becomes a function of corresponding commodity flow \( x \).
alone. The design for the prices $P_x$ implies prices do not change for constant commodity flows and we have for all $j = 1, 2, ..., n,$

$$
\sum_{i=1}^{n} \frac{\partial P_{x_j}(x(\tau))}{\partial x_i(\tau)} \dot{x}_i(\tau) = \dot{P}_{x_j}(x(\tau)).
$$

The Hamiltonian of the problem is

$$
H = \frac{\pi(\tau)}{\pi(t)} P_x(x(\tau)) x(\tau) + \lambda(\tau) \dot{K}(\tau),
$$

and the Lagrangian is

$$
L = H + \sum_{i=1}^{r} \mu_i(\tau) h_i \left( x(\tau), K(\tau), \dot{K}(\tau), \tau \right),
$$

Then we have the necessary conditions for an optimum

$$
\frac{\partial L}{\partial x_j} : \quad \frac{\pi(\tau)}{\pi(t)} P_{x_j}(x(\tau)) + \frac{\pi(\tau)}{\pi(t)} \sum_{i=1}^{n} \frac{\partial P_{x_i}(x(\tau))}{\partial x_j(\tau)} x_i(\tau)
$$

$$
= - \sum_{i=1}^{r} \mu_i(\tau) \frac{\partial h_i}{\partial x_j}, \quad j = 1, ..., n \tag{63}
$$

$$
\frac{\partial L}{\partial K_j} : \quad \lambda_j(\tau) = - \sum_{i=1}^{r} \mu_i(\tau) \frac{\partial h_i}{\partial K_j}, \quad j = 1, ..., m \tag{64}
$$

$$
\frac{\partial L}{\partial K_j} : \quad \dot{\lambda}_j(\tau) = - \sum_{i=1}^{r} \mu_i(\tau) \frac{\partial h_i}{\partial K_j}, \quad j = 1, ..., m. \tag{65}
$$

$$
\mu_i(\tau) \geq 0, \quad \mu_i(\tau) h_i(\tau) = 0 \quad i = 1, ..., r \tag{66}
$$

Following the similar procedure as the competitive sector described in Section 3, we obtain

$$
\frac{\pi(\tau)}{\pi(t)} \sum_{j=1}^{n} P_{x_j}(x_j(\tau)) \dot{x}_j(\tau) + \frac{\pi(\tau)}{\pi(t)} \sum_{i,j=1}^{n} \frac{\partial P_{x_i}(x_j(\tau))}{\partial x_j(\tau)} \dot{x}_j(\tau) x_i(\tau)
$$

$$
= - \sum_{i=1}^{m} \frac{d}{d\tau} \left[ \lambda_i(\tau) \dot{K}_i(\tau) \right] + \sum_{i=1}^{r} \mu_i(\tau) \frac{\partial h_i}{\partial \tau} \tag{67}
$$
and sectoral net investments

\[ \int_t^\infty \frac{\pi(\tau)}{\pi(t)} P_x(\tau) \dot{x}(\tau) \, d\tau \]

\[ = \int_t^\infty \frac{\pi(\tau)}{\pi(t)} \sum_{j=1}^n P_{xj}(\tau) \dot{x}_j(\tau) \, d\tau \]

\[ = -\int_t^\infty \sum_{i=1}^m \frac{d}{d\tau} \left[ \lambda_i(\tau) \dot{K}_i(\tau) \right] \, d\tau + \int_t^\infty \sum_{i=1}^r \mu_i(\tau) \frac{\partial h_i}{\partial \tau} \, d\tau - \int_t^\infty \frac{\pi(\tau)}{\pi(t)} \sum_{j=1}^n \dot{P}_{xj}(x(\tau)) x_j(\tau) \, d\tau \]

\[ = \sum_{i=1}^m \lambda_i(t) \dot{K}_i(t) + \int_t^\infty \sum_{i=1}^r \mu_i(\tau) \frac{\partial h_i}{\partial \tau} \, d\tau - \int_t^\infty \frac{\pi(\tau)}{\pi(t)} \dot{P}_x(x(\tau)) x(\tau) \, d\tau. \]

We have one more term related to the change of future prices controlled by the monopolist via commodity flows. By moving the term to the l.h.s., we notice the value of the change of future capital stocks equals the sum of net investments and price change effects defined by the sectoral income method. Since in this case, price change comes from the commodity flows alone, the price change effects can be seen as kind of re-evaluation of the sectoral capital stocks. After the price change effects is included in the value of the change of all the sectoral capital stocks, there is no other price change and thus by (25), the monopolist’s income is

\[ Y^r(t) = P_x(x(\tau)) x(\tau) + \lambda(t) \dot{K}(t) + \int_t^\infty \mu(\tau) \frac{\partial h}{\partial \tau} \, d\tau, \]

which is the sum of current cash flow and the value of the change of all the sectoral capital stocks.

In this specific case, the net investments by sectoral income method together with price change effects coincide with the value of the change of all the sectoral capital stocks. This provides a plausible indicator of minimum savings for tomorrow to keep us as well off tomorrow as today since the monopolist is in a stationary environment and the changed prices do not reflect exogenous changes.

Interested readers can apply the results to a monopolist in the market of a non-renewable resource.
Appendix D: Restricted extraction rates

In reality, it is almost impossible for a producer to produce along with the optimal path of the extraction rates. The real path of the extraction rates is always restricted by various causes. For example, the optimal extraction rate for a producer may exceed the available production ability of the producer at a given time. It may be the case that the condition of the resource reservoir do not allow to delay the extraction such that it is better to extract more than the optimal extraction rate. Hence, the real path of the extraction rate for a producer may be far from the optimal path. Then it is interesting to study the case of restricted extraction rates. The real path of the extraction rates, however, may coincides with the optimal one.

Assume the path of a one-dimensional extraction rates is chosen by the producer in the model. The extraction rate at time $0 \leq \tau \leq T$ can be expressed by

$$r (\tau) = r (0) \exp \left( \int_{0}^{\tau} \alpha (v) \, dv \right), \quad (69)$$

where $r (0) > 0$ is the initial extraction rate at time 0 and $\alpha (v)$ is the change rate of the extraction rate at time $v$. At time $T$ the resource is used up and there is no extraction later on.

Since the extractive efforts required for a given level of extraction rate is affected by other factors like the remaining resource stock and the technology, the path of the extractive efforts does not necessarily follow the pattern of the path of the extraction rates. Suppose the path of the efforts required by the chosen path of the extraction rates is also determined. Then the extractive efforts at time $0 \leq \tau \leq T$ can be expressed by

$$a (\tau) = a (0) \exp \left( \int_{0}^{\tau} \beta (v) \, dv \right), \quad (70)$$

where $a (0) > 0$ is the initial extractive efforts at time 0 and $\beta (v)$ is the change rate of the extractive efforts at time $v$.

The real market prices of the resource and the efforts are determined exogenously. Suppose the real market price of the resource at time $\tau \geq 0$ can be expressed by

$$P_r (\tau) = P_r (0) \exp \left( \int_{0}^{\tau} \gamma (v) \, dv \right), \quad (71)$$

75
where $P_r(0)$ is the initial real market price of the resource at time 0 and $\gamma(v)$ is the change rate of the price at time $v$. Similarly, the real market price of the efforts at time $\tau \geq 0$ can be expressed by

$$P_a(\tau) = P_a(0) \exp \left( \int_{0}^{\tau} \theta(v) \, dv \right),$$  \hspace{1cm} (72)

where $P_a(0)$ is the initial real market price of the efforts at time 0 and $\theta(v)$ is the change rate of the price at time $v$.

Define the value share of the efforts to the gross resource revenue at any time $\tau \geq 0$,

$$\epsilon(\tau) = \frac{P_a(\tau) a(\tau)}{P_r(\tau) r(\tau)},$$

which is assumed satisfying $0 < \epsilon(\tau) < 1$. Then by the definition of sectoral income in (3), the resource income can be expressed by

$$Y^r(t) = \int_{t}^{\infty} \frac{\pi(\tau)}{\pi(t)} R(\tau) \left[ 1 - \epsilon(\tau) \right] P_r(\tau) r(\tau) \, d\tau,$$  \hspace{1cm} (73)

which shows the resource income depends on the future value share of the efforts and gross resource revenue.

If we define

$$\hat{\pi}(\tau) = \exp \left( \int_{0}^{\tau} [\alpha(v) + \gamma(v)] \, dv \right), \quad \text{and}$$

$$\bar{\pi}(\tau) = \exp \left( \int_{0}^{\tau} [\beta(v) + \theta(v)] \, dv \right)$$

for all $\tau \geq 0$, then still by (3), the resource income can be expressed by

$$Y^r(t) = \int_{t}^{T} \frac{\pi(\tau)}{\pi(t)} R(\tau) \left[ \frac{\hat{\pi}(\tau)}{\bar{\pi}(t)} P_r(t) r(t) - \frac{\pi(\tau)}{\bar{\pi}(t)} P_a(t) a(t) \right] \, d\tau,$$  \hspace{1cm} (74)

which shows the resource income can be worked out on the basis of the current extraction variables and their future change rates.
By using (18) and (19), the net investments in (25) can be expressed by\(^9\)

\[
\int_t^\infty \frac{\pi (\tau)}{\pi (t)} \left[ P_r (\tau) \dot{r} (\tau) - P_a (\tau) \dot{a} (\tau) \right] d\tau = \int_t^T \frac{\pi (\tau)}{\pi (t)} \left[ \alpha (\tau) P_r (\tau) r (\tau) - \beta (\tau) P_a (\tau) a (\tau) \right] d\tau - \frac{\pi (T)}{\pi (t)} \left[ \dot{P}_r (T) r (t) - \frac{\pi (T)}{\pi (t)} P_a (t) a (t) \right],
\]

where the first term on the r.h.s. is the effects of the change of the resource extraction before the resource is used up and the second term, the present value of the cash flow generated at the last extraction time \(T\), is the effects of the sudden exhaustion of the resource at the time \(T\) when the resource is used up. If the extraction of the resource can last until infinity, then the second term disappears. In this sense, the second term can be taken as a result from the non-renewable property of the resource.

By using (71) and (72), the price change effects in (25) can be expressed by

\[
\int_t^\infty \frac{\pi (\tau)}{\pi (t)} \left[ \dot{P}_r (\tau) r (\tau) - \dot{P}_a (\tau) a (\tau) \right] d\tau = \int_t^\infty \frac{\pi (\tau)}{\pi (t)} \left[ \gamma (\tau) P_r (\tau) r (\tau) - \theta (\tau) P_a (\tau) a (\tau) \right] d\tau.
\]

**Special examples: constant interest rate and change rates of variables**

To simplify the expression for the resource income and its components, the following examples are based on more specific assumptions: The real interest rate is assumed to be constant all the time as \(R\); the extraction rates and corresponding efforts are assumed to change at the same constant rate over time, i.e. the parameters \(\alpha = \beta\); and their prices also change at the same constant rates over time, i.e. the parameters \(\gamma = \theta\). Then the cash flows over time change at a constant rate, i.e.

\[
\alpha + \gamma = \beta + \theta,
\]

which implies \(\dot{\pi} (\tau) = \ddot{\pi} (\tau)\) for all \(\tau \geq 0\). In addition, the the value share of the efforts also

\(^9\)To justify the argument, the path of extraction rate could be approximated near the time \(T\). e.g. let

\[
r_s = \begin{cases} 
  r_t & \text{if } t \leq s \leq T \\
  \frac{r_t (1 - (s - T)/h)}{T} & \text{if } T \leq s < T + h \\
  0 & \text{if } s \geq T + h
\end{cases}
\]

where \(h > 0\) and \(h \to 0\). The efforts \(a\) near the time \(T\) is also approximated in the same way. In fact, the result can be got directly by the application of Riemann-Stieltjes integral (or more generally Lebesgue integral).
becomes constant, i.e. \( \epsilon (\tau) = \epsilon \) for all \( \tau \geq 0 \). Then by (74) and assuming \( R > (\alpha + \gamma) \), the resource income can be worked out as\(^{10}\)

\[
Y^r (t) = (1 - \epsilon) P_r (t) \frac{R \left[ 1 - e^{-(R-\alpha-\gamma)(T-t)} \right]}{R - (\alpha + \gamma)},
\]

which is a constant share of current cash flow. The share is determined by the interest rate, the change rate of future cash flow, and the life length of the resource.

By (75), the net investments becomes

\[
(1 - \epsilon) P_r (t) r (t) \left\{ \frac{\alpha \left[ 1 - e^{-(R-\alpha-\gamma)(T-t)} \right]}{R - (\alpha + \gamma)} - e^{-(R-\alpha-\gamma)(T-t)} \right\},
\]

which is also a constant share of current cash flow. One part of the net investments results from the change of commodity flows before the time \( T \) and another part is the effects of the sudden exhaustion of the resource at the time \( T \).

By (76), the price change effects becomes

\[
(1 - \epsilon) P_r (t) \frac{\gamma \left[ 1 - e^{-(R-\alpha-\gamma)(T-t)} \right]}{R - (\alpha + \gamma)},
\]

which is also a constant share of current cash flow. The change rate of prices determines the level of the price change effects.

Numerically if \( R = 4\% \), \( \alpha = \beta = -4\% \), \( \gamma = \theta = 2\% \), and \( T - t = 50 \), then the net investments is near 70 percent, \( -(2 + e^{-3})/3 \simeq -0.683 \), of current cash flow. Meanwhile, the price change effects is over 30 percent, \( (1 - e^{-3})/3 \simeq 0.317 \), of current cash flow. Both net investments and the price change effects are quite considerable.

\[^{10}\]

\[
Y^r (t) = \int_t^\infty \frac{\pi (\tau)}{\pi (t)} \frac{\dot{\pi} (\tau)}{\pi (t)} [P_r (t) r (t) - P_a (t) a (t)] d\tau
\]

\[
= R \int_t^\infty \frac{\pi (\tau)}{\pi (t)} \frac{\dot{\pi} (\tau)}{\pi (t)} (1 - \epsilon) P_r (t) r (t) d\tau
\]

\[
= R (1 - \epsilon) P_r (t) r (t) \int_t^T \exp (- (R - \alpha - \gamma) (\tau - t)) d\tau
\]

\[
= \begin{cases} 
R (1 - \epsilon) P_r (t) r (t) (T - t) & \text{if } R = \alpha + \gamma \\
\frac{R}{\pi - \alpha - \gamma} (1 - \epsilon) P_r (t) r (t) \left[ 1 - e^{-(R-\alpha-\gamma)(T-t)} \right] & \text{otherwise} 
\end{cases}
\]

In this appendix, the \( R = \alpha + \gamma \) case is not discussed in detail. Following the similar method below in this appendix, the case can be analyzed.
Three more specific cases are discussed as follows.

- Case 1: $\alpha = \beta = 0$, $\gamma = \theta = 0$

The extraction rates, the efforts, and their prices keep constant before the time when the resource is exhausted. By using (25), (79) and (80), the resource income is expressed by

$$ Y^r (t) = \left(1 - \epsilon\right) P_r (t) r (t) - \left(1 - \epsilon\right) P_r (t) r (t) e^{-R(T-t)} + \frac{0}{\text{net investments}} + \frac{0}{\text{price change effects}}. \quad (81) $$

In this case, the net investments include only the present value of the cash flow generated at the time when the resource is exhausted\(^{11}\). There is no price change effects since all the prices keep constant over time.

- Case 2: $\alpha = \beta = 0$, $\gamma = \theta > 0$

The extraction rates and the efforts keep constant before the time when the resource is exhausted. Their prices keep increasing at the same rate over time. Then by using (25), (79) and (80), the resource income is expressed by

$$ Y^r (t) = \left(1 - \epsilon\right) P_r (t) r (t) - \left(1 - \epsilon\right) P_r (t) r (t) e^{-(R-\gamma)(T-t)} + \left(1 - \epsilon\right) P_r (t) r (t) \frac{\gamma \left[1 - e^{-(R-\gamma)(T-t)}\right]}{R - \gamma}. \quad (82) $$

Here the net investments is still interpreted as the present value of the cash flow generated at the time when the resource is exhausted since in fact it comes from the expression: $\left(1 - \epsilon\right) P_r (T) r (T) e^{-R(T-t)}$. The price change effects is determined by the increasing rate of the future prices.

To do a simple numerical exercise. If real interest rate $R = 4\%$ and the growth rate of real prices $\gamma = \theta = 2\%$, then the price change effects is exactly one half of the income, $Y^r (t) / 2$, which is quite considerable.

- Case 3: $\alpha = \beta < 0$, $\gamma = \theta = 0$

\(^{11}\)Since the cash flow before the time $T$ is constant, the net investments might be misinterpreted as the present value of the delayed cash flow if the current extraction happens at the time when the resource is exhausted instead of current time.
The extraction rates and the efforts keep decreasing at the same rate before the time when the resource is exhausted. Their prices, on the other hand, keep constant over time. Then by using (25), (79) and (80), the resource income is expressed by

\[
Y^r(t) = (1 - \epsilon) P_r(t) r(t) + (1 - \epsilon) P_e(t) r(t) \left\{ \frac{\alpha [1 - e^{-(R-\alpha)(T-t)}]}{R - \alpha} - e^{-(R-\alpha)(T-t)} \right\} + \frac{0}{\text{price change effects}}.
\]

(83)

Then net investments include two terms: the one related to the decreasing rate of the commodity flows and another one is the effects of the sudden exhaustion of the resource at the time \( T \). There is no price change effects since all the prices keep constant over time.

A numerical example may help understand better. If \( R = 4\% \), \( \alpha = -4\% \), and \( T - t = 50 \), then the net investments is over 50 percent, \(- (1 + e^{-4}) / 2 \approx -0.509\), of current cash flow.

**Appendix E: Net investments of the resource**

In this appendix, the maximization problem (26) faced by producers in the non-renewable resource sector is solved directly to show the same results as in the Subsection 4.1.

In the problem, the state variables are the remaining resource stock \( S \), and man-made capital \( k \), while the control variables are the extraction rate \( r \) and extractive efforts \( a \). The Hamiltonian of this problem is

\[
H = \frac{\pi(\tau)}{\pi(t)} \left[ P_r(\tau) r(\tau) - P_a(\tau) a(\tau) \right] - \lambda_s(\tau) r(\tau) + \lambda_k(\tau) \dot{k}(\tau),
\]

and Lagrangian is

\[
L = H + \mu(\tau) g \left( r(\tau), a(\tau), s(\tau), k(\tau), \dot{k}(\tau), t \right),
\]

where \( \lambda_s, \lambda_k, \) and \( \mu \) are the shadow prices of the remaining resource stock \( S \), man-made capital \( k \), and the constraint \( g(\cdot) = 0 \) respectively. Notice that the Hamiltonian is expressed in the terms of present value rather than current value.
Then among others, the following necessary conditions hold:

\[
\begin{align*}
\frac{\partial H}{\partial r_i} : & \quad \frac{\pi(\tau)}{\pi(t)} P_{r_i}(\tau) + \mu(\tau) g'_{r_i} = \lambda_{s_i}(\tau) \quad (84) \\
\frac{\partial H}{\partial a_i} : & \quad \frac{\pi(\tau)}{\pi(t)} P_{a_i}(\tau) = \mu(\tau) g'_{a_i} \quad (85) \\
\frac{\partial H}{\partial k_i} : & \quad \lambda_{k_i}(\tau) = -\mu(\tau) g'_{k_i} \quad (86) \\
- \frac{\partial H}{\partial S} : & \quad \lambda_{s_i}(\tau) = -\mu(\tau) g'_{s_i} \quad (87) \\
- \frac{\partial H}{\partial k_i} : & \quad \lambda_{k_i}(\tau) = -\mu(\tau) g'_{k_i} \quad (88) \\
\mu(\tau) & \geq 0, \mu(\tau) g(\cdot) = 0, \quad (89)
\end{align*}
\]

where the last condition enables us to have \( \mu(\tau) \frac{dg(\cdot)}{d\tau} = 0 \), i.e.

\[
\mu(\tau) \sum_i g'_{r_i} \dot{r}_i + \mu(\tau) \sum_i g'_{a_i} \dot{a}_i + \mu(\tau) \sum_i g'_{S_i} \dot{S}_i + \mu(\tau) \sum_i g'_{k_i} \dot{k}_i + \mu(\tau) \sum_i g'_{k_i} \ddot{k}_i + \mu(\tau) \frac{\partial g}{\partial \tau} = 0. \quad (90)
\]

Hence, the present value of the change of commodity flows at any time is

\[
\begin{align*}
\frac{\pi(\tau)}{\pi(t)} & \left[ P_r(\tau) \dot{r}(\tau) - P_a(\tau) \dot{a}(\tau) \right] \\
& = \lambda_s(\tau) \dot{r}(\tau) - \mu(\tau) \sum_i g'_{r_i} \dot{r}_i(\tau) - \mu(\tau) \sum_i g'_{a_i} \dot{a}_i(\tau) \quad \text{by (84) and (85)} \\
& = \lambda_s(\tau) \dot{r}(\tau) + \mu(\tau) \sum_i g'_{S_i} \dot{S}_i + \mu(\tau) \sum_i g'_{k_i} \dot{k}_i + \mu(\tau) \sum_i g'_{k_i} \ddot{k}_i + \mu(\tau) \frac{\partial g}{\partial \tau} \quad \text{by (90)} \\
& = \lambda_s(\tau) \dot{r}(\tau) - \lambda_s(\tau) \ddot{S} - \lambda_k(\tau) \ddot{k} - \lambda_k(\tau) \dot{k} + \mu(\tau) \frac{\partial g}{\partial \tau} \quad \text{by (87), (88), and (86)} \\
& = - \frac{d}{d\tau} \left[ \lambda_s(\tau) \dot{S} \right] - \frac{d}{d\tau} \left[ \lambda_k(\tau) \dot{k} \right] + \mu(\tau) \frac{\partial g}{\partial \tau} \quad \text{by (21), and integration by parts.}
\end{align*}
\]

and then net investments generated by the resource sector are

\[
\int_t^\infty \frac{\pi(\tau)}{\pi(t)} \left[ P_r(\tau) \dot{r}(\tau) - P_a(\tau) \dot{a}(\tau) \right] = \lambda_s(t) \dot{S} + \lambda_k(t) \dot{k} + + \int_t^\infty \mu(\tau) \frac{\partial g}{\partial \tau} d\tau \quad (91)
\]

given the transversality conditions \( \lambda_s(\tau) \dot{S} \to 0, \lambda_k(\tau) \dot{k} \to 0 \), as \( \tau \to \infty \), which is exactly the equation (28) since \( f_2 = g \).
References


Net investments and price change effects on income arising from Norwegian petroleum sector*

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Abstract

What are net investments at a sectoral level? What are price change effects on real income arising from a non-renewable resource sector? These two questions are crucial for a small open resource-rich economy since the price change of the resources may have significant effects on national income and net investments. By taking Norwegian petroleum sector as an example, the paper answers these questions based on the theory of sectoral income proposed by Ashem and Wei (2009).

Key words: non-renewable resource income, net investments, price change effects, comprehensive national accounting

JEL classification: C43, C82, O47, Q32

1 Introduction

What are net investments at a sectoral level? A commonly applied approach is to estimate the depreciation of man-made capital and the depletion of natural resources, which is then

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deducted from gross investments in the sector to obtain sectoral net investments. The sectoral net investments add up to net investments at the national level, which is used to calculate net national product (NNP). However, estimation of the value of capital depreciation and resource depletion is controversial. Hence, it is desirable to be able to estimate sectoral net investments by a direct method.

Another question is what are the price change effects on real income arising from a sector, in particular, a non-renewable resource sector? Intuitively future price change has great effects on the income arising from a non-renewable resource sector. Attention is often paid to the uncertainty of the future prices. However, even in a deterministic economy, the price change from period to period may also have great effects on the income arising from a non-renewable resource. How great are these effects?

The answers to these questions will help policy makers determine how much cash flow generated from the resource sector should be used for consumption and/or investments. Since the change of resource prices has significant effects on national income and net investments, these questions are crucial for an economy with high shares of import and/or export of the resource. For example, the increasing petroleum prices in the world market make the exporter better off and the importer worse off.

To answer these questions, the present paper takes Norwegian petroleum sector as an example by following a new developed theory of sectoral income proposed by Asheim and Wei (2009). By the theory, real income at a sectoral level can be estimated and moreover, it can be split into three terms: current cash flow, sectoral net investments and price change effects. The decomposition is particularly significant for non-renewable resources like oil and gas.

Methods to estimate income arising from non-renewable resources have been discussed extensively in the literature. A commonly suggested method is income defined as real interest on sectoral wealth, where sectoral wealth is the present value of future sectoral cash flows (eg. Aslaksen et al, 1990; Brekke, 1997b). The practical methods mentioned in SEEA (2003) are special cases of the wealth-based method, e.g. the user cost method first mentioned by El Serafy (1981). However, as pointed by Asheim and Wei (2009), the wealth-based method does not satisfy some desirable properties for the notion of sectoral income.
Uncertainty plays an important role in a non-renewable resource sector. To focus on the estimation of sectoral net investments and price change effects on real income, the present paper abstracts from uncertainty and assumes a deterministic economy\(^1\).

The present paper is structured as follows. Next section introduces a discrete-time definition of sectoral income. Section 3 describes the definition of the petroleum sector. Section 4 presents an estimate of the real income for Norwegian petroleum sector. Section 5 considers the alternative scenarios to study the sensitivity of the results in the previous section. The final section offers concluding remarks. The Appendix provides technical derivations for the definition of sectoral income in discrete-time.

2 Sectoral income method

Following Asheim and Wei (2009), national income is defined as the present value of real interest on future national consumption, and sectoral income is defined as the present value of real interest on future sectoral cash flows. In particular, sectoral income is further decomposed as three components: current cash flow, net investments, and price change effects. To apply the theory to practice, this section gives the main expressions for sectoral income and its components in discrete time. Derivations for the expressions are found in the Appendix.

Assume the current point in time is the beginning of current year \(t \geq 0\). For simplification, assume that real interest rate \(R\) is constant over time, which means that we receive \((1 + R)\) units at the end of a year if we invest 1 unit at the beginning of the year.

Let \(s \geq t\) represents any future year. Then the discount factor of the \(s\) year is defined by

\[
\pi_s = (1 + R)^{-s}. \tag{1}
\]

This leads to the following relationship between the real interest rate and the discount factor,

\[
R = \frac{\pi_s - \pi_{s+1}}{\pi_{s+1}}. \tag{2}
\]

\(^1\) As a first step towards extending the analysis of sectoral income to the uncertainty case, Wei (2009a) has provided a definition of stochastic national income in the continuous time case.
for all \( s \geq t \).

Let \( \{ X_s \}_{s=0}^{\infty} \) denote the vector of commodity flows in a sector (excluding consumption flows if applicable), and \( \{ P_{x,s} \}_{s=0}^{\infty} \) the corresponding path of real prices of these commodity flows. Then real sectoral income at each year \( t \geq 0 \) is defined by

\[
Z_t = \sum_{s=t}^{\infty} R^s \frac{\pi_{s+1}}{\pi_t} P_{x,s+1} X_{s+1} ,
\]

(3)

which is the present value of real interest on sectoral future cash flow excluding consumption flows acquired for end use\(^2\). A discussion of the definition is provided in the Appendix.

The sectoral income can be separated to be two terms as current cash flow and present value of future cash flow change. By using the definitions in (2), real sectoral income can be rewritten as

\[
Z_t = P_{x,t}X_t \quad \text{current cash flow} + \sum_{s=t}^{\infty} \frac{\pi_s}{\pi_t} (P_{x,s+1} X_{s+1} - P_{x,s} X_s) ,
\]

(4)

\[\text{present value of future cash flow change}\]

Sectoral income differs from current cash flow. The difference comes from future cash flow change and is represented by the second term on the r.h.s of (4). If all the future cash flow is the same as current one, then the sectoral income coincides current cash flow. Otherwise, the current cash flow should be adjusted to calculate sectoral income.

Since the real interest rate is assumed to be constant, then the adjustment term in (4) has another interpretation as the change of sectoral wealth\(^3\). The change of sectoral wealth

\(^2\)Since \( R \) is constant, this sectoral income can be interpreted as interest on sectoral wealth if we move \( R \) out of the sum sign in (3).

\(^3\)Since for any time \( v \geq t \), we have \( \pi_v/\pi_{v+1} = 1 + R \), then the change of sectoral wealth is

\[
= \sum_{s=t}^{\infty} \left( \frac{\pi_{s+1}}{\pi_{s+1}} \right) \frac{\pi_s}{\pi_t} P_{x,s+1} X_{s+1} - \sum_{s=t}^{\infty} \frac{\pi_s}{\pi_t} P_{x,s} X_s
\]

\[
= \sum_{s=t}^{\infty} \frac{\pi_s}{\pi_t} (P_{x,s+1} X_{s+1} - P_{x,s} X_s) .
\]

The result does not hold if the real interest rate varies over time since generally \( \pi_{s+1}/\pi_s \neq \pi_{t+1}/\pi_t \) for any \( s \neq t \). Interested readers are referred to Asheim and Wei (2009).
is associated with the value of depletion of resources in the practice of non-renewable resource accounting (ref. SEEA, 2003). This view on depletion of resources may be misleading since the change of sectoral wealth may result from price change effects besides net investments as shown below.

By decomposition of the change of future cash flow, we obtain another expression for real sectoral income as

\[
Z_t = \underbrace{P_{x,t}X_t}_{\text{current cash flow}} + \sum_{s=t}^{\infty} \frac{\pi_s}{\pi_t} P_{x,s} (X_{s+1} - X_s) + \sum_{s=t}^{\infty} \frac{\pi_s}{\pi_t} (P_{x,s+1} - P_{x,s}) X_{s+1}. \tag{5}
\]

Here the present value of future cash flow change (or the change of sectoral wealth in this case) is split to net investments and price change effects. Net investments is the present value of future commodity flow change\(^4\). Price change effects is the present value of sectoral cash flow due to price change alone. Notice that the net investments disappears if the future commodity flows keep constant and the price change effects disappears if the real prices keep constant over time. In the real world, it occasionally happens that the future cash flows keep constant in a sector. It is also a special case to have constant future commodity flows and/or constant prices in the future. However, it is interesting to see the effects of their changes on the sectoral income.

It is worth noticing that the net investments can be interpreted from the perspective of sectoral wealth. At current time \(t\), assume there are two possible commodity flow paths: \(\{X_s\}_{s=t}^{\infty}\) and \(\{\tilde{X}_s\}_{s=t}^{\infty}\). The difference of sectoral wealth corresponding to the two paths is represented by

\[
\sum_{s=t}^{\infty} \frac{\pi_s}{\pi_t} P_{x,s} \tilde{X}_s - \sum_{s=t}^{\infty} \frac{\pi_s}{\pi_t} P_{x,s} X_s,
\]

which is the net investments if the path \(\{\tilde{X}_s\}_{s=t}^{\infty}\) is equal to the "actual" commodity flow path as of the next time \(t + 1\), i.e. \(\tilde{X}_s = X_{s+1}\) for all \(s \geq t\). Hence, net investments is the change of sectoral wealth due to commodity flow change alone. On the other hand,

\(^4\)Why can we call this term as net investments? Wei (2009b) provides a theoretical interpretation: this term coincides with the value of the change of current capital stocks in a sector under stationary assumptions.

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the price change effects can be interpreted as the change of sectoral wealth due to price change alone.

Another factor we have to consider in practice is the uncertainty of future commodity flows and their prices in the sector. What we can do is to forecast the future values of these variables. The forecast can be taken as the expectation of future variables. If all commodity flows are assumed to be independently determined without considering their real prices, e.g. a sector facing perfect markets, then sectoral income can be estimated on the basis of these forecast alone. Hence, the forecast can be dealt with as deterministic.

To illustrate the definitions of sectoral income and its components and show its implication for resource accounting, we will apply it to the petroleum sector in Norway. Before turning to the estimation, the petroleum sector is discussed in the next section.

3 The petroleum sector

The petroleum sector in the paper involves all the oil and gas extraction activities in an economy. During the extraction process, the commodities flowing out of the sector are the extracted oil and gas and the commodities entering in the sector can be summarized as intermediate goods, labor, and investment goods. Suppose all the prices are exogenously given for the sector. Moreover, suppose all the future information is known for sure at current time.

As shown in the previous section, the concept of sectoral income is based on the future cash flow to a sector. For the petroleum sector, the cash inflow is the revenue due to the sale of output (oil and gas), and the cash outflow is the payout of variable costs (e.g. intermediate goods and labor wages) and gross investments. Then total cash inflow net of total cash outflow yields the cash flow to the sector.

---

5 For each scenarios that could happen, the sectoral income can be estimated. The expectation of all the possible estimation can be defined as sectoral income under uncertainty. If all commodity flows are independent on their prices and real interest rates are given, then as an expectation, the sectoral income under uncertainty can be simply estimated on the basis of the expectation of these variables.
4 Income arising from Norwegian petroleum sector

Following the previous section, the Norwegian petroleum sector is defined as all the oil and gas extraction activities in Norway.

4.1 The petroleum sector in Norway

The sector plays an important role in Norway. Petroleum activities have contributed significantly to economic growth and to financing of the welfare state. It is Norway’s largest industry today. The petroleum sector accounted for 24 percent of GDP in the country in 2007. Through taxes and direct ownership, the government is ensured a high proportion of the values created from the petroleum sector. In 2007, the state’s net cash flow from the petroleum sector amounted to approximately 31 percent of total revenues (MPE, 2008).

In 2007, Norway ranks as the world’s fifth largest oil exporter and the eleventh largest oil producer. In 2006, Norway was the third largest gas exporter and the sixth largest gas producer in the world (MPE, 2008). However, the share in the world market is small and generally it is assumed that Norway is a small open economy and has almost no market power in the world oil market and limited power in the European gas market.

The current Norwegian resource management model determines that the state controls most of the extraction activities of the petroleum resource. Then to simplify the analysis, the extraction rate is assumed to be deterministic in the future years, independent on oil and gas prices. Thus, we can do the estimation based on forecast for the future alone.

At the time when the production is stopped, there will be some remaining man-made capital like equipment. The value of these capital is difficult to estimate. Part of the capital might be sold to earn some money. On the other hand, additional cost is needed to deal with these capital. In the estimation, the benefit from the "useless" capital is assumed to cancel out these capital stocks. In this paper, these events are assumed to happen in 2050. Hence, they have trivial effects on today’s income even with a fairly small discount factor.

The fact that the petroleum sector exports the majority of its production (about 95 percent in 2006) indicates that domestic demand has trivial effects on the production
scale of the sector. Therefore, a computable general equilibrium (CGE) model focusing on
domestic market has its limitations for the purpose of estimating the petroleum income.
However, a scenario designed for such a CGE model is good enough to do the estimation
since it combines reasonable external forecast for the Norwegian economy including the
petroleum sector. Next a reference scenario applied by a CGE model, namely MSG6, is
described.

4.2 Reference scenario

The MSG6 model was developed by Statistics Norway (Heide et al., 2004) and many
studies have been done on the basis of the model (eg. Heide et al., 2006). The present
paper then adopts a reference scenario applied by the model. The scenario describes a
reasonable developing path of Norwegian economy during 2006 - 2050. Here we focus on
the petroleum sector. Notice that most of the data for the sector comes from external
sources other than the results from the MSG6 model running (Heide et al., 2006).

Norway is assumed to be a small open economy such that it is a price-taker in the
world market. Prices of crude oil and gas are assumed to decrease slightly in the beginning
years and then grow by about two percent annually (Ministry of Finance, 2004). Real
prices of other goods (including investment goods) in the world market are assumed to
grow by 2 percent annually (see Figure 1). Real wage cost per hour goes up by 4.4 percent
annually (See Figure 2).

The production of oil and gas, intermediate goods, labor, and investment commodity
flows are shown in Figures 3 and 4 respectively. Before 2030, The production of oil and
gas and the input of investment goods follows the forecast made by MPE (2008). After
2030, the production of oil and gas and the investments is assumed to decline almost
linearly. During the whole period, the labor input follows almost the same pattern as the
intermediate goods.

In this estimation, it is assumed that the petroleum extraction is stopped after 2050.
It may happen because the cash flow becomes negative in 2051 due to increasing costs.
In any case, such an assumption is arbitrarily determined. There are some reasons to
believe the extraction will continue after 2050, for example, the technical progress and
Figure 1: Prices forecast of oil, gas, and investment goods (Index 100 in 2006)

Figure 2: Price forecast of labour, intermediate goods, and investment goods (Index 100 in 2006)
Figure 3: Production forecast of oil and gas (Bil. 2006 NOK)

Figure 4: Commodity inflow forecast to the petroleum sector (Bil. 2006 NOK)
new unexpected discovery. If better data becomes available, the scenario could be altered later.

The real interest rate is assumed to be 4 percent annually. In Norway, the petroleum revenues collected by the government are invested in the Central Government Pension Fund - Global (CPF), and only the expected real return of the fund, i.e. 4 percent of the fund each year, is allowed to be used for public expenditure. As mentioned by Heide et al. (2006, p.4), "The rule is supposed to be an operational compromise balancing concerns for intergenerational distribution of the petroleum wealth, high uncertainty about the value of the petroleum resources, and a gradual phasing in of the 'oil-money' in order to avoid Dutch Disease problems associated with excessive de-industrialisation."

### 4.3 Income estimation

On the basis of the reference scenario, the petroleum income and its components at each year are estimated (see Figure 5).

The petroleum income goes down almost linearly until zero in the end even though its three terms are fluctuating over time. The income is always less than the current cash flow and greater than the price change effects. The income in 2006 is about NOK 250 billion, over half of current cash flow to the sector. The other half of the current cash flow cancels out the net investments since the price change effects is trivial this year.

Even though theoretically the net investments could be positive, the estimation for the reference scenario is always negative since the extraction activities tend to be diminishing. This implies that the ability to produce income from the sector is decreasing over time. The level of net investments follows the level of the current cash flow each year even though gross investments fluctuate greatly as shown in Figure 4. This shows the depletion of the resource has significant effects on the net investments of the resource.

The price change effects first goes up and then down almost linear until zero in the end. At the beginning they are trivial. However, they become considerable after 2008. The main cause for the pattern should be the change of the oil and gas prices: decreasing in the beginning years and then growing later as shown in Figure 1.

Since all the variables after 2030 changes at almost constant rates annually, The curves
of income and its components become almost linear after the year 2030. This shows the relatively stable relations between income and its components in the case of linear changes of all future variables in the forecast.

However, the almost linear decreasing pattern of the petroleum income should not be over-interpreted. It is just a result on the basis of current available information in 2006, which in this paper is assumed to be deterministic. Since the information is updated over time, we can not expect such a linear pattern of petroleum income in the real world\(^6\).

In fact, mainly due to the updated forecast of oil prices, there is large fluctuation of the estimated petroleum income over time, which has been shown by Aslaksen et al (1990) and Brekke (1997b).

In the following, we focus on the beginning year 2006. We will check the sensitivity of net investments and price change effects if the real interest rate or oil and gas price

\[^6\text{For example, in the year } t, \text{ the income next year } t+1 \text{ is estimated by}\]

\[
Z_{t+1} = \sum_{s=t+1}^{\infty} R_{t+1}^{s+1} P_{x,s+1}^x X_{s+1}^x.
\]

If now we stand in the year \(t+1\), the forecast of all the prices is updated as one percent higher than before, i.e. \(\hat{P}_{x,s} = 1.01 P_{x,s}\) for all \(s \geq t+1\). By using the updated information, the income of the year \(t+1\) can be updated as

\[
\hat{Z}_{t+1} = \sum_{s=t+1}^{\infty} R_{t+1}^{s+1} \hat{P}_{x,s+1}^x X_{s+1}^x = 1.01 Z_{t+1},
\]

which is one percent higher than before. For the petroleum sector, the prices of inputs are relatively stable and the update of oil prices may have great effects on the estimated income.

Besides the updated prices, the forecast of the production may be updated, too.
path differs from the one in the reference scenario.

5 Sensitivity analysis

5.1 The effects of interest rate

The effects of interest rate on petroleum income and its components is an empirical question to a large extent. By the definition of sectoral income (3), the derivative w.r.t. real interest rate \( R \) can be approximated by

\[
\frac{dZ_t}{dR} \approx \sum_{s=t}^{\infty} \left[ \left( 1 - \frac{sR}{1 + R} \right) \frac{\pi_{s+1}}{\pi_t} P_{x,s+1} X_{s+1} \right],
\]

which implies if the time when the resource is exhausted \( T \leq 1 + 1/R \), then the sectoral income definitely moves to the same direction as the real interest rate. Otherwise, the effects of the interest rate is ambiguous. The effects of the interest rate on net investments and price change effects are also ambiguous since the changes of commodity flows and theirs prices from year to year may be positive or negative.

For the Norwegian petroleum sector, the reference scenario sets the real interest rate 4 percent annually. Undoubtedly the choice of real interest rate has significant influence on the estimation of income, which has been shown in the literature. Here the focus is on the net investments and price change effects.

Corresponding to various real interest rates, the future discount factors differ largely. A higher real interest rate implies lower discount factors in the future. Figure 6 shows the change of discount factors corresponding to four selected real interest rates.

The income and its components are estimated for 2006 corresponding to various real interest rates (see Figure 7). One obvious observation is that the price change effects is decreasing slowly along with increasing real interest rates. For the given future production and price forecast, the choice of interest rate level has trivial effects on the price change effects.

When real interest rates increase from half percent to 10 percent, the income goes up from 10 percent to 70 percent of current cash flow. Since the net investments is always
Figure 6: Discount factors corresponding to selected real interest rates (%)

Figure 7: Effects of interest rate. 2006 (Bil. NOK)
negative and the price change effects is trivial, the income is always lower than current cash flow. Even though both income and net investments go up with higher real interest rates, the effect on net investments is stronger in this case.

It is also interesting to notice that the income and net investments become less sensitive when the real interest rates is higher than 4 percent, the level in the reference scenario. This shows the asymmetric effects of the real interest rates. An implication of the observation is that in practice, the choice of real interest rate in a reasonable range is not a serious problem for the estimation of sectoral income and its components.

5.2 Change of oil and gas prices

Price forecast for the future plays an important role in the estimation of sectoral income and its components. Thus, it is interesting to see the effects of changes of price forecast. The estimation involves prices of all input and output. Since input can be thought of as negative output, then it is enough to focus on prices of output. For the petroleum sector, it is more interesting to see the effect of change of oil and gas price forecast.

By the definition (3), since the cash flows are always non-negative, the petroleum income moves along with the oil and gas prices, i.e. higher value of petroleum prices implies higher income. The interesting question is the effects on net investments and price change effects.

The effects on net investments is ambiguous. If a higher value of oil and gas prices meets a large positive change of extraction rate flows in a future year, then the effects may be positive, and vice versa. For the Norwegian petroleum sector, the change of extraction rate flows is always negative or trivial positive in the reference scenario, then the net investments will probably move to the opposite direction of the oil and gas prices.

The term of price change effects is mainly determined by the value of price change. A higher oil and gas price at a year may lead to lower or higher price change effects. Next several alternative oil and gas price paths are considered to check the effects of oil and gas prices on the income and its components.
5.2.1 Scaling up (or down) change rates

The selected paths of oil prices over time are illustrated in Figure 8. The path marked as -1.75 (or 1.75) in the figure means the petroleum price growth rate is 1.75 percent lower (or higher) than the reference scenario for each year. As shown in the 3.00 percent path, the oil price in 2050 is 3.5 times of the reference (BAU) price in the same year, which might seem extreme. The paths of gas prices follow that of oil prices.

Figure 9 gives the effects with different oil and gas price forecast. If it is marked as -1.75 (or 1.75) in the horizontal axis of the figure, then it means the petroleum price growth rate is 1.75 percent lower (or higher) than the reference scenario for each year.

Price change effects increase dramatically with more rapid oil and gas price growth.
In the most optimistic case, it is more than the current cash flow. On the contrary, the price change effects in the most pessimistic case is negative and in absolute value 30 percent of current cash flow.

It is not surprising that the income becomes greater along with higher price forecast. On the contrary, net investments are moving at almost the same rate in the opposite direction. Even though the price increase implies more negative net investments, the price change effects overweighs it and makes the income increasing. Since high petroleum prices imply high value of the same amount of petroleum resource, it is reasonable to observe higher absolute value of the income and net investments.

The alternative paths of oil and gas prices here have very different values in the end year 2050. If the prices in the end keep the same, will the various paths of prices still affect the results dramatically?

5.2.2 Fixed price in the end year

In this subsection, the oil and gas prices in the last year with positive extraction are fixed. The price paths vary within the 2006-2050 period. Figure 10 shows seven possible price paths: the middle path is linear, three are lower and three are higher\textsuperscript{7}. By applying the various price paths, the petroleum income and its components can be worked out as shown in Figure 11.

Higher oil an gas price paths leads to greater real income from the sector, However, it is a bit surprising that the path with the highest prices in the case High 3 leads to real income more than the current cash flow. This happens due to price change effects exceed the net investments. Since the oil and gas prices are the same at the beginning and the end of the whole period as in the reference scenario, this special case shows the petroleum income can be higher than current cash flow\textsuperscript{8} for some acceptable scenarios.

\textsuperscript{7}The paths of the prices are assumed to follow the function: \( P(t) = at^2 + bt + P(0) \). \( P(0) \) and \( P(T) \) are assumed given. The parameters \( a \) and \( b \) are solved out by \( P(T) = aT^2 + bT + P(0) \) and the FOC \( 2at + b = 0 \), where the value of \( t \) is chosen to obtain the reasonable price path. Here corresponding to the six paths (excluding linear path), the value of \( t \) equals \( (0.17, 0.27, 0.33, 0.66, 0.73, 1.33)T \) respectively. Then \( a = \frac{P(T) - P(0)}{2T^2} \), \( b = \frac{[P(T) - P(0)]}{T} - aT \). The linear path is got by letting \( a = 0 \) and \( b = \frac{[P(T) - P(0)]}{T} \).

\textsuperscript{8}This result should be common at the beginning stage of the extraction activities since current cash flow may be very small or even negative.
Figure 10: Alternative oil and gas price paths (Index 100 in 2006)

Figure 11: Effects of oil and gas price forecast change. 2006 (Bil. NOK)
even though the extraction rates of oil and gas tend to be decreasing over time as shown in Figure 3.

When compared with the results in the previous subsection, both net investment and price change effects now look more stable even though the price paths vary greatly. If more prices within the period can be fixed, the variation of net investments and price change effects together with the petroleum income could be much smaller.

It also shows that the negligible price change effects only happens for some specific price paths. Besides the positive effects on real income, the price change also has negative effects on net investments: Higher oil and gas prices imply more negative net investments in these cases. However, the change of net investments due to price change can be overweighted by the positive price change effects.

5.3 Discussion

In all the cases discussed above, the net investments is always negative and considerable, which shows the depletion of the resource has significant effects on the sectoral investments even though there are some causes leading to positive investments, such as technical progress and returns to current man-made capital. Under certain assumptions such as competitive market, no technical progress, and constant man-made capital, it can be shown that the net investments equals Hotelling rent, which can be thought of as the value of the resource depletion (Wei, 2009b). It is a natural result since the value of the resource depletion corresponds to the diminishing resource over time.

On the other hand, the price change effects may be positive or negative under reasonable cases for Norwegian petroleum sector. By the expression in (5), the negative case becomes possible since oil and gas prices are decreasing for certain years and prices of all the inputs keep increasing over time. In addition, as shown in the previous subsection, different price paths corresponds to different value of price change effects even though the prices change at the same rate on average over the whole period.

Before the net investments and price change effects are identified, it is hard to explain the case that the sectoral income is higher than current cash flow. It is not plausible to assume a negative value of the resource depletion. As shown in the previous subsection,
if the sectoral income exceeds current cash flow, it is caused by the future change of the resource prices. After cancelling out the value of the sectoral net investments (including the value of the resource depletion), the price change effects is still positive to make the sectoral income greater than current cash flow.

6 Concluding remark

In this paper, a discrete-time version of Asheim and Wei (2009)’s sectoral income definition has been described and applied to estimate income arising from Norwegian petroleum sector. In particular, as the components of the income, the net investments and price change effects have been identified and analyzed. As a whole, by taking Norwegian petroleum sector as an example, the paper shows how to estimate sectoral net investments and price change effects of a non-renewable resource sector. The distinction between net investments and price change effects is useful for the practical non-renewable resource accounting and related policy-making issues.

It turns out that the choice of real interest rate in a reasonable range has modest effects on price change effects for Norwegian petroleum sector. Both income and net investments become larger with increasing real interest rates. Furthermore, the change of price forecast may influence dramatically the price change effects, which indicates the importance of the price change effects. In all the cases, the net investments are always negative and considerable even in case that the income or price change effects is above current cash flow.

There are several topics for further study. First, the reference scenario could be updated year by year. Then, during the estimation, the extraction activities are assumed to be independent on any prices. If this is not a valid assumption, then the relation between the production and prices should be studied. Moreover, the paper assumes a deterministic economy. It will be worth studying the income and its components for an uncertain economy.
Appendix: Discrete time version of sectoral income

First, a discrete-time version of national income is derived by following the arguments in Asheim (2007) and Asheim and Wei (2009, Appendix A). Assume initial time is 0. Suppose dynamic welfare, \( W \), is a functional \( \mathcal{G} \) of the path of consumption flows:

\[
W(t) = \mathcal{G}(\{C_{\tau}\}_{\tau=t}^{\infty}, t),
\]

(A1)

where the functional \( \mathcal{G} \) has two properties

- **Time-invariant.** For any given time \( t \geq 0 \) and any two consumption flows \( \{C_{\tau}\}_{\tau=t}^{\infty} \) and \( \{\tilde{C}_{\tau}\}_{\tau=0}^{\infty} \),

\[
\mathcal{G}(\{C_{\tau}\}_{\tau=t}^{\infty}, t) = \mathcal{G}(\{\tilde{C}_{\tau}\}_{\tau=0}^{\infty}, 0)
\]

(A2)

if \( \tilde{C}_{\tau} = C_{\tau+t} \) for all \( \tau \geq 0 \).

- **Independent future.** If any two consumption flows \( \{C'_{\tau}\}_{\tau=0}^{\infty} \) and \( \{C''_{\tau}\}_{\tau=0}^{\infty} \) coincide during the interval \([0, t] \), then

\[
\mathcal{G}(\{C'_{\tau}\}_{\tau=0}^{\infty}, 0) < \mathcal{G}(\{C''_{\tau}\}_{\tau=0}^{\infty}, 0) \iff \mathcal{G}(\{C'_{\tau}\}_{\tau=t}^{\infty}, t) < \mathcal{G}(\{C''_{\tau}\}_{\tau=t}^{\infty}, t).
\]

(A3)

Then directly by the definition, welfare change after a period is numerically equal to

\[
W(t + 1) - W(t) = \mathcal{G}(\{C_{\tau}\}_{\tau=t+1}^{\infty}, t + 1) - \mathcal{G}(\{C_{\tau}\}_{\tau=t}^{\infty}, t)
= \mathcal{G}(\{\tilde{C}_{\tau}\}_{\tau=0}^{\infty}, 0) - \mathcal{G}(\{\tilde{C}_{\tau}\}_{\tau=0}^{\infty}, 0)
\]

by (A2),

where \( \tilde{C}_{\tau} = C_{\tau+t+1} \) and \( \tilde{C}_{\tau} = C_{\tau+t} \) for all \( \tau \geq 0 \). The second equation shows the welfare change does not depend on current time \( t \) explicitly. The first-order approximation gives

\[
\mathcal{G}(\{\tilde{C}_{\tau}\}_{\tau=0}^{\infty}, 0) - \mathcal{G}(\{C_{\tau}\}_{\tau=0}^{\infty}, 0) \approx \sum_{s=0}^{\infty} \nabla \mathcal{G}_s(\{\tilde{C}_{\tau}\}_{\tau=0}^{\infty}, 0) \left( \tilde{C}_{s} - \hat{C}_{s} \right),
\]

where \( \nabla \mathcal{G}_s(\{\tilde{C}_{\tau}\}_{\tau=0}^{\infty}, 0) \) is the vector of marginal welfare w.r.t. consumption flow at time \( s \),
i.e. if consumption flow at time $s$ is a vector as $\hat{\mathbf{C}}_s = \left( \hat{C}_s(1), \hat{C}_s(2), ..., \hat{C}_s(n) \right)$, then

$$\nabla G_s(\{\hat{\mathbf{C}}_r\}_{r=0}^{\infty}, 0) = \left( \frac{\partial G}{\partial \hat{C}_s(1)}, \frac{\partial G}{\partial \hat{C}_s(2)}, ..., \frac{\partial G}{\partial \hat{C}_s(n)} \right).$$

Hence, by sequential substitution, the welfare change is approximated by

$$W(t + 1) - W(t) \approx \sum_{s=t}^{\infty} \nabla G_s(\{\mathbf{C}_r\}_{r=t+1}^{\infty}, t + 1) \left( \mathbf{C}_{s+1} - \mathbf{C}_s \right). \quad (A4)$$

Define present value price of consumption at time $t$ by

$$p_{c,s} = \nabla G_s(\{\mathbf{C}_r\}_{r=t}^{\infty}, t)$$

for all $s \geq t$. Applying the price definition to (A4), welfare change is represented by

$$W(t + 1) - W(t) \approx \sum_{s=t}^{\infty} p_{c,s} \left( \mathbf{C}_{s+1} - \mathbf{C}_s \right). \quad (A5)$$

Following the definition in equation (1) of Asheim and Wei (2009), national income is defined by current consumption value plus the present value of future consumption changes. In the discrete time case, national income is expressed by

$$y_t = p_{c,t} C_t + \sum_{s=t}^{\infty} p_{c,s} \left( \mathbf{C}_{s+1} - \mathbf{C}_s \right), \quad (A6)$$

where the second term on the right-hand side is approximately equal to welfare change by (A5) and can also be defined as national savings. This gives a sound welfare interpretation of the definition. Furthermore, under certain assumptions, the income such defined equals net national product, which gives the definition a productive interpretation. This is discussed in more detail in Sefton and Weale (2006) and Asheim and Wei (2009).

By expanding the expression of the definition in (A6) and reorganizing terms, we rewrite the national income as

$$y_t = \sum_{s=t}^{\infty} (p_{c,s} - p_{c,s+1}) \mathbf{C}_{s+1}. \quad (A7)$$
Define Paasche consumption price indices by $\pi_0 = 1$ and

$$\frac{\pi_{s+1} - \pi_s}{\pi_{s+1}} = \frac{(p_{c,s+1} - p_{c,s})C_{s+1}}{p_{c,s+1}C_{s+1}} \quad \text{(A8)}$$

for all $s \geq 0$. Let real prices of consumption

$$p_{c,s} = \frac{p_{c,s}}{\pi_s}. \quad \text{(A9)}$$

Define real interest rate at time $s + 1$ by

$$R_{s+1} = \frac{\pi_s - \pi_{s+1}}{\pi_{s+1}}, \quad \text{(A10)}$$

for all $s \geq 0$. Thus the Paasche consumption price indices can be expressed by $\pi_0 = 1$ and

$$\pi_s = \prod_{v=1}^{s} \frac{1}{1 + R_v}, \quad \text{(A11)}$$

for all $s \geq 1$. Then by sequential substitution,

$$(p_{c,s} - p_{c,s+1})C_{s+1} = \frac{\pi_s - \pi_{s+1}}{\pi_{s+1}}p_{c,s+1}C_{s+1} \quad \text{by (A8)}$$

$$= \frac{\pi_s - \pi_{s+1}}{\pi_{s+1}}(\pi_{s+1}p_{c,s+1})C_{s+1} \quad \text{by (A9)}$$

$$= R_{s+1}\pi_{s+1}p_{c,s+1}C_{s+1} \quad \text{by (A10).} \quad \text{(A12)}$$

Then substituting (A12) into (A7), real national income is expressed as

$$Y_t = \sum_{s=t}^{\infty} R_{s+1} \frac{\pi_{s+1}}{\pi_t} p_{c,s+1}C_{s+1}, \quad \text{(A13)}$$

which, in fact, is the present value of real interest on future consumption at the beginning of current time $t$. The expression under continuous time is derived by Asheim and Wei (2009) and both welfare and productive justifications.

Following the similar argument given by sections 5 and 6 in Asheim and Wei (2009), real national income can be decomposed to individual and sectoral income. Here we just give an intuitive interpretation. Since all current and future consumption contributes to national
income, then future cash flow to a sector that could be used for consumption also contributes to real national income. Hence, real sectoral income is defined as the present value of real interest on the current and future cash flow to the sector that is used for consumption.

Let \( \{ \mathbf{X}_s \}_{s=0}^{\infty} \) denote the vector of commodity flows excluding consumption for end use in the sector, and \( \{ \mathbf{P}_{x,s} \}_{s=0}^{\infty} \) be the corresponding path of real prices of these commodity flows. Then real sectoral income is defined by

\[
Z_t = \sum_{s=t}^{\infty} R_{s+1} \frac{\pi_{s+1}}{\pi_t} \mathbf{P}_{x,s+1} \mathbf{X}_{s+1},
\]

(A14)

which is the present value of real interest on sectoral future cash flow used for consumption. For a sector without consumption flows, its income is simplified as the present value of real interest on the sectoral future cash flow.

If real interest rate is assumed to be constant over time, \( R_s = R \) for all \( s \geq t \), then real sectoral income in (A14) can be rewritten as

\[
Z_t = R \sum_{s=t}^{\infty} \frac{\pi_{s+1}}{\pi_t} \mathbf{P}_{x,s+1} \mathbf{X}_{s+1},
\]

which can be interpreted as real interest on sectoral wealth at the end of the year. Then the income as interest on wealth is a special case of real income defined by (A14). The concept of income as interest on wealth is commonly suggested to measure sectoral income (Brekke, 1997a, Section 3.3). More theoretical analysis is provided by Appendix B of Asheim and Wei (2009), where the commonly suggested definition is compared with the definition of real income in (A14) for the continuous-time case.

An advantage of the definition of real sectoral income in (A14) is that it yields a decomposition that provides useful information. By using the definitions in (A10), real sectoral income can be rewritten as

\[
Z_t = \underbrace{\mathbf{P}_{x,t} \mathbf{X}_t}_{\text{current cash flow}} + \sum_{s=t}^{\infty} \frac{\pi_s}{\pi_t} \left( \mathbf{P}_{x,s+1} \mathbf{X}_{s+1} - \mathbf{P}_{x,s} \mathbf{X}_s \right). \quad \text{(A15)}
\]

If the current cash flow is one part of the real sectoral income, then the rest of the income comes from future cash flow change, which is the present value of future cash flow change. As for the
cash flow change at each year $s \geq t$, we have the equation

$$P_{x,s+1}X_{s+1} - P_{x,s}X_s = P_{x,s} (X_{s+1} - X_s) + (P_{x,s+1} - P_{x,s}) X_{s+1}.$$  

Then inserting into (A15), we obtain another expression for real sectoral income as

$$Z_t = \underbrace{P_{x,t}X_t}_{\text{current cash flow}} + \sum_{s=t}^{\infty} \underbrace{\frac{\pi_s}{\eta_t} P_{x,s} (X_{s+1} - X_s)}_{\text{net investments}} + \sum_{s=t}^{\infty} \underbrace{\frac{\pi_s}{\eta_t} (P_{x,s+1} - P_{x,s}) X_{s+1}}_{\text{price change effects}} \quad (A16)$$

Here the effects of future cash flow change is further decomposed to be net investments and price change effects. Net investments is the effects of future commodity flow change. Price change effects is the present value of sectoral cash flow due to price change.

References


Stochastic national income*

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Abstract

How can national income be adjusted to indicate welfare improvement if the future is uncertain? The paper extended the definition of national income to stochastic settings on the basis of discounted utilitarian welfare function. Real interest rate of consumption is redefined so that real national income can be interpreted as the expected present value of real interest on future national consumption. A stochastic one-good model is used to illustrate the application of the stochastic real national income. It turns out that the curvature of utility function may have dramatic effects on real national income under uncertainty and national income estimated in the deterministic manner may no longer serve as an indicator for prudent behavior.

Key words: Stochastic income, comprehensive national accounting

JEL classification: C43, D60, O47, P44

1 Introduction

Hicks (1946, p. 172) defines “...a man’s income as the maximum value which he can consume during a week, and still expect to be as well off at the end of the week as he was at the beginning.” By such a definition (Hicksian income), Hicks (1946) intends to

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make income an indicator to give people an amount "which they can consume without impoverishing themselves." The definition specifies three key elements:

- Income corresponds to a maximum value of consumption in a period satisfying certain assumptions
- There is something that should be "as well off" if the income is consumed in the period
- The level "as well off" is an expectation and hence income itself is an expectation

In the literature of comprehensive national accounting theory, there is no doubt on the first element, i.e. income is always interpreted as a maximum value of consumption satisfying certain conditions. However, there are several interpretations for the second element. What should be "as well off?"

Following the tradition of Fisher (1906) and Lindahl (1933, Section II), income is defined as interest on wealth, where wealth is the present value of future consumption. The definition suggests keeping wealth "as well off." If the interest rate is constant all the time, then constant wealth provides a constant income flow. Otherwise, the constant wealth can not ensure a constant income flow.

Then In the spirits of “Income No. 3” offered by Hicks (1946), income can be represented by the “stationary equivalent of future consumption” (Weitzman, 1976, p. 160). This concept of income intends to keep consumption "as well off" and is related to the constant level of consumption with the same present value as the actual future stream of consumption in an economy where well-being depends on a single consumption good. Unfortunately, as pointed out by Asheim (1997), Sefton and Weale (2006, Section 3.1.2) and Asheim and Wei (2009, Appendix B), such wealth equivalent income does not satisfy certain properties and is difficult to generalize to the empirically relevant case of multiple consumption goods.

The latest attempt (Pemberton and Ulph, 2001; Sefton and Weale, 2006) is to associate “as well off” with the level of dynamic welfare. Following the view, Sefton and Weale (2006) define national income as a weighted present value of future national consumption, which is further interpreted as the present value of real interest on future national
consumption by Asheim and Wei (2009).

The third key element of the Hicksian income mentioned at the beginning is ignored in the literature for a long time. When the theory is applied to practice, the uncertainty is an unavoidable problem. Then it is better to interpret the level "as well off" as an expectation and so for the concept of income.

By applying the maximum principle of control theory, Weitzman (2003, Chapter 7) introduces a stochastic process to the capital stock in a one-good stochastic model and a concept of income in a stochastic setting is defined as the return on expected wealth (Weitzman, 2003, p. 321), where wealth can be understood as the present value of future consumption discounted by constant rate. The stochastic concept of income is a natural extension of the wealth equivalent income. Since there are new concepts of national income by associating "as well off" with the level of dynamic welfare (ref. Sefton and Weale, 2006; Asheim and Wei, 2009), could we extend the new concept to the realm of uncertainty? The question is what the present paper aims to answer.

As shown in the citations from Hicks (1946), The Hicksian income is estimated for the purpose of prudent behavior. It means the income is estimated to make the decision on how much to consume for the current period. Hence, the income has to be calculated at the beginning of the period before the current decision is made, which can be called *ex ante* income. Obviously *ex post* income, which is calculated after the current decision is made, is irrelevant to the decision-making process for the current period. On the contrary, the current decision may have effects on the *ex post* income. Then, the present paper focuses on *ex ante* income. The results of the paper reminds us that national income estimated in the deterministic manner may no longer serve as an indicator for prudent behavior under uncertainty.

The basic logic of the paper is as follows. After stochastic processes are introduced, income becomes an expectation of maximums of consumption such that the expected dynamic welfare keeps constant. Then real national income is defined as the *expected* present value of real interest on future national consumption, which has good interpretation in a welfare and productive perspective. Moreover, the paper shows that the curvature of utility function may have dramatic effects on real income under uncertainty by a stochastic one-good model. The higher the curvature of utility function, the less is
the real income since more resource is required for precaution of future uncertainty.

The paper is organized as follows. Next section illustrates the latest theory of national income in the deterministic case by using the discounted utilitarian welfare function. Then stochastic national income is defined and discussed in Section 3 by introducing stochastic processes. In Section 4, a stochastic one-good model is used to illustrate the findings in the paper. The final section concludes the paper. Within each section, it begins with a brief literal statement for the contents in the section and then follows by more rigorous description by mathematical tools. The Appendix includes the mathematics applied in the analysis.

2 Deterministic national income

As mentioned above, the concept of income is a maximum of consumption level such that the dynamic welfare keeps at the same level at the end of each time. Then the first step is to know what the dynamic welfare is. In the literature, the dynamic welfare is generally related to utility and then consumption, namely all factors that have effects on utility are defined to be included in the concept of "consumption." Hence, any instantaneous change of dynamic welfare can be associated with the consumption change. It is proved that the present value of future consumption change measures welfare improvement by properly defined prices of consumption (ref. Samuelson, 1961; Sefton and Weale, 2006; Asheim and Wei, 2009). By interpreting the present value of future consumption changes as national savings and adding to current consumption (measured in the same numeraire), we get the concept of national income. Furthermore, real national income can be defined as the present value of real interest on future consumption (ref. Definition 1 below).

The rest of this section illustrates the concept of national income for a deterministic economy by applying the discounted utilitarian welfare function.

For a given unidimensional utility flow \( \{ U(s) \}_{s=0}^{\infty} \) over time, dynamic welfare at any time \( t \geq 0 \) is defined by the discounted utilitarian,

\[
W(t) = \int_{t}^{\infty} e^{-\rho(s-t)} U(s) \, ds,
\]
where $\rho$ is a given constant utility discount rate. Then all the future utility has effect on the present dynamic welfare. By Leibniz’s Formula (Theorem 6.1), the instantaneous change of welfare is represented by

$$\frac{dW(t)}{dt} = -U(t) + \rho \int_t^\infty e^{-\rho(t-s)}U(s)\,ds.$$  

Furthermore, by assuming the utility flow is a smooth curve over time, i.e. the first-order derivative of utility w.r.t. time exists almost everywhere, and the transversality condition holds: \(\lim_{s \to \infty} e^{-\rho(s-t)}U(s) = 0\), which implies the infinite future utility means nothing for current time, then we obtain by integrating by parts

$$\frac{dW(t)}{dt} = \int_t^\infty e^{-\rho(s-t)}\dot{U}(s)\,ds,$$

which shows that instantaneous change of welfare over time depends on the present value of future utility change (ref. Asheim, 2007).

Now assume the utility is a time-invariant, concave and non-decreasing function $u$ with continuous second derivatives w.r.t. an $n$-dimensional vector of consumption $C$. Suppose the utility flow is given by

$$U(s) = u(C(s)),$$  

for all $s \geq 0$, where $C(s)$ is the consumption at time $s \geq 0$. Define present value consumer prices, \(\{p_c(s)\}_{s=0}^\infty\) for all $s \geq 0$, satisfying,

$$p_c(s) = e^{-\rho(s-t)}\nabla u(C(s)),$$

where $\nabla u(C(s))$ represents the vector of marginal utility w.r.t. consumption at the time $s$, i.e.

$$\nabla u(C(s)) = \left(\frac{\partial u}{\partial c_1}(C(s)), \frac{\partial u}{\partial c_2}(C(s)), \ldots, \frac{\partial u}{\partial c_n}(C(s))\right).$$

In addition we still assume the transversality condition holds: \(\lim_{s \to \infty} e^{-\rho s}u(C(s)) = 0\).
By applying (2) to (1), we achieve that for a given smooth consumption flow over time\(^1\),

\[
\frac{dW(t)}{dt} = \int_t^\infty e^{-\rho(s-t)}\nabla u(C(s))\dot{C}(s)\, ds = \int_t^\infty p_c(s)\dot{C}(s)\, ds,
\]

which shows that instantaneous change of welfare is represented by the present value of future consumption change. As shown in the previous literature (eg. Samuelson, 1961; Sefton and Weale, 2006; Asheim, 2007), \(\int_t^\infty p_c(s)\dot{C}(s)\, ds\) can be interpreted as national savings. Then, if national income is to serve as a guide for prudent behavior such that the dynamic welfare improves if and only if national consumption is smaller than national income, we obtain a definition of national income as the sum of current consumption value plus the present value of future consumption change,

\[
y(t) = p_c(t)C(t) + \int_t^\infty p_c(s)\dot{C}(s)\, ds. \tag{3}
\]

Just by integrating by parts, we directly derive another expression of national income from (3),

\[
y(t) = \int_t^\infty -\ddot{p}_c(s)C(s)\, ds, \tag{4}
\]

since \(u(C)\) has continuous second derivatives w.r.t. consumption.

A Divisia consumer price index \(\{\pi(s)\}_{s=0}^\infty\) can be defined satisfying \(\pi(0) = 1\) and

\[
\frac{\dot{\pi}(s)}{\pi(s)} = \frac{\dot{p}_c(s)C(s)}{p_c(s)C(s)} \tag{5}
\]

for all \(s \geq 0\). Define the path of real consumption interest rates \(\{R(s)\}_{s=0}^\infty\) by

\[
R(s) = -\frac{\dot{\pi}(s)}{\pi(s)} \tag{6}
\]

for all \(s \geq 0\). Then for all \(s \geq 0\),

\[
\pi(s) = \exp\left(\int_0^s -R(v)\, dv\right). \tag{7}
\]

\(^1\)In the paper, the product of any two vectors means the inner-product of the two vectors. eg. if \(p = (p_1, p_2, ..., p_n)\) and \(c = (c_1, c_2, ..., c_n)\), then \(pc = \sum_{i=1}^n p_ic_i\). Then we always have the Commutative Law, \(pc = cp\).
The real consumption price flow \( \{P_c(s)\}_{s=0}^{\infty} \) can be defined by

\[
P_c(s) = \frac{p_c(s)}{\pi(s)} \tag{8}
\]

for all \( s \geq 0 \). Since for all \( s \geq 0 \),

\[
-p_c(s)C(s) = -\frac{\pi(s)}{\pi(s)}p_c(s)C(s) \quad \text{by (5)}
\]

\[
= R(s)p_c(s)C(s) \quad \text{by (6)}
\]

\[
= R(s)\pi(s)P_c(s)C(s) \quad \text{by (8)},
\]

Then by (4), real national income is associated with the sum of present value of real interest on future national consumption as stated in the following definition (Sefton and Weale, 2006; Asheim and Wei, 2009).

**Definition 1** Real national income at time \( t \) is defined as

\[
Y(t) = \int_t^{\infty} \frac{\pi(s)}{\pi(t)} R(s)P_c(s)C(s) \, ds \tag{9}
\]

### 3 Stochastic national income

A natural question now is what the definition of national income will be if the economy is not deterministic. This section introduces Brownian motions (or Wiener processes) into the model and expressions of national income for the stochastic case are derived by following the logic of the deterministic case. In the stochastic case, the dynamic welfare is an expectation related to utility and further consumption. The stochastic national income is an expectation of all the possible maximums of consumption while the welfare level is expected to keep constant. Since the current consumption is known, then the uncertainty of national income results from the national savings. If the real interest rate is properly redefined, the real national income can be expressed by the expected present value of real interest on future consumption.

The analysis in the section involves stochastic integrals. All the results can be expressed by the common Itô integral, which is better for the real calculation. However,
in order to interpret the results more intuitively, another type of stochastic integral, so-called Stratonovich integral, is used to express the results. The latter type of integral exhibits the useful property of chain rule, which enables us to derive expressions in the stochastic case similar to the deterministic case. The basic mathematical knowledge on these stochastic integrals is listed briefly in the Appendix.

3.1 Welfare and national income

In the subsection, the concept of national income is derived from a dynamic welfare function under uncertainty. The section shows that instantaneous improvement of welfare can be represented by the expected present value of future utility change. By assuming the utility is determined completely by consumption, the instantaneous improvement of welfare can be represented by the expected present value of future consumption change, which is associated with national savings. Then national income is defined by the current consumption plus the national savings.

Suppose a unidimensional utility flow \( \{U(s)\}_{s=0}^{\infty} \) with given initial value \( U(0) \) is an Itô process, i.e. the utility has the form

\[
dU(s) = a(s, \omega) ds + b(s, \omega) dB(s)
\]

for all \( s \geq 0 \), where \( \omega \) is random variables, \( B \) is a unidimensional Brownian motion, and \( a(s) = E[\frac{dU(s)}{ds}] / ds \), the expected instantaneous change of utility at time \( s \). Dynamic welfare at any time \( t \geq 0 \) is defined by the expected discounted utilitarian,

\[
W(t) = E^t \left[ \int_t^\infty e^{-\rho(s-t)} U(s) ds \right],
\]

where \( \rho \) is a constant utility discount rate. If instantaneous welfare improvement is represented by the expected instantaneous change of welfare at any time \( t \geq 0 \),

\[
AW(t) = \lim_{h \to 0^+} \frac{E^t [W(t+h)] - W(t)}{h},
\]
then we have

**Proposition 1** If utility in the infinite future is assumed to mean nothing for current economy, i.e. \( \lim_{s \to \infty} e^{-\rho(s-t)} E^t [U(s)] = 0 \) for any given point in time \( t \geq 0 \), then instantaneous welfare improvement, i.e. the expected instantaneous change of welfare at any time \( t \geq 0 \), \( AW(t) \), is represented by the expected discounted value of future utility change,

\[
AW(t) = E^t \left[ \int_t^\infty e^{-\rho(s-t)} dU(s) \right].
\]

(12)

Proof. \( e^{-\rho(s-t)} \) is a deterministic process over time and \( U(s) \) is an Itô process, then by Theorem 6.3 in the Appendix,

\[
d \left[ e^{-\rho(s-t)} U(s) \right] = e^{-\rho(s-t)} dU(s) - \rho e^{-\rho(s-t)} U(s) ds,
\]

which is the representation of the integral

\[
\int_t^\infty d \left[ e^{-\rho(s-t)} U(s) \right] = \int_t^\infty e^{-\rho(s-t)} dU(s) - \int_t^\infty \rho e^{-\rho(s-t)} U(s) ds,
\]

where the first term on the r.h.s. involves Itô integral. By rearranging terms, taking the expectation on both sides and applying the definition of welfare, we obtain that

\[
E^t \left[ \int_t^\infty e^{-\rho(s-t)} dU(s) \right] = -U(s) + \rho W(t),
\]

(13)

since \( \lim_{s \to \infty} E \left[ e^{-\rho(s-t)} U(s) \right] = 0 \).

Let \( h \geq 0 \) be a very small time interval, then by the definition of welfare (10), the welfare at time \( t + h \),

\[
W(t + h) = E^{t+h} \left[ \int_{t+h}^\infty e^{-\rho(s-(t+h))} U(s) ds \right].
\]

(14)
On the other hand, the current welfare at time $t$ can be rewritten as

$$W (t) = E^t \left[ \int_t^{t+h} e^{-\rho (s-t)} U (s) \, ds + \int_{t+h}^{\infty} e^{-\rho (s-t)} U (s) \, ds \right]$$

$$= E^t \left[ \int_t^{t+h} e^{-\rho (s-t)} U (s) \, ds \right] + E^t \left[ \int_{t+h}^{\infty} e^{-\rho (s-t)} U (s) \, ds \right]$$

$$= E^t \left[ \int_t^{t+h} e^{-\rho (s-t)} U (s) \, ds \right] + e^{-\rho h} E^t \left[ \int_{t+h}^{\infty} e^{-\rho (s-(t+h))} U (s) \, ds \right]$$

$$= E^t \left[ \int_t^{t+h} e^{-\rho (s-t)} U (s) \, ds \right] + e^{-\rho h} E^t [W (t + h)] \quad \text{by (14)} \quad (15)$$

for any $h \geq 0$. By the definition of the limit, we know that the limit of the first term on the r.h.s. of (15)

$$\lim_{h \to 0^+} \frac{E^t \left[ \int_t^{t+h} e^{-\rho (s-t)} U (s) \, ds \right]}{h} = E^t \left[ e^{-\rho (s-t)} U (s) \right] \big|_{s=t} = U (t). \quad (16)$$

Now directly by (15) and (16), the following expression holds

$$\lim_{h \to 0^+} \frac{E^t [W (t + h)] - W (t)}{h}$$

$$= - \lim_{h \to 0^+} \frac{E^t \left[ \int_t^{t+h} e^{-\rho (s-t)} U (s) \, ds \right]}{h} + \lim_{h \to 0^+} \frac{(1 - e^{-\rho h}) E^t [W (t + h)]}{h}$$

$$= -U (s) + \lim_{h \to 0^+} \frac{(1 - e^{-\rho h})}{h}$$

$$= -U (s) + \rho W (t). \quad (17)$$

Combine (11), (13) and (17), the proposition is proven. [Q.E.D.]

Now let the utility at any future time defined as a time-invariant, concave and non-decreasing function $u$ with continuous second derivatives w.r.t. an $n$-dimensional vector of consumption $\mathbf{C}$, i.e. $U (t) = u (\mathbf{C} (t))$ for all $t \geq 0$, where consumption is an Itô process with given initial value $\mathbf{C} (0)$. i.e. the vector of consumption has the form

$$d \mathbf{C} (t) = \mathbf{\alpha} (t, \omega) \, dt + \mathbf{\beta} (t, \omega) \, dB_t$$

for all time $t \geq 0$, where $\mathbf{\alpha}$ is an $n$-dimensional vector, $\omega$ random variables, $\mathbf{\beta}$ an $n \times m$
dimensional vector, and $B$ is an $m$-dimensional Brownian motion. Then the utility flow 
\{U (t)\}_{t=0}^{\infty}$ is also an Itô process by the Itô formula (Theorem 6.2). Still let $\nabla u(C(t))$
 denote the vector of marginal utility w.r.t. consumption at any time $t \geq 0$, and the vector 
of the present value prices of consumption is defined satisfying
\[ p_c(s) = e^{-\rho(s-t)} \nabla u(C(s)) \] (18)
for all $s \geq t$. Then Proposition 1 can be applied to obtain one corollary:

**Corollary 1** If the utility due to consumption in the infinite future is assumed to mean 
nothing for the current economy, i.e. $\lim_{s \to \infty} E^t \left[ e^{-\rho(s-t)} u(C(s)) \right] = 0$ for any given point 
in time $t \geq 0$, then the instantaneous welfare improvement is indicated by the expected 
present value of future consumption change,
\[ AW(t) = E^t \left[ \int_{t}^{\infty} p_c(s) \circ dC(s) \right], \] (19)
where $\circ d$ indicates Stratonovich integral.

**Proof.** Since the consumption \{C(s)\}_{s=0}^{\infty} is an Itô process, the property of chain rule of 
Stratonovich integral (Theorem 6.4.a) tells us
\[ du(c(s)) = \nabla u(c(s)) \circ dC(s) = e^{\rho(s-t)} p_c(s) \circ dC(s) \]
by (18). Since the utility \{U(t)\}_{t=0}^{\infty} is also an Itô process under the settings, then Proposition 
1 is valid. By (12), the *instantaneous welfare improvement*,
\[ AW(t) = E^t \left[ \int_{t}^{\infty} e^{-\rho(s-t)} \left( e^{\rho(s-t)} p_c(s) \circ dC(s) \right) \right] 
= E^t \left[ \int_{t}^{\infty} p_c(s) \circ dC(s) \right] \]
since $e^{\rho(s-t)}$ is a deterministic process over time. [Q.E.D.]

Since the r.h.s. of expression (19) is a Stratonovich integral, the consumption prices 
used for evaluation can be called the *Stratonovich-like prices of consumption.*
Corollary 1 gives a welfare foundation for interpreting $E^t \left[ \int_t^\infty p_c (s) \circ dC (s) \right]$ as national savings. Hence, if national income is supposed to be a guide for prudent behavior such that the dynamic welfare improves if and only if national consumption is smaller than national income, then national income equals an expectation, $p_c (t) C (t) + E^t \left[ \int_t^\infty p_c (s) \circ dC (s) \right]$. By using the chain rule of Stratonovich integral (Theorem 6.4.b) on $p_c (s) C (s)$, we have

$$\lim_{s \to \infty} p_c (s) C (s) - p_c (t) C (t) = \int_t^\infty p_c (s) \circ dC (s) + \int_t^\infty C (s) \circ dp_c (s).$$

If the consumption in the infinite future means nothing for the current economy, i.e.

$$\lim_{s \to \infty} E^t [p_c (s) C (s)] = 0,$$

Then we obtain

$$\int_t^\infty C (s) \circ (-d p_c (s)) = p_c (t) C (t) + \int_t^\infty p_c (s) \circ dC (s).$$

By taking the expectation on both sides, we find another expression for national income,

$$E^t \left[ \int_t^\infty C (s) \circ (-d p_c (s)) \right] = p_c (t) C (t) + E^t \left[ \int_t^\infty p_c (s) \circ dC (s) \right]. \quad (20)$$

The equation is the main result of the subsection. Under uncertainty, national income is an expectation. Since current consumption is known, then the uncertainty comes from the national savings.

A one-dimensional example for national savings

If the consumption is one-dimensional and follows the form,

$$dC (t) = \alpha (t) \, dt + \beta (t) \, dB_t,$$

where $\alpha$ and $\beta$ are given paths of constants over time, then by the relations between Itô and
Stratonovich integrals (Theorem 6.5), national savings can be expressed by the Itô integral,

$$
E^t \left[ \int_t^\infty p_c(s) \circ dC(s) \right] = E^t \left[ \int_t^\infty p_c(s) \alpha(t) \, ds \right] + \frac{1}{2} E^t \left[ \int_t^\infty \frac{dp_c(s)}{dc}(C(s))(\beta(t))^2 \, ds \right],
$$

where the first term on the r.h.s. can be calculated on the basis of the expected instantaneous change of future consumption and their expected prices. The second term is an adjustment term, which depends on the correlations between future consumption and their prices. For instance, if the demand curve is given by a downward-sloping linear function,

$$p_c(s) = e^{-\rho(s-t)} [\bar{c}_c - b_c C(s)],$$

where $b_c > 0$, then the adjustment term equals

$$\frac{1}{2} E^t \left[ \int_t^\infty e^{-\rho(s-t)} (-b_c)(\beta(s))^2 \, ds \right] = -\frac{1}{2} b_c E^t \left[ \int_t^\infty e^{-\rho(s-t)} (\beta(s))^2 \, ds \right] \leq 0,$$

which implies national savings should be adjusted downwards and so should national income.

### 3.2 National income and ex ante NNP

So far the definition of the stochastic national income by (20) has a sound welfare interpretation. By adding more assumptions, we can show that stochastic national income coincides with ex ante net national product (NNP). Ex ante NNP is defined as the sum of current consumption value plus the expected value of current capital change evaluated at a Stratonovich-like capital price, the latter is also called genuine savings (the terminology introduced by Hamilton, 1994). The NNP is called ex ante because it is evaluated after the investment is decided and just before the stochastic factor on capital stock is revealed at current time.

Suppose the state of the economy is represented by an $m$-dimensional vector of capital stock $K$, which is known at current time and used to produce more goods. Besides consumption, commodities produced can be invested to change the capital stock flow over time. The consumption - net investment pair $(C, I)$ is called attainable if $(C(s), I(s)) \in S(K(s))$ for time $s \geq 0$, where $S(K)$ is the production probabilities sets given capital $K$. If the level of capital $K$ at any two points in time is the same, then the allocated consumption and net investment are the same for these two points.
Thus, both consumption and net investment at each point in time are functions of current capital alone, i.e. $C(s) = C(K(s))$ and $I(s) = I(K(s))$ hold for any time $s \geq 0$. Such an arrangement is called a resource allocation mechanism (RAM) (as introduced by Dasgupta and Mäler, 2000; Dasgupta, 2001; Arrow, et al, 2003).

Suppose the capital change at each time is affected by two factors: the net investment and a random variable with zero expectation. Then let a RAM decided at the beginning of current time 0, the net investment depends on capital alone and the expected instantaneous capital change can be expressed by a function of the net investments alone, i.e. $E [dK(s)]/ds = \mu_K (I(K(s)))$ for all $s \geq 0$. Thus, let the capital stock flow $\{K(s)\}_{s=0}^{\infty}$ with given initial value $K(0)$ represented by an autonomous Itô diffusion of the form$^2$

$$dK(s) = \mu_K (I(K(s))) \, ds + \sigma (K(s)) \, dB(s), \quad (21)$$

for all $s \geq 0$, where $B$ is an $m$-dimensional Brownian motion over time. The Brownian motion represents the stochastic factor with zero expectation that has effects on capital stock. Suppose the stochastic differential equation has a unique solution given the initial capital. Hence, the capital stock at any point in time $s \geq 0$ is determined by the initial capital stock and the Brownian motion up to the point in time $s$, i.e.

$$K(s) = f(K(0),\{B(v)\}_{v=0}^{s}) = f(K(t),\{B(v)\}_{v=t}^{s})$$

where the second equation holds for any $0 \leq t \leq s$.

Then given the RAM in the economy, the consumption flow $\{C(s)\}_{s=t}^{\infty}$ is determined by the initial given capital stock $K_t$ and the Brownian motion. If further the utility is still assumed to be a function of consumption alone, i.e. $U(t) = u(C(t))$ for all time $t \geq 0$, then the dynamic welfare in the economy is uniquely determined by the initial condition of the capital stock since

$$W(t) = E_t \left[ \int_t^\infty e^{-\rho(s-t)} U(s) \, ds \right] = E_t \left[ \int_t^\infty e^{-\rho(s-t)} \tilde{u}(K(t),\{B(s)\}_{s=t}^{s}) \, ds \right] , \quad (22)$$

$^2$An intuitive interpretation for the stochastic capital is provided by Weitzman (2003, Chapter 7).
where

$$\tilde{u}(K(t), \{B(s)\}_{s=t}^u) = u(C(f(K(t), \{B(v)\}_{v=t}^u))).$$

Thus, the dynamic welfare expressed by (22) can be rewritten as a function of the current capital,

$$W(t) = \tilde{w}(K(t)),$$

which is assumed to be twice continuously differentiable w.r.t. capital $K$. In addition, the vector of prices of net investment at time $t$ is defined to equal the vector of marginal welfare w.r.t. current capital $K$, i.e. $p_k(t) = \nabla \tilde{w}(K(t))$. By the chain rule of Stratonovich integral (Theorem 6.4.a), the instantaneous change of welfare can be expressed by

$$dW(t) = \nabla \tilde{w}(K(t)) \circ dK(t) = p_k(t) \circ dK(t),$$

(23)

which shows that instantaneous change of welfare is represented by the present value of current capital change evaluated at the Stratonovich-like capital prices. Then national income can be shown to equal ex ante NNP, the sum of current consumption value plus the genuine savings.

**Proposition 2** By specifying a resource allocation mechanism (RAM), dynamic welfare is defined as a function with respect to the initial capital alone, which is assumed to be twice continuously differentiable. Then, national income coincides with the ex ante NNP\(^3\), i.e.

$$E^t \left[ \int_t^\infty C(s) \circ (-dp_c(s)) \right] = p_c(t)C(t) + E^t \left[ p_k(t) \circ dK(t) \right],$$

(24)

where the second term on the r.h.s. is the expected value of net investments ex ante, which is evaluated just before the uncertainty on current capital is revealed.

Proof. By (23), the expected instantaneous change of welfare

$$AW(t) = \lim_{h \to 0^+} \frac{E^t [W(t + h)] - W(t)}{h} = \frac{E^t [dW(t)]}{dt} = \frac{E^t [\nabla \tilde{w}(K(t)) \circ dK(t)]}{dt} = \frac{E^t [p_k(t) \circ dK(t)]}{dt}.$$\(^3\)

\(^3\)The formula in Weitzman (2003, Chapter 7.) can be taken as a special expression of ex ante NNP defined here.
Since the consumption $C$ and the utility $U$ are also Itô processes by assumptions, then Corollary 1 is valid. By (19) and (20), the proposition is proved. [Q.E.D.]

A one-dimensional example for ex ante NNP

Assume the capital is one-dimensional. By the relations between Itô and Stratonovich integrals (Theorem 6.5), national income can be expressed as ex ante NNP,

$$y(t) = p_c(t)C(t) + p_k(t)I(t) + \frac{1}{2} \frac{dp_k}{dk} (K(t)) \sigma^2(t). \quad (25)$$

If the capital and its price is deterministic, i.e. $\sigma(t) = 0$ for all $t \geq 0$, then the last term on the r.h.s. of (25) disappears. Hence, if the capital is a stochastic process, national income ex ante calculated should be adjusted to include the value of all types of capital changes, including the uncertain part. The adjustment term depends on the correlations between future capital and its prices. For instance, if for any given capital level, its price is given by a linear function, $p_k(s) = \bar{p}_k - b_kK(s)$, where $b_k > 0$, then the adjustment term equals

$$\frac{1}{2} \frac{dp_k}{dk} (K(t)) \sigma^2(t) = -\frac{1}{2} b_k \sigma^2(t) \leq 0,$$

which implies the national savings (and also the national income) should be adjusted downwards to precaution the uncertainty of the capital.

3.3 Real national income

In this subsection, after national income in real terms is defined by introducing the Divisia consumption price index and the redefined real interest rate, some related facts are presented. First the instantaneous change of real prices of consumption does not change the value of the corresponding consumption, which is the key property of the Divisia price index. Then the real national income defined here can not be interpreted as the expectation of weighted present value of future consumption. The last is that the real interest rate redefined here can be expressed by three elements: the expected instantaneous change rate of the Divisia consumption price index $\pi$, an adjustment due to the correlations between consumption and the Divisia price index, and an adjustment due to the correlations between consumption and their real prices.
Define a Divisia price index $\{\pi (s)\}_{s=0}^{\infty}$ satisfying initial value $\pi (0) = 1$ and

$$\frac{d\pi (s)}{\pi (s)} = \frac{C (s) d\mathbf{p}_c (s)}{C (s) \mathbf{P}_c (s)} \quad (26)$$

for all $s \geq 0$. Notice that the index is always a stochastic process since future consumption is stochastic and so their prices are uncertain. The real consumption price $\mathbf{P}_c$ is defined by

$$\mathbf{P}_c (s) = \frac{\mathbf{P}_c (s)}{\pi (s)} \quad (27)$$

for all $s \geq 0$.

To derive the national income in real terms, we further define real interest rate of consumption ($R$) as the expected instantaneous change rate of the present value of consumption due to price change alone, i.e.

$$R (s) = \frac{1}{C (s) \mathbf{P}_c (s)} \lim_{h \to 0^+} \frac{1}{h} E^t \left[ \int_s^{s+h} C (v) \circ (-d\mathbf{p}_c (v)) \right] \quad (28)$$

for all $s \geq 0$. By using (28) and (27), we have national income\(^4\)

$$E^t \left[ \int_t^{\infty} C (s) \circ (-d\mathbf{p}_c (s)) \right] = E^t \left[ \int_t^{\infty} \pi (s) R (s) \mathbf{P}_c (s) C (s) ds \right] \quad (29)$$

for all time $t \geq 0$. Thus, we have

**Definition 2** Real national income equals the expected present value of real interest on future national consumption,

$$Y (t) = E^t \left[ \int_t^{\infty} \frac{\pi (s)}{\pi (t)} R (s) \mathbf{P}_c (s) C (s) ds \right], \quad (30)$$

which can be split as

$$Y (t) = \mathbf{P}_c (t) C_t + E^t \left[ \int_t^{\infty} \frac{\pi (s)}{\pi (t)} \mathbf{P}_c (s) \circ dC (s) \right]. \quad (31)$$

\(^4\)Roughly speaking, the real interest rate of consumption can be thought of as $R (s) = E^t [C (s) \circ (-d\mathbf{p}_c (s))]/ds$. Then it can be rewritten as $E^t [C (s) \circ (-d\mathbf{p}_c (s))] = R (s) C (s) \mathbf{P}_c (s) ds = \pi (s) R (s) \mathbf{P}_c (s) C (s) ds$. Hence, $E^t [C (s) \circ (-d\mathbf{p}_c (s))] = E^t \{E^* [C (s) \circ (-d\mathbf{p}_c (s))]\} = E^t \{\pi (s) R (s) \mathbf{P}_c (s) C (s) ds\}$. 

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The second part of Definition 2 is obtained by substituting (27) to the r.h.s. of (20).

Below we show some important facts.

**The key property of Divisia consumption price index**

By using the definition of real consumption price (27) and Theorem 6.3 in the Appendix, we know

\[
C(s) \, dp_c(s) = C(s) \, d(\pi(s) \, p_c(s)) \\
= C(s) \, p_c(s) \, d\pi_s + C(s) \, \pi(s) \, dP_c(s) + C(s) \, dP_c(s) \, d\pi(s).
\]

(32)

Hence by substituting (27) and (32) into (26), we derive

\[
\frac{C(s) \, dP_c(s)}{C(s) \, p_c(s)} + \frac{d\pi(s)}{\pi(s)} \frac{C(s) \, dP_c(s)}{C(s) \, p_c(s)} \equiv 0
\]

(33)

for all \( s \geq 0 \), which is the property of Divisia consumption price index\(^5\) in the stochastic setting.

**Real national income: Expectation of weighted present value of future national consumption?**

By (26) and (27), we have

\[
C(s) \, dp_c(s) = C(s) \, p_c(s) \, d\pi(s).
\]

(34)

If another real interest rate is defined as the expected instantaneous change rate of the Divisia consumption price index \( \pi \),

\[
\hat{R}(s) = -\frac{1}{\pi(s)} \lim_{h \rightarrow 0^+} \frac{E^s[\pi(s + h) - \pi(s)]}{h}
\]

(35)

for all \( s \geq 0 \), then by assuming \( \pi^t[\pi(s)] \rightarrow 0 \) as \( s \rightarrow \infty \) for any given time \( t \geq 0 \), we have the fact

\[
E^t \left[ \int_t^\infty \frac{\pi(s)}{\pi(t)} \hat{R}(s) \, ds \right] = \frac{1}{\pi(t)} E^t \left[ \int_t^\infty -d\pi(s) \right] = 1
\]

(36)

\(^5\)If there is no uncertainty, then the second term on the l.h.s. disappears and equation (33) is collapsed to be \( C(s) \, dP_c(s) = 0 \), which is the desired property of Divisia price index under certainty.
for all $t \geq 0$. By (34) and (35), we know the equation

$$E^t \left[ \int_t^\infty C(s) (-dp_c(s)) \right] = E^t \left[ \int_t^\infty \pi(s) \tilde{R}(s) P_c(s) C(s) \, ds \right], \tag{37}$$

for all $t \geq 0$. As long as there are correlations between consumption and their prices, we have

$$E^t \left[ \int_t^\infty C(s) \circ (-dp_c(s)) \right] \neq E^t \left[ \int_t^\infty C(s) (-dp_c(s)) \right],$$

which implies the r.h.s. of (29) differs from the r.h.s. of (37) and then real national income

$$Y(t) = E^t \left[ \int_t^\infty \frac{\pi(s)}{\pi(t)} R(s) P_c(s) C(s) \, ds \right] \neq E^t \left[ \int_t^\infty \frac{\pi(s)}{\pi(t)} \tilde{R}(s) P_c(s) C(s) \, ds \right]. \tag{38}$$

Since the last term of (38) can be interpreted as the expectation of weighted present value of future real consumption by noticing (36), then the inequality of (38) implies real national income can not be interpreted in the same way.

**Decomposition of the redefined real interest rate**

To express the real interest rate in real terms, by the relations between Itô and Stratonovich integrals (Theorem 6.5), we know for any time $v$,

$$C(v) \circ dp_c(v) = C(v) dp_c(v) + \frac{1}{2} dC(v) dp_c(v)$$

$$= C(v) P_c(v) d\pi(v) + \frac{1}{2} dC(v) dp_c(v) \quad \text{by (34)}$$

$$= C(v) P_c(v) d\pi(v) + \frac{1}{2} P_c(v) dC(v) d\pi(v)$$

$$+ \frac{1}{2} \pi(v) dC(v) dP_c(v) \quad \text{by (27),} \tag{39}$$

By substituting (39) in (28), the real interest rate of consumption can be expressed by

$$R(s) = -\frac{1}{\pi(s)} \lim_{h \to 0^+} \frac{E^s s+h \pi(s+h)}{h} - \pi(s) + \frac{1}{2} \lim_{h \to 0^+} \frac{1}{h} E^s \left[ \int_s^{s+h} \left( \frac{P_c(v)}{P_c(s) C(s)} - \pi(v) \right) \right]$$

$$- \frac{1}{2} \lim_{h \to 0^+} \frac{1}{h} E^s \left[ \int_s^{s+h} \frac{dC(v) dP_c(v)}{C(s) P_c(s)} \right], \tag{40}$$

where the first term on the r.h.s. is interpreted as the expected instantaneous change
rate of the Divisia consumption price index \( \pi \), i.e. \( \tilde{R} \) defined by (35). The second term is an adjustment due to the correlations between consumption and the Divisia price index. The last term is another adjustment due to the correlations between consumption and their real prices. Since the two adjustment terms can not be ignored, it is not plausible to estimate the real interest rate \( R \) only by the first term, i.e. the Divisia consumption price index \( \pi \) alone.

If the Divisia price index is assumed to be deterministic, which is common in practice, then the second term on the r.h.s. of (40) disappears. However, the last adjustment term is still there as long as there are correlations between consumption and their real prices.

If the vector of consumption follows the form,

\[
d\text{C}(t) = \alpha(t) \, dt + \beta(t) \, dB_t,
\]

where \( \alpha \) and \( \beta \) are given paths of constants over time. Further, if the demand curve is given by a downward-sloping linear function, \( P_c(s) = \tilde{P}_c - b_cC(s) \), where \( b_c > 0 \), then the last term in (40) equals

\[
- \frac{1}{2} \lim_{h \to 0^+} \frac{1}{h} E_s \left[ \int_s^{s+h} \frac{d\text{C}(v) \, dP_c(v)}{\text{C}(s) \, P_c(s)} \right] = \frac{1}{2} \frac{b_c(\beta(s))^2}{\text{C}(s) \, P_c(s)} \geq 0
\]

and real interest rate becomes

\[
R(s) = \tilde{R}(s) + \frac{1}{2} \frac{b_c(\beta(s))^2}{\text{C}(s) \, P_c(s)} \geq \tilde{R}(s)
\]

(41)

for all \( s \geq 0 \). In the sense of (41), if \( \tilde{R}(s) \) is wrongly used to estimate real national income by (30), then real national income is underestimated. On the other hand, if some bads like pollutants are consumed, then the prices may go in the same direction as the consumption change, which then implies some elements of \( b_c \) might be negative such that the difference between \( R(s) \) and \( \tilde{R}(s) \) is alleviated.

However, the assumption of the deterministic price index may be problematic. The Divisia price index might be correlated to future consumption. e.g. if consumption in a future point in time is estimated to be enormous, the price index should be expected to differ from the case of very little consumption at the same point.
4 A stochastic one-good model

This section illustrates the findings of the paper by a stochastic one-good model. A Cobb-Douglas production function with two inputs, labor and capital, is adopted. The labor is normalized to be unity. At the beginning the capital is supposed to be constant all the time and the uncertainty comes from the technological change. At each point in time, the value of the technological change follows a normal distribution. In this case, an approximation is calculated for the real national income. Later the technological change is assumed to be constant and the capital becomes a stochastic variable, which is used to illustrate the relations between real national income and ex ante NNP. Under both cases, the real national income is estimated to be the same *ceteris paribus*. Then the redefined real interested rate is compared with the expected capital price to show their difference.

4.1 Stochastic technological change

The dynamic welfare is expected discount utilitarian and utility function exhibits the property of constant relative risk aversion (CRRA). The RAM is determined such that all the current production is used for consumption and not for investment at all. Under these settings, the real interest rate used for income calculation is a constant all the time. On the contrary, the capital prices are stochastic and its expectation differs from the constant real interest rate. It shows that the curvature of utility function may haves dramatic effects on the estimation of real income under uncertainty (ref. Figure 1).

Assume the output from the production depends also on a normal-distributed technological variable $z_t$ besides the constant capital $k$ at each time $t$ (no depreciation for the capital). Hence, production, $q(t)$ at time $t$ is given by

$$q(t) = e^{z_t}k^\alpha$$

where the available labor $\ell$ is constant and normalized to one (i.e., $\ell(t) = 1$ for all $t$), which is ignored in the function. We can let $z_t = \sigma B_t$, a stochastic process with initial value $z_0 = 0$, where the constant $\sigma > 0$ and $B_t$ is a Brownian motion. We also assume that $0 < \alpha < 1$. 

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Suppose the resource allocation mechanism (RAM) is determined as follows. Given any time $t \geq 0$, all the production is consumed and nothing is invested, $dk(t) = 0$, which is consistent with the constant capital assumption. Then consumption

$$C(s) = q(s) = e^{\sigma B_s K^a} \tag{42}$$

for all $s \geq 0$.

Next assume the utility function is given by

$$U(s) = u(C(s)) = \frac{C(s)^{1-\eta}}{1-\eta} \tag{43}$$

where the curvature of the function $\eta \geq 0$ and $\eta \neq 1$\textsuperscript{6}. Given the discounted utilitarian welfare function (10), by letting the current consumption price $p_c(0) = 1$, we have the present value prices of consumption

$$p_c(s) = e^{-\rho s} \frac{\partial u}{\partial c}(C(s)) = C_0^\eta e^{-\rho s} C(s)^{-\eta} \tag{44}$$

for all $s \geq 0$. Then on the basis of (42) and (44), by Itô integral (Theorem 6.2), we know

$$dC(s) = C(s) \left[ \frac{1}{2} \sigma^2 ds + \sigma dB(s) \right]$$

$$dp_c(s) = -\rho p_c(s) ds + C_0^\eta e^{-\rho s} d(C(s)^{-\eta})$$

$$= -\rho p_c(s) ds + C_0^\eta e^{-\rho s} (-\eta) C(s)^{-\eta - 1} dC(s) + \frac{1}{2} C_0^\eta e^{-\rho s} (-\eta) (-\eta - 1) C(s)^{-\eta - 2} (dC(s))^2$$

$$= p_c(s) \left[ \left( \frac{1}{2} \eta^2 \sigma^2 - \rho \right) ds - \eta \sigma dB(s) \right]$$

and

$$C(s) \circ (-dp_c(s)) = C(s) p_c(s) \left( \rho - \frac{1}{2} \left( \eta^2 - \eta \right) \sigma^2 \right) ds + C(s) p_c(s) \eta \sigma dB_s \tag{45}$$

$$p_c(s) \circ dC(s) = C(s) p_c(s) \frac{1}{2} (1-\eta) \sigma^2 ds + C(s) p_c(s) \sigma dB_s. \tag{46}$$

\textsuperscript{6}If $\eta = 1$, then the utility function is elapsed to be the logarithmic form, $u(C(s)) = \log C(s)$.
Hence, by definition (28), the real interest rate over time,

\[ R(s) = \rho - \frac{1}{2} \left( \eta^2 - \eta \right) \sigma^2 = R, \] (47)

which is a constant and we suppose it is positive. Then if \( \eta > 1 \), the uncertainty implies smaller expectation of future real interest rate. Since this is a one-good economy, the present value prices of consumption can serve as the Divisia consumption price index, i.e. \( \pi(s) = p_c(s) \) for all \( s \geq 0 \). Then the real prices of consumption is a constant, \( P_c(s) = 1 \) for all \( s \geq 0 \). Hence, by (40), the real interest rate includes two terms: the expected instantaneous change rate of the Divisia consumption price index \( \pi \) and the adjustment due to the correlations between consumption and the Divisia price index.

\[ R(s) = \underbrace{\rho - \frac{1}{2} \eta^2 \sigma^2}_{\hat{R}, \text{expected change rate of } \pi} + \underbrace{\frac{1}{2} \eta \sigma^2}_{\text{adjustment due to correlation between } C \text{ and } \pi}. \]

The adjustment term could be considerable. For example, if \( \eta = 1 \) and \( \sigma = 0.1 \), then the adjustment term \( \frac{1}{2} \eta \sigma^2 \) is half percent, which should not be ignored if real interest rate \( R \) is small, eg. 4 percent.

Before the calculation of real national income, we notice that

\[
E^0 \left[ \int_0^\infty C(s) p_c(s) \, ds \right] = C(0) \eta E^0 \left[ \int_0^\infty e^{-\rho s} C(s)^{1-\eta} \, ds \right] \quad \text{by (44)}
\]

\[
= \kappa \eta E^0 \left[ \int_0^\infty e^{-\rho s} (e^{\sigma B_1} k^\alpha)^{1-\eta} \, ds \right] \quad \text{by (42)}
\]

\[
= \kappa \eta k^\alpha (1-\eta) E^0 \left[ \int_0^\infty e^{-\rho s} e^{(1-\eta)\sigma B_1} \, ds \right]
\]

\[
= \kappa^\alpha \int_0^\infty e^{-\rho s} E^0 \left[ e^{(1-\eta)\sigma B_1} \right] \, ds
\]

\[
= \kappa^\alpha \int_0^\infty e^{-\rho s} e^{\frac{1}{2} (1-\eta)^2 \sigma^2} \, ds
\]

\[
= \frac{\kappa^\alpha}{\rho - \frac{1}{2} (1-\eta)^2 \sigma^2} \quad \text{if } \rho > \frac{1}{2} (1-\eta)^2 \sigma^2.
\]
Hence by (31) and (46), real national income is

\[
Y (0) = P_c (0) C (0) + E^0 \left[ \int_0^\infty \frac{p_c (s)}{P_c (0)} P_c (s) \circ dC (s) \right] \\
= k^\alpha + \frac{1}{2} (1 - \eta) \sigma^2 E^0 \left[ \int_0^\infty C (s) p (s) \, ds \right] + E^0 \left[ \int_0^\infty C (s) p (s) \sigma d B (s) \right] \\
= \frac{k^\alpha}{\rho} + \frac{1}{2} (1 - \eta) \sigma^2 k^\alpha \frac{1}{\rho - \frac{1}{2} (1 - \eta)^2 \sigma^2} \quad \text{if } \rho > \frac{1}{2} (1 - \eta)^2 \sigma^2 \\
= \frac{\rho - \frac{1}{2} (\eta^2 - \eta) \sigma^2}{\rho - \frac{1}{2} (1 - \eta)^2 \sigma^2} k^\alpha, \quad (48)
\]

where national savings is negative if \( \eta > 1 \). Otherwise, if \( \eta < 1 \), national savings is positive.

We can obtain the same result by (30),

\[
Y (0) = E^0 \left[ \int_0^\infty \frac{p_c (s)}{P_c (0)} R (s) P_c (s) C (s) \, ds \right] \\
= \left( \rho - \frac{1}{2} (\eta^2 - \eta) \sigma^2 \right) E^0 \left[ \int_0^\infty C (s) p (s) \, ds \right] \quad \text{by (47)} \\
= \frac{\rho - \frac{1}{2} (\eta^2 - \eta) \sigma^2}{\rho - \frac{1}{2} (1 - \eta)^2 \sigma^2} k^\alpha \quad \text{if } \rho > \frac{1}{2} (1 - \eta)^2 \sigma^2 .
\]

Notice that if \( \rho \leq \frac{1}{2} (1 - \eta)^2 \sigma^2 \), then the income goes to infinity over time in this case. In the following we always assume that \( \rho > \frac{1}{2} (1 - \eta)^2 \sigma^2 \).

By setting \( \rho = 0.04 \), Figure 1 shows the rate of real national income to the current production corresponding to various values of the curvature of utility function \( \eta \) and the uncertainty level of production \( \sigma \). Under certainty, i.e. \( \sigma = 0 \), the real national income coincides with the current production irrespective of the curvature of utility function. Under uncertainty, i.e. given certain level of \( \sigma \neq 0 \), higher curvature of utility implies lower real national income. Notice that the real national income is just half of the current production if \( \eta = 3 \) and \( \sigma = 0.1 \). However, given the curvature of utility \( \eta \), the real national income may become higher or lower along with higher level of uncertainty \( \sigma \).

It turns out national income is estimated to differ from current production because of the uncertain future production and the curvature of utility function. For example, if we
Figure 1. The change rates of real national income w.r.t. the curvature of utility function and the uncertainty level of production

apply the linear utility function $U(s) = C(s)$, i.e. $\eta = 0$, then national income becomes

$$Y(0) = \frac{\rho}{\rho - \frac{1}{2}\sigma^2} k^\alpha,$$

which is higher than the current production. Since utility is exactly equal to consumption all the time, this higher estimation shows the future output (and so consumption) is expected to increase, i.e. $E^0[q(t)] = E^0[e^{\gamma t} k^\alpha] = e^{\frac{1}{2}\sigma^2 t} k^\alpha \geq k^\alpha$. It can be shown that if future consumption is expected to be the same as current consumption, the linear utility function implies that national income equals current production (this can be got by assuming $dz_t = -\frac{1}{2}\sigma^2 dt + \sigma dB_t$).

If instead, we assume the utility function is given by the logarithmic form,

$$U(s) = u(C(s)) = \log C(s),$$

Then the national income becomes

$$Y(0) = k^\alpha,$$

which is exactly the current production since the effect of curvature of utility function
cancels out the expectation of higher future production.

Since the real interest rate is constant, I would like to approximate the real national income by constructing a discount factor satisfying $\tilde{\pi} (0) = 1$ and

$$\tilde{\pi} (s) = \exp \left( -\int_0^s Rdv \right) = e^{-Rs}. $$

Still assume the real prices of consumption is constant as unity all the time. By replacing $\pi$ with $\tilde{\pi}$ in (30), the real national income can be approximated by

$$\tilde{Y} (0) = E^0 \left[ \int_0^\infty \frac{\tilde{\pi} (s)}{\tilde{\pi} (0)} R (s) \ P_c (s) \ C (s) \ ds \right] $$

$$= E^0 \left[ \int_0^\infty e^{-Rs} R e^{\sigma Bs} k^\alpha ds \right] $$

$$= \frac{R}{R - \frac{1}{2}\sigma^2} k^\alpha. \quad (49)$$

Notice that by (47), the exact real national income in (48) can be rewritten as

$$Y (0) = \frac{R}{R - \frac{1}{2}\sigma^2} k^\alpha. \quad (50)$$

Then by (49) and (50), the approximation always overestimates the real national income. The rate of the error is given by

$$\frac{\tilde{Y} (0) - Y (0)}{Y (0)} = \frac{\frac{1}{2}\eta\sigma^2}{R - \frac{1}{2}\sigma^2},$$

which shows the approximation exactly coincides the real national income if there is no uncertainty, i.e. $\sigma = 0$, or the utility function is linear, i.e. $\eta = 0$. Otherwise, the approximation differs from the exact real national income.

By using the same values of parameters as in Figure 1, we calculate the rates in Figure 2\textsuperscript{7}. Under certainty, i.e. $\sigma = 0$, the approximation coincides with the exact real national income. Under uncertainty, i.e. given certain level of $\sigma \neq 0$, higher curvature of utility function implies higher rates of the approximation error. On the other hand, given the curvature of utility function $\eta$, the rates of the error is also increasing with higher level

\textsuperscript{7}In Figure 2, the value of $\eta$ is within the interval $[0, 2]$ to highlight the tendency.
of uncertainty $\sigma$. Notice that the rate of the error is 40 percent of the exact real national income if $\eta = 2$ and $\sigma = 0.1$. It shows that the approximation is unacceptable under rather high curvature of utility function and uncertainty level. However, if the uncertainty level is small, then the approximation is quite good even though the curvature of utility function is rather high. For example, the rate of error is less than two percent of real national income if $\sigma = 0.01$ and $\eta = 10$.

After introducing a simple stochastic process into the production function, we find out that the theory under certainty is correct only under some special cases, e.g. the utility function takes some special forms. The linear utility function can not justify the extension of the theory from certainty to uncertainty. In our specific model, due to the expectation of higher future consumption, the real national income under uncertainty is higher than current production even though they are the same under certainty.

### 4.2 Stochastic capital stock

In the above model, the capital keeps constant over time even though the national savings may differ from zero. It seems contradictory the fact that the net investments is zero, but in fact it is correct. In the model, the dynamic welfare is a function not only of capital, but also of the random variable of technological process. Thus, the national savings comes
from the value of technological change over time. If the technological process is thought of as one kind of capital, we would find out that the net investment is produced due to the correlation between technological "capital" and its marginal welfare. In this sense, the scope of "capital" has to be extended to include any time-dependent factors that have effect on dynamic welfare in order to make national income coincide with ex ante NNP.

We can slightly modify the above model to check if the national income equals ex ante NNP. Assume the technical variable is constant, e.g. \( z(t) = 0 \) all the time. On the other hand, let the capital becomes a stochastic process due to unanticipated events like earth quake or new discovery of exhaustible resources. Let capital evolves according to the path given by

\[
k(t) = ke^{\sigma_B t / \alpha}
\]

or in the differential form

\[
dk(t) = k(t) \left[ \frac{\sigma^2}{2\alpha^2} dt + \frac{\sigma}{\alpha} dB_t \right]
\]

(51)

for all \( t \geq 0 \). Then the above analysis on national income is the same. However, the modification of the model allows us to calculate the ex ante NNP. It can be shown this is a special case of Proposition (2):

**Corollary 2** If all uncertainty is caused by the capital in a one-good economy, i.e. (51) holds, then real national income (and national savings) coincides with ex ante NNP (and national net investments):

\[
Y(0) = \alpha + \left. \frac{E_t \left[ p_k(t) \circ dK(t) \right]}{dt} \right|_{t=0}
\]

Proof. The discounted utilitarian welfare function (10) can be rewritten as

\[
W(t) = \bar{w}(k(t)) = E_t \left[ \int_0^\infty e^{-\rho(s-t)} \left( e^{\sigma_B s} k(t)^\alpha \right)^{(1-\eta)} ds \frac{1}{1-\eta} \right]
\]

\[
= \frac{1}{1-\eta} \left( \frac{1}{\rho} - \frac{1}{2} \frac{1}{(1-\eta)^2} \frac{1}{\sigma^2} \right)
\]

Then the present value price of capital at any time \( t \geq 0 \),
\[ p_k(t) = \frac{d\hat{w}}{dk}(k(t)) \left/ \frac{\partial u}{\partial \varepsilon}(C_0) \right. = \frac{k(0)^\alpha \eta \alpha k(t)^{\alpha(1-\eta)-1}}{\rho - \frac{1}{2} (1 - \eta)^2 \sigma^2} \]

and

\[ dp_k(t) = \frac{(\alpha (1 - \eta) - 1) p_k(t) dK(t)}{k(t)} + \frac{1}{2} \frac{(\alpha (1 - \eta) - 1) (\alpha (1 - \eta) - 2) p_k(t)}{k(t)^2} [dK(t)]^2 \]

\[ = p_k(t) \left[ \frac{\sigma^2}{2\alpha^2} (1 - \eta)^{2} dt + \frac{\sigma}{\alpha} dB_t \right] \]

We can calculate

\[ p_k(t) \circ dK(t) = p_k(t) dK(t) + \frac{1}{2} dp_k(t) dK(t) \]

\[ = p_k(t) k(t) \left[ \frac{\sigma^2}{2\alpha} (1 - \eta) dt + \frac{\sigma}{\alpha} dB_t \right]. \]

Then

\[ E' \left[ p_k(t) \circ dK(t) \right] \left|_{t=0} \right. = p_k(0) k(0) \frac{\sigma^2}{2\alpha} (1 - \eta) \]

\[ = \frac{1}{2} (1 - \eta) \sigma^2 k^\alpha \]

\[ = \frac{1}{2} (1 - \eta)^2 \sigma^2 / \rho \]

which is the same as national savings on the r.h.s. of (48). [Q.E.D.]

### 4.3 Real interest rate and capital price

It is interesting to notice the difference between real interest rate used for income calculation and the capital prices. In the one-good model, \( R \) is constant while \( P_k \) is a stochastic process and its expectation is not constant since

\[ P_k(t) = \alpha k^{\alpha-1} e^{\sigma B_t} \]

and

\[ E^0 [P_k(t)] = \alpha k^{\alpha-1} e^{\frac{1}{2} \sigma^2 t} = P_k(0) e^{\frac{1}{2} \sigma^2 t} \]

for all \( t \geq 0 \) even though the capital is assumed to be constant all the time.

If there is no uncertainty, i.e. \( \sigma = 0 \), then both \( R \) and \( P_k \) are constant, \( R = \rho \) and
\( P_k = P_k(0) \). Furthermore, if the initial capital stays at the steady state, then \( R \) and \( P_k \) coincide with each other, i.e. \( R = P_k \). Hence, to consume all the current production is the optimal RAM. However, under uncertainty, the strategy is just suboptimal since there is difference between \( R \) and expected \( P_k \). It would be better to invest/disinvest certain amount at some points in time. For example, if at current time 0, \( E^0 [P_k(t)] > R \), then it would be better to make positive investment at time \( t \). However, if the uncertainty comes from the technology, then positive investment means more capital and lower capital price in the instantaneous future, which may induce disinvestment activity to drive the capital price going up. Hence, the optimal capital level under uncertainty fluctuates around a capital level satisfying \( E [P_k] \to R \).

5 Concluding remark

The paper studied the concept of national income (Sefton and Weale, 2006; Asheim and Wei, 2009) in the stochastic case. If the prices of consumption and net investment are properly defined, then it turns out that stochastic national income defined in the paper has plausible interpretation from both a welfare and productive respective. The concept of national income does not assume autonomous Itô diffusion for consumption even though it is required to show that real national income coincides with ex ante NNP.

Furthermore, stochastic real national income can be explained as the expectation of present value of real interests on future national consumption if a real consumption interest rate is properly defined, which now involves two adjustment terms related to the correlations of consumption with other two variables: real prices of consumption and the Divisia consumption price index.

A simple one-good model is used to illustrate the theory. The model shows that whether real national income equals current production or not depends on the future uncertainty and the curvature of utility function. The curvature of utility function has dramatic effects on real national income under uncertainty. Higher production does not necessarily imply higher real income if it is associated with more uncertainty.

In the real world, there are great uncertainty that is not directly led by the capital stock. Then the deviation between real national income and ex ante NNP commonly
exists.

By the one-good model, it becomes clear that real interest rate relates to marginal welfare and has no direct relations to the production or capital prices. It means the income of an individual would be positive if she were able to borrow to live without repayment since her cash flow used for consumption is always non-negative.

Notice that the paper does not tell anything on how to find the proper prices and real consumption interest rate in the real world. Arrow, et al (2003) has given some useful guidelines for practical calculation of these prices in the deterministic case. In addition, national income defined in the paper does not assume the optimal economic path.
6 Appendix

Definition 3 (Itô integral). The Itô integral of a function $f$ is defined by

$$
\int_0^T f(t)dB_t = \lim_{\Delta t_j \to 0^+} \sum_j \left[ B_{t_{j+1}} - B_{t_j} \right] f \left( t_j \right),
$$

where $\Delta t_j = t_{j+1} - t_j$ and $B$ is a Brownian motion.

Definition 4 (Stratonovich integral). The Stratonovich integral of a function $f$ is defined by

$$
\int_0^T f(t) \circ dB_t = \lim_{\Delta t_j \to 0^+} \sum_j \left[ B_{t_{j+1}} - B_{t_j} \right] f \left( t_j^* \right),
$$

where $\Delta t_j = t_{j+1} - t_j$, $t^* = (t_j - t_{j+1})/2$ and $B$ is a Brownian motion. In the paper, the sign "\( \circ \)" represents the Stratonovich integral.

Definition 5 (Itô process). Let $X$ a random variable over time with given initial value $X_0$ and assume that

$$
\frac{dX_t}{dt} = a(t, X_t) + b(t, X_t)W_t,
$$

where the "white noise" $W_t$ satisfies properties (i) for all $t_1 \neq t_2$, $W_{t_1}$ and $W_{t_2}$ are independent; (ii) $\{W_t\}$ is stationary, i.e. the joint distribution of $\{W_t\}$ does not depend on $t$; (iii) for all time $t$, $E[W_t] = 0$. Then the stochastic process $X$ is a Itô process and can be represented by

$$
dX_t = a(t, X_t)dt + b(t, X_t)dB_t,
$$

which is the short form of the integral

$$
X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dB_s,
$$

where $B_t = W_t dt$ is a Brownian motion and the second term on the r.h.s is Itô integral of function $b(t, X_t)$. Then $a(t, X_t) = \frac{E[dX_t]}{dt}$ represents the expected instantaneous change of the random variable. $b(t, X_t)$ is the variation of the "white noise" (ref. Øksendal 2005).

Theorem 6.1 (Leibniz’s Formula). Suppose that $f(x, t)$ and $f'_x(x, t)$ are continuous over the rectangle determined by $a \leq x \leq b, c \leq t \leq d$, that $u(x)$ and $v(x)$ are $C^1$
functions over \([a, b]\), and that the ranges of \(u\) and \(v\) are contained in \([c, d]\). Then

\[
F(x) = \int_{u(x)}^{v(x)} f(x, t) \, dt \implies \\
F'(x) = f(x, v(x)) v'(x) - f(x, u(x)) u'(x) + \int_{u(x)}^{v(x)} \frac{\partial f(x, t)}{\partial x} \, dt
\]

**Theorem 6.2 (A version of Itô formula).** Let \(X_t\) be an \(n\)-dimensional Itô process given by

\[
dX_t = a(t, X_t)dt + b(t, X_t)dB_t.
\]

Let \(g(x) : \mathbb{R}^n \to \mathbb{R}\) be twice continuously differentiable. Then the process

\[
Y_t = g(X_t)
\]

is again an Itô process, which is given by

\[
dY_t = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(X_t) \, dX_i + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 g}{\partial x_i \partial x_j}(X_t) \, dX_idX_j,
\]

where \(X_i\) is the element \(i\) of the vector \(X_t\) and computed according to the rules

\[
dtdt = dtdB_i = dB_idt = 0, \quad (dB_i)^2 = dt.
\]

(ref. Øksendal 2005, page 49, Theorem 4.2.1).

**Theorem 6.3** Let \(X_t, Y_t\) be Itô processes in \(\mathbb{R}^n\). Then

\[
d(X_tY_t) = X_t dY_t + Y_t dX_t + dX_t \cdot dY_t,
\]

which is the short form of Itô integral

\[
X_tY_t = X_0Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \int_0^t dX_s \cdot dY_s.
\]

Furthermore, if \(X_t\) is a deterministic process, i.e. \(X_t\) has the form of \(dX_t = a(t, X_t) dt\)
Then
\[ d(X_t Y_t) = X_t dY_t + Y_t dX_t, \]
which is the short form of
\[ X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s. \]

**Theorem 6.4 (the Chain rule of Stratonovich integral).**

(a) If \( K_t \in \mathbb{R}^n \) is an \( n \)-dimensional autonomous Itô diffusion given by
\[ dK_t = a(K_t) dt + b(K_t) dB_t, \]
and let \( w(K_t) : \mathbb{R}^n \to \mathbb{R} \) be twice continuously differentiable, then the process

\[ W_t = w(K_t) \]

is again an Itô process with the form,

\[ dW_t = \nabla w(K_t) \circ dK_t, \]

which is the short form of the Stratonovich integral,

\[ W_t = W_0 + \int_0^t \nabla w(K_s) \circ dK_s, \]

where \( \circ dK_s \) indicates the Stratonovich integral and \( \nabla W(K_t) \) is the vector of the first order derivatives w.r.t. \( K_t \), i.e.

\[ \nabla w(K_t) = \left( \frac{\partial w}{\partial k_1}(K_t), \ldots, \frac{\partial w}{\partial k_n}(K_t) \right). \]

(ref. Rogers and Williams 1987).

(b) Let \( X_t, Y_t \) be Itô processes in \( \mathbb{R}^n \). Then
\[ d(X_t Y_t) = X_t \circ dY_t + Y_t \circ dX_t, \]
which is the short form of the Stratonovich integral

\[ X_t Y_t = X_0 Y_0 + \int_0^t X_s \circ dY_s + \int_0^t Y_s \circ dX_s. \]

**Theorem 6.5 (The relations between Itô and Stratonovich integrals).** Let \( X_t \) be an \( n \)-dimensional Itô process given by

\[ dX_t = a(X_t)dt + b(X_t)dB_t . \]

Let \( g(x) : \mathbb{R}^n \to \mathbb{R} \) be twice continuously differentiable. Let \( Z(x) = \nabla g(x) \) be the vector of derivative w.r.t. \( x \). Then the process

\[ Y_t = g(X_t) \]

is again an Itô process, which can be expressed by both Itô and Stratonovich integrals,

\[ dY_t = Z(X_t) \circ dX_t = Z(X_t) dX_t + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 g}{\partial x_i \partial x_j} (X_t) dX_i dX_j. \]

**References**


