Long-run behavior in games

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Preface

This paper is the final work in my 5-year masters degree in Economics at the University of Oslo.
I would like to thank my supervisor Atle Seierstad for helping me pick the subject and for always making time when I had questions. In addition his superior knowledge and the ability to explain complex things in a simple manner have made this paper possible. For this, I sincerely thank you!
All possible mistakes and inconsistencies are totally my responsability.

Summary

This paper has two purposes. The first is to describe the existing theory of long-run behavior of populations playing a normal-form game. In this paper the emphasis is on symmetric 2×2 games which is the most analytical convienent. The methods here relies on that agents are not fully rational, they can make mistakes when playing. The reason for this possible mistakes or perturbations are not specified but it can be random experementation or just ignorance. When studying these processes the theory of Markov chains becomes useful. When assuming that there is one population playing, the dynamics of the distribution of strategies is a one-dimensional Markov chain. By using standard theory or the *Markov chain tree theorem* we can deduce a limiting distribution for the process(we let time go towards infinity). This limiting distribution will be a function of the perturbations mentioned earlier. Then, if we let the pertubations tend to zero, it will often be the case that the probability of the process being in a specific state is much higher than for all the other states. This idea leads to the concept of *stochastic stability* which was introduced be Foster and Young,1993. This concept gives a prediction of how the behavior of the population will be in the long-run i.e which strategies of the game they are most likely to play. In 2×2 games there is a link between the *risk dominant equlibrium* and the *stochastically stable state* and this is used to verify the results when examples of the use of the theory is presented.

The second purpose of this paper is to extend the existing theory. The extension here is that we let two populations play against each other in the game. We assume that one population operates the rows and the other the columns. This calls for a different theoretical approach but still the theory of Markov chains is important. The best-response dynamics is affected by the distribution of strategies in the other population. When we let the time horizon tend to infinity we can again compare the probabilities of being in the different states as the perturbations go to zero. The *stochastically stable state*, which will be pairs of distributions in the two populations, shows which strategy both populations will play in the long-run. This state corresponds to the predicton we got in the one population case.

The approach here is totally theoretical. The methods used is game theory, mathematics and statistics. When new concepts or theorems needed to find the *stochastically stable states* are presented there are examples to show how these easily can be used. The reason for this is that it should be simple to use the theory in this paper in other economic applications.

The paper is written in LaTeX.

Contents

1 Introduction

In economics the strategic behavior of individuals has since the 1950's been of upmost interrest among the students within the field. The work of John Nash is in this respect one of the most important contributions, and gave matemathicians and economists the analytical tool to matemathically explain different strategic settings within an economy. Nash's theory can also be used to explain previous works on especially duopoly, like the Bertrand- and Cournot solutions from the 17th century. This paper will present a theory of stochastic adaptive dynamics, which relies on game theory but also elements from stochastic modelling. This theory stems from the pioneering work of Kandori, Mailath, Rob(1993 [11]) and Young(1993 [3]). The core of this theory is that one investigate the strategic behavior of large number of agents playing a normal-form game. An important feature is that the agents are assumed to not be fully rational, as in much of the other existing game theory used in economics. They can make mistakes when playing the game. This possibility of a "trembling hand" causes the long-run behavior of the system to be substantially different from a deterministic process.

The stochastic shocks,or perturbations, may be of different types, first the way that agents interact may be stochastic when agents are randomly paired up to play the game, second the agents can be using mixed strategies when playing the game and third the agent may observe the distribution of the population and change his type since this is a best response. All these things can be reasons for the stochastic element in the decitionmaking, but there can also be other sources.

This theory differs from the theory of evolutionary stable equilibria in games, where a small mutant group can invade the population and may cause the behavior of the agents to change. But in the evolutionary framework the disturbance to the process is an isolated event, which clearly can not be a realistic assumption when studying economic systems where shocks constantly hits the system. When working with this kind of processes one have to combine game theory and the theory of Markov chains, that is, when time is assumed to be discrete. In continuous time the stochastics are often assumed to follow a *Browninan motion*. In this paper we will limit ourself to discussing dicrete time. When modelling the behavior i.e the evolution of the strategies or types in the population, as a Markov chain one can predict what happens to the system in the long-run. This leads to the idea of *stochastic stability*(Foster and Young,1990 [1]).

The rest of the paper is compiled in the following way, chapter 2 presents the standard game theory needed, chapter 3 presents the theory of Markov chains, chapter 4 will introduce the concept stochastic adaptive dynamics, chapter 5 gives a possible extension to the theory, chapter 6 is a discussion of the assumptions underlying the theory and chapter 7 concludes.

2 Game Theory

2.1 Introduction

Game thory provides a systematic way to study strategic interaction between agents. This is useful when trying to model how economic agents behave in different situations. This chapter will briefly go through the needed concepts for the later discussion of stochastic adaptive dynamics. The chapter starts with some useful definitions, and then discusses 2×2 games that will be widely used later because of its analytic convinience. Lastly the standard theory of evolutionary stable strategies(ESS) are presented so the differences between these and stochastically stable states(SSS) can be pointed out later.

2.2 Definitions

Definition 1 *A strategy is a complete contigent plan for a player in a game. Given a game* $\boldsymbol{G}$*, let* S_i *denote the strategy space of player i. Let* s_i *denote a single strategy. A strategy profile* $s = (s_1, s_2, .., s_n)$ *shows all individual strategies.* S *is the set of all strategy profiles, where* $S = S_1 \times S_2 \times \cdots \times S_n$[1]*.*

Definition 2 *Let* $u_i : S \rightarrow \mathbb{R}$ *be defined as a payoff function.* $u_i(s_1, s_2, ...s_n)$ *is the payoff to player i when the strategy profile is* $s = (s_1, s_2, .., s_n)$

Definition 3 *A belief of player i is a probability distribution, denoted* $\mu_{-i} \in \Delta S_{-i}$ *over the strategies of the other players.* ΔS_{-i} *is the set of probability distributions over the strategies of all players except player i. The belief of player i about the behavior of player j is* $\mu_j \in \Delta S_j$*,where* ΔS_j *is the set of all probability distributions for player j, such that for each* $s_j \in S_j$ *of player j,* $\mu_j(s_j)$ *is the probability that player i thinks player j will play* s_j*.*

Properties:

$$\mu_j(s_j) \geq 0 \qquad s_j \in S_j \tag{1}$$

$$\sum_{s_j \in S_j} \mu_j(s_j) = 1 \tag{2}$$

[1]S is the cartesian product of the players individual strategy space $S_i, i = 0, .., n$.

A mixed strategy is the act of selecting a strategy according to the above probability distribution. In other words the players choose the probability that they will play the different strategies. Extending the definition of a payoff function to mixed strategies and beliefs can be done by using the concept *expected value*. When player i has beliefs μ_{-i} about the actions of others and therefore plan to use strategy s_i, then the *expected value* is:

$$u_i(s_i, \mu_{-i}) = \sum_{s_{-i} \in S_{-i}} \mu_{-i}(s_i) u_i(s_i, s_{-i}) \tag{3}$$

Assume that a rational agent wish to maxmize the payoff that the agent expect to recieve. The agent should then select the strategy that yields the greatest expected payoff against his or her beliefs.

Definition 4 *Suppose player i have belief $\mu_{-i} \in \Delta S_{-i}$ about the strategies played by other players. Player i's strategy $s_i \in S_i$ is a best response if:*

$$u_i(s_i, \mu_{-i}) \geq u_i(s_i', \mu_{-i}) \qquad \forall_i s_i \in S_i \tag{4}$$

Definition 5 *The formal definition of weak and strong dominance is the following. A pure strategy s_i of player i is dominated if there is a pure or mixed strategy denoted ς_i, where $\varsigma_i \in \Delta S_i$ such that;*

$$u_i(\varsigma_i, s_{-i}) \geq u_i(s_i, s_{-i}) \tag{5}$$

for all strategy profiles $s_{-i} \in S_{-i}$ of the other players.

The strategy is weakly or strongly dominated when the inequality is weak or strict respectively.

Definition 6 *A strategy profile is a Nash equilibrium if and only if each player's prescribed strategy is a best response against the strategies of others.*

Formally a strategy profile $s_i \in S$ is a Nash equilibrium *if and only if,*

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \tag{6}$$

for each $s_i' \in S_i$ and each player i.

Definition 7 *A* mixed-strategy Nash equilibrium *is an equilibrium where the players use a probability distribution as a strategy. Formally, a strategy profile $\sigma = (\sigma_1, \sigma_2, .., \sigma_n)$,where $\sigma_i \in \Delta S_i, \forall i$ is a* mixed-strategy Nash equilibrium *if and only if;*

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(s_i', \sigma_{-i}) \tag{7}$$

for each $s_i' \in S_i$ and each player i.

Theorem 1 *(**Nash,1950** [9]) In the n-player normalform game **G**, if n and S_i is finite for every i then there exists at least one Nash equilibrium possibly involving mixed strategies.*

2.3 Symmetric two-player games

A game G is a symmetric two-player game if $S_1 = S_2$ and $u_1(s_2, s_1) = u_2(s_1, s_2)$ for all $(s_1, s_2) \in S_1 \times S_2 = S$

To classify symmetric 2×2 games we can use a graphical treatment. Consider the payoff matrix **A**.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{8}$$

This is eqvivalent to:

$$\mathbf{A}' = \begin{pmatrix} a_{11} - a_{12} & 0 \\ 0 & a_{22} - a_{21} \end{pmatrix} \tag{9}$$

$$= \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \tag{10}$$

This eqvivalence relation holds because the *Nash equilibria*, both pure and mixed, are the same after this normalization. It should be mentioned that this eqvivalence only holds when we have a symmetric game. It follows that any symmetric 2×2 game after this normalization can be identified by a point $a = (a_1, a_2) \in \mathbb{R}^2$

Let Δ^{NE} be the set of *Nash Equilibria*. s^1 refers to the first row and the first column while s^2 to the second row and the second column.

Category **I**: In this quadrant strategy 2 stricly dominates strategy $1(a_1 < 0, a_2 > 0)$. Hence all such games are stricly dominance solvable. $\Delta^{NE} = \{s^2\}$ Ex.Prisoners Dilemma.

Category **II**: All games in this category$(a_1 > 0, a_2 > 0)$ have two symmetric strict *Nash equilibria*, and one *mixed-strategy Nash equilibrium*. $\Delta^{NE} = \{s^1, s^2, \sigma\}$. Ex. Coordination game.

Category **III**: No strategy is dominated$(a_1 < 0, a_2 < 0)$, but the best reply to a pure strategy is the other pure strategy. These games have two asymmetric strict *Nash Equilibria*, and one symmetric *mixed-strategy Nash equilibrium.* $\Delta^{NE} = \{\sigma\}$. Ex. Hawk-Dove Game.

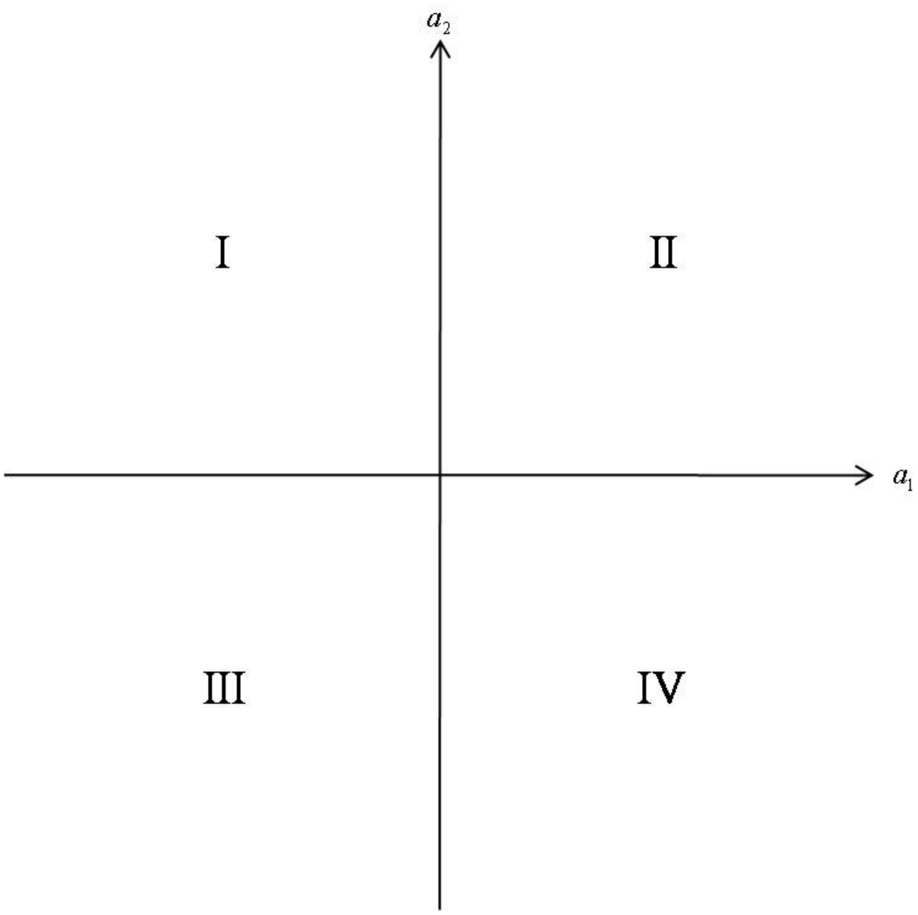

Figure 1: Classification of symmetric 2×2 games

Category **IV**: The games in this category($a_1 > 0, a_2 < 0$) are symmetric to category **I**, but strategy 1 stricktly dominates strategy 2. $\Delta^{NE} = \{s^1\}$

In the later discussion games in category **II** and **III** will be used in examples since these games are analytically tractable. The other two categories can be solved by using elimination of stricktly dominated strategies and are not interresting within the framework this paper presents.

2.4 Risk dominant equilibrium

Consider the 2×2 game **G**.

$$\mathbf{G} = \begin{array}{|c|c|} \hline a_{11}, b_{11} & a_{12}, b_{12} \\ \hline a_{21}, b_{21} & a_{22}, b_{22} \\ \hline \end{array}$$

There are two pure strategy *Nash Equilibria* $\Delta^{NE} = \{s^1, s^2\}$ if: $a_{11} > a_{21}, b_{11} > b_{12}, a_{22} > a_{12}, b_{22} > b_{21}$. Again s^1 refers to the first row and the first column while s^2 to the second row and the second column.

The equilibrium (s^1) is risk dominant if:

$$(a_{11} - a_{21})(b_{11} - b_{12}) \geq (a_{22} - a_{12})(b_{22} > b_{21}) \tag{11}$$

It is strictly risk dominant if equation (11) holds with strong inequality. (Harsanyi and Selten,1988 [6]).

It is not always the case that players adapt to the Pareto-dominant equilibrium when playing the game. The Pareto-dominant equilibrium is the one yielding highest payoff for the players. This comes from the tradeoff between efficency and strategic risk. Efficiency here means that the players know which payoff that gives them the highest utility and when realizing this should play such that this is achieved. The strategic risk stems from the possibility that the other player(s) defects from the efficient strategy and therefore inflict a loss to the other player(s).

2.5 Evolutionary stability criteria

The first step to explore evolutionary behavior in games formally was conducted in the 1970's by Maynard Smith and Price(Maynard Smith and Price,1973 [8]). The idea is that a large population is playing a game,$\mathbf{G}$. Most of the individuals are in a sense programmed to play the same strategy $s \in S$. While there is a small group of mutants in the same population programmed to play some other mutant strategy $s' \in S$.

Formally, let us assume that the share of mutants is $\epsilon \in (0, 1)$. Then pairs of individuals are repeatedly drawn at random to play $\mathbf{G}$(each with equal probability). Assume then that for the symmetic 2×2 game, $\mathbf{G}$, with payoff matrix,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{12}$$

a strategy s^* is said to be *evolutionary stable* if it for all $s \neq s^*$ exists an $\bar{\epsilon}$ such that:

$$s\mathbf{A}(\epsilon s + (1 - \epsilon)s^*) < s^*\mathbf{A}(\epsilon s + (1 - \epsilon)s^*) \tag{13}$$

for all positive $\epsilon < \bar{\epsilon}$. (Maynard Smith and Price,1973 [8];Maynard Smith,1974 [7]).

Let Δ^{ESS} be the set of evolutionary stable equlibrias, then:

$$\Delta^{ESS} = \{s \in \Delta^{NE} : u(s', s') < u(s, s')\}. \tag{14}$$

Where $s, s' \in S$. This condition states that every evolutionary stable strategy has to be a Nash equlibrium.

We can use the same kind of classification scheme for 2×2 games as in section 2.3 to show the evolutionary stable equilibrium(equilibria).

Category **I**: In this quadrant we have ($a_1 < 0$ and $a_2 > 0$). Hence in all such games $\Delta^{NE} = \{s^2\}$. The equilibrium is strict and symmetric so it have one ESS. $\Delta^{ESS} = \Delta^{NE} = \{s^2\}$ Ex.Prisoners Dilemma.

Category **II**: All games in this category ($a_1 > 0, a_2 > 0$) have two symmetric strict *Nash equilibria*, and one *mixed-strategy Nash equilibrium.* The two pure strategy equlibrias are evolutionary stable. $\Delta^{ESS} = \{s^1, s^2\}$. Ex. Coordination game.

Category **III**: These games have two asymmetric strict *Nash Equilibria*, and one symmetric *mixed-strategy Nash equilibrium* ($a_1 < 0, a_2 < 0$). Therefore $\Delta^{ESS} = \{\sigma\}$. Ex. Hawk-Dove Game.

Category **IV**: The games in this category($a_1 > 0, a_2 < 0$) are symmetric to category **I**. $\Delta^{ESS} = \Delta^{NE} = \{s^1\}$ (For a more extensive representation of evolutionary game theory see Weibull,1996 [10])

3 Markov Chain Theory

3.1 Introduction

A Markov process $\{X_t\}$ is a stochastic process with the property that, given the value of X_t, the values of X_s for $s > t$ are not influenced by the values of X_u for $u < t$. This discussion limits itself to stationary transition probabilities i.e the probabilities are independent of time.

The Markov property is formally,

$$Pr(X_{n+1} = j|X_0 = i, ..., X_n = i) = Pr(X_{n+1} = j|X_n = i) = P_{ij} \tag{15}$$

It is customary to arrange these probabilities,P_{ij}, in a transition probability matrix.

$$\mathbf{P} = \begin{pmatrix} P_{11} & \dots & \dots & \dots & P_{1n} \\ \vdots & \ddots & & & \vdots \\ \vdots & & P_{ij} & & \vdots \\ \vdots & & & \ddots & \vdots \\ P_{n1} & \dots & \dots & \dots & P_{nn} \end{pmatrix} \tag{16}$$

This matrix shows the probability of going from one state to the same or a another state in one transition. For example P_{ij} shows the probability of moving from state i to state j.

Each transition probability satisfies the conditions:

$$P_{ij} \geq 0 \qquad \forall i, j = 1, .., n \tag{17}$$

$$\sum_{j=0}^{n} P_{ij} = 1 \qquad i = 1, .., n \tag{18}$$

The probability that the process goes from state i to state j in n transitions are

$$P_{ij}^{(n)} = Pr(X_{m+n} = j | X_m = i) \tag{19}$$

or eqvivalently, the n-step transition probabilities $P_{ij}^{(n)}$ are the entries in the matrix $\mathbf{P}^n$.

3.2 One-Dimensional Movement in a Markov Chain

What is meant by an *One-Dimensional Movement in a Markov Chain* is that when the process is in state i, it can in a single transition only stay in state i or move to one of the neighboring states *i-1* or *i+1*.

$$\mathbf{P} = \begin{pmatrix} s_0 & r_0 & 0 & \dots & \dots & \dots & 0 \\ l_1 & s_1 & r_1 & 0 & \dots & \dots & 0 \\ 0 & l_2 & s_2 & r_2 & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & l_{n-1} & s_{n-1} & r_{n-1} & \vdots \\ 0 & \dots & \dots & \dots & \dots & l_n & s_n \end{pmatrix} \tag{20}$$

where $Pr(X_{n+1} = i + 1 | X_n = i) = r_i$, $Pr(X_{n+1} = i - 1 | X_n = i) = l_i$ and $Pr(X_{n+1} = i | X_n = i) = s_i, i = 0, ..., n$.

These kind of processes will become important later when studying the adaptive play in games. It is often the case that the movement in the system is one-dimensional and therefore follows such a Markov Chain.

3.3 Long-run Behavior of Markov Chains

We often want to know what happens with the stochastic process $\{X_t\}$ when we let time tend to infinity i.e. the behavior of the entries in the transition probability matrix $\mathbf{P}^n$ as $n \to \infty$. This gives us an idea of how likely it is that the process will be in the different states in the long-run.

Before moving to the limiting behavior theory some concepts have to be defined.

Definition 8 *If $P_{ij}^{(n)} > 0$ for some $n \geq 1$, we say that j is accesible from i, or in shorter notation $(i \rightarrow j)$.*

This means that there is positive probability that state j can be reached starting from state i in some finite number of transitions.

Definition 9 *Two states i and j, each accessible from the other is said to communicate $(i \leftrightarrow j)$.*

Communicating states have three properties:

Reflexitivity,$(i \leftrightarrow i)$.

Symmetry, if $(i \leftrightarrow j)$ then $(j \leftrightarrow i)$.

Transitivity, if $(i \leftrightarrow j)$ and $(j \leftrightarrow k)$ then $(i \leftrightarrow k)$.

Definition 10 *A set of states $\mathbf{C}$ is closed if $P_{ij} = 0$, for $i \in \mathbf{C}$ and $j \notin \mathbf{C}$. No state in $\mathbf{C}$ is accessible from any state outside $\mathbf{C}$.*

Definition 11 *A subset $\mathbf{C}$ of $\mathbf{S}$,where $\mathbf{S}$ is the whole state space, is* ***irreducible*** *if all states in $\mathbf{C}$ communicates.*

Definition 12 ***Periodicity of a Markov Chain*** *The period $d(i)$ of a state i is the greatest common divisor of all numbers $P_{ii}^{(n)} > 0$ for all $n \geq 1$. We say that it is* ***aperiodic*** *if $\mathbf{d(i) = 1}$. That is $P_{ii}^{1} > 0$ or $P_{ii}^{(n)} > 0$, where n is a prime number.*

We can classifiy states by deciding whether they are recurrent or transient. For this we need some notation.

$$f_{ii}^{(n)} = Pr(X_n = i, X_\nu \neq i, \nu = 1, 2, ..., n-1 | X_0 = i) \tag{21}$$

$f_{ii}^{(n)}$ is the probability of starting in state i, and then the process is in a state $i' \neq i$ for $n-1$ periods and at the n^{th} period we return to state i.

Definition 13 ***Recurrent and transient states*** *State i is recurrent if the probability, f_{ii}, that we at some point in time return to state i is equal to 1. A state i is transient if $f_{ii} < 1$.*

Theorem 2 *A state i is recurrent if and only if,*

$$\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty \tag{22}$$

Equvivalently, state i is transient if and only if,

$$\sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty \tag{23}$$

(See Karlin and Taylor for a proof [2]). It is noteworthy that the definiton and the theorem states the same thing, but in different ways.

Definition 14 ***Stationary distribution*** *The distribution $\pi = (\pi_0, \pi_1, .., \pi_n)$ is stationary if $\pi = \pi\mathbf{P}$, where $\boldsymbol{P}$ is the transition probability matrix.*

Definition 15 ***Positive recurrence*** *If $\lim_{n\to\infty} P_{ii}^{(n)} > 0$ for one i in an aperiodic recurrent class, then $\pi_j > 0$(π_j is a stationary distribution) for all j in the class of i. The class is then called* positive reccurent.

Definition 16 ***Limiting distribution*** *The distribution $\pi_j^* = \lim_{n\to\infty} P_{ij}^{(n)}$ is a limiting distribution if the limit exists, and it is independent of i and $\sum \pi_j^* = 1$.*

The next three thorems that are presented is crucial for determining if a limit distribution exists, and they will be used implisitly in later chapters.

Theorem 3 *A limit distribution is stationary.*

Theorem 4 [2] *If the statespace* **S** *is finite, then:*

1. *Some state is recurrent.*
2. *All recurrent states are positive.*
3. *A stationary distribution always exist.*
4. *A limit distribution exists if the chain is irreducible and aperiodic.*

Theorem 5 *In a positive recurrent aperiodic Markov chain with states* j=0,1,... *we have that,*

$$\lim_{n\to\infty} P_{jj}^{(n)} = \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, \qquad \sum_{i=0}^{\infty} \pi_i = 1 \tag{24}$$

and the π's are uniquely determined by the set of equations,

$$\pi_i \geq 0, \qquad \sum_{i=0}^{\infty} \pi_i = 1, \qquad \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, j = 0, 1.. \tag{25}$$

[2]In addition the following hold: A class is recurrent ⇔ the class is closed. A state is transient if and only if for some closed set **C**, $P_{ij}^n > 0$ for some $j \in \mathbf{C}$, some n and $i \notin \mathbf{C}$

4 Stochastic Adaptive Dynamics

4.1 Introduction

In economics we often want to study how individual decitionmaking affects the economy at the aggregate level. The behavior of individuals are often illustrated by assuming that they are rational and that they does not make any mistakes when engaging each other in a strategic situation. In this section we leave that assumption in a sense, and show how irrational behavior on the individual level still can give rational behavior on the aggregate level. The reason that causes this is that we let the behavior of individuals be affected by (small) persistent stochastic disturbances, which may alter the long-run behavior. In this stochastic setting, conventional wisdom like evolutionary stable strategies becomes obsolete. Instead of letting a small mutant group of players invade the population we here assume that it is the stochastic disturbances and the distribution of the population that is the factors determining the behavior and therefore the equlibrium selection. The method used in this chapter leads to a "new" equlibrium concept, namely *stochastic stability*, first introduced by Young and Foster(Foster and Young,1990 [1]). Much of this and the next chapter will mainly contain theory but there will be presented some simple examples of how this theory can be used.

This theory can be applied to many fields within economics. It can be used for explaining bargaining problems like sharecropping in agriculture, the existence of a pension system or the long-run behavior of firms in a duopoly. It is especially suited for explaining *conventions* within a society. This was only a few examples and there are many additional fields where we can use this theory(for more examples see Bowles,2004 [12]). To gain further perspective, this theory is also used in biology, where it is used to model a *Darwinian* strategic survival-of-the-fittest situation.

The stochastic disturbances may stem from different sources. First, the agents encounters with each other could happen at random. Second, the agents behavior in the game may be intentionally random when using mixed strategies. Third, there could be some kind of mutation process such that some agents do mistakes in their course of action. Fourth, if agents are assumed to have memory this can introduce uncertainty into the model.

In evolutionary game dynamics, when the population is large, random shocks at the individual level will average out. This is just an implication of the *law of large numbers*. This will induce that the process at the aggregate level will have a deterministic direction of motion. This approach can be reasonable in the short- and medium-run, but it may fail when in the case of a long time horizon.

The most important insight when analysing the following processes is that when the aggregate stochastic effects are "small" the resulting Markov process satisfies certain conditions. The conditions are that the process have a finite state space, is irreducible and aperiodic. If fullfilled the process will have a long-run distribution which is often concentrated on a single state when the probability of making errors go to zero. This is the stochastically stable state (or set if there are more states which occur with high probability). In many cases we can say more about which equlibrium is selected when considering *stochastic stability* versus *Nash or evolutionary stable equlibria*. When there are two *Nash equlibrias* in a game we can not tell which of these will be played in a real world situation, only that it is rational to play them. Using the concept stochastic stability we can say more i.e we can predict the strategic behavior of a population playing the game when the time horizon becomes long. The *stochastically stable equilibrium* need not correspond to the *Nash equlibrium* or the *evolutionary stable equilibrium*.

In this chapter there will be presented the analytical tools to find the stochastically stable state(s) in games. The games analysed here will be 2×2 games for analytical conviniance. When a theorem or new concept is introduced there will be examples to illustrate them.

4.2 The General Idea

Consider a population of size n playing the game $\mathbf{G}$.

$\mathbf{G} =$

a_{11}, b_{11}	a_{12}, b_{12}
a_{21}, b_{21}	a_{22}, b_{22}

At the beginning of each period one agent from the population is chosen at random. Time is discrete. The state of the process at time t is the current number of agents playing the first row and the first column, from now on referred to as strategy A, denoted $z_t \in Z = \{0, 1, ..., n\}$. Agents playing the second row and the second column is reffered to as playing strategy B.

The best-response to the current distribution is given by the value of z_t, that is, whether it is smaller or bigger than the *critical value*,

$$z_t^c = \frac{a_{22} - a_{12}}{a_{11} - a_{12} + a_{22} - a_{21}} n \tag{26}$$

Here it is assumed that the player includes himself in the assessment of the current distribution.

With high probability, $1 - \epsilon$, the agent chooses a best-response to the current distribution of strategies. With probability, $\epsilon/2$, he chooses strategy A at random. Similarly he chooses strategy B with probability, $\epsilon/2$. This leads to what is called a perturbed Markov Process.

Example 1 *Consider a population of n agents playing the* Pareto Coordination game.

2, 2	0, 0
0, 0	1, 1

The state of the process at time t *is the current number of agents playing the first row or the first column, from now on reffered to as strategy A and players playing the second row or the second column is referred to as playing strategy B, denoted* $z_t \in Z = \{0, 1, ..., n\}$. *Time is discrete.*

If $z_t \geq \frac{n}{3}$ *strategy A is a best-response.*

If $z_t \leq \frac{n}{3}$ *strategy B is a best-response.*

With high probability, $1 - \epsilon$, *the agent chooses a best-response to the current distribution of strategies. With probability,* $\epsilon/2$, *he chooses strategy A at random. Similarly he chooses strategy B with probability,* $\epsilon/2$. *The process is one-dimensional and the states can be illustrated as a directed tree. The only movement we can have is either to stay in the same state or move one step to the right or one step to the left.*

Lets assume that $n = 6$. *This imply that the critical value is 2.*

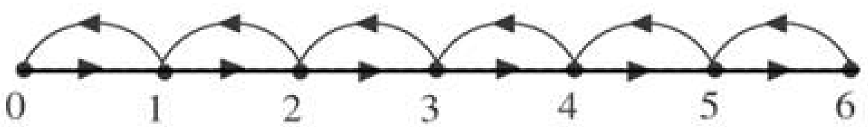

Figure 2: 6-tree

The probabilities of the different alternatives then becomes:

$z < 2$: $R_z = (1-z/6)(\epsilon/2)$, $L_z = (z/6)(1-\epsilon/2)$ *and* $S_z = (1/6)(6-z+z\epsilon-3\epsilon)$

$z > 2$: $R_z = (1-z/6)(1-\epsilon/2)$, $L_z = (z/6)(\epsilon/2)$ *and* $S_z = (1/6)(z-z\epsilon+3\epsilon)$

$z = 2$: $R_2 = 1/3$, $L_2 = 1/6$ *and* $S_2 = 1/2$

The intuition behind the above transition probabilities are quite transparent. If we are in a state z *to the right of the critical point* $z = 2$ *the process moves left if one less agent plays strategy A. This can happen if an agent currently playing A is drawn(which happens with probability* $z/6$*) and this agent make a* *mistake and plays strategy B(which happens with probability $\epsilon/2$). A symmetric argument holds when considering right transitions. The probability of staying in a state is of course one minus the probabilities of moving left and right. If we are in the critical point it is assumed that $\epsilon = 1$ so the probability to play the same or the new strategy are both one half. This leads to a perturbed Markov chain,$\mathbf{P}^\epsilon$. The entries in the matrix show the perturbed probabilities of moving between the different states. If we are in $z_t = 0$, the probability of staying in the same state is $1 - \epsilon/2$ and the probability that the agent playing changes his strategy(type) to strategy A is $\epsilon/2$. The perturbed transition matrix is then,*

$$\mathbf{P}^\epsilon = \begin{pmatrix} S_0 & R_0 & 0 & \dots & \dots & \dots & 0 \\ L_1 & S_1 & R_1 & 0 & & & \vdots \\ 0 & L_2 & S_2 & R_2 & 0 & & \vdots \\ \vdots & 0 & L_3 & S_3 & R_3 & 0 & \vdots \\ \vdots & & 0 & L_4 & S_4 & R_4 & 0 \\ \vdots & & & 0 & L_5 & S_5 & R_6 \\ 0 & \dots & \dots & \dots & 0 & L_6 & S_6 \end{pmatrix} \tag{27}$$

which in this case is equal to,

$$\mathbf{P}^\epsilon = \begin{pmatrix} 1-\frac{\epsilon}{2} & \frac{\epsilon}{2} & 0 & \dots & \dots & \dots & 0 \\ \frac{(1-\epsilon/2)}{6} & \frac{(5-2\epsilon)}{6} & \frac{5\epsilon}{12} & 0 & & & \vdots \\ 0 & \frac{1}{6} & \frac{1}{2} & \frac{1}{3} & 0 & & \vdots \\ \vdots & 0 & \frac{\epsilon}{4} & \frac{1}{2} & \frac{(1-\epsilon/2)}{2} & 0 & \vdots \\ \vdots & & 0 & \frac{\epsilon}{3} & \frac{(4-\epsilon)}{6} & \frac{(1-\epsilon/2)}{3} & 0 \\ \vdots & & & 0 & \frac{5\epsilon}{12} & \frac{(5-2\epsilon)}{6} & \frac{(1-\epsilon/2)}{6} \\ 0 & \dots & \dots & \dots & 0 & \frac{\epsilon}{2} & 1-\frac{\epsilon}{2} \end{pmatrix} \tag{28}$$

4.3 Perturbed Markov Processes

Let $\mathbf{P^0}$ be a Markov process defined on a finite state space $\mathbf{S}$. A perturbation of $\mathbf{P^0}$ is a process whose transition probabilities are slightly perturbed versions of the transition probabilities P^0_{ij}. An example of such a process is the matrix $\mathbf{P}^\epsilon$ in *Example 1* above.

Let $\mathbf{P}^\epsilon$ be a Markov process on $\mathbf{S}$. $\mathbf{P}^\epsilon$ is a regular perturbed process if it is irreducible for every $\epsilon \in [0, \epsilon^*]$. Formally,

$$\lim_{\epsilon \to 0} P^\epsilon_{ij} = P^0_{ij} \tag{29}$$

and if $P_{ij}^{\epsilon} > 0$ for some $\epsilon > 0$, then

$$0 < \lim_{\epsilon \to 0} \frac{P_{ij}^{\epsilon}}{\epsilon^{r(i,j)}} < \infty \tag{30}$$

for some $r(i,j) \geq 0$.

$r(i,j)$ is the resistance of the transition from i to j, and is unique. Some more explaining is needed on this. The resistance shows how difficult it is to move from one state to another in a perturbed process i.e how many ϵ's that we meet in the path in question. $P_{ij}^{0} > 0$ if and only if $r(i,j) = 0$. This is because transitions in the unperturbed process do not have any resistance. Next, assume that $r(i,j) = \infty$ if $P_{ij}^{\epsilon} = P_{ij}^{0} = 0$ for every ϵ.

Since the perturbed transition matrix $\mathbf{P}^{\epsilon}$ is irreducible and aperiodic for every ϵ, it has an unique stationary distribution,π^{ϵ}. This distribution is important when we want to find the stochastically stable state. It is the case that when we let ϵ go to zero and the stationary probability of being in a specific state is greater than zero this will be the stochastically stable state.

Theorem 6 ***(Young,1993a [4])*** *A state i is stochastically stable if,*

$$\lim_{\epsilon \to 0} \pi_i^{\epsilon} > 0 \tag{31}$$

Example 2 *To illuminate* Theorem 6 *let us now consider for simplicity a population of 3 individuals playing the Pareto coordination game as above. The critical point now becomes 1. The transition matrix then becomes,*

$$\mathbf{P}^{\epsilon} = \begin{pmatrix} S_0 & R_0 & 0 & 0 \\ L_1 & S_1 & R_1 & 0 \\ 0 & L_2 & S_2 & R_2 \\ 0 & 0 & L_3 & S_3 \end{pmatrix} = \begin{pmatrix} 1-\frac{\epsilon}{2} & \frac{\epsilon}{2} & 0 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} & 0 \\ 0 & \frac{\epsilon}{3} & \frac{(2-\epsilon/2)}{3} & \frac{(1-\epsilon/2)}{3} \\ 0 & 0 & \frac{\epsilon}{2} & 1-\frac{\epsilon}{2} \end{pmatrix} \tag{32}$$

We then use standard Markov Chain theory if we want to find the stationary distribution. This process is irreducible because we can with positive probability reach all states when we start in an arbitrary state i. It is also aperiodic since $P_{ii} > 0$ and therefore the greatest common divisor is of course 1. Because of this we know that a limiting ditribution exists. The equality that has to hold is $\pi^{\epsilon} = \pi^{\epsilon}\mathbf{P}^{\epsilon}$. This leads to the following equation system,

$$S_0\pi_0^{\epsilon} + L_1\pi_1^{\epsilon} = \pi_0^{\epsilon} \tag{33}$$

$$R_0\pi_0^{\epsilon} + S_1\pi_1^{\epsilon} + L_2\pi_2^{\epsilon} = \pi_1^{\epsilon} \tag{34}$$

$$R_1\pi_1^{\epsilon} + S_2\pi_2^{\epsilon} + L_3\pi_3^{\epsilon} = \pi_2^{\epsilon} \tag{35}$$

$$R_2\pi_2^{\epsilon} + S_3\pi_3^{\epsilon} = \pi_3^{\epsilon} \tag{36}$$

$$\pi_0^\epsilon + \pi_1^\epsilon + \pi_2^\epsilon + \pi_3^\epsilon = 1 \tag{37}$$

The solution to the system is,

$$\pi_0^\epsilon = \varphi\epsilon, \pi_1^\epsilon = 3\varphi\epsilon^2, \pi_2^\epsilon = 3\varphi\epsilon, \pi_3^\epsilon = \varphi(2-\epsilon), \varphi = \frac{1}{3\epsilon^2 + 3\epsilon + 2} \tag{38}$$

From this we clearly see that the only state that has $\lim_{\epsilon\to 0}\pi_i^\epsilon > 0, i = 0, 1, 2, 3,$ *is state 3,which is infact the stochastically stable state. In the long run all three agents will be playing* (A, A)*, which is the Pareto-efficient solution of the game.*

There are other methods to find the stochastically stable state(s), as the next theorem shows.

Theorem 7 ***(Young,1993a [4])*** *Let* $\mathbf{P}^\epsilon$ *be a regular perturbed Markov process, and let* π^ϵ *be the unique stationary distribution of* $\mathbf{P}^\epsilon$ *for each* $\epsilon > 0$*. Then,*

$$\lim_{\epsilon\to 0}\pi^\epsilon = \pi^0 \tag{39}$$

exists, and π^0 *is a stationary distribution of the unperturbed process,*$\mathbf{P^0}$*. The stochastically stable states are precisely those states that are contained in the reccurrent class(es) of* $\mathbf{P^0}$ *having minimum stochastic potential,denoted* γ_i*.*

The concept minimum stochastic potential is explained in the next example.

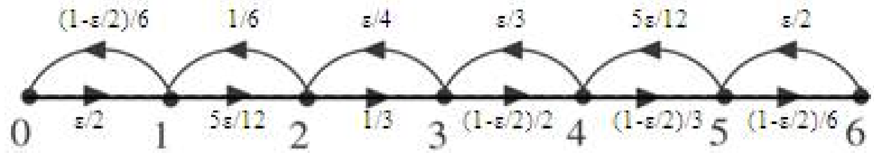

Figure 3: Perturbed tree

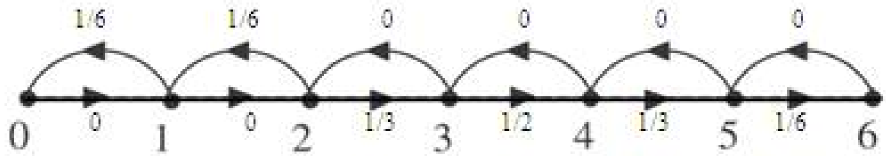

Figure 4: Unperturbed tree($\epsilon = 0$)

Example 3 *In figure 4 we observe that the unperturbed process have two recurrent classes. $C_1 = \{0\}$ and $C_2 = \{6\}$. In the pertubed process the path with least resistance from C_1 to C_2 is:*

$$\frac{\epsilon}{2}\frac{5\epsilon}{12}\frac{(1-\epsilon/2)}{2}\frac{(1-\epsilon/2)}{3}\frac{(1-\epsilon/2)}{6} = \frac{5\epsilon^2(1-\epsilon/2)^3}{2592} \Rightarrow r_{12} = 2 \tag{40}$$

Symmetrically the least resistant path from C_2 to C_1 is:

$$\frac{\epsilon}{2}\frac{5\epsilon}{12}\frac{\epsilon}{3}\frac{\epsilon}{4}\frac{1}{3}\frac{(1-\epsilon/2)}{6} = \frac{5\epsilon^4(1-\epsilon/2)}{5184} \Rightarrow r_{21} = 4 \tag{41}$$

Therefore the stochastic potential for the two reccurent classes are $\gamma_1 = 4$ and $\gamma_2 = 2$.

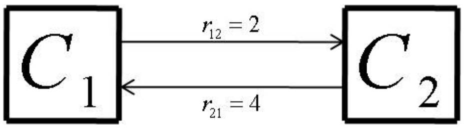

Figure 5: Resistance between two recurrent classes

We conclude that C_2 has the minimum stochastic potential and is therefore the stochastically stable state.

The stochastic potential is the minimum of the resistances between different paths starting in state j and ending in a recurrent state(or set) of the system.

The above theorem states that the long-run probability of the process will be concentrated around some state when we lack a estimate of ϵ, but know that it is "small". If ϵ was known we could in theory estmate it by computing the actual distribution using the same method as above, namely find the solution to the equationsystem $\pi^\epsilon = \pi^\epsilon \mathbf{P}^\epsilon$.

In the discussion and examples so far it has been assumed for simplicity that the population,n, has been small. If n becomes large, which is a resonable assumption, the method of finding a perturbed stationary distribution becomes very timeconsuming so another approach is more convinient to use. This is called the Markov chain tree theorem.

4.4 Markov chain tree theorem

Let $\mathbf{P}$ be any irreducible Markov process defined on a finite state space $\mathbf{S}$. Take one state i, and consider the directed tree T_i which consist of all right transitions from states to the left of i and all left transitions from states to right of i. This is called a i-tree. A directed tree is a graph that consists of $\mathbf{S} - 1$ edges and from every vertex $j \neq i$ there exists an uniqe path from j to i. Figure 3 is an example of a tree. (for more on graph theory see Wiitala,1987 [13]).

Theorem 8 ***(Markov chain tree theorem [5])*** *For one-dimensional processes, the long-run probability of being in state i is proportional to the product of the probabilities on the edges of the directed tree T_i,*

$$\pi_i = k \prod_{y<i} R_y \prod_{y>i} L_y \tag{42}$$

, here k is a factor of proportionality such that $\sum_{i=0}^{n} \pi_i = 1$.

This theorem applied offers a tractable way to calculate the probabilities for being in different states, and when applied it can be used to see which state(s) is the stochastically stable one. One can just compare the different probabilities i.e to which power ϵ the stationary distribution have in all the states and see what happens as the perturbation tend to zero($\epsilon \to 0$). Usually one state is much more likely than the other states.

To check this theorem one can start out with the regular theory about stationary distributions in finite Markov chains, and insert in the above theorem. Assume that the state is i. The subscipt i on the transition matrix,$\mathbf{P}$, means the *i'th* column of the matrix.

$$\begin{aligned}
\pi_i &= \pi_i \mathbf{P}_i \\
&= R_{i-1}\pi_{i-1} + S_i\pi_i + L_{i+1}\pi_{i+1} \\
&= R_{i-1}k \prod_{y<i-1} R_y \prod_{y>i-1} L_y + S_i k \prod_{y<i} R_y \prod_{y>i} L_y \\
&+ L_{i+1}k \prod_{y<i+1} R_y \prod_{y>i+1} L_y \\
&= R_{i-1}k \prod_{y<i-1} R_y \prod_{y>i-1} L_y + (1 - L_i - R_i)k \prod_{y<i} R_y \prod_{y>i} L_y \\
&+ L_{i+1}k \prod_{y<i+1} R_y \prod_{y>i+1} L_y \\
&= k \prod_{y<i} R_y \prod_{y>i} L_y \equiv \pi_i
\end{aligned} \tag{43}$$

This holds for every state i. In addition the condition that $\sum_i \pi_i = 1$ must hold. In this way we have shown that this approach is equvivalent to solving the equation system $\pi = \pi\mathbf{P}$.

Example 4 *cont. of Example 1. This example will show how the relative probabilities differ in which power the ϵ's are, from this it is easy to find the stochastically stable state in the game in Example 1. The relative probabilities of the six different states are:*

$$\pi_0 = k \prod_{y<0} R_y \prod_{y>0} L_y = kL_1L_2L_3L_4L_5L_6 = k\frac{5\epsilon^4(1-\epsilon/2)}{10368} \tag{44}$$

$$\pi_1 = k\prod_{y<1} R_y \prod_{y>1} L_y = kR_0L_2L_3L_4L_5L_6 = k\frac{5\epsilon^5}{3456} \quad (45)$$

$$\pi_2 = k\prod_{y<2} R_y \prod_{y>2} L_y = kR_0R_1L_3L_4L_5L_6 = k\frac{5\epsilon^6}{6912} \quad (46)$$

$$\pi_3 = k\prod_{y<3} R_y \prod_{y>3} L_y = kR_0R_1R_2L_4L_5L_6 = k\frac{5\epsilon^5}{5184} \quad (47)$$

$$\pi_4 = k\prod_{y<4} R_y \prod_{y>4} L_y = kR_0R_1R_2R_3L_5L_6 = k\frac{5\epsilon^4(1-\epsilon/2)}{3456} \quad (48)$$

$$\pi_5 = k\prod_{y<5} R_y \prod_{y>5} L_y = kR_0R_1R_2R_3R_4L_6 = k\frac{5\epsilon^3(1-\epsilon/2)^2}{864} \quad (49)$$

$$\pi_6 = k\prod_{y<6} R_y \prod_{y>6} L_y = kR_0R_1R_2R_3R_4R_5 = k\frac{5\epsilon^2(1-\epsilon/2)^3}{2592} \quad (50)$$

When $\epsilon \to 0$ state 6 where all the players play strategy A is the most probable state of the system, and therefore is the stochastically stable state. *The intuition behind this is that π_6 include ϵ^2 while in the other states epsilon is in a higher power than 2. Therefore as $\epsilon \to 0$ the probabilities of the other states will go much faster towards zero than the probability of state 6. From this result we can draw the conclusion that this must be the stochastically stable state. State 6 i.e all coordinate on playing (A,A) is also the Pareto-optimal solution of the game. In this case the stochastically stable state is also a Nash equilibrium in the game. This refinement gives us more information about what will happen when a population plays a game than standard game theory. There are caveats in relation to the assumption about the long time horizon that is needed to reach this equlibria, but this will be discussed in chapter 6.*

4.5 A interesting example

Consider the Hawk-Dove game,**G**,

G =

$\frac{w-l}{2}, \frac{w-l}{2}$	$w, 0$
$0, w$	$\frac{w}{2}, \frac{w}{2}$

Here w is the reward of winning the fight. The cost of losing is l. We assume that there is an equal probability of winning if there are a fight. In addition it is assumed that $\frac{w-l}{2} < 0$. The first row and the first column is denoted strategy *fight* while the second row and the second column is denoted strategy *flight*.

There is two asymmetric pure strategy Nash equilibria in this game, namely *(fight,flight)* and *(flight,fight)*. There is also one mixed strategy Nash equlibrium where the players play the *fight* strategy with probability $\frac{w}{l}$. This is also the evolutionary stable equlibrium in this game.

Next assume that there are n agents in a population playing the game above. The state of the system, at period t, is the number of agents playing the *fight*-strategy denoted by z_t. There is also trembling of the same kind as in the examples above. When deciding on which strategy to play the agents compare the expected payoff of the two strategies. The critical point is therefore, $z_t = \frac{w}{l}n = \delta n$, where $\delta \in (0,1)$. So if $z_t > \delta n \Rightarrow fight$ is a best response, and if $z_t < \delta n \Rightarrow flight$ is a best response.

For illustrative purposes now assume that $w = 4, l = 6$ and that $n = 15$. The critical point then becomes $z_t = 10$. The transitions probabilities are of course, $z < 10$: $R_z = (1 - z/15)(\epsilon/2)$, $L_z = (z/15)(1 - \epsilon/2)$ and $S_z = 1 - R_z - L_z$ $z > 10$: $R_z = (1 - z/15)(1 - \epsilon/2)$, $L_z = (z/15)(\epsilon/2)$ and $S_z = 1 - R_z - L_z$ $z = 10$: $R_{10} = 1/3$, $L_{10} = 1/6$ and $S_{10} = 1/2$

Using the Markov chain tree theorem we can find the relative probabilities of the different states when $t \to \infty$.

$$\pi_0 = k \prod_{y<0} R_y \prod_{y>0} L_y = kL_1 \dots L_{15} = k(\epsilon/2)^5(1-\epsilon/2)^9 \tag{51}$$

$$\pi_{15} = k \prod_{y<15} R_y \prod_{y>15} L_y = kR_0 \dots R_{14} = k(\epsilon/2)^{10}(1-\epsilon/2)^4 \tag{52}$$

Here the only two states we need to compare the probability of is the left- and the right endpoint, since these are the possible candidates. From the calculation it is clear that the probability that no one play *fight* is the greatest, which imply that the stochastically stable state(SSS) is *(flight,flight)*. The probability of this state is larger by a factor of $1/\epsilon^5$ when comparing the two alternatives. This differs from the predictions of other equilibrium concepts. The long-run aggregate behavior in the population is to "cooperate" such that the Pareto-optimal equilibrium is played.

4.6 Link between Stochastic Stability and Risk Dominance

There is a direct link between stochastic stability and risk dominance in 2×2 games. In this kind of game the stochastically stable state is the risk dominant equilibrium. This was first showed by Kandori, Mailath and Rob(1993) [11]. The conditions for this result to hold is that the mutation rate is uniform and that the population is large enough. This eqvivalence relation can be illustrated by using the results from *Example 2*.

Example 5 *cont. of Example 2 The Pareto Coordination game is the following,*

$$\mathbf{G} = \begin{array}{|c|c|} \hline a_{11}, b_{11} & a_{12}, b_{12} \\ \hline a_{21}, b_{21} & a_{22}, b_{22} \\ \hline \end{array}$$

with $a_{11} = b_{11} = 2$,$a_{12} = b_{12} = a_{21} = b_{21} = 0$ *and* $a_{22} = b_{22} = 1$. *Using the the method in chapter 2.4,*

$$(a_{11} - a_{21})(b_{11} - b_{12}) \geq (a_{22} - a_{12})(b_{22} > b_{21}) \tag{53}$$

which in this case is the same as,

$$(2 - 0)(2 - 0) > (1 - 0)(1 - 0) \Rightarrow 4 > 1 \tag{54}$$

Therefore the equilibrium (A,A) is the stricktly risk dominant equilibrium, and because of the eqvivalence relation also is the stochastically stable state as showed in Example 2.

The eqvivalence between the two concepts holds only for 2×2 games. When each player have more than two strategies in their strategy space this relation breaks down.

4.7 Remarks

When trying to find the *stochastically stable state (or set)* there are "many roads that leads to Rome". When the different methods should be used depends on which situation one is in. Theorem 5 is not very usefull when the population becomes large since this demands tedious calculations to find the (perturbed) stationary distribution of the system. Theorem 6 is easier to use when n increases because it is often simple to find the recurrent classes and after this to find the state with the minimum stochastic potential. The Markov chain tree theorem is tractable when working with one-dimensional Markov chains, but becomes difficult to use if we leave this assumption. Next there will be presented an extension to the existing theory using a different method. The Markov chain tree theorem could possibly also be used, but that is omitted in this paper.

5 Extension of the Concept

5.1 Games with two populations playing agains each other

Consider the game $\mathbf{G}$,

$$\mathbf{G} = \begin{array}{|c|c|} \hline a_{11}, b_{11} & a_{12}, b_{12} \\ \hline a_{21}, b_{21} & a_{22}, b_{22} \\ \hline \end{array}$$

Assume that there are m players in population A operating the rows, and that there are n players in population B operating the columns. Let i be the number of agents in population A that plays the first row, denoted as strategy a, and let j be the number of agents in population B that plays the first column, also denoted strategy a. The second row and the second column is both denoted strategy b. Assume that time is discrete, and that in period t one player from both populations is drawn at random. In addition the two agents drawn faces a probability equal to 1/2 to be allowed to play i.e. in each period there is only one of the drawn agents playing. The probability that the agent from population A plays strategy a is $p_a^A = i/m$ and that an agent from population B plays strategy a is $p_a^B = j/n$. If the player from population A is allowed to play the agent will choose strategy a if this yields a higher payoff than playing strategy b. The payoffs are determined as follows: The opponent is a randomly drawn player from population B which play strategy a or strategy b with the probabilities $p_a^B = j/n$ and $p_b^B = 1 - j/n$ respectively. The agent from population A then compares the expected payoffs,

$$E_A(a) = a_{11}p_a^B + a_{12}(1 - p_a^B) = a_{11}(j/n) + a_{12}(1 - j/n) \tag{55}$$

$$E_A(b) = a_{21}p_b^B + a_{22}(1 - p_b^B) = a_{21}(j/n) + a_{22}(1 - j/n) \tag{56}$$

This gives a critical point for the agents behavior,

$$j^* = \frac{a_{22} - a_{12}}{a_{11} - a_{12} + a_{22} - a_{21}} n = \delta_A n, \delta_A \in (0, 1) \tag{57}$$

This gives the following rule for playing the game: If $j > j^* = \delta_A n \Rightarrow$play strategy a, if $j < j^* = \delta_A n \Rightarrow$play strategy b and if $j = j^* = \delta_A n \Rightarrow$the agent is indifferent between strategy a and strategy b.

If the agent is a a-player and $j < j^* = \delta_A n$, the agent changes strategy to being a b-player hence the state changes from (i/m) to $((i - 1)/m)$. On the other hand if the player is a b-player and $j > j^* = \delta_A n$ the state will change to $((i+1)/m)$. In the other cases we get no change in the state. Assume that there is no change in j/n. In addition to this there is trembling. With probability $1 - \epsilon$ the agent follows a best-response scheme, with probability $\epsilon/2$ the agent goes left by mistake and with probability $\epsilon/2$ the agent goes right by mistake. Since we allow trembling the system will be similar to the cases in chapter 4, but the difference here is that the states will be pairs (i, j) in the perturbed transition probability matrix.

$$\mathbf{P}^{\epsilon} = \begin{pmatrix} P_{(0,0)(0,0)} & \cdots & \cdots & \cdots & P_{(0,0)(m,n)} \\ \vdots & \ddots & & & \vdots \\ \vdots & & P_{(i,j)(i,j)} & & \vdots \\ \vdots & & & \ddots & \vdots \\ P_{(m,n)(0,0)} & \cdots & \cdots & \cdots & P_{(m,n)(m,n)} \end{pmatrix} \tag{58}$$

If it is an agent from the B population playing it is naturally symmetric so,

$$i^* = \frac{b_{22} - b_{12}}{b_{11} - b_{12} + b_{22} - b_{21}} n = \delta_B m, \delta_B \in (0,1) \tag{59}$$

is the critical point and the behavior rules are, if $i > i^* = \delta_B m \Rightarrow$play strategy a, if $i < i^* = \delta_B m \Rightarrow$play strategy b and if $i = i^* = \delta_B m \Rightarrow$the agent is indifferent between strategy a and strategy b.

The probabilities in the above transitions matrix,$\mathbf{P}^{\epsilon}$ can be calculated explicitly. If we are in a state (i,j), there are five possible transitions that can happen. A agent from population A drawn with probability 1/2. The probabilities of the transitions to the left or to the right will be affected by the state in population B, since the best-response dynamics for the population A player is a affected by the value of j. It will be symmetric for a agent from the B population. If a agent from population A is drawn the transitions probabilities are:

$$P_{(i,j)(i-1,j)} = \frac{1}{2}(\frac{i}{m})(1 - \frac{\epsilon}{2}), P_{(i,j)(i+1,j)} = \frac{1}{2}(1 - \frac{i}{m})(\frac{\epsilon}{2}), \qquad j < j^*$$

$$P_{(i,j)(i-1,j)} = \frac{1}{2}(\frac{i}{m})(\frac{\epsilon}{2}), P_{(i,j)(i+1,j)} = \frac{1}{2}(1 - \frac{i}{m})(1 - \frac{\epsilon}{2}), \qquad j > j^*$$

$$P_{(i,j)(i-1,j)} = \frac{1}{4}(\frac{i}{m}), P_{(i,j)(i+1,j)} = \frac{1}{4}(1 - \frac{i}{m}), \qquad j = j^*$$

, and symmetrically if a agent from population B are drawn:

$$P_{(i,j)(i,j-1)} = \frac{1}{2}(\frac{j}{n})(1 - \frac{\epsilon}{2}), P_{(i,j)(i,j+1)} = \frac{1}{2}(1 - \frac{j}{n})(\frac{\epsilon}{2}), \qquad i < i^*$$

$$P_{(i,j)(i,j-1)} = \frac{1}{2}(\frac{j}{n})(\frac{\epsilon}{2}), P_{(i,j)(i,j+1)} = \frac{1}{2}(1 - \frac{j}{n})(1 - \frac{\epsilon}{2}), \qquad i > i^*$$

$$P_{(i,j)(i,j-1)} = \frac{1}{4}(\frac{j}{n}), P_{(i,j)(i,j+1)} = \frac{1}{4}(1 - \frac{j}{n}), \qquad i = i^*$$

The probability of the transition to the same state is of course $P_{(i,j)(i,j)} = 1 - P_{(i,j)(i-1,j)} - P_{(i,j)(i+1,j)} - P_{(i,j)(i,j-1)} - P_{(i,j)(i,j+1)}$.

Next, assume that the states $(0,0)$ and (m,n) are both recurrent and absorbing. Then any stationary distribution is a convex combination of π_0 which equals 1 for $(0,0)$ and 0 for all other states, and π_1 which equals 1 for (m,n) and zero for all other states. Then if π^ϵ is the (unique) stationary distribution of $\mathbf{P}^\epsilon$, and π is a cluster point of π^ϵ when $\epsilon \to 0$, then π is a stationary distribution of the unperturbed process $\mathbf{P^0}$ and therefore equals a convex combination of π_0 and π_1. Let π^{ϵ_k} be a convergent sequence where $\epsilon_k \to 0$ as $k \to \infty$ such that,

$$\lim_{k\to\infty} \pi^{\epsilon_k} = \pi \tag{60}$$

For any $\gamma > 0$, $\pi^{\epsilon_k}(h,k') < \gamma$ for all $(h,k') \notin \{(0,0),(m,n)\}$. Moreover, $\pi^{\epsilon_k}_{(0,0)} = \lim_{q\to\infty} P^q_{(h,k')(0,0)}$ for all (h,k'), in particular for $(h,k') = (m,n)$ so $\pi^{\epsilon_k}_{(0,0)} = \lim_{q\to\infty} P^q_{(m,n)(0,0)}$ and symmetrically $\pi^{\epsilon_k}_{(m,n)} = \lim_{q\to\infty} P^q_{(0,0)(m,n)}$

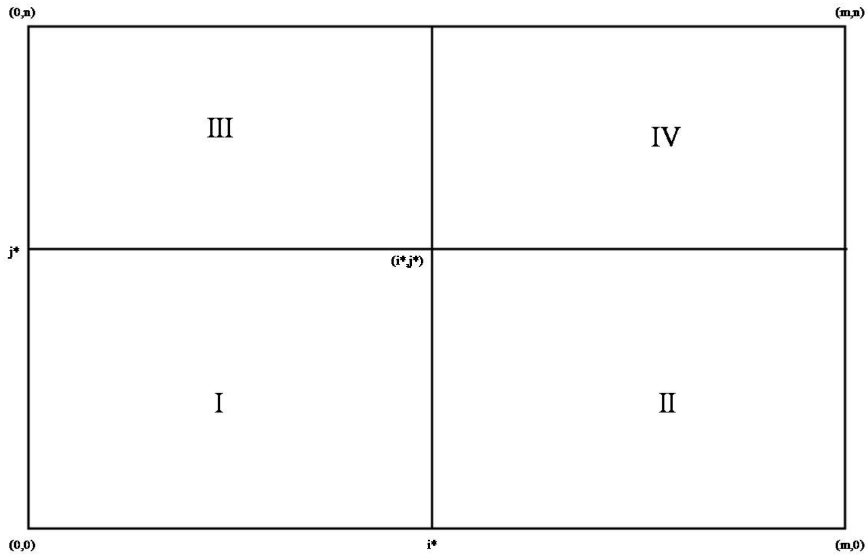

Figure 6: Illustration of the process

We know that j^* is the critical value for population A. So if $j > j^* \Rightarrow$ play strategy a and if $j < j^* \Rightarrow$play strategy b. Similarly i^* is the critical value for population B and if $i > i^* \Rightarrow$play strategy a or if $i < i^* \Rightarrow$play strategy b. When considering the movement in the graph above these values become important, and we can divide the behavior of the Markov process into four categories. The movement in the process can be shown using a dynamic "best-response-diagram".

I: $j < j^*$ and $i < i^* \Rightarrow$ strategy b is a best-response for both players since $(1/2)P_{(i,j)(i+1,j)} < (1/2)P_{(i,j)(i-1,j)}$ for the A population and $(1/2)P_{(i,j)(i,j+1)} < (1/2)P_{(i,j)(i,j-1)}$ for the B population i.e movement to the right is more likely than movement to the left for both populations so the movement will be downward and to the left.

II: $j < j^*$ and $i > i^* \Rightarrow$ strategy b is a best-response for players in population A and strategy a is a best-response for players in population B since $(1/2)P_{(i,j)(i+1,j)} < (1/2)P_{(i,j)(i-1,j)}$ for the A population and $(1/2)P_{(i,j)(i,j+1)} > (1/2)P_{(i,j)(i,j-1)}$ for the B population i.e the movement of the process is upward and to the left.

III: $j > j^*$ and $i < i^* \Rightarrow$ strategy a is a best-response for players in population A and strategy b is a best-response for players in population B since $(1/2)P_{(i,j)(i+1,j)} > (1/2)P_{(i,j)(i-1,j)}$ for the A population and $(1/2)P_{(i,j)(i,j+1)} < (1/2)P_{(i,j)(i,j-1)}$ for the B population i.e the movement of the process is downward and to the right.

IV: $j > j^*$ and $i > i^* \Rightarrow$ strategy a is a best-response for both players since $(1/2)P_{(i,j)(i+1,j)} > (1/2)P_{(i,j)(i-1,j)}$ for the A population and $(1/2)P_{(i,j)(i,j+1)} > (1/2)P_{(i,j)(i,j-1)}$ for the B population i.e movement to the right is more likely than movement to the left for both populations so the movement will be upward and to the right.

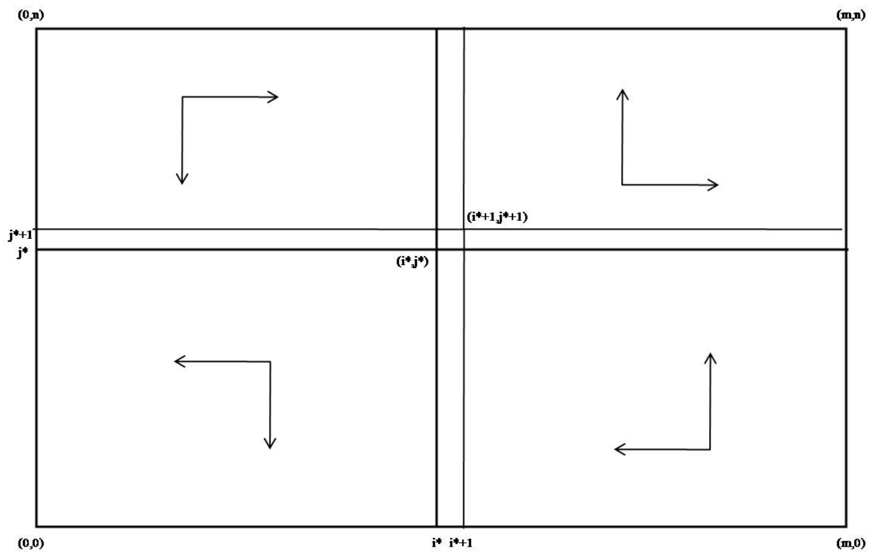

Figure 7: Law-of-motion in the process

We will show the order of magnitudes,with respect to ϵ, of the various probabilities. To calculate order of magnitudes, we can replace probabilities connected with "nontrembling" by one i.e the cases where we have $1 - \epsilon/2 \approx 1$. We use $\sim$ when order of magnitude is described. From the figure we see that $P^{i^*+j^*}_{(i^*,j^*)(0,0)} \sim 1$. This means that moving from the point (i^*, j^*) to $(0, 0)$ in the process happens without meeting resistance since the movement of the process goes in this direction,when assuming that this is a directed path from (i^*, j^*) to $(0, 0)$. The reason why the probability has i^*+j^* as power is that it will take i^*+j^* steps to reach from $(0,0)$ from (i^*,j^*) and vice versa. When we go from $(0,0)$ to (i^*,j^*) there will be resistance since we are going uphill against the movement of the process. The probability of this is $P^{i^*+j^*}_{(0,0)(i^*,j^*)} \sim \epsilon^{\min\{i^*+1,j^*+1\}}$. If we consider the other recurrent state ,(m,n), the probability of moving from (i^*,j^*) to (m,n) is $P^{m-i^*+n-j^*}_{(i^*,j^*)(m,n)} \sim 1$ and the probability of moving from (m,n) to (i^*,j^*) is equal to $P^{m-i^*+n-j^*}_{(m,n)(i^*,j^*)} \sim \epsilon^{\min\{m-i^*,n-j^*\}}$. Therefore the probabilities of moving from one recurrent state to the other is:

$$P^{m+n}_{(m,n)(0,0)} \sim \epsilon^{\min\{m-i^*,n-j^*\}} \tag{61}$$

$$P^{m+n}_{(0,0)(m,n)} \sim \epsilon^{\min\{i^*+1,j^*+1\}} \tag{62}$$

These probabilities are proportional and by introducing a proportionality variable we can find the probabilities more exact. The next to last step before ending in the recurrent state (m,n) or $(0,0)$ can originate from three different positions namely $(m-1,n),(m,n-1)$ and (m,n) or $(1,0),(0,1)$ and $(0,0)$ respectively. We can therefore make the proportionality variable a convex combination of these three possibilities. Let $\alpha_{(0,0)}$ be the variable corresponding to the path from (m,n) to $(0,0)$ and let $\beta_{(m,n)}$ correspond to the path $(0,0)$ to (m,n). Then the probabilities becomes:

$$P^{m+n}_{(m,n)(0,0)} = \alpha_{(0,0)}\epsilon^{\min\{m-i^*,n-j^*\}} \tag{63}$$

$$P^{m+n}_{(0,0)(m,n)} = \beta_{(m,n)}\epsilon^{\min\{i^*+1,j^*+1\}} \tag{64}$$

Here $\alpha_{(0,0)} = kco\left\{\{\alpha_{(0,0)},\alpha_{(1,0)},\alpha_{(0,1)}\}\right\}$ and $\beta_{(m,n)}=$ $kco\left\{\{\beta_{(m,n)},\beta_{(m-1,n)},\beta_{(m,n-1)}\}\right\}$. k is a factor which is the same for both $\alpha_{(0,0)}$ and $\beta_{(m,n)}$ and stems from the number of addends in P^{m+n}. $k \in [1,3)$.

Here it is assumed that when the state is exactly in i^* the downward movement in the third quadrant is in effect and likevise when the state is exactly in j^* the dominant effect is the left movement in the second quadrant. This is the case when we me move from (m,n) to $(0,0)$. When we have the opposite path, namely from $(0,0)$ to (m,n), we say that the upward movement in the second quadrant kicks in at i^*+1 and that the right movement in the third quadrant is in effect at j^*+1.

The explaination of why the probabilities is propotional to $\epsilon^{\min\{i^*+1,j^*+1\}}$ or $\epsilon^{\min\{m-i^*,n-j^*\}}$ is that this must be the paths of least resistance. Within this framework resistance means how difficult it is to move against the motion of the process i.e how many ϵ's one encounter when going from one state to another. The intuition is that when starting at $(0,0)$ the movement up and to the right meets more resistance when we move along an arbitrary path to (i^*+1, j^*+1) than if we follow the most efficient of the two possible paths. There are two paths that potentially can have the least resistance moving from $(0,0)$ to (m,n). The first of these paths goes from $(0,0)$ to $(i^*+1,0)$ which causes i^*+1 steps that meets resistance and when we get to $(i^*+1,0)$ the movement of the system is upward and this imply no resistance to reach (i^*+1, j^*+1). From this point the movment is upward and to the right so we can go from (i^*+1, j^*+1) to (m,n) meeting no more resistance. The other path first go from $(0,0)$ to $(0, j^*+1)$ meeting j^*+1 steps with resistence and then the right movment in the third quadrant gives no resistance to reach (i^*+1, j^*+1) and also here there are no more resistance from (i^*+1, j^*+1) to (m,n) as above. Taking the minimum of these two paths gives the path from $(0,0)$ to (m,n) that meets the least resistance and is therefore the most efficient path. When we start at (m,n) the two paths that are candidates for being the one with the least resistance. The first is the path (m,n) to $(m-i^*, n)$,with $m-i^*$ steps meeting resistance, and then move downwards to $(m-i^*, n-j^*)$ without resistance in the third quadrant. When in the point $(m-i^*, n-j^*)$ the process meets no more resistance on its way down to $(0,0)$. The second path is (m,n) to $(m, n-j^*)$, with $n-j^*$ steps meeting resistance, and then move without resistance down to $(0,0)$ from $(m, n-j^*)$. We use the minimum function in both cases to decide which of the two paths are the one with the least resistance.

By induction, assume that for all *m,n* and any given $q \geq m+n$ the probabilities are proportional to,

$$P^{q'}_{(m,n)(0,0)} \sim \epsilon^{\min\{m-i^*, n-j^*\}} \tag{65}$$

$$P^{q'}_{(0,0)(m,n)} \sim \epsilon^{\min\{i^*+1, j^*+1\}} \tag{66}$$

and $q \geq q' \geq m+n$.
Or,

$$P^{q'}_{(m,n)(0,0)} = \alpha_{(q',0,0)} \epsilon^{\min\{m-i^*, n-j^*\}} \tag{67}$$

$$P^{q'}_{(0,0)(m,n)} = \beta_{(q',m,n)} \epsilon^{\min\{i^*+1, j^*+1\}} \tag{68}$$

Like above $\alpha_{(q',0,0)} = kco\left\{\{\alpha_{(q',0,0)}, \alpha_{(q',1,0)}, \alpha_{(q',0,1)}\}\right\}$ and $\beta_{(q',m,n)} =$ $kco\left\{\{\beta_{(q',m,n)}, \beta_{(q',m-1,n)}, \beta_{(q',m,n-1)}\}\right\}$. k is a factor which is the same for both $\alpha_{(q',0,0)}$ and $\beta_{(q',m,n)}$ and stems from the number of addends in $P^{q'}$. $k \in [1,3)$.

In the next induction argument these variables are omitted for convinence since they wont affect the result.

Let us show this formula for $q'+1$. For any path from $(0,0)$ to (m,n) or (m,n) to $(0,0)$, if at step q' the state is in (m,n) or $(0,0)$ respectively there is nothing to prove since $P^{q'+1}_{(0,0)(m,n)} \sim \epsilon^{\min\{i^*+1,j^*+1\}}$ and $P^{q'+1}_{(m,n)(0,0)} \sim \epsilon^{\min\{m-i^*,n-j^*\}}$. If the next to last state is either $(m-1,n)$ or $(m,n-1)$ the products up to step q' are, respectively, $P^{q'}_{(0,0)(m-1,n)} \sim P^{q'+1}_{(0,0)(m,n)} \sim \epsilon^{\min\{i^*+1,j^*+1\}}$ and $P^{q'}_{(0,0)(m,n-1)} \sim P^{q'+1}_{(0,0)(m,n)} \sim \epsilon^{\min\{i^*+1,j^*+1\}}$, since there is no resistance for this movement(the last step has a probability ~ 1). But for the reversed path there will be alterations, here the probabilities will be,$P^{q'}_{(m-1,n)(0,0)} \sim \epsilon^{\min\{m-1-i^*,n-j^*\}} \Rightarrow P^{q'+1}_{(m,n)(0,0)} \sim \epsilon\epsilon^{\min\{m-1-i^*,n-j^*\}}$ and
$P^{q'}_{(m,n-1)(0,0)} \sim \epsilon^{\min\{m-i^*,n-1-j^*\}} \Rightarrow P^{q'+1}_{(m,n)(0,0)} \sim \epsilon\epsilon^{\min\{m-i^*,n-1-j^*\}}$. This is because the last step imply one more unit of resistance i.e one more ϵ. This holds if we require that the path should go from (m,n) to $(0,0)$ in $q'+1$ steps. It will be totally symmetric when at step q' we are in state $(1,0)$ or $(0,1)$ and require that we go from state $(0,0)$ to (m,n) in $q'+1$ steps.

To decide which of these paths has the least resistance we have to compare the transition probabilities. We now consider the case when we at step q' are in state $(m-1,n)$ or $(m,n-1)$. If, $P^{q'}_{(m-1,n)(0,0)} \sim \epsilon^{\min\{m-1-i^*,n-j^*\}} < \epsilon^{\min\{m-i^*,n-1-j^*\}} \sim P^{q'}_{(m,n-1)(0,0)}$ then, $\min\{m-1-i^*,n-j^*\} <$
$\min\{m-i^*,n-1-j^*\}$. Here $m-1-i^*$ is clearly less than $m-i^*$, and $n-j^*$ is bigger than $n-1-j^*$ so we have to compare $m-1-i^*$ and $n-1-j^*$. This gives, $m-1-i^* < n-1-j^* \Rightarrow m-i^* < n-j^*$. So if $m-i^* < n-j^*$ the best path is to the left from (m,n) to $(0,0)$.

Lets check if this holds for $q'+1$, $P^{q'+1}_{(m,n)(0,0)} \sim \epsilon\epsilon^{\min\{m-1-i^*,n-j^*\}} = \epsilon\epsilon^{m-1-i^*} = \epsilon^{m-i^*} = \epsilon^{\min\{m-i^*,n-j^*\}}$. When the opposite inequality holds we get, $P^{q'+1}_{(m,n)(0,0)} \sim \epsilon\epsilon^{\min\{m-i^*,n-1-j^*\}} = \epsilon\epsilon^{\min\{m-i^*,n-1-j^*\}} = \epsilon\epsilon^{n-1-j^*} = \epsilon^{n-j^*} = \epsilon^{\min\{m-i^*,n-j^*\}}$. Again we have that because of symmetry this holds for the paths from $(1,0)$ and $(0,1)$ to (m,n).

The induction argument above shows that the transition probabilities are only affected by the value of q in the variables $\alpha_{(q,0,0)}$ and $\beta_{(q,m,n)}$. If we assume that these are bounded i.e $0 < \lim_{q\to\infty} \alpha_{(q,0,0)} < \infty, \lim_{q\to\infty} \alpha_{(q,0,0)} = \overline{\alpha}$ and $0 < \lim_{q\to\infty} \beta_{(q,m,n)} < \infty, \lim_{q\to\infty} \beta_{(q,m,n)} = \overline{\beta}$, we can say the following,

$$\pi^{\epsilon_k}_{(0,0)} = \lim_{q\to\infty} \alpha_{(q,0,0)} P^q_{(m,n)(0,0)} = \overline{\alpha}\epsilon^{\min\{m-i^*,n-j^*\}} \tag{69}$$

and,

$$\pi^{\epsilon_k}_{(m,n)} = \lim_{q\to\infty} \beta_{(q,m,n)} P^q_{(0,0)(m,n)} = \overline{\beta}\epsilon^{\min\{i^*+1,j^*+1\}} \tag{70}$$

So $\lim_{\epsilon\to 0} \overline{\alpha}\pi^{\epsilon_k}_{(0,0)} \succ \lim_{\epsilon\to 0} \overline{\beta}\pi^{\epsilon_k}_{(m,n)}$ if, $\epsilon^{\min\{m-i^*,n-j^*\}} < \epsilon^{\min\{i^*+1,j^*+1\}}$ which imply the following conditions for $(0,0)$ to be the equilibrium, $m-i^* < i^*+1 \Leftrightarrow$

$m < 2i^*+1$, $n-j^* < j^*+1 \Leftrightarrow n < 2j^*+1$, $m-i^* < j^*+1 \Leftrightarrow m < i^*+j^*+1$ and $n-j^* < i^*+1 \Leftrightarrow n < i^*+j^*+1$. If we have $\lim_{\epsilon\to 0}\overline{\alpha}\pi^{\epsilon_k}_{(0,0)} \prec \lim_{\epsilon\to 0}\overline{\beta}\pi^{\epsilon k}_{(m,n)}$ the conditions is symmetric. So the equilibrium is given from the size of the populations and the distribution of payoffs in the game.

Example 6 *Consider the Hawk-Dove game as in section 4.5, with* $w = 4$ *and* $l = 6$*. The payoffs in the game then becomes,*

$-1,-1$	$4,0$
$0,4$	$2,2$

We now want to check which predictions the new concept gives in this game. Calculating the critical points in each population is done as above and yields, $j^* = \frac{2}{3}n$ *and symmetrically* $i^* = \frac{2}{3}m$*. Now assume that the A and B populations are of different sizes, say* $n = 9$ *and* $m = 6$*, which imply* $j^* = 6$ *and* $i^* = 4$*. The probability of moving from one recurrent state to the other is:* $P^{15}_{(0,0)(6,9)} \sim \epsilon^{\min\{m-i^*,n-j^*\}} = \epsilon^{\min\{2,3\}} = \epsilon^2$ *and* $P^{15}_{(6,9)(0,0)} \sim \epsilon^{\min\{i^*+1,j^*+1\}} = \epsilon^{\min\{5,7\}} = \epsilon^5$*. Using this result we can find the limiting distributions:* $\pi^{\epsilon_k}_{(0,0)} = \lim_{q\to\infty}\alpha_{(q,0,0)}P^q_{(6,9)(0,0)} = \overline{\alpha}\epsilon^{\min\{m-i^*,n-j^*\}} = \overline{\alpha}\epsilon^2$ *and* $\pi^{\epsilon_k}_{(6,9)} = \lim_{q\to\infty}\beta_{(q,6,9))}P^q_{(0,0)(6,9)} = \overline{\beta}\epsilon^{\min\{i^*+1,j^*+1\}} = \overline{\beta}\epsilon^5$*. To find the stochastically stable state we have to compare how the two limiting distributions evolves when* $\epsilon \to 0$*, which is:* $\lim_{\epsilon\to 0}\pi^{\epsilon_k}_{(0,0)} = \lim_{\epsilon\to 0}\overline{\alpha}\epsilon^2$ *and* $\lim_{\epsilon\to 0}\pi^{\epsilon_k}_{(6,9)} = \lim_{\epsilon\to 0}\overline{\beta}\epsilon^5$*. Since the long-run probability of state* $(0,0)$ *involves* ϵ *to the power of 2 while the probability for state* $(6,9)$ *involves* ϵ *to the power of 5 it should be obvious that, as* $\epsilon \to 0$*, the state* $(0,0)$ *is much more likely than state* $(6,9)$*. Therefore state* $(0,0)$ *is the stochastically stable state i.e all the agents in population A and population B plays the second row and the second column respectively. This equilibrium strategy is flight. This corresponds to the predicted stochastically stable state in section 4.5, but now the only difference is that we have two populations playing against each other in contrast to the single population earlier.*

6 Discussion of the assumptions underlying the theory

An economy is a extremly complex system and to be able to investigate the different aspects it is crucial to make simplifying assumption. In this section we have a closer look at the assumptions of the models and methods used in this paper.

First, when we want to find the stochastically stable state (set) we let the number of time periods tend to infinity. Therefore it takes a very long time before the system reaches the equilibrium and we can not say anything about the strategic interaction in the short and medium-run. So stochastic stability is a concept only to be used when the time period we are interested in are long. Many will be tempted to say that this makes the concept unusable because of the known fact that "in the long-run, we're all dead", but if the time periods are measured in short intervals i.e the strategic interactions happens very frequently, we can probably reach the stochastically stable state in a shorter time horizon.

Second, the trembling in the models is assumed to be stationary since ϵ is a given small number. This means that all agents in the population(s) are equaly likely to make an error, so the population is homogenous in this respect. In a real world population this is clearly not the case, but if the probability of making an error was to be different for each agent(it could have been normally distributed) it would make the model more complex and therefore is omitted here.

Third, in the above models we have not said anything about the exact value of ϵ, only that it is a small number. It is a resonable assumption that the probability of making errors are small, but if ϵ increases for some reason this may alter the behavior of the agents (See Young,1998 [4] or Young,2009 [5]).

Fourth, here it has been the case that the players have made their decitions on the basis of the number of agents in the population playing the different strategies. This can be a reasonable assumption if it is possible to do so. In the litterature there exist other factors that the agents base their decitions on, for example letting the agents have memory. In this kind of models the agent draw a memory combinaton,of what happened in the m last periods, and base their decition on this(See Young,1998 [4]).

7 Conclusion

The aim of this paper have been to present and refine the theory of *stochastic stability* in normal-form games. The existing theory is here stated without proofs, but references to the litterature is given. When the theorems have been presented there are examples to make the theory more applicable to economic problems. The new extension give an alternative way to find the stochastically stable state(s) in games with two populations. The results from this theory gives reasonable predictions when compared with the one population case.

References

[1] Foster D. and H.P. Young. Stochastic evolutionary game dynamics. *Theoretical Population Biology*, 38:219–32, 1990.

[2] Taylor H.M. and S. Karlin. *An Introduction to Stochastic Modeling*. Academic Press, third edition, 1998.

[3] Young H.P. The evolution of conventions. *Econometrica*, 61:57–84, 1993.

[4] Young H.P. *Individual Strategy and Social Structure*. Princeton University Press, second edition, 1998.

[5] Young H.P. Stochastic adaptive dynamics. *New Palgrave Dictionary of Economics*, second edition, 2009.

[6] Harsanyi J. and R. Selten. *A General Theory of Equlibrium Selection in Games*. MIT Press, 1988.

[7] Maynard Smith J. The theory of games and the evolution of animal conflict. *Journal of Theoretical Biology*, 47:209–21, 1950.

[8] Maynard Smith J. and G. R. Price. The logic of animal conflict. *Nature*, 246:15–18, 1973.

[9] Nash John. Equlibrium points in n-person games. *Proceedings of the National Academy of Sciences*, (36):48–49, 1950.

[10] Weibull J.W. *Evolutionary Game Theory*. MIT Press, second edition, 1996.

[11] Kandori M. Mailath, G. and R. Rob. Learning, mutation and long-run equlibria in games. *Econometrica*, 61:29–56, 1993.

[12] Bowles S. *Microeconomics: Behavior, instutuitions and evolution*. Princeton University Press, first edition, 2004.

[13] Wiitala S.A. *Discrete mathematics: A Unified Approach*. McGraw-Hill Book Co., first edition, 1987.