Effects of taxes on the after-tax cost of capital: 
*A simulation approach for multi-period models*

Zongwei Lu

Master of Philosophy in Economics
Department of Economics

UNIVERSITY OF OSLO

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Preface

I can not express more gratitude to my supervisor, Diderik Lund. That is the only thing I have in my mind after the completion of this thesis. The topic seems to be a “Mission impossible” for me at the beginning due to my ignorance and slow progression. But with his extraordinarily patient and insightful instructions I gradually see the clear sky behind the cloud. Along the rough road, I also feel warm from his kind feedbacks every time, which is very important for the writing process. While his invaluable inputs completely exceed my expectation and are critical for the completion of the thesis, needless to say, all remaining mistakes are entirely my own responsibility.

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1 Introduction

1.1 Background

Effect of various elements of tax system on project valuation and capital budgeting is a less explored area and receives less attention in practice. Project valuation concerning risk usually relies on two main methods; the Adjusted Present Value (APV) of Myers (1974) and the Weighted Average Cost of Capital (WACC) method of Modigliani and Miller (1963) (see also Miles and Ezzell (1980)). Since Hamada (1972) a method has been established for deriving required expected rates of return for a company to be applied to potential projects based on the observable beta of shares of the company. The commonly used WACC formula based on Capital Asset Pricing Model (CAPM) only adjusts for interest payment deduction. However, other elements such as depreciation schedule and loss offset should also be considered. Unfortunately, these are ignored even in certain textbooks.\(^1\) This thesis uses the APV for valuation and derives the adjusted equity betas which can be used to find the correct betas for the WACC.

With the APV one does not need to bother with the WACC. But why should one be interested in the WACC, and thus the beta of equity, even when the APV is available and more reasonable, which looks at different elements of the cash flow instead of the net cash flow only? According to Lund (2011), one of the reasons is that the correctly “unlevered and untaxed” beta should be used to improve the method of Hamada (1972) under different tax systems and different production technologies.\(^2\) Second, the WACC is still widely used by firms and it is interesting to quantify the magnitude of mistakes being made for the purpose of future practices.

More importantly, the effect from riskiness of being in tax position was not observed. Lund (2002) provides a unified framework and explicitly points out that the patterns of risk-sharing between the firms and the tax authorities are different under different tax systems.\(^3\) This is

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\(^1\) E.g., in Brealey et al (2008, p.561), it states: Depreciation tax shields contribute to project cash flow, but they are not valued separately; they are just folded into project cash flows along with dozens, or hundreds, of other specific inflows and outflows. The project's opportunity cost of capital reflects the average risk of the resulting aggregate.

\(^2\) Practically, e.g., in oil extraction sector as considered by Jacoby and Laughton (1992), different size of field could represent different technologies.

\(^3\) Derrig (1994) and Galai (1988) also observe the point but they do not solve for marginal project.
illustrated clearly in Lund (2011) by taking a pure cash flow tax (Brown, 1948) as a starting point. With negative net cash flow, negative taxes are paid, which makes the tax authority similar to a shareholder. Hence it does not affect the beta of the after-corporate-tax cash flow. If negative taxes are not paid out and the negative tax base is carried forward to a future period with positive net cash flow, the negative taxes act as a loan to tax authority from the firm. This “loan”, being the opposite of the traditional leverage, decreases the systematic risk of the net cash flow and hence the effect should be taken into account for investment decisions. In particular, the beta of the marginal project gives the correct cost of capital which is defined to be the minimum required rate of return of the asset and will be different under different tax systems. Therefore the analysis of riskiness of being in tax position has particularly important implications for capital budgeting of a multinational (facing many different tax systems) or the resource extraction industry (facing high uncertainty and high tax rates). There are also policy implications for a government which is considering a tax reform because, as Lund (2002) finds out, the distortion from the tax system can be substantial in reducing the investment size and output of the firms.

Lund (2011) further recognizes that there is a difference between marginal and average betas and discusses the relationship of them. In Lund (2011), the average beta is defined to be the beta associated with the project at a size maximizing the market value by choosing investment optimally, which is found by the first order condition (FOC) of the maximization problem. The marginal project is defined to be the project with a size such that the market value of the project is equal to the after-tax financing need and the project earns a zero expected “profit”. There are two versions of marginal beta, one for stand-alone marginal case and one for infra-marginal case. The latter case is referred to the last invested monetary unit in the average project, assuming a decreasing return to scale (DRS) technology. Since the infra-marginal project is taxed together with the average project as a whole, the riskiness of the tax position of this marginal unit of investment depends on the infra-marginal project revenue. The difference between the betas is due to the fact that they are associated with different

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4 At least, it should be taken into account when a firm bases its decisions on its net expected cash flows, applying the WACC as the discount rate.

5 Lund (2002) only sets up models for marginal project and solve for marginal beta.

6 In reality, firm may behave differently. The average beta in observed data would depend on the actual behavior of firms.
investment-size decisions due to the existence of the tax system. In other words, if there are no taxes, the betas are the same.

Lund (2002) derives analytical results of marginal beta for one-period models on both certain and uncertain tax position, and results for a multi-period model but only for certain tax position case. Lund (2011) illustrates the ideas of average and marginal betas by incorporating depreciation schedule in one-period models. It is more realistic and interesting to extend the analysis to a multi-period model under uncertainty, but then analytical results cannot be obtained for certain tax systems (e.g., carryforwards).\textsuperscript{7} However, by using Monte Carlo simulation, the decision making problems can be solved approximately and the betas can be derived. Such Monte Carlo simulations are the main topic of this thesis.

1.2 Review of project valuation

The crucial part of the multi-period models is the appropriate method of project valuation. The tax system for resource extraction industry usually varies greatly across time and countries. According to Lund (1991), e.g., the change of tax rate for petroleum sector is substantial in 1987 tax reform of Norway. Therefore, the consideration of tax effect is rather interesting in such a sector and the thesis exemplifies the valuation method with the aid of simulation technique for the resource extraction industry.

Capital Asset Pricing Model (CAPM) is widely applied in financial market because it is an equilibrium pricing model built on microeconomic foundation and because its application is simple but elegant and intuitive. It asserts the expected rate of return of an asset can be estimated through the historical data of a broad-based market portfolio. Although the standard theory is only for a static one-period model, a simple-minded extension to multiple periods could be argued under some reasonable assumptions (see Assumption 5), at least for practical purpose.\textsuperscript{8}

For a long time the discounted cash-flow (DCF) has been the dominating method in practice of project valuation, but it has been criticized for the bias induced for long-lived

\textsuperscript{7} Under the unrealistic assumption of no-loss-offset tax system for each period, the analytical solution can be obtained and is shown in section 4.

\textsuperscript{8} Merton (1973) develops an intertemporal model in continuous time, and allows for more complex intertemporal variations than the simple-minded, discrete-time model that is used here. And it is more useful theoretically.
projects and for inability of dealing with the flexibility aspect of project choices. In particular, the valuation of the risk with a non-linear form in current context deserves a more reasonable treatment. In current context, the tax claim possessed by the tax authority is protected from downwards risk under imperfect loss offset and obviously is more valuable for the tax authority for riskier future cash flows. The option pricing theory developed by Black and Scholes (1973) originally for derivative assets can be applied to valuation of such tax claims with option like cash flows. Lund (2009) gives an overview of other works in this area. Particularly, Lund (1991) applies the valuation model using numerical method to analyze the incentive effect of Norwegian petroleum taxation system. Jacoby and Laughton (1992), Salahor (1998), Bradley (1998) do the similar works for the purpose of project evaluation.

Using a mixture of the CAPM method and the option pricing method, the thesis assumes a fully equity-financed firm and seeks to obtain results in multi-period models in order to exemplify the applications of valuation methods and show the degree of the effects of taxation on the risk characteristic of the after-tax cash flows, i.e., the betas, and indirectly the cost of capital. The thesis is organized as follows: Section 2 presents the theoretical preparation and the assumptions of the simulation. Section 3 gives the results in a one-period model to test consistency with the analytical results in order to control for the validity of the simulation. Section 4, with a constant production profile, extends to multi-period models where different tax rules (no-loss-offset and carryforwards) are treated separately and different depreciation schedules (constant and regressive) are also considered. Interesting sensitivity tests are also conducted and comparisons are made. Section 5 discusses the weakness of the simulation. Section 6 summarizes. The simulation codes are attached in appendices.

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10 Such a mixture is standard in parts of the options literature, see, e.g., McDonald and Siegel (1984), and in particular in the real options literature, see, e.g., Jacoby and Laughton (1992).

11 Risk-free debt financing can be easily incorporated.
2 Assumptions of simulation and theoretical preparation

2.1 The tax system and the production technology

Assumption 1: The project is fully equity-financed and requires only an initial investment \( I \) in period 0. The production technology is decreasing-return-to-scale (DRS), represented by a production function \( Q = f(I) = wI^\alpha \) where \( w \) is a positive constant and \( \alpha \) is a constant between zero and one. The project operates in \( T \) periods and the production in period \( t \), is denoted by \( Q_t \). The time profile of production \( \left( \frac{Q_1}{Q}, ..., \frac{Q_T}{Q} \right) \) is constant, i.e., \( \frac{Q_t}{Q} = \frac{Q_s}{Q} = \frac{1}{r} \). In addition, there is no production flexibility after the project has been initiated.

The investment is limited to be in period 0. This may look too simple in a multi-period model but it is not less reasonable as one might think because the multi-period investments can be reduced to one-period investment since we assume that the investment is a deterministic quantity. There is no operating cost assumed here for the same reason.

A multi-factor model is more realistic but it is not of interest of current context. The crucial ingredient is the constant elasticity and decreasing-return-to-scale which gives a solution to the optimization problem.

For simplicity the ratios \( \frac{Q_t}{Q} \) are assumed to be fixed and equal for all \( t \), independent of the scale of the project. The use of percentage will allow us to adjust the elasticity coefficient \( \alpha \) easily without losing the comparability.

Assumption 2: The product price \( P_t \) follows a Geometric Brownian Motion (GBM) with a drift \( \mu \) and a volatility coefficient \( \sigma \) and the period-0 price is denoted \( P_0 \), i.e.,

\[
(1) \quad P_t = P_0 e^{\left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma B(t)}
\]

or in stochastic differential equation form

\[
(2) \quad dP_t = \mu P_t dt + \sigma P_t dB(t)
\]
where $B(t)$ is a standard Brownian motion starting from $B(0) = 0$, i.e., $B(t)$ follows a standard normal distribution with mean 0 and variance equal to $t$ and $dB(t+s)$ is independently distributed with $dB(t)$.

The GBM assumption implies: $E(P_t) = P_0e^{\mu t}$, and $P_t$ has a lognormal distribution, i.e., $\ln(P_t) \sim N[\ln(P_0) + (\mu - \frac{1}{2}\sigma^2)t, \sigma^2t]$, where $N[m, v]$ is a normal distribution with mean $m$ and variance $v$. To simulate a sample of the GBM, we can first generate a sample of a standard Brownian motion $B(t)$ and then apply equation (1).

The GBM assumption is standard in Black-Scholes theory and originally is used to describe the behaviour of stock prices, but it may still be a good approximation to the price of natural resources, e.g., oil price, provided there is a market with many traders and the resources are approximately continuously traded. In particular, we need to distinguish the difference between an investment asset and a consumption asset, i.e., $P_t$ may be equal to the equilibrium price at time $t$ or not for a claim to a unit of oil available at $t + 1$ in the CAPM environment. This will be discussed in section 2.3.

While it is assumed that financial and product markets are open continuously, productions and taxes "happen" only once per period.

**Assumption 3**: Under a no-loss-offset tax system, the firm pays a tax at a rate $\tau$ on the net (after depreciation deduction) cash flow for each period if it is positive, i.e., $\tau \times \max(P_tQ_t - \delta_tL, 0)$ where $\delta_tL$ is the part of total depreciation $L$ at time $t$. The tax rate is constant over time. The tax system does not affect the equilibrium of the capital market.

**Assumption 4**: Under a carryforwards tax system, the firm pays a tax at a rate $\tau$ on the net (after depreciation deduction) cash flow in period $t$ ($t > 0$) if it is positive, i.e., $\tau \times \max(P_tQ_t - \delta_tL, 0)$; if $(P_tQ_t - \delta_tL) < 0$, the firm pays no tax and the negative tax base is carried forwards to next period so that the tax base next period is reduced and this rule still applies until the last period. The tax rate is constant over time. The tax system does not affect the equilibrium of the capital market.
2.2 The CAPM environment

Although the price change of a product usually does not necessarily satisfy the CAPM, the market value of a claim to one unit of product in the future can be assumed to satisfy the CAPM. Since we concern only about the latter, the assumption of CAPM environment is as follows.

Assumption 5: The one-period required expected rate of return of an asset is constant over time. At period $t$, a claim to one unit of product in period $t+1$ is evaluated as $V_t(P_{t+1})$ which satisfies the Capital Asset Pricing Model. For convenience of verbal discussion, it is assumed without loss of generality that the covariance of the rate of return of the claim and the market portfolio is positive.

This assumption requires that the market portfolio and the market condition are stationary over time so that the (market) required expected rate of return of an asset is the same over time. More specifically, the one-period risk-free interest rate and the one-period expected rate of return of the market portfolio are constant over time. And the ratio of the covariance between the one-period expected rate of return of the claim and the market portfolio to the variance of the one-period rate of return of the market portfolio is stationary over time. Thus,

$$E(r_t) = \frac{E(P_{t+1})}{V_t(P_{t+1})} - 1 = E\left(\beta_p (E[r_m] - r_f)\right) = E(r_{t+1}),$$

where $r_f$ is the risk free interest rate, $r_m$ is the rate of return of the market portfolio, and $r_p$ is the rate of return of the claim.

$$\beta_p = \frac{\text{cov}(r_p, r_m)}{\sigma_m^2},$$

where $\sigma_m^2$ is the variance of the one-period rate of return of the market portfolio. There are two useful one-period formulae below if a market portfolio is used in simulation.

\[ (3) \quad V_0(P_1) = \frac{E(P_1) - \lambda \text{cov}(P_1, r_m)}{1 + r_f}, \text{where} \quad \lambda = \frac{E(r_m) - r_f}{\sigma_m^2} \]

\[ (4) \quad \beta_p = \frac{\sigma_m^2 E(P_1)}{\sigma_m^2 \text{cov}(P_1, r_m) - (E(r_m) - r_f)^2} \]

The assumption is rather restrictive. Constantinides (1980) (1982) sets up a minimum set of assumptions to allow for non-stationarities of the rate of return of the market portfolio, so that the one-period CAPM can be extended to a multi-period model. Kazemi (1991) provides an alternative multi-period model where a macroeconomic variable represented by the market
price of a default-free bond is used to replace the role of market portfolio, but it does not provide a similar beta to the previous work of Lund (2011). However, for the purpose of the thesis the assumption is a convenient way and can be regarded as a baseline to formulate the theoretical environment. In this way, combined value consistency argument, it can be mitigated with the derivative pricing theory so that the risk-neutral valuation method can be a unified way of pricing for both linear and non-linear cash flows.

2.3 The derivative pricing and the risk-neutral valuation method

The tax claim of the government $\tau \ast \max (P_t Q_t - \delta_I, 0)$ is intrinsically the cash flow from a European call option under the assumption of GBM for $P_t$. According to Cox and Ross (1976), the Black-Scholes solution can be obtained by a risk-neutral valuation method. In the real world, the risk-averse preferences of investors push the price of the underlying asset growing at a drift $\mu$ and the risk of the option is compensated by an expected rate of return determined by $\mu$. However, in Black-Scholes formula which is based on the assumption of no-arbitrage, the market value of the option does not depend on the risk preference of investors. Therefore, under no-arbitrage assumption, the investors can be assumed to be risk-neutral in an artificial world. Then in the risk-neutral world, the price of the underlying asset grows at a rate equal to the risk free interest rate and the option also earns an expected rate of return equal to the risk free interest rate. This suggests a convenient valuation approach, replacing the drift $\mu$ by the risk free interest rate $\gamma$, calculating the corresponding cash flow at expiration date and discounting at $\gamma$.\(^{12}\)

According to stochastic calculus (Ito’s lemma), $P_t Q_t$ also follows a GBM with the same drift and volatility as $P_t$ but a different period-0 value $P_0 Q_t$.\(^{13}\) Therefore, a direct application of option pricing to the tax claim is valid.

McDonald and Siegel (1984) shows that when the underlying asset exhibits a rate-of-return shortfall the correct valuation should adjust for the rate. The rate-of-return shortfall is sometimes called convenience yield for a consumption asset. This is particularly important for risk-neutral valuation method.

\(^{12}\) The corresponding cash flow is exactly $\tau \ast \max (P_t Q_t - \delta_I, 0)$ with $P_t$ replaced by the risk-neutral price.

\(^{13}\) $d(P_t Q_t) = \left(\frac{\delta(P_t Q_t)}{\partial_t} \mu P_t + \frac{\delta(P_t Q_t)}{\partial t} + \frac{1}{2} \frac{\delta^2(P_t Q_t)}{\partial t^2} \sigma^2 P_t^2 \right) dt + \frac{\delta(P_t Q_t)}{\partial t} \sigma P_t dB(t) = \mu P_t Q_t dt + \sigma P_t Q_t dB(t)$
From the derivative (e.g., forward contract) perspective, an asset is an investment asset if
the current price is equal to the no-arbitrage (current) price implied by its forward price. If the
former is higher than the latter, then the asset is a consumption asset which gives a
convenience yield for the holder of the asset. And because of the convenience yield the holder
does not have the incentive of arbitraging so that the actual price can be kept higher than the
theoretical (no-arbitrage) price. For example, the holder saves the transportation costs of re-
buying the commodities in large quantity or may be able to profit from selling the
commodities in the local market when there is a possibility of temporal shortage within
certain periods.\textsuperscript{14} Under the CAPM, when the current price is higher than the equilibrium
valuation of a claim to a unit product, $1 - \frac{V_0(P_t)}{P_0}$ can be regarded as the convenience yield rate
provided by the asset. Therefore, when simulating the risk-neutral sample of the underlying
asset price, one should adjust for convenience yield by discounting the current asset price $P_0$
by $e^{-yt}$, where $y$ is the convenience yield.\textsuperscript{15}

The assumption for tax claim valuation is as follows.

**Assumption 6:** The cash flow of a tax claim $\tau \cdot \max(P_tQ_t - \delta_tI, 0)$ has a market value at
period 0 equal to

$$\tau \cdot (V_0(P_t)Q_t \Phi(z_1) - \delta_tI\Phi(z_2)e^{-rt})$$

where $z_1 = \frac{\ln\left(\frac{V_0(P_t)Q_t}{\delta_tI} + \left(r + \frac{\sigma^2}{2}\right)t\right)}{\sigma\sqrt{t}}$ and $z_2 = z_1 - \sigma\sqrt{t}$.

$\Phi(.)$ is the cumulative distribution function (CDF) of a standard normal variable.

### 2.4 The combination of CAPM and Black-Scholes

Up to now, two ways of valuation for different components of the cash flow for each period
are presented. Two natural questions arise: Is it valid to put them together to evaluate the cash

\textsuperscript{14} For stocks, there are no such benefits since the stock market can be assumed as efficient and there are no such storage
costs, transportation costs and profits from shortage.

\textsuperscript{15} This gives a risk-neutral (GBM) sample of the future price with a period-0 value equal to the (CAPM) market value of a
claim to one unit of the product.
flow as a whole, since one component is a CAPM value and the other is a Black-Scholes value? Is the risk characteristic (beta) of the option-like cash flow compatible with the CAPM environment? The answers are both yes as long as the current price of the underlying asset is equal to (or adjusted to) the (CAPM) market value of a claim to a unit of the asset. The crucial assumption implicitly embedded in the option pricing theory, the no-arbitrage assumption, leads to the important principle of value consistency (or value additivity) which underlies the adjusted present value (APV) method. Hence, the total value of the cash flow is just the sum of the values of each component. And according to McDonald and Siegel (1984), the use of the CAPM value of the underlying asset will enable the option earn an expected rate of return satisfying the CAPM-type equilibrium. Hence, the risk characteristic of the option can be measured by the CAPM beta. This can be seen from the pricing formula directly, which is illustrated in the appendix of Lund (2011). Moreover, these two points can be confirmed by a simulation (see appendix 2.1). Intuitively, the option is itself also an asset and the cash flow of the option is also a cash flow intrinsically same as the cash flow we usually meet even though it is non-linear.

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16 Equation (B.8) in p.39.
3 The one-period model

3.1 The simulation procedure

The one-period models are presented in Lund (2011), which gives analytical solutions and can be used as a control for multi-period extensions. More importantly, the one-period model is relatively simpler and the procedure of simulation can be better explained using it as an example. In addition, the results are also meaningful for comparisons with multi-period models.

Although the risk-neutral method is applied for project evaluation, which uses a sample of risk-adjusted prices, i.e., a sample of GBM with a drift rate equal to the risk free interest rate, the unadjusted prices are still needed to find the expected rate of return of the project to derive the desired betas.\(^\text{17}\)

In one-period model, there is no loss offset and the period-1 cash flow is \( Pf(I) - \tau \* \max(Pf(I) - \delta I, 0) \), where \( P \) is the price and \( \tau \) is the tax rate and \( \delta \) is the depreciation rate. The following shows how to simulate an average beta.

1. Simulate a sample of unadjusted prices. This can be done using equation (1) \( P_{unad} = P_0 e^{(\mu - \frac{1}{2} \sigma^2) + \sigma B(1)} \), where \( B(1) \) is the simulated pseudo-observations from a standard Brownian motion. When simulating the adjusted prices, the convenience yield should be reflected by adjusting \( P_0 \). The convenience yield can be calculated by \( y = 1 - \frac{V_0(P_S)}{P_0} \), which means a correlated sample of the rate of return of market portfolio should also be simulated.\(^\text{18}\) \( V_0(P_{unad}) \) can be calculated by equation (4). On the other hand, one may just use an experience-based number, and then no market portfolio is needed.\(^\text{19}\) For simplicity, here we use the second approach and a number 0.04 for convenience yield is used. The

\(^{17}\) The idea of the simulation here is almost the same as Jacoby and Laughton (1992) except that here a simple (more stylized) production function is used while they try to use more realistic functions for oil extraction projects. The simple production function here is for theoretical purpose and the simulation results can be compared to the analytical results in Lund (2011).

\(^{18}\) The formula of convenience yield for continuous case is \( e^{-y} = \frac{V_0(P_S)}{P_0} \) which implies \( y = -\ln \frac{V_0(P_S)}{P_0} \). If we take approximation, \( y = -\ln \frac{V_0(P_S)}{P_0} \approx -\left( \frac{V_0(P_S)}{P_0} - 1 \right) \), which gives the formula in discrete case.

\(^{19}\) Still, \( E(r_m) \) is needed for valuation and deriving betas. But if we are only interested in the ratio of the betas, there is indeed no need of \( E(r_m) \).
adjusted prices is derived by $P_{ad} = e^{-y P_0 e^{(r_f - q/2)^2 + 2qB(1)}}$. The same sample of Brownian motion should be used to avoid undesired randomness. $P_0$ can be normalized to 1. Incidentally, a sample of 10000 observations is sufficiently for the purpose.

2. Then, since we want to trace out the betas for different elasticity, set the elasticity coefficient $\alpha = 0.01$ (or e.g., 0.1 if we do not need that many numbers and just want to reduce the load of the CPU).

3. And then choose a particular $I$ (typically starting from a low value of 0.001 or 1 etc., since iterations for $I$ are going to be executed to search for the optimal investment).

4. For each adjusted price, subtract the corresponding depreciation amount $\delta I$ from $P_{ad} f(I)$, which gives a positive value or zero (if it is negative, change it to 0 by the assumption of no loss offset), i.e., $\max(P_{ad} f(I) - \delta I, 0)$.

5. Calculate the net cash flow $P_{ad} f(I) - t \max(P_{ad} f(I) - \delta I, 0)$.

6. Find the mean of the net cash flow and discount by the risk free interest rate to find the market value. Then subtract the after-tax financing need $I$. This gives expected net value.

7. Do step 3-6 again for a higher $I$, and higher $I$s again, and so on. By such iteration, a optimal $I_{Aver}$ which maximizes the expected net value will be found, denoted by $V_{Aver}$.

8. Calculate $P_{unad} f(I_{Aver}) - t \max(P_{unad} f(I_{Aver}) - \delta I_{Aver}, 0)$. Find the mean of this unadjusted-price cash flow, denoted by $E(CF1)$. Then use $-V_{Aver} + E(CF1)e^{-\rho} = 0$ to find out the expected rate of return $\rho$.

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20 The after-tax financing need is $(1 - \tau)I$ in Lund (2011), where $\tau$ is the weight of immediate deduction. Since we assume the deduction only happens in the future, $\tau = 0$ here.

21 This relies on an assumption that the net value is a concave function of $I$, with a unique maximum. Current simple assumptions are sufficient. Since Lund (2011) derives the analytical solutions, the simulation results can be and is verified.

22 This is the same method used by Jacoby and Laughton (1992). The $\rho$ here is what they call ECDR on p. 40 (and they also extend the equation to a multi-period setting).
9. The expected rate of return \( \rho \) satisfies the CAPM equilibrium by assumption. Therefore, the beta can be found by the CAPM formula \( \beta_{RA} = \frac{\rho - r_f}{E(r_m) - r_f} \), where \( r_m \) and \( r_f \) are assumed in step 1.

10. Then do step 2-9 for \( \alpha = 0.02, \ldots, 1 \) to find out \( \beta_{RA} \) for different \( \alpha \).

11. To find out the ratio of this simulated \( \beta_{RA} \) to \( \beta_P \), \( \beta_P \) is needed and calculated by equation (4) if the market portfolio is assumed and the rate of return is simulated in step 1. Since we use experience-based number for \( y \), \( \beta_P = \frac{E(r) - r_f}{E(r_m) - r_f} \), where \( E(r) = \ln \left( \frac{E(P_{unad})}{V_0(P_{unad})} \right) \) and \( V_0(P_{unad}) = E(P_{ad})e^{-r_f} \). So the ratios \( \beta_{RA}/\beta_P \) for different \( \alpha \) are found.

There are two versions of marginal beta. One is marginal for stand-alone case and is defined to be the beta associated with the marginal project with a market value equal to the after-tax financing need which in current context is equal to the investment \( I \). To simulate for this beta, one need only adjust the criterion for the optimal investment to a criterion with the expected net value equal to zero in step 7, i.e., \( V_0[P_{adf}(I) - t * \max(P_{adf}(I) - \delta I, 0)] = I \)

The infra-marginal beta is defined to be the beta associated with the infra-marginal project created by the last marginal increment of investment with a market value equal to the marginal increment of investment itself. Denote the future unadjusted-price cash flow \( P_{unadf}(I) - \tau * \max(P_{unadf}(I) - \delta I, 0) \) by \( X_{RM} \), then it means \( \Delta V_0(X_{R_M}^{unad}) = \Delta I \) at a certain investment level. However, it turns out that the particular investment level is exactly \( I_{Aver} \). Thus, to simulate for this beta, one needs only to modify the simulation from step 8. The new procedure is following.

8. Calculate \( P_{unadf}(I_{Aver}) - t \max(P_{unadf}(I_{Aver}) - \delta I_{Aver}, 0) \). Find the mean of this unadjusted-price cash flow, denoted by \( E\left(X_{RM}^{unad}(I_{Aver})\right) \). Also

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23 Intrinsically it is the same to choose the particular initial investment to maximize the net market value of the project, since it means the infra-marginal project created by the last dollar investment has a net value equal to one dollar (the last dollar). If the firm invests more, then the marginal project created by the further marginal increment of investment will have a market value lower than the marginal increment of investment itself and the net value of the whole project will decline. And this is confirmed by the simulation. In the figure below, \( I_{RA} \) is almost identical to \( I_{RM} \). We can also do the simulation by definition and the accuracy is almost the same. But the insight here has a benefit of being able to combine all the three betas in one program which reduces the simulation time greatly. Indeed, greatly. And life is much easier! Thanks to Diderik Lund for the suggestion of the insight.
calculate \( E \left( X_{RM}^{unad} (I_{Aver} - \Delta I) \right) \). And then \( E[\Delta X_{RM}^{unad}] = E[X_{RM}^{unad} (I_{Aver})] - E[X_{RM}^{unad} (I_{Aver} - \Delta I)] \). Then use \( \Delta I = E[\Delta X_{RM}^{unad}] e^{-\rho} \) to find out the expected rate of return \( \rho \) for this infra-marginal project. By construction this \( \rho \) also satisfies the CAPM.

3.2 The results

Using the same parameters as Lund (2011), the simulation produces the similar results. This is reassuring, and gives us a control that the simulation is reasonable at least in the situations when an analytical result also exists. To repeat, among others, the betas depend on the two parameters: the tax rate and the volatility of the GBM. The tax rate in the simulation is set to 0.35 and the volatility 0.3. And the present value of tax shield \( \frac{\delta}{1+r_f} \) is equal to 1/1.05.

Table 1 Parameters used in the simulation

<table>
<thead>
<tr>
<th>( P_0 )</th>
<th>( \mu )</th>
<th>( \sigma )</th>
<th>( r_f )</th>
<th>( E(r_m) )</th>
<th>( y )</th>
<th>( \tau )</th>
<th>( \delta )</th>
<th>( w^{24} )</th>
<th>( \Delta I )</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>0.1</td>
<td>0.3</td>
<td>0.05</td>
<td>0.16</td>
<td>0.04</td>
<td>0.35</td>
<td>1</td>
<td>1</td>
<td>0.0001</td>
</tr>
</tbody>
</table>

The simulation results are shown in Figure 1 below.\(^{25}\)

Figure 1

Betals, 1-period, no loss offset

Investments, 1-period, no loss offset

\(^{24}\) The coefficient \( w \) does not affect the results. But in simulation, a specific number is needed in order to do calculations.

\(^{25}\) Through the whole thesis, the results for betas and investment are shown in separate figures but always are put together in the same line.
Figure 1 above shows the ratio of the equity betas to pre-tax asset beta, $\beta_{R^*}/\beta_p$, as functions of production elasticity, $\alpha$.\textsuperscript{26} $\beta_{R^*}/\beta_p$ are denoted in the figure B\textsubscript{R*}_P. B stands for beta ratio and RA\textsubscript{P} stands for the average beta to the beta of a claim to one unit product. RM corresponds to infra-marginal beta and RC corresponds to marginal beta which is from a stand-alone project with constant-return-to-scale (CRS) property.\textsuperscript{27} The investment decisions can also be found, which are also shown below in Figure 2. Investments are denoted by I\textsubscript{R*}. The notations are the same as the betas above. An additional curve I\textsubscript{no_tax} for optimal investment under the ideal situation of no tax distortion at all, which is meant to be compared with I\textsubscript{RA} or I\textsubscript{RM} since they are the same by construction, is also presented in the figure for investment.\textsuperscript{28}

The zigzag path of infra-marginal beta indicates the less accuracy in the simulation. And in particular for $\alpha$ close to 0 and 1, the simulated results for infra-marginal betas are completely wrong. This is because that the marginal project for such $\alpha$ is rather “marginal” and requires high degree of accuracy. However, for $0.1 \leq \alpha \leq 0.9$ the results are nice enough. Later on we will focus on the results for $\alpha$ within this range. The following interpretation is according to Lund (2011).

The marginal beta, B\textsubscript{RC_p} is approximately equals to 0.8 and independent of $\alpha$ because by definition the marginal project is defined to be with a market value equal to the financing need, here equal to initial investment always.\textsuperscript{29}

The average beta decreases from $\beta_p$ when $\alpha$ increases from 0 to approximately 0.75, and then increases back to the marginal beta B\textsubscript{RC_p}. Denote the cash flow by $P_1 f(I) - \tau \times \max(P_1 f(I) - \delta l, 0)$. When $\alpha \to 0$, because the expected operating income $P_1 f(I)$ is so large relative to $\delta l$, the project is almost certain in tax position.\textsuperscript{30} But also due to the high

\textsuperscript{26} For convenience of verbal discussion, we can normalize $\beta_p$ to be 1 so that the beta ratios are exactly the beta values themselves.

\textsuperscript{27} The marginal beta for different elasticity always equals to the marginal beta when the elasticity equals to one, i.e., a CRS project.

\textsuperscript{28} I\textsubscript{no_tax} is derived as follows. The cash flow without tax is $P f(I)$. The valuation is $V(P) f(I)$. The firm is supposed to maximize $V(P) f(I) - l$. The FOC gives $I_{\text{no_tax}} = (V(P) \alpha)^{1/\alpha}$.

\textsuperscript{29} The expected rate of return are constant for different $\alpha$.

\textsuperscript{30} This may sound counter-intuitive. But the investment decision is based on the FOC and the decision equation B(7) in Lund (2011) suggests the point.
weight of \( P_1 f(I) \), the average beta is close to \( \beta_P \). When \( \alpha \) increases, the ratio \( f(I)/I \) decreases. Hence the tax shields \( \delta I \) have more weight which leads to a lower beta. At the same time, the probability of not being in tax position increases and the tax shields will less likely be used, which increases the risk of the total cash flow, the beta value. However, the effect from decreasing ratio \( f(I)/I \) dominates until \( \alpha \) is close to 0.75 in current project of parameters. This is the reason of the non-monotonicity of the average beta. As can be seen from below, the non-monotonicity could disappear when we move to multi-periods, which agrees with the statement “nonmonotonicity … may not be true for all parameter configurations” from Lund (2011).\(^{31}\) In the limit of \( \alpha \rightarrow 1 \), the project is approaching CRS, therefore the beta approaches \( B_{RC_p} \) since the marginal project also approaches CRS. The convexity indicates that the increases in \( \alpha \) have a stronger risk-reducing effect, decreasing the ratio \( f(I)/I \), when \( \alpha \) is smaller.

Since the infra-marginal project is created by the last dollar invested in and taxed together with the average project, the explanation to the infra-marginal beta is almost the same as the average beta. \( B_{RM_p} \) is increasing and convex in \( \alpha \). When \( \alpha \rightarrow 0 \), \( B_{RM_p} \) is approximately equal to the beta of the risk-free tax position \( \beta_{FM} \) derived in Lund (2011), here it is not focused, due to the same reason that the operating income is so large relative to the tax shields being used and the probability of being in tax position is close to 1. When \( \alpha \rightarrow 1 \), again the infra-marginal project also approximately has the CRS property, which leads the associated beta to approach \( B_{RC_p} \).

The nonmonotonicity of the optimal investment is due to the particular value of \( V_0(P_1) \), lower than 1 in current setting.\(^{32}\) The figure shows the optimal investment is significantly less than no-tax case when \( \alpha \) is large. It is a result of distortion effect from tax shields which is discussed extensively in Lund (2002).

Corresponding to the sensitivity tests for tax rate equal to 0.7 and volatility 0.2 in Lund (2011), the results are shown separately below. Throughout the figures in rest of the text, the

\(^{31}\) At least within the range of \( 0.1 < \alpha < 0.9 \) (and if the non-monotonicity remains, the minimum point moves to an \( \alpha \) higher than 0.9).

\(^{32}\) If \( V_0(P_1) > 1 \), then the optimal investment will be increasing and convex. But then it will be difficult to derive accurate results because the optimal investment will increase exponentially to infinity for large \( \alpha \). However, for current purpose, we are more interested in the response of the optimal investment to changes in tax rules. It will be beneficial for the analyses to limit the optimal investment lower than 1.
curves to be compared with are dashed and added a “_0” to the name, i.e., the tax rate equal to 0.35 and volatility 0.3 cases here. The same type of curves is in the same colour.

As can be seen from below, the results again agree with Lund (2011). The betas decrease significantly when the tax rate doubles and the volatility decreases (for large \( \alpha \), due to the same reason that the ratio \( f(I)/I \) is small) since the tax shields have a risk reducing effect and the volatility is a source of the risk. The corresponding investment decreases as tax rate increases and volatility increases (from 0.2 to 0.3). These are expected. The higher tax rate has a higher distortion effect leading to a lower investment. The effect of volatility change can be explained by option pricing theory. A higher volatility increases the value of tax claim held by the government since the tax claim is protected from downwards risk. Such an increase in asymmetry of risk sharing thus has a stronger disincentive effect on the firm’s investment decision.
Observe also that the directions of shifts (or changes) of beta and investment are not the same in the two tests, which indicates there is no exact relationship between the direction of shift of beta and investment.

The results above confirm the validity of the simulation approach. It is more interesting and important to explore the behaviour of the betas and the investments in the more realistic multi-period models which we shall turn to now.
4 The multi-period models

When it comes to a multi-period project, it is more realistic to consider the carryforwards tax rule since it is more common in most countries. However, no-loss-offset can still be a starting point for comparisons between different tax systems. And it is also possible and realistic to take different depreciation schedules into account, such as constant depreciation and regressive depreciation. This section first derives the results for a two-period model, because it can be a bridge to comparisons between results of a multi-period model with more periods and the one-period model. Then the results for a ten-period model will be simulated and compared to the two-period model. It will be shown that the time-span (the number of periods) has effects on the value of betas and on the sensitivity of betas to changes in the tax rate and volatility. In each model, the aspects mentioned above will be considered. As assumed in section 2, the total production is the same for different time-span models and is divided equally into each period so that the comparisons between them are reasonable.

4.1 The two-period model

4.1.1 No-loss-offset

Take average beta as an example. The main changes of simulation in multi-period project are now that the valuation by risk-neutral method in each period needs to consider the compound discount factor (using risk-free interest rate). Then by the value-additivity principle they can be summed up, denoted by \( V_{sum} \). To put the unadjusted-price cash flows of different periods (denoted by \( CF_t \)) on the Security Market Line, the expected one-period return (denoted by \( \rho \)) will satisfy \( V_{sum} = \sum_{t=1}^{T} E(CF_t) e^{-\rho t} \) and can be found again by iteration. Observe this is under the assumption of stationary market condition and the required expected rates of return by investors for each period are the same over time. Then the average beta is again found by \( \beta_{RA} = \frac{\rho - r_f}{E(r_m) - r_f} \). The infra-marginal and marginal beta are found by similar changes.

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33 A progressive schedule is rare in reality and not interesting.

34 The \( \rho \) here is called equivalent constant discount rate (ECDR) in Jacoby and Laughton (1992) which can be regarded as a measure of the riskiness of the cash flows.
The analytical solution for no-loss-offset still can be obtained. For example, under the assumptions of constant production profile and constant depreciation schedule, using the same method in Lund (2011) and the parameters assumed here, the average beta for a T-period project is found to be $\beta_{RA}^T = \frac{1-e^{-rT}}{1-e^{-rT(1-\alpha)}} \beta_p$, where $\beta_{RA}^T$ is the T-period average beta, and $z_2^T$ is determined by $T + 1$ equations with $T + 1$ unknown variables ($z_2^T$ and the optimal $I$):

$$ z_2^T = \frac{\ln\left(\frac{V_0(P_T)f(I)}{I}\right) + (r - \frac{\sigma^2}{2})t}{\sigma \sqrt{t}} \text{ and the FOC} $$

$$ \sum_{t=1}^{T} V_0(P_t)f(I) \left[ \frac{1}{T} - \frac{1}{T} N(z_2^T + \sigma \sqrt{T}) \right] = \frac{1}{\alpha} \left[ 1 - \frac{T}{T} \sum_{t=1}^{T} e^{-rT} N(z_2^T) \right]. $$

When $T = 1$, it collapses to the $\beta_{RA}$ in equation (29) in Lund (2011). When $T > 1$, we still need iterations to find out the optimal investments and the beta values. From the solution, it can be verified by numerical methods that for larger $T$, the beta is larger. For given $T$, the behaviour for different $\alpha$ can also be traced out. The sensitivity test can also be obtained. Unfortunately, the analytical results for carryforwards models can not be obtained and we can only rely on simulation methods. However, the simulation method verified from no-loss-offset models can provide guidance for simulation used in carryforwards models. It turns out that the analytical results from one-period model under certain and uncertain tax positions can be rough estimates for carryforwards models, which is discussed in the ten-period models.

4.1.1.1 The effects of different depreciation schedule

In practice, the two most common forms of depreciation schedules are linear and exponentially decreasing, here called constant and regressive respectively.

Case 1: Constant depreciation schedule

Like constant production profile, the constant depreciation means the equal depreciation amount in each period and the sum is equal to the initial investment, i.e., $\delta_1 = \delta_2 = 0.5$. This

35 See appendix 1.

36 This is also the case for marginal and infra-marginal betas.

37 The intuition is given below in the analysis of the results for constant depreciation schedule.
The project is least realistic but most comparable to the one-period model, although the total production (equal to one-period model) is divided evenly into two periods and the risk-free discounting matters. It can also be a baseline for comparisons across depreciation schedules and tax rules.

Using the same tax rate 0.35 and volatility 0.3 and other parameters for one period model, the results for a two-period project are shown below.\(^{38}\)

Figure 4

As can be seen from above, the patterns of two-period betas are the same as one-period ones and the values are a little higher. The higher betas are due to the less weight on the depreciation tax shields. This can be seen from the option like cash flow in the two periods, \(\tau \cdot \max \left( \frac{1}{2} P_t f(I) - \frac{1}{2} I, 0 \right) = \frac{1}{2} \tau \cdot \max (P_t f(I) - I, 0), t = 1, 2\). But the possible tax shields in the second period are discounted more heavily so that the present value of total deduction becomes lower. The results and the analysis are also consistent with the results of the multi-period marginal beta for certainty tax position considered in Lund (2002). The “\(A\)” in equation (30) there is smaller in a more-period model (if we divide the depreciation amount evenly into each periods), which leads to a higher marginal beta. And since the direction of shift in average beta is the same as marginal beta, the average beta is also higher in a more-period model. And the difference between betas with certainty tax position and uncertain tax position is only quantitative, not qualitative. Hence the risk-reducing effect from tax shields is reduced and the betas increase. In the ten-period project, the betas are even higher than the

\(^{38}\) In more-than-one-period model, the load of computer is increased greatly, especially for a 10-period setting. Hence only the results for scale elasticity \(\alpha = 0.1, 0.2, \ldots, 0.9\) are simulated. Still, the patterns are roughly preserved.
two-period project due to the tax shields are discounted more heavily again than the two-period project.

The analysis above is important for the interpretation of later results. It turns out that the present value of the total tax shields in the multi-period models even under uncertainty is the key for the different directions of the shifts of the betas in different settings. Roughly speaking, ceteris paribus, the beta values are decreasing in the present value of the total tax shields for no loss offset tax rule.

The investments are lower relative to one-period and in the ten-period project are even lower, which reflects higher distortion of the tax system in more-period projects.

**Case 2: Regressive depreciation schedule**

The regressive schedule is most often modeled as exponentially decreasing, i.e., a fixed percentage of the remaining capital value from the previous period. Here for simplicity and intuition, the regressive schedule is assumed to be a decreasing sequence of numbers, \((\delta_1, \delta_2) = (0.8, 0.2)\) for the two-period project. The results are shown below and compared to the constant schedule.

As can be seen from below, the betas are now increased compared to the constant schedule, due to the fact that it is more likely that the project is out of tax position in the first period which has a higher weight than the second period in which the project is more likely in the tax position. Hence the present value of the total possible tax shields is lower and the betas are higher. The investments as expected decrease again due to the higher distortion effect of the tax system.

Figure 5
4.1.1.2 Sensitivity tests

Similar to one-period model, the sensitivity tests for changes in tax rate and volatility are conducted separately here for the constant depreciation schedule.

First the tax rate is increased from 0.35 to 0.7 and the results are shown below.

Figure 6

Similar to the one-period project, the increase in tax rate has a significant risk-reducing effect and the investment reduces as the distortion from the tax system is greater. But the degree of the effects is a little lower than the one-period project, which is also due to the fact that the present value of the total tax shields is lower than the one-period project.

Second, the volatility is decreased from 0.3 to 0.2, keeping the tax rate at 0.35. The results are shown below.

Figure 7
Similar to one-period project again, the decrease in the volatility of prices reduces the risk but it is hard to conclude whether the effect is smaller or larger than one-period project. However, it will be shown in the ten-period setting that the increased number of periods makes the effect less significant, which is similar to the sensitivity test for tax rate. The investment also increases due to less distortion and the effect is also smaller than one-period project.

4.1.2 Carryforwards

Under carryforwards tax rule the negative tax base is carried forwards to next period for deduction. Different countries allow for different time-span of carryforwards. For simplicity, it is assumed here that it is always allowed to carry forwards up to the end of the project (the carryforwards at the last period is assumed to be lost). Sometimes, it is possible that the carryforwards is accompanied with (risk free) interest payment from tax authority. Here we consider both without-interest and with-interest cases. Under carryforwards tax rule the analytical solutions can not be derived and we can only rely on the simulation method. The only change in the simulation is that the composition of the cash flow is different.

4.1.2.1 Carryforwards without interest

Without accompanied interest payment, the cash flow in period $t$ for a $T$-period project is in the form of

$$P_t * \left( \frac{1}{T} f(I) \right) - \tau * \max \left( P_t * \left( \frac{1}{T} f(I) \right) - \frac{1}{T} I - carryf_t, 0 \right),$$

where

$$carryf_t = \max \left( \frac{1}{T} I + carryf_{t-1} - P_{t-1} * \left( \frac{1}{T} f(I) \right), 0 \right),$$

with $carryf_0 = 0$ for adjusted-prices in risk-neutral valuation and unadjusted-price in actual cash flows respectively. The risk neutral valuation based on option pricing theory is still valid even though there is an extra element $carryf_t$ in the max$(\ldots)$ term because $carryf_t$ is determined by the information from period $t-1$ which has been revealed in period $t$ and can be thought as an increase in the strike price.

The results for the two-period model under carryforwards without interest compared with no-loss-offset are shown below.
The figures below show that the carryforwards rule compared to no-loss-offset has a risk-reducing effect in the two-period project. As explained in introduction section, carryforwards acts like a risk-free “loan” to tax authority, here without interest. Thus, it has an effect opposite to the traditional leverage and reduces the risk. In contrast, no-loss-offset rule exaggerates the risk of reimbursement of the depreciation deduction. Formally, the allowance of carrying negative tax base forwards increases the present value of the tax shields fundamentally. At the same time, the extra term $carryf_t$ in the option like component also increases the probability of being out of tax position and the deduction will be less likely used, which decreases the present value of the tax shields. This effect turns out to be important for interpretation of the indifference between carryforwards without interest and with interest, as shown below for both the two- and ten-period project. Moreover, the assumed GBM process also contains autocorrelation, which means a lower price in one period will make the price to be very likely lower for a long time and the deduction will be even less likely used. This can be seen in the ten-period project, where the betas for no loss offset rule are the same as the carryforwards rule. However, in the two-period project the “loan” effect dominates and the risk is reduced.

4.1.2.2 Carryforwards with interest

With an interest payment accompanying the carryforwards tax deduction, one needs only to multiply $(1 + r_f)$ to the carryforwards equation in previous case in order to adjust for the interest payment. The results are shown below, compared to the without-interest rule.
It is surprising that the interest payments from tax authority do not affect either the risk or the investments decisions, compared to without-interest rule. And the results in the ten-period project are the same. As explained in the previous case, carryforwards without interest, here the extra interest payment from tax authority increases the value of tax shields but at the same time also increases the probability of being out of tax position which decreases the present value of the tax shields. The two opposite effects cancel each other and the betas remain unchanged compared to the carryforwards without interest case.

Through the results in this section, the simulations show that the two-period project behaves roughly the same as one period project. It is interesting and important to see how the number of periods affects the degree of the risk-affecting effect and we now turn to the ten-period project.

4.2 The ten-period results

Because the simulation in the ten-period model is in principle the same as the two-period model except the number of the periods shall be adjusted, the results are given without repeating the procedure.

4.2.1 No-loss-offset

4.2.1.1 The effects of different depreciation schedule
Case 1: Constant depreciation schedule

The results for constant depreciation schedule are shown below.

Figure 10

As can be seen from above, the ten-period betas are significantly higher and the corresponding investments are lower than two-period project and they exhibit the same patterns except one new feature for the average beta. It seems that now the average beta becomes a monotonically decreasing function of $\alpha$ although the convexity is unchanged, which may confirm the statement in Lund (2011, p.24): ...a non-monotonic function... may not be true for all parameter configurations.

Similar to the two-period project, the possible tax shields in the more distant future periods are discounted more heavily so that the present value of total deduction becomes even lower than the two-period project. Hence the risk-reducing effect from tax shields is reduced, although the initial investments also decrease due to higher distortion of the tax system in more-period projects. Thus betas increase further relative to the two-period project.

Case 2: Regressive depreciation schedule

The regressive schedule now is assumed to be (20, 15, 13, 13, 8, 8, 8, 5, 5, 5)/100. The results compared to constant schedule are shown below.

The results are similar to the two-period project and now the risk-increasing effect is less significant. Obviously, although the regressive schedule in ten-period project is not comparable to the two-period project ((80, 20)/100), the relative weight of early periods is reduced in this specific ten-period regressive schedule. Thus, the probability of not being in
tax position in early periods is lower and the present value of the possible tax shields is higher relative to the two-period schedule. Hence the risk-increasing effect is reduced relative to the two-period schedule.

4.2.1.2 Sensitivity tests

The results from sensitivity tests similar to the two-period project are shown as follows.\(^{39}\)

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\(^{39}\) The abnormal decrease for infra-marginal beta at \(\alpha = 0.9\) is because of the lack of simulation accuracy for \(\alpha\) closed to 1.
When the tax rate increases from 0.35 to 0.7, the distortion effect from tax system increases and the investments decrease. The degree of risk-reducing effect is even lower than the two-period project.

As can be seen from below, the decrease in volatility has the similar effect as in two-period project; the betas decrease and the investments increase, although even less.

**Figure 13**

There is an aspect from the sensitivity tests needed to be emphasized. The degree of responses of the betas and the investments to the change in tax rate and volatility is smaller in more-period projects. The less sensitive responses can be explained again by the discounting effect from the more distant future periods and thus the responses are “diluted” in the more-period projects.

### 4.2.2 Carryforwards

#### 4.2.2.1 Carryforwards without interest

The results for carryforwards without interest payment compared to no-loss-offset tax rule are shown below.

As explained in the two-period project under carryforwards without interest tax rule, there are three aspects that should be considered when the tax rule changes from no loss offset to
carryforwards without interest. The carryforwards increases the value of tax shields on the one hand. On the other hand, the extra term \( carryf \) in the option like component also increases the probability of being out of tax position and the deduction will be less likely used, which decreases the present value of the tax shields. Moreover, the assumed GBM process also contains autocorrelation, which means a lower price in one period will make the price to be very likely lower for a long time and the deduction will be even less likely used. When the number of periods increases, the latter two effects become more significant and tend to cancel out the first. In this ten-period project, carryforwards without interest seems to have no benefits for the project and the betas remain unchanged compared to no loss offset tax rule.

Figure 14

4.2.2.2 Carryforwards with interest

The results for carryforwards with interest compared to without interest are shown below in Figure 15.

It seems that in this special setting the interest payments from tax authority do not reduce the risk further, when the carryfowards rule changes from without interest to with interest. As explained in the two-period case, the interest payments increase the present value of the total tax shields and the overall tax shields become even less risky. However, at the same time the increased deduction amount in each period also increases the probability of being out of tax position. The two opposite effects almost cancel each other and the changes in the betas are insignificant. So are the investments.
It is interesting to observe that the betas here are still very close to the betas for one-period no-loss-offset project respectively. This is also only due to the fact that it is impossible to carry the possible negative tax base in the last period further, thus the cumulated negative tax base in the last period is again faced a no-loss-offset situation. When the number of periods decreases, it converges to the one-period no-loss-offset project. If it is allowed to pay out the cumulated negative base in the last period for certainty, then the corresponding betas will converge to the certainty-tax-position betas $\beta_{FM}$ and $\beta_{FA}$ in Lund (2011) when the number of periods decreases. Incidentally, such simulated results thus also confirm the validity of the simulation method.

Figure 15

![Betas, carryforwards-with-interest VS without-interest, 10-period](image)

![Investments, carryforwards-with-interest VS without-interest, 10-period](image)
5 Discussion

To derive the most comparable betas to Lund (2011), the multi-period models are based on simplified assumptions on several accounts. The simple-minded multi-period CAPM used here relies on the assumptions that the market condition is stationary and the beta of a claim to one unit of product is constant over time. These are certainly not realistic. Constantinides (1980) (1982) provide a set of assumptions for extensions of one-period CAPM to multi-period setting where the non-stationarity of the variables, such as the risk free interest rate, the rate of return of the market portfolio and the security betas can be incorporated. However, the application is complex and of little practical use. Kazemi (1991) sets up a model where macroeconomic variables, the risk-free bond prices (and the aggregate consumption), are used for valuation. But then the betas are not comparable to Lund (2011) and the focus of the current context is not to explore the effect of the non-stationarity of the variables. For both theoretical attempt and practical use in current context, the stationary market condition could still be a baseline of discussion.

It is assumed that the change in tax system does not affect the equilibrium of capital market. Thus it is a partial equilibrium analysis. As stated in Lund (2011), this will be a good approximation if the project is in a small sector of the economy or the change is in a small abroad economy. Even for large abroad economy we can still assume the change does not affect international capital market.

The volatility change is assumed not to affect the beta of a claim to one unit product. If the alternative approach, that a market portfolio is assumed and the correlation coefficient of the price and the market portfolio is unchanged, is used, then $\beta_p$ changes when the volatility is changed. But it turns out the results are still close to Lund (2011) in one-period model since the ratios of betas are robust to such a change.

The assumption of GBM for the price process may not be realistic. Different processes may be considered, e.g., mean reversion. In a multi-period model, a GBM will make tax deductions more risky than many other processes, since a negative shock to the output price in one period will, for GBM, also result in the same proportional negative shock to all subsequent prices. If $P_tQ_t$ is so low that the tax base becomes negative, then all subsequent $P_t$ values will also tend to be low, so that the carryforwards will go on for a long time, and perhaps there will never be effective deduction for the carryforwards. With a different
process, this tendency is not so strong and the betas may be reduced further. The risk-neutral valuation is however robust for no-arbitrage pricing.

Operating costs are not included for multi-period model because they are considered as deterministic. Multi-period investments are not included for the same reason. If they are included, the directions of affecting the risk of not being in tax position are opposite and the effects will cancel out each other (although not completely). In real applications it is easy to incorporate them.

Debt financing is ignored, in order to be able to compare with the analytical results. The interest deduction has a similar effect to depreciation deduction. According to Lund (2002, 2011), while it is easy to incorporate risk free debt financing, it will require another model if the debt is risky.

Production flexibilities are not considered here because it requires an alternative model of real option which is not the intention of current context.

Different production profiles are not considered because the effect is opposite to the depreciation schedules with the same weights over periods. It is also easy to deal with such changes in production profiles.
6 Summary

Using a simulation method based on risk-neutral valuation the thesis derives results of the after-tax asset betas in multi-period settings when depreciation tax shields are incorporated. The qualitative results are summarized as follows.

The number of periods of the project increases the betas. However, even for a ten-period project the after-tax betas are still significantly lower than the pre-tax asset beta, $\beta_p$. Thus the commonly used formula of the WACC is not only qualitatively but also quantitatively misleading. The results are consistent with the conclusion of Jacoby and Laughton (1992) that oil-extraction projects of larger fields are undervalued if the same discount structure is applied. The optimal investment is also affected and is decreasing in the number of periods.

Across tax systems, no-loss-offset in every period exaggerates the risk of not being in tax position. The multi-period case with no loss offset is the most complex model we have been able to solve analytically. The solution is found in Appendix 1, and represents an improvement compared with the analytical results of Lund (2011). When shifting to carryforwards rule, the betas decreases hence the cost of capital is even lower when the number of periods is small. However, when the project is operated in more periods, the benefits from carryforwards either without or with interest are small and the betas change little when shifting from no loss offset to carryforwards tax rule. Still, the betas are significantly lower than the pre-tax asset beta.

The sensitivity test for tax rate shows that the change of tax rate has a significant effect on the betas and investment decisions. The distortion effect is increasing in tax rate and this should be considered when the tax authority considers reforms. The application of the same discount rate under different tax systems for the project is misleading for firms.

Compared to the unrealistic constant depreciation schedule, the stylized regressive schedule increases the betas but the effect is small when the project is operated in more periods. The betas are still significantly lower than the pre-tax asset beta.

The results for investments also confirm an intuitive economic idea that when the tax rule is more favourable for firms, e.g., a lower tax rate, a lower price volatility, and carryforwards compared to no loss offset, firms invest more.
Clearly, the results are important for capital budgeting and valuation of potential projects, especially for a multi-national firm. One needs not only to “unlever” but also to “untax” and “unaverage” betas when the WACC is in use for different tax systems.

References


Appendices

1 Derivation of T-period average beta under no-loss-offset

Assuming constant production profile and constant depreciation schedule, the after-tax cash flow in period t is

\[
\frac{1}{T} [P_t f(I) - \tau \cdot \max(P_t f(I) - I, 0)]
\]

The valuation of this cash flow is

\[
\frac{1}{T} [V_0(P_t) f(I) - \tau (V_0(P_t) f(I) N(z_1^t) - IN(z_2^t) e^{-\tau r_t})]
\]

Where \(z_2^t = \frac{\ln(\frac{V_0(P_t) f(I)}{I}) + (r_f - \frac{\sigma^2}{2})t}{\sigma \sqrt{t}}\) and \(z_1^t = z_2^t + \sigma \sqrt{t} \) .

Because of the existence of convenience yield, \(V_0(P_t) \neq V_0(P_{t+1})\). Then by summing up the valuations the total after-tax value of the future cash flows is

\[
\frac{1}{T} \sum_{t=1}^{T} V_0(P_t) f(I) - \frac{\tau}{T} \sum_{t=1}^{T} (V_0(P_t) f(I) N(z_1^t) - IN(z_2^t) e^{-\tau r_t})
\]

The firm is supposed to maximize

\[
\pi = \frac{1}{T} \sum_{t=1}^{T} V_0(P_t) f(I) - \frac{\tau}{T} \sum_{t=1}^{T} (V_0(P_t) f(I) N(z_1^t) - IN(z_2^t) e^{-\tau r_t}) - I
\]

Denote the valuation of the option-like component by

\[40] z_1^t \text{ is not a exponential function of } z_t. \text{ The superscripts refer to time period } t.\]
It can be shown that
\[ \frac{\partial c_t}{\partial (v_0(P_t)f(I))} = N(z_1^t) \text{ and } \frac{\partial c_t}{\partial l} = e^{-\gamma t} N(z_2^t) \]

Thus, the FOC of the maximization problem is
\[ \frac{dx_t}{dl} = \frac{1}{T} \sum_{t=1}^{T} V_0(P_t)f'(I) - \frac{\tau}{T} \sum_{t=1}^{T} \left( \frac{\partial c_t}{\partial (v_0(P_t)f(I))} \frac{\partial (v_0(P_t)f(I))}{\partial l} + \frac{\partial c_t}{\partial l} \right) - 1 = 0. \]

This is
\[ \frac{1}{T} \sum_{t=1}^{T} V_0(P_t)f'(I) - \frac{\tau}{T} \sum_{t=1}^{T} [N(z_1^t)V_0(P_t)f'(I) + e^{-\gamma t} N(z_2^t)] = 1 \]

Introducing the assumption \( f(I) = wI^\alpha \), the FOC becomes
\[ \sum_{t=1}^{T} V_0(P_t)f(I) \left[ \frac{1}{T} - \frac{\tau}{T} N(z_1^t) \right] = \frac{l}{\alpha} \left[ 1 - \frac{\tau}{T} \sum_{t=1}^{T} e^{-\gamma t} N(z_2^t) \right] \]

Then the average beta is the weighted beta of the two components, i.e.,
\[ \beta_{RA}^T = \frac{\sum_{t=1}^{T} V_0(P_t)f(I) \left[ \frac{1}{T} - \frac{\tau}{T} N(z_1^t) \right]}{\sum_{t=1}^{T} V_0(P_t)f(I) \left[ \frac{1}{T} - \frac{\tau}{T} N(z_1^t) \right] + \frac{\tau}{T} \sum_{t=1}^{T} e^{-\gamma t} N(z_2^t)} \beta_p \]

Using the FOC for this equation and the equation for \( z_2^t \), we have the result.
\[ \beta_{RA}^T = \frac{1 - \frac{\tau}{T} \sum_{t=1}^{T} e^{-\gamma t} N(z_2^t)}{1 - \frac{\tau}{T} (1 - \alpha) \sum_{t=1}^{T} e^{-\gamma t} N(z_2^t)} \beta_p \], where \( \beta_{RA}^T \) is the T-period average beta, and \( z_2^t \) is determined by \( T + 1 \) equations with \( T + 1 \) endogenous variables (\( z_2^t \) and the optimal \( I \)).

\[ z_2^t = \frac{\ln(V_0(P_t)f(I)) + \gamma - \frac{\sigma^2}{2} t}{\sigma \sqrt{t}} \] and the FOC
\[ \sum_{t=1}^{T} V_0(P_t)f(I) \left[ \frac{1}{T} \frac{\tau}{T} N(z_2^t + \sigma \sqrt{t}) \right] = \frac{l}{\alpha} \left[ 1 - \frac{\tau}{T} \sum_{t=1}^{T} e^{-\gamma t} N(z_2^t) \right]. \]
2 Programming codes (Stata 11)

2.1 The beta and valuation from option pricing are consistent with the CAPM

```stata
quietly {
    #delimit ;
    drop _all; sca drop _all; set varabbrev off; set more off;
    sca T = 1; sca sigma = 0.3; sca rf = 0.05; sca p0 = 10; sca rp = 1+rf; sca mu = 0.15;
    sca a = 0.5; // we can change the parameters above to any reasonable values we want.
    mat A = (1, a
        a,1); mat B = cholesky(A);
    set obs 10000; gen col1 = invnormal(uniform()); gen col2 = invnormal(uniform());
    gen j = B[1,1]*col1 + B[1,2]*col2; gen m = B[2,1]*col1 + B[2,2]*col2; gen rm = 0.3*m + 0.12;
    gen beta_ratio = 0; gen V_ratio1 = 0; gen V_ratio2 = 0;
    gen p = p0*exp((mu -(sigma^2)/2)*T + sigma*sqrt(T)*j);  // generate GBM p with drift mu
    sum p; sca Ep = r(mean); sum rm; sca Erm = r(mean); corr p rm, covariance;
    sca var_rm = r(Var_2); sca var_p = r(Var_1); sca cov_p rm = r(cov_12);
    sca Vp = (Ep - (Erm - rf)*cov_p rm/var_rm)/rp;         // find present value of p by CAPM
    sca beta_p = cov_p rm/Vp/var_rm;       // find beta of p
    #delimit cr
}
```

// BS valuation of xR using a GBM with current equil price Vp and risk-free drift , formula
```stata
sca z1 = ln(Vp/(K/rp))/sigma + sigma/2;    // equation B4 in Lund (2011)
```

// BS valuation of xR using a GBM with current equil price Vp and risk-free drift , risk neutral simulation
```stata
gen p_Vp = Vp* exp((rf -(sigma^2)/2)*T + sigma*sqrt(T)*j);  // generate GBM p with drift rf and Vp
```

// CAPM Valuation of xR , treat xR just as a usual cash flow
```stata
sca cov_xRrm = max(p_Vp - K, 0); sum xR_sim; sca V_Vp = r(mean)/rp;
```

// CAPM beta of xR , treat xR just as a usual cash flow
```stata
sca beta_CAPM = rp/(ExR*var_xRm - Erm + rf);
```

// BS weighted beta from BS formula with S0=V(p)
```stata
sca beta_BS = (Vp*normal(z1)/Vp)*beta_p;
```

// save valuation and beta ratios for diff K
```stata
replace V_ratio1 = V_BSVxR in `k'; replace V_ratio2 = V_VpVxR in `k';
```

```stata
} ;

list V_ratio1 V_ratio2 beta_ratio in 1/10; sum V_ratio1 V_ratio2 beta_ratio in 1/10;
#delimit cr
```

/* the means of all the ratio are close to 1, and so are the max and min of these ratios. */

```
2.2 The standard Brownian motion observation generator

#delimit ;
quietly {
    drop _all; set varabbrev off; set more off; set obs 10000; gen B0 = 0;
    // sca mu = 0.1; sca sigma = 0.3;
```
forvalues t = 1/10 {;
    local T = `t' - 1;
    gen d_B`t' = invnormal(uniform());
    gen B`t' = B`T' + d_B`t';
};
save "the desired path of the file containing the Standard Brownian motion observations ";
// for example C:\Users\Public\Documents\B_t.dta

2.3 The betas under no loss offset in each period

quietly {
    drop _all; sca drop _all; local drop _all; set varabbrev off;
    use "the standard Brownian motion observations file"; // e.g., C:\Users\Public\Documents\B_{t}.dta
    set more off;
    // mat production = (10, 10, 10, 10, 10, 10, 10, 10, 10, 10); mat depreciation = (10, 10, 10, 10, 10, 10, 10, 10, 10, 10);
    // mat production = (20, 20, 20, 20, 20); mat depreciation = (20, 20, 20, 20, 20);
    // mat production = (50, 50); mat depreciation = (50, 50);
    mat production = (100);
    mat depreciation = (100);
    local period = colsof(prod);
    sca rf = 0.05; sca p0 = 1; sca tax = 0.35; sca sigma = 0.3; // parameters
    set obs 10000; gen index = _n;
    sca CY = 0.04; sca Erm = 0.16;
    gen B_RA_p = . ; gen I_RA = . ; gen B_RM_p = . ; gen I_RM = . ;
    gen B_RC_p = . ; gen I_RC = . ;
    // generate p_unadj, p_adj and find beta_p
    forvalues t = 1/10 {;
        sca T = `t';
        gen p_un`t' = p0*exp((mu - (sigma^2)/2)*T + sigma* B`t');
        gen p_adj`t' = p0*exp((rf - CY - (sigma^2)/2)*T + sigma* B`t');
    };
    sum p_un1; sca Ep1 = r(mean); sum p_adj1; sca V_p1 = r(mean)/exp(rf);
    sca rho_p = ln(Ep1/V_p1); sca beta_p = (rho_p - rf)/(Erm - rf); // Ep1/V_p1 - 1,
    noisily di "This is RA RM RC no offset" " tax = " tax " sigma = " sigma " CY = " CY;
    quietfully {
        forvalues alf = 1/99 {;
            sca Alf = `alf'/100;
            capt drop S* ; capt drop x* ; capt drop min* ; capt drop pai_xRA;
            gen Sum_VxRA = 0; gen pai_xRA = -100000; gen min_I = . ;
            forvalues I = 1/1000 {;
                sca i = `I'/1000;
                forvalues t = 1/period' {;
                    sca T = 't';
                    sca Q`t' = prod[1,`t']*i^Alf ;
                    gen xRA_adj`t' = p_adj`t'*Q`t' - tax* max(p_adj`t'*Q`t' - depreci[1,`t']*i,0);
                    sum xRA_adj`t'; sca ExRA`t' = r(mean); sca VxRA`t' = ExRA`t'/exp(rf*T);
                    sum VxRA = Sum_VxRA + VxRA`t';
                };
                replace Sum_VxRA = Sum_VxRA in 'I';
                replace pai_xRA = Sum_VxRA - i in 'I';
                replace min_I = abs(Sum_VxRA - i) in 'I';
            };
            sort pai_xRA; sca I_Aver = index[N]/1000; sca Vstar = Sum_VxRA[N]; sort index;
            sort min_I; sca I_MC = index[1]/1000; sca V_MC = Sum_VxRA[1]; sort index;
            sca I_Marg = I_Aver; sca d_{I} = 0.0001; sca I_marg = I_Marg - d_{I};
        }
    }

forvalues t = 1/period' { 
    sca T = `t';
    sca Q = prod[1,`t']*I_Marg^Alf; // for average and infra-marginal
    gen xRM_un = p_un*Q - tax*max(p_un*Q - depr[1,`t']*I_Marg , 0);
    sum xRM_un; sca ExRM_t = r(mean);
    sca q = prod[1,`t']*I_Marg^Alf; // for infra-marginal
    gen xRM_un = p_un*Q - tax*max(p_un*Q - depr[1,`t']*I_Marg , 0);
    sum xRM_un; sca ExRM_t = r(mean);
    sca d_ExRM_t = ExRM_t - ExRm_t; // now we have d_ExRM_t, we want to find 
    // a rho to make the sum of present value equal to d_I
    sca Q_MC = prod[1,`t']*I_MC^Alf; // for marginal
    gen xMC_un = p_un*Q - tax*max(p_un*Q - depr[1,`t']*I_MC , 0);
    sum xMC_un; sca ExMC_t = r(mean);
}

gen mini_A = .; gen mini_M = .; gen mini_C = .; 
forvalues rho = 1/1000 { 
    sca Rho = `rho'/1000; sca V_d_ExRM = 0; sca Sum_VECF = 0; sca Sum_VECF_C = 0;
    forvalues t = 1/period' { 
        sca T = `t';
        sca V_d_ExRM = V_d_ExRM + d_ExRM_t/exp(Rho*T); 
        sca VECF_t = ExRM_t/exp(Rho*T); sca Sum_VECF = Sum_VECF + VECF_t; // exp(Rho*T)
        sca VECF_C_t = ExMC_t/exp(Rho*T); sca Sum_VECF_C = Sum_VECF_C + VECF_C_t; // ((1+Rho)^T)
    }
    replace mini_M = abs(V_d_ExRM - d_I) in `rho';
    replace mini_A = abs(Sum_VECF - Vstar) in `rho';
    replace mini_C = abs(Sum_VECF_C - V_MC ) in `rho';
}

sort mini_M; sca rho_Marg = index[1]/1000; sort index;
sort mini_A; sca rho_Aver = index[1]/1000; sort index;
sort mini_C; sca rho_MC = index[1]/1000; sort index;

sca beta_xRA = (rho_Aver - rf)/(Erm - rf); sca beta_ratio_A = beta_xRA /beta_p;
sca beta_xRM = (rho_Marg - rf)/(Erm - rf); sca beta_ratio_M = beta_xRM /beta_p;
sca beta_xRC = (rho_MC - rf)/(Erm - rf); sca beta_ratio_C = beta_xRC /beta_p;

noisily di "alf = " Alf " I_Aver = " I_Aver " beta_xRA/beta_p = "beta_ratio_A;
noisily di "alf = " Alf " I_Marg = " I_Marg " beta_xRM/beta_p = "beta_ratio_M;
noisily di "alf = " Alf " I_MC = " I_MC " beta_xRC/beta_p = "beta_ratio_C;

noisily di " ";
replace B_RA_p = beta_ratio_A in `alf'; replace I_RA = I_Aver in `alf';
replace B_RM_p = beta_ratio_M in `alf'; replace I_RM = I_Marg in `alf';
replace B_RC_p = beta_ratio_C in `alf'; replace I_RC = I_MC in `alf';
replace alpha = Alf in `alf';
}

keep B_* I* alpha;
// twoway(line B_RA_p alpha in 1/99, ylabel(0(0.1)1)) (line B_RM_p alpha in 1/99) (line B_RC_p alpha in 1/99);
#delimit cr
2.4 The betas under carryforwards
#delimit ;
quietly { 
    drop _all; sca drop _all; local drop _all; set varabbrev off; set more off;
    use "the standard Brownian motion observation file"; // e.g., C:\Users\Public\Documents\B_t.dta
    // mat production = (10, 10, 10, 10, 10, 10, 10, 10, 10, 10); mat depreciation = (10, 10, 10, 10, 10, 10, 10, 10, 10, 10);
    // mat production = (20, 20, 20, 20, 20, 20, 20, 20, 20, 20); // mat production = (50, 50); mat depreciation = (50, 50);
    // mat production = (100); mat depreciation = (100); 
    mat prod = production/100; mat depr = depreciation/100; 
    local period = colsof(prod);
    sca rf = 0.05; sca p0 = 1; sca mu = 0.1; sca tax = 0.35; sca sigma = 0.3; // parameters
    set obs 10000; gen index = _n; sca CY = 0.04; sca Erm = 0.16;
    gen B_RA_p = .; gen I_RA = .; gen B_RM_p = .; gen I_RM = .; gen alpha = .;
// generate p_unadj, p_adj and find beta_p
forvalues t = 1/10 { 
  sca T = t;
  gen p_un\t' = p0*exp((mu
  (sigma^2)/2)*T + sigma* B\t');
  gen p_adj\t' = p0*exp((rf - CY
  (sigma^2)/2)*T + sigma* B\t');
};

sum p_un1; sca Ep1 = r(mean); sum p_adj1; sca V_p1 = r(mean)/exp(rf);
sca rho_p = ln(Ep1/V_p1); sca beta_p = (rho_p
  rf)/(Erm
  rf);

noisily di "This is RA RM RC carryf tax = " tax " sigma = " sigma;
noisily mat list prod; noisily mat list depr;
}

quietly forvalues alf = 1/9 { ; set more off;
  sca Alf = alf/10;
capt drop S* ; capt drop x* ; capt drop min* ; capt drop pai_xRA;
gen Sum_VxRA = 0; gen pai_xRA = -100000; gen min_I = . ; // find I_Aver and I_MC
forvalues I = 1/1000 { ; // search for I<1
  sca i = I/1000; sca Sum_VxRA = 0; capt drop xRA*; capt drop carr*; gen carryf_1 = 0;
  forvalues t = 1/period' { 
    sca T = T';
    sca Q't' = prod[1,\t']i*\t';
    gen xRA_adj\t' = p_adj\t'Q't' - tax* max(p_adj\t'*Q't' - depr[1,\t']i - carryf_1,0); 
    sum xRA_adj\t'; sca ExRA\t' = r(mean); sca VxRA\t' = ExRA\t'/exp(rf*T);
    sca Sum_VxRA = Sum_VxRA + VxRA\t';
    replace carryf_1 = max(depr[1,\t']*i + carryf_1 - p_adj\t'*Q't', 0);
  }
  replace Sum_VxRA = Sum_VxRA in `I';
  replace pai_xRA = Sum_VxRA - i in `I';
  sort pai_xRA; sca I_Aver = index[_N]/1000; sca Vstar = Sum_VxRA[index[_N]]; sort index;
  sort min_I; sca I_MC = index[1]/1000; sca V_MC = Sum_VxRA[1]; sort index;
  sca I_Marg = I_Aver; sca d_I = 0.0001; sca I_marg = I_Marg - d_I;
capt drop carr*; gen carryf_1 = 0; gen carryf_2 = 0; gen carryf_3 = 0;
  forvalues t = 1/period' { 
    sca T = T';
    sca Q't' = prod[1,\t']i*I_Marg^Alf ; // for average and infra-marginal
    gen xRM_un\t' = p_un\t'*Q't' - tax* max(p_un\t'*Q't' - depr[1,\t']i - carryf_1,0); 
    replace carryf_1 = max(depr[1,\t']i + carryf_1 - p_un\t'*Q't', 0);
    sca q't' = prod[1,\t']i*I_Marg^Alf ; // for infra-marginal
    gen xRM_un\t' = p_un\t'*q't' - tax* max(p_un\t'*q't' - depr[1,\t']i - carryf_2,0); 
    replace carryf_2 = max(depr[1,\t']*i + carryf_2 - p_un\t'*q't', 0);
    sum xRM_un\t'; sca ExRM\t' = r(mean); 
    sum xRM_un\t'; sca ExRM\t' = r(mean);
    sca d_ExRM\t' = ExRM\t' - ExRm\t'; // now we have d_ExRM\t' , we want to find 
    // a rho to make the sum of present value equal to d_I
    sca Q_MC\t' = prod[1,\t']i*I_MC^Alf ; // for marginal
    gen xMC_un\t' = p_un\t'*Q_MC\t' - tax* max(p_un\t'*Q_MC\t' - depr[1,\t']i - carryf_3,0); 
    replace carryf_3 = max(depr[1,\t']*i + carryf_3 - p_un\t'*Q_MC\t', 0);
    sum xMC_un\t'; sca ExMC\t' = r(mean);
  }
};

gen mini_A = . ; gen mini_M = . ; gen mini_C = . ; 
forvalues rho = 1/1000 { ;
  sca Rho = rho/1000; sca V_d_ExRM = 0; sca Sum_VECF = 0; sca Sum_VECF_C = 0;
  forvalues t = 1/period' { 
    sca T = t';
    sca V_d_ExRM = V_d_ExRM + d_ExRM\t'/exp(Rho*T);
    sca VECF\t' = ExRM\t'/exp(Rho*T); sca Sum_VECF = Sum_VECF + VECF\t'; // ((1+Rho)^\t')
    sca VECF_C\t' = ExMC\t'/exp(Rho*T); sca Sum_VECF_C = Sum_VECF_C + VECF_C\t';
  }
replace mini_M = abs(V_d_ExRM - d_I) in `rho';
replace mini_A = abs(Sum_VECF - Vstar) in `rho';
replace mini_C = abs(Sum_VECF_C - V_MC ) in `rho';
}
sort mini_M; sca rho_Marg = index[1]/1000; sort index;
sort mini_A; sca rho_Aver = index[1]/1000; sort index;
sort mini_C; sca rho_MC = index[1]/1000; sort index;
sca beta_xRA = (rho_Aver - rf)/(Erm - rf); sca beta_ratio_A = beta_xRA /beta_p;
sca beta_xRM = (rho_Marg - rf)/(Erm - rf); sca beta_ratio_M = beta_xRM /beta_p;
sca beta_xRC = (rho_MC - rf)/(Erm - rf); sca beta_ratio_C = beta_xRC /beta_p;

noisily di "alf = " Alf " I_Aver = " I_Aver "   beta_xRA/beta_p = " beta_ratio_A;
noisily di "alf = " Alf " I_Marg = " I_Marg "   beta_xRM/beta_p = " beta_ratio_M;
noisily di "alf = " Alf " I_MC   = " I_MC   "   beta_xRC/beta_p = " beta_ratio_C;
replace B_RA_p = beta_ratio_A in `alf'; replace I_RA = I_Aver in `alf';
replace B_RM_p = beta_ratio_M in `alf'; replace I_RM = I_Marg in `alf';
replace B_RC_p = beta_ratio_C in `alf'; replace I_RC = I_MC in `alf';
replace alpha = Alf in `alf';
}

keep B_* I* alpha;

2.5 The betas under carryforwards with interest

drop _all; sca drop _all; local drop _all; set varabbrev off; set more off;
use "the standard Brownian motion observation file";

set obs 10000; gen index = _n; sca CY = 0.04; sca Erm = 0.16;

sum p_un1; sca Ep1 = r(mean); sum p_adj1; sca V_p1 = r(mean)/exp(rf);
sca rho_p = ln(Ep1/V_p1); sca beta_p = (rho_p - rf)/(Erm - rf);

noisily di "This is RA RM RC carryf with rf tax = " tax " sigma = " sigma;
noisily mat list prod; noisily mat list depr;

quietly forvalues alf = 1/9 { ; set more off;
sca Alf = `alf'/10;
} ;
capture S*; capture x*; capture min*; capture pai_xRA;
generate Sum_VxRA = 0; generate pai_xRA = -100000; generate min_I = .; // find I_Aver and I_MC

forvalues i = 1/1000 { ; // search for I<1
    scalar i = `i'/1000; scalar Sum_VxRA = 0; capture drop xRA*; capture drop carr*; generate carryf_1 = 0;
    forvalues t = 1/`period' { ;
        scalar T = `t';
        scalar Q_t = prod[1,`t']^r^Alf ;
generate xRA_adj_t = p_adj_t^*Q_t - tax* max(p_adj_t^*Q_t - depr[1,`t']^r - carryf_1, 0);
scalar xRA_adj_t = scalar ExRA_t = r(mean); scalar VxRA_t = ExRA_t/exp(r^T);
scalar Sum_VxRA = Sum_VxRA + VxRA_t ;
        replace carryf_1 = (1+r^f)* max(depr[1,`t']^r + carryf_1 - p_adj_t^*Q_t, 0);
    } ;
    replace Sum_VxRA = Sum_VxRA in `i';
    replace pai_xRA = Sum_VxRA - i in `i';
}

sort pai_xRA; scalar I_Aver = index[_N]/1000; scalar Vstar = Sum_VxRA[_N]; sort index; scalar I_MC = index[1]/1000; scalar V_MC = Sum_VxRA[1]; sort index;

capture I_Marg = I_Aver; capture d = d_I = 0.0001; capture I_marg = I_Marg - d_I;
capture drop carr*; generate carryf_1 = 0; generate carryf_2 = 0; generate carryf_3 = 0;

forvalues t = 1/`period' { ;
    scalar T = `t';
    scalar Q_t = prod[1,`t']^I_Marg^r^Alf ; // for average and infra-marginal
    generate xRM_un_t = p_un_t^*Q_t - tax* max(p_un_t^*Q_t - depr[1,`t']^I_Marg - carryf_1, 0);
    replace carryf_1 = (1+r^f)* max(depr[1,`t']^I_Aver + carryf_1 - p_un_t^*Q_t, 0);
    scalar q_t = prod[1,`t']^I_marg^r^Alf ; // for infra-marginal
    generate xRM_un_t = p_un_t^*Q_t - tax* max(p_un_t^*Q_t - depr[1,`t']^I_marg - carryf_1, 0);
    replace carryf_1 = (1+r^f)* max(depr[1,`t']^I_Aver + carryf_1 - p_un_t^*Q_t, 0);
    scalar xRM_un_t; scalar ExRM_t = r(mean);
scalar sum_xRM_un_t; scalar ExRM_t = r(mean);
scalar Q_MC_t = prod[1,`t']^I_MC^r^Alf ; // for marginal
    generate xMC_un_t = p_un_t^*Q_MC_t - tax* max(p_un_t^*Q_MC_t - depr[1,`t']^I_MC - carryf_3, 0);
    replace carryf_3 = (1+r^f)* max(depr[1,`t']^I_MC + carryf_3 - p_un_t^*Q_MC_t, 0);
    scalar xMC_un_t; scalar ExMC_t = r(mean);
scalar sum_xMC_un_t; scalar ExMC_t = r(mean);
} ;

generate mini_A = . ; generate mini_M = . ; generate mini_C = . ;
forvalues rho = 1/1000 { ;
    scalar rho = `rho'/1000; scalar V_d_ExRM = 0; scalar Sum_VECF = 0; scalar Sum_VECF_C = 0;
    forvalues t = 1/`period' { ;
        scalar T = `t';
        scalar V_d_ExRM = V_d_ExRM + d_d_ExRM_t/exp(Rho^T);
        scalar VECF_t = ExRM_t/exp(Rho^T); scalar Sum_VECF = Sum_VECF + VECF_t ; // ((1+Rho)^T)
        scalar VECF_C_t = ExMC_t/exp(Rho^T); scalar Sum_VECF_C = Sum_VECF_C + VECF_C_t ;
    } ;
    replace mini_M = abs(V_d_ExRM - d_I) in `rho';
    replace mini_A = abs(Sum_VECF - Vstar) in `rho';
    replace mini_C = abs(Sum_VECF_C - V_MC) in `rho';
}

sort mini_M; scalar rho_Marg = index[1]/1000; sort index; scalar mini_A; scalar rho_Aver = index[1]/1000; sort index; scalar mini_C; scalar rho_MC = index[1]/1000; sort index;
sca beta_xRA = (rho_Aver - rf)/(Erm - rf); sca beta_ratio_A = beta_xRA /beta_p;
sca beta_xRM = (rho_Marg - rf)/(Erm - rf); sca beta_ratio_M = beta_xRM /beta_p;
sca beta_xRC = (rho_MC - rf)/(Erm - rf); sca beta_ratio_C = beta_xRC /beta_p;

noisily di "alf = " Alf " I_Aver = " I_Aver  " beta_xRA/beta_p = " beta_ratio_A;
noisily di "alf = " Alf " I_Marg = " I_Marg  " beta_xRM/beta_p = " beta_ratio_M;
noisily di "alf = " Alf " I_MC   = " I_MC    " beta_xRC/beta_p = " beta_ratio_C;

replace B_RA_p = beta_ratio_A in `alf'; replace I_RA = I_Aver in `alf';
replace B_RM_p = beta_ratio_M in `alf'; replace I_RM = I_Marg in `alf';
replace B_RC_p = beta_ratio_C in `alf'; replace I_RC = I_MC in `alf';
replace alpha = Alf in `alf';
}

keep B_* I* alpha;

#delimit cr

2.6 Graph commands

// beta ratios
twoway(line B_RA_p B_RA_p_0 alpha in 1/9, ylabel(0.3(0.1)1) xlabel(0(0.1)1) ///
title("Betas, desired title") ///
color(black black) lpattern(solid shortdash) ///
legend(title("desired legend title", size(m.large))) ///
(line B_RM_p B_RM_p_0 alpha in 1/9, color(dkorange dkorange) lpattern(solid shortdash)) ///
(line B_RC_p B_RC_p_0 alpha in 1/9, color(lime lime) lpattern(solid shortdash))

// investments
twoway(line I_RA I_RA_0 alpha in 1/9, ylabel(0(0.1)1) xlabel(0(0.1)1) ///
title("Investments, desired title") ///
color(dknavy dknavy) lpattern(solid shortdash) ///
legend(title("desired legend title", size(m.large))) ///
(line I_RM I_RM_0 alpha in 1/9, color(dkorange dkorange) lpattern(solid shortdash)) ///
(line I_RC I_RC_0 alpha in 1/9, color(lime lime) lpattern(solid shortdash))