Essays on Consumption and Risk Sharing

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## Contents

1 Introduction .................................................. 1

2 Insure the Uninsurable by Yourself: Accounting for Consumption Insurance in a Life-cycle Model ....... 5
   2.1 Introduction ............................................. 5
   2.2 Structured Facts ........................................ 8
      2.2.1 BPP’s measure ...................................... 8
      2.2.2 Theoretical counterpart of BPP’s measure ........... 9
      2.2.3 BPP’s findings ...................................... 10
   2.3 Model ..................................................... 11
      2.3.1 Environment ......................................... 11
      2.3.2 Equilibrium ......................................... 13
      2.3.3 Estimation .......................................... 14
   2.4 Results ................................................... 14
      2.4.1 Calibration in the benchmark setup ................. 14
      2.4.2 Accounting for partial insurance .................. 16
      2.4.3 Self-insurance over the life cycle .................. 17
      2.4.4 The role of wealth income ratio ................... 21
   2.5 Robustness ............................................... 22
      2.5.1 Sensitivity .......................................... 22
      2.5.2 Misspecification of income process ................. 25
      2.5.3 Alternative measures of insurance ................ 27
   2.6 Conclusion ............................................... 33
   2.7 Appendix ............................................... 34
      2.7.1 Numerical algorithm ................................ 34
      2.7.2 Some properties of the estimators ................. 35

3 Consumption Inequality and Discount Rate Heterogeneity ....... 41
   3.1 Introduction ............................................. 41
   3.2 Model .................................................... 44
      3.2.1 Calibration .......................................... 46
      3.2.2 Estimation of the distribution of $\beta$ ........... 48
   3.3 Results ................................................... 52
      3.3.1 Age profile of consumption inequality ............ 52
      3.3.2 Role of pension ..................................... 56
4 Complete Markets Strikes Back: Revisiting Risk Sharing Tests under Preference Heterogeneity

4.1 Introduction .................................................. 71
4.2 The Risk Sharing Tests ...................................... 75
  4.2.1 Methodology ............................................. 75
  4.2.2 Evidence ................................................. 76
4.3 Complete Market Models ................................. 79
  4.3.1 Environment ............................................. 79
  4.3.2 Complete markets with stochastic income .......... 80
  4.3.3 Bond economy with predictable income change .... 83
4.4 Matching the Data ...................................... 84
  4.4.1 Identification ......................................... 85
  4.4.2 Evaluating the models ................................ 87
4.5 Is $\text{cov}(\beta_i, \theta_i) > 0$ plausible? .......... 90
  4.5.1 Education choice .................................... 90
  4.5.2 Mortality and health ................................. 91
4.6 Conclusion ............................................... 92
4.7 Appendix .................................................. 93
  4.7.1 Proofs .................................................. 93
Chapter 1

Introduction

This thesis consists of three related essays on consumption and risk sharing. It contributes to the growing literature which focuses on the importance of consumption inequality. Compared with wage, income and wealth, consumption reflects the abundance of the lifetime resource of a household and is a more direct measure of economic welfare. In particular, the age profile of consumption inequality provides us with valuable information for the nature of idiosyncratic labor income processes, the degree of consumption insurance or risk sharing, and, ultimately, the inequality of welfare. In other words, the welfare implication of income inequality depends on how much income inequality is transmitted into consumption inequality. This leads to the most intriguing question: how well do people insure against income risks? The answer to this question is crucial in both model choices and policy choices. Moreover, it should be noticed that the questions are usually addressed with the implicit assumption of homogeneous preferences. With the presence of ex ante heterogeneity, however, we might have complete different answer to this question. Using quantitative economic theory, I will attempt to answer this important question of risk sharing in the following three chapters.

In Chapter 2, "Insure the Uninsurable by Yourself: Accounting for Consumption Insurance in a Life-cycle Model", I lay down the foundation for further investigation of consumption and risk sharing. It is a theoretical and quantitative response to the recent empirical research by Blundell, Pistaferri and Preston (2008 AER) (BPP hereafter), who find that individual consumption does not fully respond to permanent income shocks and thus conclude that there must be partial insurance beyond self-insurance. In this chapter, I investigate the degree of consumption insurance by BPP’s measure, in a standard overlapping-generations life-cycle model calibrated to the U.S. data. To directly confront the model predictions with the recent empirical evidence of BPP, I apply the same estimation procedure proposed by BPP to the model generated
1. Introduction

data. I find that the self-insurance model can account for almost all the insurance for transitory shocks found in the data, which is in line with the prediction of PILCH. As to the insurance against the permanent shocks, however, my result is far from the conventional wisdom. I find that 80% of the partial insurance for the permanent shocks found by BPP can be accounted for by a model with only self-insurance available. This result does not vary much throughout a variety of parameters. If there were model misspecifications that income process was less persistent and agents had advance information of income shocks, the self-insurance model would perform even better. This indicates that the previous perception that one cannot insure against permanent shocks by wealth is misleading.

I also find heterogeneity of the degree of insurance in age and wealth. The consumption insurance increases over the life cycle and increases sharply in the last 10 years before retirement, whereas BPP find that insurance follows a non-significant linear trend. This is mainly due to the wealth effect: as the old have accumulated a large amount of life-cycle wealth, they have enhanced their ability of self-insurance and their consumption becomes less sensitive to income shocks. This result reconfirms the puzzle of the shape of consumption inequality profile in life-cycle models, which motives the studies in the next two chapters.

In Chapter 3, "Consumption Inequality and Discount Rate Heterogeneity", I investigate the role of discount rate heterogeneity in consumption inequality. The U.S. data for consumption inequality have two salient features: the consumption inequality significantly increases over the life cycle and the profile is approximately linear, or at least non-concave. This evidence gives us valuable information of identifying different labor income process, namely, (1) the Restricted Income Profiles (RIP) model with highly persistent shocks and (2) the Heterogeneous Income Profiles (HIP) model where shocks are less persistent. In this chapter, I make a small modification to the standard model by only relaxing the assumption of homogeneous discount factors. After estimating the distribution of the discount factor, I use the model to study consumption inequality in both RIP and HIP models. The target of estimation I chose is the fraction of agents holding zero or negative wealth, which cannot be explained by a standard homogeneous discount rate life-cycle model with realistic social security.

The main finding of this chapter is that the quantitative model with discount rate heterogeneity can be successful in matching the empirical age profile of consumption inequality. A HIP model with discount rate heterogeneity matches the data well in both the magnitude of increase and the shape of consumption inequality profile, while RIP model does not. If there exists partial insurance, however, a RIP model can also do a good job in matching the consumption profile. Therefore, consumption data alone is not sufficient for us to distinguish between HIP model with only self-insurance and
1. Introduction

RIP model with partial insurance. I also find that without taking into account discount rate heterogeneity, the consumption insurance in the model is up-ward biased and the importance of borrowing constraints is underrated. Finally, the life-time inequality is mostly due to initial conditions (96%), among which the discount rate heterogeneity is the most important and the fixed effect of income is the least important.

In Chapter 4, "Complete Markets Strikes Back: Revisiting Risk Sharing Tests under Preference Heterogeneity", I go one step further than Chapter 2 and Chapter 3, and ask a more general question about market completeness and heterogeneity. I see this as potentially the starting point of a new agenda of structural estimation of ex ante heterogeneity in a general framework. Previous risk sharing tests reject the hypothesis of complete markets because in the data the consumption dispersion increases over the life cycle and individual consumption co-moves with income. In this chapter I show that, if there is positive correlation between heterogeneous discount factors and heterogeneous income growth rates, all these tests arrive at the conclusion of imperfect risk sharing even if the markets are actually complete. I first demonstrate that it is theoretically possible for a class of complete market models to pass these risk sharing tests. I then show that it is quantitatively admissible for a simple parameterized complete market model to account for the empirical evidence on both the consumption dispersion and the co-movement of consumption and income. Using the results from Chapter 3, I evaluate the complete market (CM) model and the standard incomplete market (SIM) models in a calibrated framework with heterogeneous preferences and heterogeneous income profiles. I find that the CM model outperforms the SIM models in matching the data. Finally, I demonstrate that the positive correlation between discount factors and income growth rates can be justified by a model of education choice or a model of health.

The result from this chapter has two main implications. On the one hand, even if we do not take the extreme case of complete market, it suggests that there could exist more risk sharing, or more insurable risks, than we previously thought. On the other hand, the complete market model highlights the importance of ex ante heterogeneity. Due to its simplicity, the CM model provides us with a useful framework for risk sharing: by understanding the success and failure of the calibrated CM model in matching the data, we can learn more about the market structure, the income process and the preference heterogeneity.
Bibliography

Chapter 2

Insure the Uninsurable by Yourself: Accounting for Consumption Insurance in a Life-cycle Model

2.1 Introduction

Compared with wage, income and wealth, consumption reflects the abundance of the life-time resource of a household and is a more direct measure of economic welfare. To understand the determination of consumption inequality, the degree of consumption insurance is important: it determines how much income inequality is transmitted into consumption inequality. In the real world, income shocks are mitigated by a variety of insurance mechanisms. In economic theory, different hypotheses of market structure provide different levels of consumption insurance. Figuring out consumption insurance in the class of quantitative models and test the model hypotheses by the data is, in my view, an indispensable step towards our better understanding of consumption inequality.

This paper has two main goals. The first is to account for the consumption insurance found in the data, using a (simplest possible) calibrated model. It is motivated by the recent empirical findings of Blundell, Pistaferri, and Preston (2008) (BPP here-after) where they find the evidence of “some partial insurance for permanent shocks and almost complete insurance of transitory shocks”. On the one hand, BPP’s finding is consistent with the previous work using micro data (Attanasio and Davis 1996) that soundly rejects the complete market hypothesis under which individuals’ income risks
are completely insured. On the other hand, BPP’s finding is also consistent with the “consumption excessive smoothness puzzle” in the macro literature (Campbell and Deaton 1989), where the consumption reacts too little to the permanent shocks as predicted by the Permanent Income/Life Cycle Hypothesis (PILCH).

There are a few attempts to explain the consumption inequality and risk sharing found in the data using quantitative models: e.g. Storesletten, Temler and Yaron (2004) in an overlapping generations model, Krueger and Perri (2006) in a limited enforcement model, Attanasio and Pavoni (2008) in a dynamic moral hazard model with hidden savings, etc. Usually, an applied theorist tests the model by evaluating the distance between some of the model generated moments and those found in the data. To confront the model with “structured facts” as found by BPP, however, we must keep in mind that we impose the same structure as the empirical work does. In order to test the model’s prediction on the degree of consumption insurance, the applied theorist has to work as an applied econometrician to estimate the measure of consumption insurance by a series of artificial data generated by the model. This is what this paper will do.

The second main goal of this paper is the flip side of the first one. I ask: how much is the level of self-insurance in the stationary equilibrium of an incomplete market overlapping-generations life-cycle model and how does individual’s ability of self-insurance vary across age and wealth? Before we explicitly model any insurance market, it is useful to start from a scenario where all insurance markets are shut down and only a risk-free bond is traded. This scenario, where agents can only smooth consumption by self-insurance through borrowing and saving, dates back to PILCH (Friedman 1957, Brumberg and Modigliani 1954) as one of the main workhorses in macroeconomics. Partial equilibrium versions of PILCH with precautionary savings motive and/or liquidity constraint include the work of Deaton (1991), Carroll (1997), Gourinchas and Parker (2002), Cagetti (2003), among others. As a heterogeneous-agent general equilibrium generalization of PILCH, Bewley (1986), Imrohoroglu (1989), Huggett (1993), Aiyagari (1994) develop a class of incomplete market models with the same assumption that the exogenous income shocks are idiosyncratic and uninsurable. In a number of quantitative studies, researchers also incorporate life-cycle and overlapping generations feature formulated in Rios-Rull (1994) into the Bewley-Imrohoroglu-Hugget-Aiyagari framework (e.g. Huggett 1996, Castaneda et al. 2003, Storesletten et al. 2004).

As Deaton (1992) put it, self-insurance may allow inter-temporal consumption smoothing against the transitory shocks, but it cannot insure against the permanent shocks without violating the budget constraint. The conventional wisdom of insurance under PILCH models is that agents can obtain almost complete insurance against the
transitory shocks and almost no insurance against the permanent shocks. However, when there is positive wealth, the income shocks might become less substantial, so that the consumption may not respond completely even if the shock is permanent. After all, how much insurance it is in a life-cycle model is a quantitative question.

To achieve these two main goals, I construct a standard overlapping-generations model calibrated to the U.S. data, estimate the degree of consumption insurance using the same measure proposed by BPP, and confront the model results with BPP’s empirical evidence directly. My findings are the following. The self-insurance model can account for almost all the insurance for transitory shocks found in the data, which is in line with the prediction of PILCH. As to the insurance against the permanent shocks, however, my result is far from the conventional wisdom. I find that 80% of the partial insurance for the permanent shocks found by BPP can be accounted for by a model with only self-insurance available. This result does not vary much throughout a variety of parameters. If there were model misspecifications that income process was less persistent and agents had advance information of income shocks, the self-insurance model would perform even better.

I also find heterogeneity of the degree of insurance in age and wealth. The consumption insurance increases over the life cycle and increases sharply in the last 10 years before retirement, whereas BPP find that insurance follows a non-significant linear trend. This is mainly due to the wealth effect: as the old have accumulated a large amount of life-cycle wealth, they have enhanced their ability of self-insurance and their consumption becomes less sensitive to income shocks. This result reconfirms the puzzle of the shape of consumption inequality profile in life-cycle models\(^1\): the age profile of consumption inequality is concave in the model, whereas consumption inequality in data (e.g. Deaton and Paxson 1994) grows linearly over the life cycle.

The rest of the paper is organized as follows. Section 2 provides the structured facts of consumption insurance found by BPP and discusses the rationale of BPP’s approach. Section 3 presents a standard incomplete market overlapping-generations life-cycle model. Section 4 calibrates the model to the U.S. data and then reports the partial insurance parameters estimated by BPP approach using the artificial data set generated by model simulation. Section 5 explores the robustness of the result by using different model parameters, different specifications of income process and different measures of insurance. Section 6 concludes. Proofs and the technical details of solution method are in Appendix.

\(^1\)See Storesletten et al. (2004) for a discussion and a tentative resolution for this puzzle.
2. Insure the Uninsurable by Yourself: Accounting for Consumption Insurance in a Life-cycle Model

2.2 Structured Facts

In the data, how does household’s consumption respond to income shocks? It is not a trivial empirical question. Firstly, we need a panel data set of both consumption and income. Unfortunately, so far there is no large raw panel data set for broad measures of consumption\(^2\). The lack of data source is arguably one of the reasons why joint distribution of consumption and income has not been well documented. Secondly, a measure of consumption insurance must be constructed. Finally, we need to estimate the measure of insurance by imposing some structure on the joint distribution of income and consumption process. In other words, the empirical evidence we get are “structured” facts.

2.2.1 BPP’s measure

BPP provide the structured facts of the consumption insurance in the U.S. from late 1970s to early 1980s. After mapping food data into consumption data using the estimates of a demand function for food that are present in both the PSID and the CEX, they create an unbalanced panel data series of consumption and income.

BPP denote partial insurance as the degree of transmission of income shocks to consumption and construct an empirical measure of it. They adopt a widely used income process in which the unexplained log income can be decomposed into a unit root permanent part and an i.i.d. transitory part

\[
\begin{align*}
\log y_t^i &= z_t^i + \varepsilon_t^i, \\
z_t^i &= z_{t-1}^i + \eta_t^i,
\end{align*}
\]

where \(t\) indexes time, \(\varepsilon_h^i \sim N(0, \sigma_{\varepsilon}^2), \eta_t^i \sim N(0, \sigma_{\eta}^2)\). BPP further assume that the growth of unexplained log consumption is linear in the innovation of permanent shocks and transitory shocks

\[
\Delta \log c_t^i = \phi_t^i \eta_t^i + \psi_t^i \varepsilon_t^i + u_t^i, 
\]

where \(\eta_t^i\) is the innovation of the permanent shock and \(\varepsilon_t^i\) is the transitory shock, \(u_t^i\) is the error term. The loading factors \(\phi_t^i\) and \(\psi_t^i\) are BPP’s measure of consumption insurance, which are called partial insurance parameters\(^3\). The closer the coefficient is to zero, the higher the degree of insurance. With the panel data set of consumption and income they have created, the partial insurance parameters can be identified.

\(^2\)For example, the Panel Study of Income Dynamics (PSID) contains longitudinal income data, but its consumption items are restricted to food. On the other hand, the Consumer Expenditure Survey (CEX) contains a variety of measures of consumption, but it is a repeated cross-section data set.

\(^3\)Although BPP allow \(\phi_t^i\) and \(\psi_t^i\) to vary across types of households, BPP assume away any individual heterogeneity in their main estimation and use \(\phi_t\) and \(\psi_t\).
2.2.2 Theoretical counterpart of BPP’s measure

Before we discuss about BPP’s empirical findings, it is useful to ascertain what the \( \phi_i^j \) and \( \psi_i^j \) are in some standard theories. To single out the effect of consumption insurance, we assume that agent’s utility is separable in consumption and leisure and thus the marginal utility of consumption is independent of leisure.

**Complete Market**: \( \phi_i^j = 0, \psi_i^j = 0 \). When the insurance market is complete in the sense that there is a full set of contingent claims against any income uncertainty, each agent’s consumption does not respond to income shocks.

**Autarchy**: \( \phi_i^j = 1, \psi_i^j = 1 \). Since consumption is nondurable and there is no trade available, consumption of each agent tracks individual income in each period.

**PILCH with linear quadratic preference** In PILCH, only a risk-free bond is available and the agents can only insure against the shocks by self-insurance through saving and borrowing. In the text book version of PILCH where the preference is linear quadratic, the certainty equivalence is attained and thus consumption is a martingale process (Hall 1978) which follows\(^4\)

\[
\Delta c_t^j = \eta_t^j + \frac{r - 1}{1 + r} \xi_t^j, \tag{2.3}
\]

where \( \zeta_t = (1 - \frac{1}{1 + r} r) \) is the annuitization factor, \( T \) is the time horizon, \( r > -1 \) is the interest rate.

It is tempting to think that \( \phi_i^j = 1 \) in this case. However, it is not true. Comparing (2.3) with (2.2), we notice that \( \Delta c_t \) is the difference of consumption, not log consumption as in BPP’s measure. More generally, if the growth of log consumption is not linear in income innovations, then the theoretical prediction of partial insurance parameters is not clear.

**PILCH with CARA preference** The PILCH model with Constant Absolute Risk Aversion (CARA) permits precautionary saving. Caballero (1990) proves that

\[
\Delta c_t^j = \eta_t^j + \frac{r - 1}{1 + r} \xi_t^j + \Gamma_t^j, \tag{2.4}
\]

where \( \Gamma_t^j \) is the term representing precautionary savings but independent of the realization of income shocks. The level of consumption follows a random walk with a drift, while the growth of log consumption is not necessarily linear in income innovations. Therefore, the theoretical counterpart of BPP’s measure cannot be derived in the CARA case, either.

**PILCH with CRRA preference** In the PILCH model with Constant Relative Risk Aversion (CRRA) preference where precautionary saving motive is present, BPP

\(^4\) Notice that this and the following expressions of consumption require that the borrowing constraints are not binding.
prove that the growth of log consumption can be approximated as:  
\[ \Delta \log c_t^i \simeq \pi^i_t \eta^i_t + \pi^i_t \frac{r I_{t-1}}{1+r} + \xi^i_t. \] 

(2.5)

where \( \xi^i_t \) is the innovation independent of the income shocks, \( \pi^i_t = \sum_{s=0}^{T} (1+r)^{-s} I_{t-s} \), \( I^i_t \) is the income, \( W^i_t \) is the (nonhuman) wealth. This nice expression shows that in the PILCH with CRRA preference, the degree of insurance can be measured by the share of human wealth (discounted future labor income) in the current total wealth. Notice that \( \phi^i_t \) exceeds unit if agent \( i \) is in debt.

Equation (2.5) is the rationale behind BPP’s approach. The model prediction of partial insurance parameters of \( \phi^i_t \) and \( \psi^i_t \) depends on the time span, the interest rate, and the distribution of the wealth and income in the economy. When there is no wealth, \( \pi^i_t = 1 \) implies there is no insurance against the permanent shock. On the other hand, if \( r \) is a small positive number, then \( \psi_{i,t} \simeq 0 \), implying the agent can obtain almost full insurance against transitory shock. Most of the previous empirical works assume a one-for-one response of log consumption to the permanent income shocks (e.g. Blundell and Preston 1998, Primiceri and Rens 2009). However, if there is non-negligible positive wealth, \( \phi^i_t \) is not necessarily close to 1 even in the CRRA case with infinite horizon.

2.2.3 BPP’s findings

BPP’s main finding is that, in the whole sample, the estimate of \( \phi \) and \( \psi \) is 0.6423 and 0.0533, respectively. In other words, a 10 percent permanent (transitory) shock of disposable income induces a 6.4(0.5) percent change in household’s nondurable consumption. This result provides the evidence of the partial insurance against permanent shocks, while the hypothesis of complete insurance against transitory shocks cannot be rejected. BPP also do the estimation by two subgroups of age and wealth, and more insurance is found in the subgroup of older cohorts and wealthier households. Finally, they find a non-significant linear trend of decreasing partial insurance parameter for permanent shocks over the life cycle. BPP conclude: “Neither of these [self-insurance and complete market] models were found to accord with the evidence.”

In order to find out how much partial insurance is over and above self-insurance, or, in other words, how much the self-insurance models fail to explain, one needs to take

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5This results is consistent with the finding of Constantinides and Duffie (1996) that in a bond economy where only permanent shock are uninsurable and there is no initial wealth, the equilibrium allocation is autarchy.

6See theoretical analyses of the unimportance of transitory shocks in Yaari (1975), Levine and Zame (2002).
2. Insure the Uninsurable by Yourself: Accounting for Consumption Insurance in a Life-cycle Model

the first step to quantitatively investigate the degree of insurance in a self-insurance model using BPP’s measure and estimation procedure. Moreover, the rationale of BPP’s measure is based on the self-insurance model with CRRA preference; any other models where the growth of log consumption is not approximately linear in the income innovations can not be directly compared with BPP’s empirical evidence.

2.3 Model


2.3.1 Environment

Time is discrete. The economy is populated by $T$ overlapping generations, each of which consists of a continuum of agents. Each agent is born at age 1 and can live a maximum of $T$ periods. Agents face mortality risks. The probability of surviving between age $t-1$ and age $t$ is denoted by $\xi_t$, with $\xi_1 = 1$ and $\xi_{T+1} = 0$. The unconditional probability of being alive at age $t$ is $s_t = \prod_{\tau=1}^{t} \xi_\tau$. The measure of the new born agents is denoted by $\mu_1$ and the population grows at a constant rate $n$, implying a stable population structure with $\mu_t = \mu_1 s_t (1+n)^{1-t}$.

Agents’ preference over the stream of consumption is given by:

$$E \sum_{t=1}^{T} \beta^{t-1} s_t u(c_t),$$

where $\beta$ is the time discount factor and $c_t$ is the consumption at age $t$. I further assume that the period utility is in CRRA\textsuperscript{7} form with risk aversion $\sigma$:

$$u(c_t) = c_t^{1-\sigma}/(1-\sigma).$$

Agents enter the labor market at age 1 and the mandatory retirement age is $R$. At working age $t < R$, the agents supply inelastically one unit of labor, while they differ in the efficient unit of labor. The exogenous uninsurable labor income of agent $i$ is

$$y_i^t = (1-\tau) w e_i^t,$$

\textsuperscript{7}I stick to the CRRA utility for three reasons: 1. It makes a consistent comparison with BPP’s estimation; 2. It gives us the stationary equilibrium under balanced growth path; 3. It simplifies the numerical computation by dropping the permanent shock as a state variable.
2. Insure the Uninsurable by Yourself: Accounting for Consumption Insurance in a Life-cycle Model

where $\tau$ is the pension tax, $w$ is the wage rate identical to all the agents in a given cross-section, and $e$ is the efficient unit of labor which is assumed to follow

$$\log c_i^t = \frac{\kappa_t + \gamma_i^t + \theta_i^t t + z_i^t + \varepsilon_i^t}{\text{predictable part}} + \frac{\rho z_i^t + \eta_i^t}{\text{idiosyncratic shocks}}, \quad (2.7)$$

where $\rho \in (0, 1], \eta_i^t \sim N(0, \sigma_h^2), \varepsilon_i^t \sim N(0, \sigma_e^2), z_0^i = 0$. The income can be decomposed into a predictable part and an idiosyncratic shocks part. $\kappa_t$ is the income profile which is identical to all the agents of the same age. In the stationary equilibrium where there is only a cross-section of overlapping generations, there is no time effect and the cohort and age coincide. The next two terms are heterogeneous in agents: $\gamma_i^t$ is the fixed effect which is predetermined before the agent enters the labor market; $\theta_i^t$ is the heterogeneity in the growth rate of individual income. The idiosyncratic shocks part consists of a permanent (or AR(1) when $\rho < 1$) part $z_i^t$ and a transitory (i.i.d.) part $\varepsilon_i^t$. If BPP’s income model is not misspecified, $\rho = 1$ and $\sigma_h^2 = 0$.

After retirement, the agent receives pension $B_t$ which is funded by a Pay-As-You-Go system through the pension tax $\tau$. For simplicity, $B_t$ is assumed as a constant fraction of income at one year before retirement\(^9\)

$$B_t^i = b y_{R-1}^i.$$

The market is incomplete in the sense that agents can only have access to a risk-free bond which yields the net interest rate $r$. In the benchmark setup, I assume that there exist perfect annuity markets for mortality risks, so that the return of asset is interest rate plus a survival premium. The agent’s budget constraint is given by

$$c_t + a_{t+1} \leq a_t (1 + r) / \xi_t + y_t, \quad (2.8)$$

where $a_t$ is the asset or financial wealth. The agent cannot leave negative asset at year $T$ and faces a borrowing constraint $a_{t+1} \geq a$, where $a$ is an ad hoc borrowing constraint which can be set as low as the natural borrowing constraint.

To close the model, I adopt the Cobb-Douglas aggregate production:

$$Y = AK^\alpha L^{1-\alpha}, \quad (2.9)$$

\(^8\)To be compared with BPP’s estimation using net family income, no redistribution tax is considered here.

\(^9\)I assume away the redistribution function of the pension system. Including concave pension benefit would deliver more insurance in the model. Technically, if $B_t$ is independent of the previous income and is proportional to the last working year’s income, then the individual’s problem can be reformulated to eliminate the permanent component of income as a state variable.
2. Insure the Uninsurable by Yourself: Accounting for Consumption Insurance in a Life-cycle Model

where $K$ is the aggregate capital, $L$ is the aggregate efficient labor, $A$ is the aggregate productivity level which grows at a constant rate $g$. The law of motion of capital is $K' = Y - C + (1 - \delta)K$, where $C$ is the aggregate consumption and $\delta$ is the depreciation rate of capital. In the robustness test, I will assume away the production side by simply considering a small open economy where the asset is supplied with infinite elasticity at world interest $r$.

2.3.2 Equilibrium

For CRRA utility function, we can obtain the balanced growth path by dividing all the quantities by the accumulated productivity growth. Given constant $r$ and $w$, each agent’s decision problem can be written recursively as

$$V(\gamma, \theta, a, \varepsilon, z; t) = \max_{a'} \left\{ \frac{c^{1-\sigma}}{1-\sigma} + \beta \xi_{t+1} (1 + g)^{1-\sigma} E[V(\gamma, \theta, a', \varepsilon', z'; t + 1)|z] \right\}$$

subject to

$$c + (1 + g)a' \leq a(1 + r)/\xi_t + \begin{cases} y_t & t \leq R \\ B_t & t > R \end{cases}$$

$$a' \geq \frac{a}{a_{T+1}} \geq 0$$

(2.10)

The terminal period value function is set to $V(\cdot; T + 1) = 0$. The equilibrium we study is a stationary recursive competitive equilibrium where the factor prices are constant over time and the age-wealth distribution is stationary. Formally, denote $(X, B(X), \Psi_t)$ as the probability space, where $X$ is the domain of state variables, $B(X)$ is the Borel $\sigma$-algebra on $X$, and $\Psi_t$ is the probability measure. Denote $P(x, t, B)$ as the probability that an age $t$ agent transit to set $B$ given the agent’s current state is $x$. This transition function is derived from the individual’s decision rule $a'(\cdot)$.

Definition 1 A stationary recursive competitive equilibrium is a pair of prices \{r, w\}, a value function and a decision rule \{V(\cdot), a'(\cdot)\}, such that

(i) Individual optimization: $V(\cdot), a'(\cdot)$ solve the agent’s Bellman equation (2.10).

(ii) Competitive firms maximize profits: $w = (1-\alpha)AK^\alpha L^{-\alpha}; r = \alpha AK^{\alpha-1}L^{1-\alpha} - \delta$.

(iii) Markets clear: $\sum_{t=1}^{T} \mu_t \int_X a_t^0 d\Psi_t = K; \sum_{t=1}^{R-1} \mu_t \int_X e_t^0 d\Psi_t = L$.

(iv) The distribution is consistent with individual’s behavior: $\Psi_{t+1} = \int_X P(x, t, B)d\Psi_t$, for all $t$ and $B \in B(X)$.
2. Insure the Uninsurable by Yourself: Accounting for Consumption Insurance in a Life-cycle Model

\[(v)\text{Pension is funded by a Pay-As-You-Go system: } \tau \sum_{t=1}^{R-1} \mu_t \int_X y_t^i \Psi_t = \sum_{t=R}^{T} \mu_t \int_X B_t^i d\Psi_t.\]

Under this standard setup, the stationary equilibrium is attained immediately. The individual’s problem can be solved numerically by backward induction. Technical details are in Appendix.

2.3.3 Estimation

After simulating the model, we would obtain an artificial panel data set of $y_t^i$ and $c_t^i$. It is a panel data of $(N \times T)$ observations of one cohort, where $N$ is the number of agents in this cohort. In BPP, the unexplained consumption and income are the residuals of the first stage regression on observable individual characteristics, the year dummies and the cohort dummies. There is no observable individual characteristics in the simulated data. The residual of the regression on year and cohort dummies \((\tilde{y}_t^i, \tilde{c}_t^i)\) can be simply obtained by subtracting the average log income and log consumption at each age:

\[
\tilde{y}_t^i = \log y_t^i - \overline{\log y_t} \\
\tilde{c}_t^i = \log c_t^i - \overline{\log c_t}
\]

We can calculate the partial insurance parameters for each age by BPP approach,

\[
\hat{\phi}_t^{BPP} = \frac{E(\Delta \tilde{c}_t (\Delta \tilde{y}_{t-1} + \Delta \tilde{y}_t + \Delta \tilde{y}_{t+1}))}{E(\Delta \tilde{y}_t (\Delta \tilde{y}_{t-1} + \Delta \tilde{y}_t + \Delta \tilde{y}_{t+1}))}
\]

\[
\hat{\psi}_t^{BPP} = \frac{E(\Delta \tilde{c}_t \Delta \tilde{y}_{t+1})}{E(\Delta \tilde{y}_t \Delta \tilde{y}_{t+1})}
\]

and the \(\hat{\phi}^{BPP}\) and \(\hat{\psi}^{BPP}\) of whole sample are estimated by pooled estimation in the same sample as BPP’s.

2.4 Results

2.4.1 Calibration in the benchmark setup

Demography The model period is 1 year. Agents begin to work at age 22, which coincides with age 1 in the model. Conditional on surviving, they then work for 45 years, retire at age 66 and die at age 100. Agents are interpreted as households in the

\footnote{By the law of large numbers, we can construct a cross-section of agents by adjusting the effect of wage growth, mortality and population growth. The sample size of each generation is equal in the estimation.}
data, and hence we chose the conditional surviving rate from the U.S. life table for females in 1989-1991. The annual population growth rate is set to \( n = 1.0\% \) per year.

**Preference** The utility function is CRRA with risk aversion set to \( \sigma = 3 \). Since wealth income ratio is the key determinant in the degree of insurance, I calibrate the time discount factor \( \beta \) to match the wealth income ratio in the U.S. Note that \( \beta \) not only serves as the time discount factor but also captures all the factors outside the model that determine the wealth income ratio. I use the wealth income ratio of 4.56, which is the average wealth to earnings ratio of the 99\% wealth quantile from SCF 1992 and 1998\(^{11} \). It gives \( \beta = 0.997 \) in the benchmark model.

**Production** The secular productivity growth rate is set to \( g = 1.5\% \) per year. The capital share is set to \( \alpha = 0.33 \) and the depreciation rate is \( \delta = 0.06 \). This parameterization is standard and consistent with Cooley (1995). In the general equilibrium, it generates \( r = 4.96\% \) and the capital output ratio \( K/Y \simeq 3 \).

**Income process** Income in BPP’s data is family net income. To make a reliable comparison with the empirical finding, I use the average variance of transitory shocks and permanent shocks estimated by BPP (2008 Table VI) of 1979–1992 in PSID and get \( \sigma_{\eta}^2 = 0.0188 \) and \( \sigma_{\varepsilon}^2 = 0.0407 \). The variance of the fixed effect is set to \( \sigma_{\gamma}^2 = 0.2105 \) as estimated by Storesletten et. al (2004) from PSID. The average age profile of income \( \kappa_t \) is chosen to match the average income in the U.S. Census 1990. In the benchmark model, I set \( \rho = 1 \) and assume away the profile heterogeneity by setting \( \theta^i = 0 \), implying the income process BPP use is not misspecified.

**Pension** The coefficient \( b \) is calibrated to match the average replacement ratio (48\%) in the U.S. In the benchmark setup, it generates \( b = 0.393 \). In a PAYG system, it requires a pension tax of 13.3\%, which is fairly close the U.S. contribution rate of 12\%.

**Borrowing constraint** In the benchmark setup, I consider the self-insurance in a strict sense: households are excluded from any borrowing, i.e. \( a = 0 \). This no-borrowing assumption might seem stark. Yet we will see below that the borrowing constraint is not quantitatively important for the households in the sample for estimation.

**Initial Wealth** No initial wealth is considered in the benchmark setup.

The key parameters used in benchmark setup are listed in Table 1.

\(^{11}\)Neither the standard incomplete market model nor the PSID data captures the behavior of the households with top 1\% wealth quintile. These ratios are computed from Diaz-Gimenez, Quadrani and Rios-Rull (1997, 2002).
2. Insure the Uninsurable by Yourself: Accounting for Consumption Insurance in a Life-cycle Model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Target</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time discount factor</td>
<td>$\beta = 0.997$</td>
<td>SCF: 92.98 wealth income ratio = 4.56$^a$</td>
</tr>
<tr>
<td>Replacement rate</td>
<td>$b = 0.393$</td>
<td>U.S. average replacement ratio = 0.48$^b$</td>
</tr>
<tr>
<td>Variance of permanent shock</td>
<td>$\sigma^2_{\eta} = 0.0188$</td>
<td>PSID: BPP 1980–1992</td>
</tr>
<tr>
<td>Variance of transitory shock</td>
<td>$\sigma^2_{\epsilon} = 0.0407$</td>
<td>PSID: BPP 1980–1992</td>
</tr>
<tr>
<td>Variance of fixed effect</td>
<td>$\sigma^2_{\alpha} = 0.2105$</td>
<td>PSID: Storesletten et.al 1969–1992</td>
</tr>
<tr>
<td>Capital share</td>
<td>$\alpha = 0.33$</td>
<td>Cooley (1995)</td>
</tr>
<tr>
<td>Capital depreciation rate</td>
<td>$\delta = 0.06$</td>
<td>Cooley (1995)</td>
</tr>
<tr>
<td>Population growth rate</td>
<td>$\gamma = 1%$</td>
<td>U.S. postwar</td>
</tr>
<tr>
<td>Productivity growth rate</td>
<td>$g = 1.5%$</td>
<td>U.S. postwar</td>
</tr>
<tr>
<td>Relative Risk Aversion</td>
<td>$\sigma = 3$</td>
<td>-</td>
</tr>
</tbody>
</table>


$^b$ In the model, it generates the PAYG pension tax = 13.3%

2.4.2 Accounting for partial insurance

Table 2 reports the main results of the paper. In the model, 10 percent permanent (transitory) shock of the household’s income induces 7.1 (0.6) percent change of the household’s consumption. If we measure the degree of insurance by $1 - \phi$ and $1 - \psi$, these results suggest that the model accounts for 80% of the insurance for permanent shocks and 99% of the insurance for transitory shocks. Almost complete insurance for transitory shocks is a well-known feature of the class of self-insurance models.

The degree of insurance for permanent shocks, however, is stunning. Different from the conventional wisdom, the overall response of consumption to permanent income shocks is far less than one-for-one in the self-insurance model$^{12}$. Although the existence of partial insurance over and above self-insurance can not be rejected, it’s degree is quantitatively small (20%). The result from the calibrated overlapping generations life-cycle model is different from that in the simulation study of a reduced form model, where $\phi$ is close to 0.8 (Blundell et al. 2004), and the study for the marginal propensity of consumption (MPC) in a stationary infinite horizon model (Carroll 2001), where $\phi$ is between 0.85 and 0.95.

I also run the estimations for different age and wealth subgroups as BPP do in their empirical work. In both the model and the data, households in older and wealthier subgroups have obtained more insurance. The model performs even better in accounting for the insurance in age subgroups, especially for the old. As for the wealth

$^{12}$ As we see in Table 2, when the households under age 30 is included in the estimation, $\phi$ is much higher. Yet it is still significantly lower than 1.
2. Insure the Uninsurable by Yourself: Accounting for Consumption Insurance in a Life-cycle Model

subgroups, the model performs well in explaining the high wealth subgroup, whereas the model generates much more insurance in the low wealth subgroup. The reason for “overshooting” is that the benchmark model has too few households with little wealth after age 30 and does not match the fraction of population with zero or negative wealth that constitutes the majority of the 20% lowest wealth household in the U.S. economy.

<table>
<thead>
<tr>
<th>Table 2: Partial Insurance Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Permanen</td>
</tr>
<tr>
<td>Model</td>
</tr>
<tr>
<td>Whole Sample</td>
</tr>
<tr>
<td>By Age</td>
</tr>
<tr>
<td>Young</td>
</tr>
<tr>
<td>Old</td>
</tr>
<tr>
<td>Including 30-</td>
</tr>
<tr>
<td>By Wealth</td>
</tr>
<tr>
<td>Low Wealth</td>
</tr>
<tr>
<td>High Wealth</td>
</tr>
</tbody>
</table>

The column 'Model' contains the estimates from model generated data using BPP approach. The column 'Data' contains the results reported by BPP. 'Share' represents the explaining power of model and is measured by (1- Model)/(1 - Data).

Whole sample is BPP’s sample from 30 - 65 in both the model and the data.

Young (Old) subgroup corresponds to the cohort born in 1940s (1950s) in the data. Since BPP’s data start from 1979, this translates into age 34 - 47 (44 - 57) in the model.

Including 30- is the estimate for the whole agents in the model. Note the households aging from 22 to 29 do not exist in BPP’s sample.

Low wealth subgroup is the lowest 20% wealth households both in model and the data. High wealth is the highest 80% wealth households both in model and the data.

2.4.3 Self-insurance over the life cycle

Figure 1 and Figure 2 show the estimates of partial insurance parameters over the life cycle, using both the BPP and the OLS approach (I will discuss it later). The partial insurance parameter for permanent shocks decreases over the life cycle, more sharply when approaching the retirement age. In a separate experiment, BPP find that $\phi$ is more likely to follow a decreasing linear trend. Confronting the empirical facts with

\footnote{BPP find some evidence of a decline in the value of $\phi$ by age, but the slope is small and the estimates are not very precise.}
the model prediction, I argue that the insurance over and above self-insurance decreases over the life cycle and decreases sharply in the last few years before retirement. This result is reminiscent of the puzzle of the shape of the consumption inequality profile in life-cycle models: the age profile of consumption inequality is concave in the model, whereas consumption inequality in data grows linearly over the life cycle.

![Graph](image1.png)

**Figure 1:** $\phi_t$ Benchmark Setup

![Graph](image2.png)

**Figure 2:** $\psi_t$ Benchmark Setup
2. Insure the Uninsurable by Yourself: Accounting for Consumption Insurance in a Life-cycle Model

There are two possible explanations for the concave age profile in the model. First, the duration of permanent shocks is shorter when approaching the terminal period. In fact, the effect of permanent shocks is exactly like a transitory one in the penultimate period. I call it the *age effect*. Second, when approaching the retirement age, households have accumulated a large amount of life-cycle wealth that makes the effect of income shocks less substantial. I call it the *wealth effect*. Is the concavity of the age profile mainly due to the age effect or the wealth effect? In this model, the pension depends only on the income of the last working year, so that the effect of permanent shocks lasts for a very long periods until death (35 years of retirement) and the age effect seems not likely to increase sharply when the approaching retirement age. On the other hand, most of the life-cycle wealth has been increasingly accumulated from age 50 to age 65. Thus I conclude that the wealth effect is a more plausible explanation of the concavity profile than the age effect.

To see this point clearly, I make experiments of changing life-cycle wealth using different pension system with different $b$. As we observe in Figure 3, lower pension (and high life-cycle wealth) increases the concavity of the age profile.

![Partial Insurance Parameter Over the Life Cycle: Permanent](image)

**Figure 3:** $\phi_l$ Effect of Life-cycle Wealth

I also make experiments of changing precautionary wealth by changing risk aversion. Increasing risk aversion has two effects on $\phi$: one indirect and one direct. In the language of dynamic programming, the indirect effect is the change of the state variable and the direct effect is the change of the decision rule.
2. Insure the Uninsurable by Yourself: Accounting for Consumption Insurance in a Life-cycle Model

The indirect effect comes from precautionary (buffer-stock) wealth. With a higher \( \sigma \) in CRRA preference, all the households accumulate more precautionary wealth. Since the precautionary savings constitute the most part of the wealth for the younger households, it affects the young more than the old. Keeping the overall wealth income ratio unchanged, this effect is like shifting the wealth from the old to the young, which changes the distribution of wealth and the heterogeneity of partial insurance parameters. Since the young are less wealthy and more sensitive to the increase of wealth, this “redistribution” of wealth also results in the drop of the overall \( \phi \).

The direct effect comes from consumption smoothing. In CRRA preference, a higher \( \sigma \) implies higher risk aversion and lower intertemporal substitution, so that the households are more willing to smooth consumption across time and states and therefore current consumption reacts less. This effect causes all the individual \( \phi_t \) go down; the overall \( \phi \) drops.

In Figure 4, I plot the life-cycle profile with different risk aversion. The profiles of the young are much more affected than the old. Note that there are no intersections of the profiles: \( \phi_t \) decreases with higher risk aversion for all ages. It suggests that the direct effect always dominates the indirect effect for the old. In the last working year, the direct and indirect effect cancel each other out and therefore the lowest \( \phi_t \) would not change with \( \sigma \).

![Figure 4: \( \phi_t \) Effect of Precautionary Wealth](image-url)
2. Insure the Uninsurable by Yourself: Accounting for Consumption Insurance in a Life-cycle Model

2.4.4 The role of wealth income ratio

Why a plain-vanilla model like this is able to account for a big chunk of insurance found in the data? The wealth income ratio is the key. In the calibration, I adjust the time discount factor to match the wealth income ratio in the U.S. data. In other words, the discount factor captures all the factors outside the model that may have effects on the determination of households’ wealth. As long as the wealth income ratio is right, the simple model performs well. The general equilibrium framework is important, because it makes it possible to pin down the equilibrium wealth income ratio.

<table>
<thead>
<tr>
<th>Table 3: Wealth Income Ratios</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Model</strong></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>W/I = 2</td>
</tr>
<tr>
<td>W/I = 3</td>
</tr>
<tr>
<td>W/I = 4</td>
</tr>
<tr>
<td>W/I = 4.56</td>
</tr>
<tr>
<td>W/I = 5</td>
</tr>
<tr>
<td>W/I = 6</td>
</tr>
<tr>
<td>W/I = 7</td>
</tr>
</tbody>
</table>

How to accurately map the wealth and income in the model to the wealth and income reported in the data is still an open question. To investigate the role of the wealth income ratio, I report in Table 3 the results of the experiments using different levels of wealth income ratios. As a rule of thumb, increasing the wealth income ratio by 1 unit increases the \( \phi \) by 5 percentage points. The empirical findings of partial insurance for both permanent shocks and transitory shocks could be fully explained if the wealth income ratio was between 6 and 7. The previous results of Blundell et. al (2004) and Carroll (2001), where \( \phi \) is higher than 0.8, corresponds to the case where wealth income ratio is lower than 3.

The role of wealth income ratio can also be illustrated in the approximation of a partial equilibrium infinite horizon model\(^4\). From the expression for \( \pi_t \) we can easily derive \( \phi \) in the infinite horizon by assuming that the expected future income is constant and equal across agents

\[
\phi_{\text{inf}}^t \approx \frac{1}{1 + \frac{r}{1+r} \frac{W}{T}}
\]

\(^4\)The standard infinite horizon Bewley-Imrohoroglu-Hugget-Aiyagari economy does not have the stationary equilibria if income shocks are permanent. An interesting infinite horizon benchmark is to populate this economy with perpetual youth agents. It is on the agenda of this project.
2. Insure the Uninsurable by Yourself: Accounting for Consumption Insurance in a Life-cycle Model

The relation between $\phi_{inf}^i$ and wealth income ratio with different interest is graphed in Figure 5. The lower the wealth income ratio, the closer $\phi_{inf}^i$ is to unit. In this simple infinite horizon model, wealth income ratio has to be doubled to generate the same $\phi$ as in a life-cycle model. At the wealth income ratio used benchmark setup, the infinite horizon model gives $\phi_{inf}^i = 0.83$, which is not far from the findings of Blundell et al. (2004) and Carroll (2001).

![Figure 5: $\phi$ Role of Wealth Income Ratio](image)

2.5 Robustness

2.5.1 Sensitivity

I explore the sensitivity of my result to a number a parameters. In each of the experiment, I recalibrate $\beta$ to keep the wealth income ratio unchanged. The results of the sensitivity test are reported in Table 4.
2. Insure the Uninsurable by Yourself: Accounting for Consumption Insurance in a Life-cycle Model

<table>
<thead>
<tr>
<th>Table 4: Robustness</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data</td>
</tr>
<tr>
<td>Model</td>
</tr>
<tr>
<td>Benchmark</td>
</tr>
<tr>
<td>Model</td>
</tr>
<tr>
<td>Benchmark</td>
</tr>
<tr>
<td>Risk Aversion</td>
</tr>
<tr>
<td>$\sigma = 0.5$</td>
</tr>
<tr>
<td>$\sigma = 2$</td>
</tr>
<tr>
<td>$\sigma = 10$</td>
</tr>
<tr>
<td>Small Open Economy</td>
</tr>
<tr>
<td>$r = 0.03$</td>
</tr>
<tr>
<td>$r = 0.06$</td>
</tr>
<tr>
<td>Income Risk</td>
</tr>
<tr>
<td>Storesletten et al. (2004)</td>
</tr>
<tr>
<td>Guvenven (2007)</td>
</tr>
<tr>
<td>Pension System</td>
</tr>
<tr>
<td>$b = 0$</td>
</tr>
<tr>
<td>$b = 0.6$</td>
</tr>
<tr>
<td>Natural Borrowing Constr.</td>
</tr>
<tr>
<td>Initial Wealth Distr.</td>
</tr>
<tr>
<td>No Annuity Market</td>
</tr>
</tbody>
</table>

**Risk aversion** I reset the risk aversion to $\sigma = 2$, which is widely used in the macro literature, and I find $\phi$ increases by 1%. When $\sigma$ is as low as 0.5, which is used by Gourinchas and Parker (2002), I find $\psi$ is very close to that in the data and $\phi$ increases to 0.76. These results show that $\phi$ decreases with $\sigma$.

An interesting question is: how much risk aversion would be needed to account for all the insurance found in the data? I find it is $\sigma = 10$.

**Small open economy** The parameters on the production function affect the interest rate through the channel of general equilibrium. To single out the effect of interest rate, I turn to the assumption of small open economy with interest rate exogenously given. By setting the world interest rate to 3% and 6%, I find that the higher the interest rate, the lower the $\phi$ is. Higher interest rate decreases the discounted value of future income and thus the share of human wealth in total wealth. Although higher interest may potentially induce more savings, time discount factor is adjusted in the opposite direction to keep the wealth income ratio unchanged; therefore the inter-temporal motive of saving does not vary much. Quantitatively, the effect of changing interest rate is small.
Income risks Different empirical studies using PSID differ in their estimates. I adopt the estimates from Storesletten et al. (2004) \( \sigma_\eta^2 = 0.0161, \sigma_\gamma^2 = 0.063 \), where they get a higher variance of transitory shock, and from Guvenen (2007) \( \sigma_\eta^2 = 0.058, \sigma_\gamma^2 = 0.015, \sigma_\gamma^2 = 0.061 \), where he gets a higher variance of fixed effect. Neither of these two sets of estimates change the result significantly.

Pension In the life-cycle model, the salient feature is that agents’ income is much less after retirement. The generosity of the pension system does affect the life-cycle savings of the agents. I consider two alternative pension systems. One extreme is to exclude any pension system where the agent has to accumulate a great amount of life-cycle wealth. The other extreme is to make the pension very close to the expected net income when working. Neither of these two extreme pension systems changes the result much. Counter-intuitively, more generous pension leads to less consumption insurance. It is because after lowering the \( \beta \) to keep the wealth income ratio unchanged, the total effect is like the redistribution of the wealth from the young to the old and it works exactly in the opposite direction of increasing the precautionary wealth as mentioned above. Going back to Figure 3, we find that the profiles of the old, especially those with 10 working years left, are much more affected than the young. The profiles of different pension systems intersect because of the wealth “redistribution”.

Borrowing constraint In the benchmark setup, agents are excluded from any borrowing. In another extreme case, I rule out any ad hoc borrowing constraints except for the terminal condition that agents cannot die in debt at age \( T + 1 \). In other words, I set the borrowing constraint as low as the natural borrowing constraint which is not binding. And I find that it makes the \( \phi \) increase by 1%.

The borrowing constraint has two effects on increasing the wealth level: it forces some agents to save, when their current borrowing constraints are binding; it makes all the agents more willing to save, when they expect the borrowing constraint might be binding in the future.

The young are more likely to be borrowing constrained. Keeping wealth income ratio unchanged, the total effect of a tighter borrowing constraint is like shifting the wealth from the old to the young, whereby it lowers \( \phi \). Comparing with the previous case of increasing risk aversion, it reminds us of the fact that borrowing constraint is exactly another source of precautionary saving. Since the sample of our estimation starts from age 30 when most of the agents who are previously constrained have already accumulated some positive wealth, different tightness of borrowing constraint does not affect the result very much.

Initial wealth The initial wealth distribution is calibrated to mimic the wealth distribution of households at age 25 and under in SCF 1992 and 1998 (Diaz-Gimenez, Quadrini, Rios-Rull 1997, 2002). The average wealth earnings ratio and average wealth
2. Insure the Uninsurable by Yourself: Accounting for Consumption Insurance in a Life-cycle Model

Gini in these two surveys is 1.07 and 0.9, respectively. I approximate the initial wealth distribution by a log normal distribution with mean zero and variance calibrated to the wealth Gini. I generate a random sample of size 100/99 of the agents in the model simulation, discard the top 1% wealthiest, and then re-scale the mean to match the average wealth income ratio under 25 in the data.

Adding initial wealth distribution has two effects. One is to increase the dispersion of wealth. The other is to give the youngest agents more asset to run down. The effect of the former shifts the wealth to the wealthier, whereas the effect of the latter works in the opposite way. Our result shows that the first effect dominates, though the difference is not significant.

Annuity market When there is no annuity market, the rate of return of asset is lower. It is like the effect of lowering interest rate and it is what we see in the results.

The partial insurance parameters vary throughout a variety of alternative parameters, while the main result of this paper preserves: within the range of all these parameters, the highest φ we get is 0.77, which can still account for two thirds of the insurance found in the data. Throughout all the parameterization, the BPP estimates for permanent shocks are lower than the OLS estimates but the differences are quantitatively small.

2.5.2 Misspecification of income process

If the income process was less persistent than a unit root and/or a proportion of the income “shocks” was known in advance to the agent, the consumption would respond less. Could the empirical finding of partial insurance over and above self-insurance simply be an outcome of (income) model misspecification?

Less persistent shock

Although there is plenty of evidence that the permanent component of income follows a random walk or the permanent shocks are very persistent. (e.g. Gottschalk and Moffitt 1994, Storesletten et al. 2004), some studies estimate a less persistent AR(1) process (Lillard and Weiss 1979, Guvenen 2007).

If ρ < 1 in equation (2.7) of income process, then the income model in BPP is misspecified. Nevertheless, the rationale for BPP’s consumption estimation equation still holds. In the approximation under PILCH with CRRA preference, the coefficient for permanent shocks now becomes φ = ϑ(ρ)τρ, where 0 < ϑ(ρ) < 113. Lower persistence decreases ϑ(ρ), since the effect of persistent shock dies out over time.

13 In a simple case without retirement, ϑ = \frac{r[1-(\frac{r}{\sigma})^{T-t+1}]}{(r+1-\rho)[1-(\frac{r}{\sigma})^{T-t+1}]}.
2. Insure the Uninsurable by Yourself: Accounting for Consumption Insurance in a Life-cycle Model

The OLS estimator of \( \phi \) is still consistent, whereas the BPP estimator is not. The accuracy of BPP estimates can be verified quantitatively when the two estimates are compared. Table 5 reports the results under model misspecification. Since the results of the BPP estimates are close to those of the OLS estimates, BPP could still be a valid measure even if the shocks were less persistent.

**Table 5: Misspecification of Income Process**

<table>
<thead>
<tr>
<th>Data</th>
<th>Permanent (( \phi = 0.642 ))</th>
<th>Transitory (( \psi = 0.053 ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model</td>
<td>BPP OLS</td>
<td>BPP OLS</td>
</tr>
<tr>
<td>Benchmark</td>
<td>0.714 0.709</td>
<td>0.063 0.062</td>
</tr>
<tr>
<td>Less Persistent shock</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \rho = 0.99 )</td>
<td>0.713 0.700</td>
<td>0.063 0.061</td>
</tr>
<tr>
<td>( \rho = 0.97 )</td>
<td>0.701 0.699</td>
<td>0.070 0.059</td>
</tr>
<tr>
<td>( \rho = 0.95 )</td>
<td>0.622 0.623</td>
<td>0.088 0.110</td>
</tr>
<tr>
<td>Advance information</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \rho = 0.97, \sigma^2_\theta = 0.0001 )</td>
<td>0.688 0.687</td>
<td>0.068 0.059</td>
</tr>
<tr>
<td>( \rho = 0.97, \sigma^2_\theta = 0.00038 )</td>
<td>0.675 0.676</td>
<td>0.066 0.058</td>
</tr>
<tr>
<td>( \rho = 0.96, \sigma^2_\theta = 0.00038 )</td>
<td>0.650 0.648</td>
<td>0.064 0.062</td>
</tr>
</tbody>
</table>

As we see in the results, lower \( \rho \) does help explain more insurance. With \( \rho \) as low as 0.95, both the BPP and OLS estimates can account for all the insurance for permanent shocks. However, the insurance against transitory shock is significantly lower when \( \rho \) is low. It suggests that less persistent shock in a standard life-cycle model can not account for both the permanent and transitory shock at the same time.

**Advance information**

If parts of the innovations were known in advance to the agent, the income shocks would have less effect on the adjustment of household’s consumption, since they had already taken it into account (Cunha, Heckman and Navarro 2005). Empirically, it is difficult to separate advance information from the consumption insurance for permanent shocks (Primiceri and Rens 2006).

Assume, for simplicity, agents have advance information for only permanent shocks. BPP’s estimation equation becomes

\[
\Delta \log c_t = \tilde{\phi}_t \Omega \eta_t + \psi_t \varepsilon_t + \zeta_t
\]

where \( 1 - \Omega \) is the measure of advance information known to the agent. BPP argue that \( \tilde{\phi}_t \) would be underestimated by the information factor \( \Omega \). To deliver a sense of the
magnitude of $\Omega$. I consider a special case called the Heterogeneous Income Profiles (HIP) process in Guvenen (2007) with $\sigma_\theta^2 > 0$ in equation (2.7) of income process. A fraction of permanent (persistent) change of income is due to the heterogenous income growth which is observable to the agents, whereas this income change is not observable to the econometricians and treated as income “shock”. Suppose $\rho = 0.97$, I use $\sigma_\theta^2 = 0.00038$ as estimated by Guvenen (2007) and use a very small heterogeneity of $\sigma_\delta^2 = 0.0001$. Compared with the true $\hat{\phi}$ in the pure AR(1) case, $1 - \Omega$ is uncovered to be between 3% and 6%.

I find that adding profile heterogeneity to AR(1) shock lowers both the $\phi$ and $\psi$. When $\rho = 0.96$ and $\sigma_\theta^2 = 0.00038$, the model can account for both the insurance for the permanent shocks and the insurance for the transitory shocks in the data. If this was the true income process, the standard self-insurance model could not be rejected. However, there are some potential problems in this explanation. Firstly, as the empirical variance of log income is linear in the data and the variance of profile heterogeneity grows convexly over time, $\rho$ is required to be significantly less than 1 even with a small variance of $\theta^t$. For example, Guvenen (2007) estimates $\rho = 0.82$. As noted above, any $\rho$ lower than 0.95 makes the insurance in the model higher than that in the data. Secondly, a standard incomplete market model with profile heterogeneity generates a consumption profile much less steep than that in the data. Guvenen (2007) reconciles this discrepancy by assuming optimal learning behavior of the agents. It would be interesting to estimate the insurance parameters in a life-cycle model with learning, and I leave it to the future work\footnote{Another possible model misspecification might be that the labor supply is inelastic and therefore the income is not exogenous. Heathcote et al. (2008) find that consumption inequality is much less in a life-cycle self-insurance model with endogenous labor supply. It is also interesting to know how much insurance, in BPP’s measure, is in this class of models.}.

### 2.5.3 Alternative measures of insurance

**OLS Approach**

In the model, the true values of the realizations of both permanent and transitory innovations of income, $\eta_t$ and $\varepsilon_t$, are known. Thus we can estimate $\phi_t$ and $\psi_t$ simply by the Ordinary Least Square (OLS) approach.

\[
\hat{\phi}_t^{OLS} = \frac{\text{Cov}(\Delta \tilde{\eta}_t, \tilde{\eta}_t)}{\text{Var}(\tilde{\eta}_t)} \quad (2.17)
\]

\[
\hat{\psi}_t^{OLS} = \frac{\text{Cov}(\Delta \tilde{\varepsilon}_t, \tilde{\varepsilon}_t)}{\text{Var}(\tilde{\eta}_t)} \quad (2.18)
\]

Using the BPP or the OLS estimators might deliver different results. Since the OLS estimators have exploited the information of the realization of the income shocks,
presumably it could perform better. Nevertheless, we should not substitute the OLS estimators for the BPP estimators when measuring the degree of insurance. The reasons are as follows.

First, the relation between the BPP estimators and the OLS estimators can be derived as:

\[
\hat{\phi}_t^{\text{BPP}} \overset{N \to \infty}{\lim} = \hat{\phi}_t^{\text{OLS}} + \frac{\text{cov}(\Delta \hat{\xi}_t, \eta_{t-1}) - \text{cov}(\Delta \hat{\xi}_t, \xi_{t-2})}{\text{Var}(\eta_t)}. \quad (2) \quad \hat{\psi}_t^{\text{BPP}} \overset{N \to \infty}{\lim} = \hat{\psi}_t^{\text{OLS}}
\]

(3) Both the BPP estimators and the OLS estimators are consistent estimates of \( \phi_t \) and \( \psi_t \).

**Proof.** See Appendix. ■

From Lemma 1, we know that both the BPP and the OLS estimators are consistent estimators of the same true partial insurance parameters, given that the econometric model is not misspecified. If the number of observations goes to infinity, the BPP estimators are identical with the OLS estimators with probability 1.

Second, the invalidity of the BPP estimator may arise from the model misspecification of consumption equation (4.2). If the growth of log consumption is nonlinear in income innovations, neither the BPP estimators nor the OLS estimators are the consistent estimators of the “true” partial insurance parameters for permanent and transitory shocks.

Third, if the growth of log consumption is linear in the income innovations but \( \text{cov}(\Delta \hat{\xi}_t, \eta_{t-1}) \neq \text{cov}(\Delta \hat{\xi}_t, \xi_{t-2}) \), then the OLS estimators are consistent while from Lemma 1 we know that the BPP estimators for permanent shocks are not. In this case, the OLS estimators do perform better than the BPP estimators. Nevertheless, as we see from equation (2.5), in the self-insurance model with CRRA preference without borrowing constraint, the inaccuracy caused by model misspecification is only a matter of approximation error. Quantitatively, if most of the observations come from the agents who do not choose to borrow, BPP will still be a valid measure of the consumption insurance. As we will see in the next section, the difference between the BPP and the OLS estimates from the model simulation after age 30, when most of the agents in the model are not borrowing constrained, is quantitatively small. Since BPP’s empirical work restrict their sample to be above age 30, whether using the BPP or the OLS estimators in the estimation does not matter significantly.

From Figure 1 and Figure 2, we see that the BPP estimate of transitory shocks tracks the OLS estimate closely as Lemma 1 predicts. In the early stage of life, the BPP estimate of permanent shocks is much higher than the OLS estimate. It is mainly due to the existence of borrowing constraints. Recall from Lemma 1 that the difference between the BPP and the OLS estimate is determined by the difference
between $\text{cov}(\Delta c_t, \eta_{t-1})$ and $\text{cov}(\Delta c_t, \varepsilon_{t-2})$. Intuitively, the consumption growth is positively correlated with previous permanent and transitory shocks and consumption may respond more to $\eta_{t-1}$ than to $\varepsilon_{t-1}$, because $\eta_{t-1}$ has enlarged the expected value of future income and represents a more recent effect than $\varepsilon_{t-2}$. In the early periods of life when there is much less accumulated asset, the permanent shock has much stronger effect on the wealth level. Since we are only interested in the estimation of households above age 30, the problem of borrowing constraint is not severe. The BPP estimate remains slightly higher than the OLS estimate in most of the subsequent periods. Their difference is mainly due to the approximation error in equation (2.5) and is quantitatively negligible.

The life-cycle profiles of $\phi_t$ and $\psi_t$ with natural borrowing constraint are shown in Figure 6 and Figure 7. Without ad hoc borrowing constraint, the BPP estimate tracks the OLS estimate for permanent shocks more closely in the early working life. After age 30, the profile is very close to that in the benchmark setup.

![Figure 6: $\phi_t$ Natural Borrowing Constraint](image-url)
2. Insure the Uninsurable by Yourself: Accounting for Consumption Insurance in a Life-cycle Model

![Graph](image)

Figure 7: $\psi_t$ Natural Borrowing Constraint

Finally and most importantly, the OLS estimators can not be obtained from any ordinary data set, because econometricians do not know the true values of $\eta_t$ and $\varepsilon_t$. To make a reasonable comparison with the empirical findings by BPP, we have to “pretend” that we do not know the innovations of shocks and undertake the same procedure as BPP do.

**Variance approach**

Although the BPP approach is valid in measuring consumption insurance in a self-insurance model with CRRA preference, it requires panel data set of consumption to find empirical evidence and, in particular for the data set BPP use, it hinges on the validity of the imputation methodology. Before BPP’s construction of a new panel data set, many researchers use the repeated cross-section data set of CEX to study consumption inequality and risk sharing. Intuitively, the discrepancy between income inequality and consumption inequality implies the degree of consumption insurance. To formalize this implicit idea as emphasized by many authors, I define the measure of insurance by the variance approach as

$$
\phi_t^{Var} \equiv \frac{Var(\log c_t)}{Var(\log y_t)}
$$

(2.19)

It is easy to get the theoretical counterpart of the Var measure in the two extreme cases of insurance:
Complete Market: $\phi_t^{Var} = 0$ iff agents are ex ante homogenous.

Autarchy: $\phi_t^{Var} = 1$.

Note that only cross-sectional data sets are needed. As a simple measure, the Var estimator does not distinguish between the effect of the permanent shocks and that of the transitory shocks.

Difference of variance approach

Suppose we have the panel data on income, say, PSID, then we can identify the variance of permanent shocks, given the income process is not misspecified. In order to exploit this useful information, I construct the measure of insurance for permanent shocks by the difference of variance approach as

$$\hat{\phi}_t^{DVar} = \frac{\sqrt{\Delta Var(log c_t)}}{\sigma_{nl}}$$

(2.20)

The DVar measure has nice properties in standard theories:

Complete Market: $\phi_t^{DVar} = 0$.

Autarchy: $\phi_t^{DVar} = 1$, iff the variance of transitory shocks is unchanged.

PILCH with CRRA preference: We know that the BPP measure is valid in this case. The link between the BPP estimator for permanent shocks and the DVar estimator can be derived as

Lemma 3 $\lim_{N \rightarrow \infty} \hat{\phi}_t^{DVar} \simeq \sqrt{\left(\lim_{N \rightarrow \infty} \hat{\phi}_t^{BPP}\right)^2 + \frac{\psi_t^2 \sigma_{ul}^2 + \sigma_{nl}^2}{\sigma_{nl}^2}}$.

Proof. See Appendix. ■

First, we notice that the DVar estimate is not a consistent estimator of $\phi_t$ and is upwards biased. Second, since $\psi_t^2$ is a small number in self-insurance models, the approximation error depends mainly on $\sigma_{ul}^2/\sigma_{nl}^2$. If $\sigma_{ul}^2$ does not vary much over the life-cycle, then the shape of the profiles of these two estimates are similar. When using this approach, we still do not need panel data of consumption.

Figure 8 plots the consumption insurance against permanent shocks over the life-cycle using different measures. After age 30, both the Var and DVar estimate are highly positive correlated with the BPP estimate (the correlation coefficient is 0.983 and 0.981, respectively), which legitimizes these two estimators as qualitative measures. Quantitatively, the DVar estimate is higher than the BPP estimate and the difference is around 10 percentage points after age 30. Though different from the BPP estimate, the DVar estimate has a nice property of “shape preservation”, which to some extent legitimizes our previous study of self-insurance using only the BPP measure.
2. Insure the Uninsurable by Yourself: Accounting for Consumption Insurance in a Life-cycle Model

Figure 8: $\phi_i$: Different Measures

These four measures of consumption insurance are summarized in Table 6, where the results in both model and data are reported. I apply the DVar measure to the data in Deaton and Paxson (1994 Table 1). The slope of the variance of log consumption in data is 0.0084. The variance of permanent shocks in the benchmark setup is used. By DVar measure, the self-insurance model can explain 60% of the consumption insurance in the data.

<table>
<thead>
<tr>
<th>Estimator</th>
<th>Results</th>
<th>Panel Data?</th>
<th>Consistent?</th>
<th>$\eta_i$ known?</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Model</td>
<td>Data</td>
<td>Share</td>
<td>$y_i^j$</td>
</tr>
<tr>
<td>OLS</td>
<td>$\phi_{OLS}$</td>
<td>0.709</td>
<td>0.642</td>
<td>81%</td>
</tr>
<tr>
<td>BPP</td>
<td>$\phi_{BPP}$</td>
<td>0.714</td>
<td>0.642</td>
<td>80%</td>
</tr>
<tr>
<td>DVar</td>
<td>$\phi_{DVar}$</td>
<td>0.786</td>
<td>0.606</td>
<td>60%</td>
</tr>
<tr>
<td>Var</td>
<td>$\phi_{Var}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

These findings have implications for future research. When applied theorists are to test the model but there is no reliable panel data set of consumption at hand, they can alternatively use measures by the difference of variance approach in both data and model as proxies for consumption insurance.
2. Insure the Uninsurable by Yourself: Accounting for Consumption Insurance in a Life-cycle Model

2.6 Conclusion

This paper revisits the power of self-insurance models in explaining consumption insurance. To directly confront the implication of the model with recent structured facts found by BPP, I apply BPP’s measure of partial insurance and their estimation procedure to the model generated data. The main message of the paper is that a standard general equilibrium version of self-insurance model, when calibrated to match the wealth income ratio, can account for the major part of the insurance for permanent shocks and almost all the insurance for transitory shocks found in the data. In short, self-insurance could be the most important channel of consumption insurance.

I also find that self-insurance increases with age and wealth, which is consistent with the implications of previous studies of self-insurance models and reconfirms the shape of consumption inequality profile puzzle (concave in model while linear in data). I show that BPP is a valid measure when most of the agents are not borrowing constrained and the main results are maintained throughout a variety of reasonable parameters. In the calibration I use the time discount factor to capture all the other factors outside the model, so that the success of this plain-vanilla model should be viewed as the first step for the study of consumption insurance in a class of life-cycle models with various extensions. It would be interesting to apply the methodology used in this paper to the life-cycle models with labor supply (Heathcote et al. 2008), endogenous retirement (Rust and Phelan 1997), employment risk (Low et al. 2009), human capital (Hugget et al. 2006) and female labor force participation (Attanasio et al. 2007), etc.

As a response to the research agenda for estimating consumption insurance over and above self-insurance proposed by Hayashi, Altonji and Kotlikoff (1996), this paper, together with BPP’s findings, shows that the partial insurance over and above self-insurance could be quantitatively small (20%) and decrease over the life-cycle. It has two implications for future research. On the one hand, as emphasized by recent studies (e.g. Attanasio and Pavoni 2007, Abraham and Pavoni 2008), the agents’ free access to self-insurance should probably not be assumed away, if we are to make this class of dynamic contract models quantitatively match the empirical evidence of consumption insurance. In the extreme case, the bond economy with self-insurance can be viewed as the decentralization of the constraint efficient allocation with hidden information and hidden saving (Cole and K ocherlakota 2001). On the other hand, this fact favors the dynamic contract models of adverse selection and/or moral hazard with the property of increasing consumption inequality. This class of models typically generate the “immiseration” results as mentioned in Atkeson and Lucas (1992). In a life-cycle setup (e.g. Ales and Maziero 2008), the main intuition is that the old have
less insurance ability simply because it is harder to provide insurance by a shorter-run contract.

2.7 Appendix

2.7.1 Numerical algorithm

In the benchmark setup with unit root process, the individual’s state variables can be reduced to only two: current asset and the transitory shock. Define \( \tilde{x}_t = \frac{x_t}{(\gamma \Pi_{i=1}^{t-1} G_i e^{z_i})} \), where \( x_t = (c_t, a_t, y_t, B_t) \), \( G_t = (1 + g)e^{\kappa t - \kappa t - 1} \) if \( t < R \) and \( G_t = (1 + g) \) if \( t \geq R \). \( G_t \) is the total income growth rate, consisting of the time effect and the age effect. The Euler equation can be written as

\[
\tilde{c}_t - \sigma \leq \beta (1 + r) G_t^{-\sigma} E_t [(e^{\eta t+1} \tilde{\alpha}_{t+1})^{-\sigma} - (e^{\eta t} \tilde{\alpha}_t)^{-\sigma}]
\]

\[
= \beta (1 + r) G_t^{-\sigma} E_t [(e^{\eta t+1} \tilde{\alpha}_{t+1})^{-\sigma}] \text{ if } \tilde{\alpha}_t > \tilde{\alpha}
\]

The budget constraint becomes

\[
\tilde{c} + G_t e^{\eta t} \tilde{\alpha}_{t+1} \leq \tilde{\alpha}_t (1 + r) / \xi_t + \begin{cases} 
\frac{\tilde{y}_t}{\tilde{y}_t} & t < R \\
\frac{\tilde{B}_t}{\tilde{B}_t} & t \geq R
\end{cases}
\]

At time \( T \), the optimal consumption rule is \( c_T = a_T \). For \( t < T \), the consumption rule can be solved backwardly.

The computation and calibration procedure is as follows:

1. The fraction of last period income \( \gamma \) and pension tax \( \tau \) can be solved directly by summing up the total income and pension in the economy and matching the average replacement ratio.

2. Given wealth income ratio \( W/I \), the capital income ratio and thus the interest rate can be solved by \( r = \frac{1 - \alpha}{\alpha^2} \frac{1}{W/I} - \delta \).

3. Guess \( \beta \).

4. Solve for the decision rule \( \tilde{c}_t(\tilde{\alpha}) \) from the Euler equation using linear interpolation.

5. Simulate the model.

6. Recover the state variable by \( x_t = \tilde{x}_t \gamma \Pi_{i=1}^{t-1} G_i e^{z_i} \).

7. Iterate on \( \beta \) to match the wealth income ratio.

8. Compute BPP estimates using simulated data series after removing the time and age effects.

I use two discrete states for each of the exogenous state variables \( (\gamma, \theta, \varepsilon, \eta) \). Grids on asset are formed triple exponentially to make more grids where asset level is lower. I use 101 grids for the asset and use linear interpolation for the point between the
2. Insure the Uninsurable by Yourself: Accounting for Consumption Insurance in a Life-cycle Model

50,000 agents are used in the simulation. The model generates a panel data series with the same age span as BPP’s sample, so that we have 1,800,000 observations.

In the AR(1) case, we cannot reduce the state variable of permanent shock. Instead, I discretize the persistent shock by a 63 state Markov chain using the method suggested by Hussey and Tauchen (1991). Notice that using too few states for discretized Markov chain is not able to generate a process with very high persistence and consequently makes the estimated insurance parameters for permanent shocks severely downwards biased.

2.7.2 Some properties of the estimators

Proof of Lemma 1. We first derive the expression of the variance of shocks:

\[
E(\Delta \hat{y}_t (\Delta \hat{y}_{t-1} + \Delta \hat{y}_{t+1})) = E((\eta_t + \varepsilon_t - \varepsilon_{t-1})(\eta_t + \eta_{t-1} + \eta_{t+1} + \varepsilon_{t+1} - \varepsilon_{t-2}))
\]

\[
= \text{Var}(\eta_t)
\]

\[
- E(\Delta \hat{y}_t \Delta \hat{y}_{t+1}) = - E((\eta_t + \eta_{t+1} + \varepsilon_{t+1} - \varepsilon_{t-2}))
\]

\[
= \text{Var}(\varepsilon_t)
\]

Therefore,

\[
p\lim_{N \to \infty} \hat{\phi}_{t}^{BPP} - p\lim_{N \to \infty} \hat{\phi}_{t}^{OLS} = \frac{E(\Delta \hat{c}_t (\eta_t + \eta_{t-1} + \eta_{t+1} + \varepsilon_{t+1} - \varepsilon_{t-2}))}{\text{Var}(\eta_t)} - \frac{E(\Delta \hat{c}_t (\eta_{t+1} + \eta_{t+1} + \varepsilon_{t+1} - \varepsilon_{t-2}))}{\text{Var}(\eta_t)}
\]

\[
= \frac{\text{Cov}(\Delta \hat{c}_t, \eta_t) - \text{Cov}(\Delta \hat{c}_t, \varepsilon_{t-2})}{\text{Var}(\eta_t)}
\]

\[
p\lim_{N \to \infty} \hat{\psi}_t^{BPP} - p\lim_{N \to \infty} \hat{\psi}_t^{OLS} = \frac{E(\Delta \hat{c}_t (\eta_{t+1} + \varepsilon_{t+1} - \varepsilon_t))}{-\text{Var}(\varepsilon_t)} - \frac{\text{Cov}(\Delta \hat{c}_t, \varepsilon_t)}{\text{Var}(\varepsilon_t)}
\]

\[
= \frac{E(\Delta \hat{c}_t (\eta_{t+1} + \eta_{t+1} + \varepsilon_{t+1} - \varepsilon_{t-2}))}{\text{Var}(\eta_t)}
\]

\[
= 0
\]

The consistency of OLS estimator is obvious. From the fact that \(\text{cov}(\Delta \hat{c}_t, \hat{\eta}_{t-1}) = \text{cov}(\Delta \hat{c}_t, \hat{\varepsilon}_{t-2}) = 0\) if the econometrics model is not misspecified, we conclude BPP estimators are also consistent.

Proof of Lemma 2. If there is no heterogeneity in \(\phi_t\), we have

\[
\log c_t = \log c_{t-1} + \phi_t \eta_t + \psi_t \varepsilon_t + u_t
\]

(2.21)
Take variance on both sides,
\[
\phi_t^2 \eta_t^2 = \Delta Var(\log c_t) - \psi_t^2 \sigma_{\varepsilon,t}^2 - \sigma_{ut}^2
\]
\[= \lim_{N \to \infty} DVar_{\phi_t} \psi_t^2 - \psi_t^2 \sigma_{\varepsilon,t}^2 - \sigma_{ut}^2
\]
Applying the fact \( \lim_{N \to \infty} \tilde{BPP}_{\phi_t} = \phi_t \) and rearranging, we get the needed result. ■
Bibliography


BIBLIOGRAPHY


Chapter 3

Consumption Inequality and Discount Rate Heterogeneity

3.1 Introduction

Recent studies have emphasized the importance of consumption inequality\(^1\). In particular, the age profile of consumption inequality provides us valuable information for the nature of idiosyncratic labor income processes, the degree of consumption insurance, and, ultimately, the inequality of welfare.

The workhorse of studying consumption inequality over the life cycle is the class of incomplete market models developed by Bewley(1986), Imrohoroglu(1989), Huggett(1993) and Aiyagari(1994). In these "heterogeneous agents" models, agents may differ in income profiles or initial wealth, but in all the other aspects they are ex-ante homogeneous. In general, this restriction on the initial heterogeneity makes the model simple and therefore highlights the effect of income shocks when markets are incomplete.

One dimension of ex-ante heterogeneity which the standard models assume away is the discount rate heterogeneity. After the seminal paper by Krusell and Smith (1998), it is well-known in the macroeconomic literature that discount rate heterogeneity can potentially play an important role in wealth distribution. However, there is very little, if any, emphasis on the implication of discount rate heterogeneity on consumption inequality.

The U.S. data for consumption inequality have two salient features: the consumption inequality significantly increases over the life cycle and the profile is approximately linear, or at least non-concave\(^2\). This evidence gives us valuable information of identi-


\(^2\)The empirical studies using CEX data include: Deaton and Paxson 1994, Heathcote et al. 2005, Guvenen 2007, Heathcote et al. 2009. All of these results have the same feature of increase of life-
3. Consumption Inequality and Discount Rate Heterogeneity

fying different labor income process, namely, (1) the Restricted Income Profiles (RIP) model with highly persistent shocks and (2) the Heterogeneous Income Profiles (HIP) model where shocks are less persistent. Storesletten et al. (2004) find that the standard life-cycle incomplete market model can generate an increase of consumption inequality when labor market shocks are highly persistent, and therefore they claim the evidence favors the RIP model. However, their model generates a concave consumption profile, which is at odds with the data. It is because as the old agents accumulate a considerable amount of life-cycle wealth for retirement, they are more capable of self-insuring against even the permanent labor income shocks. This concavity result is robust in the standard life-cycle incomplete market models, given that the variance of income shocks does not increase and the discount factor is well calibrated to a reasonable wealth-income ratio.

Guevenen (2007) emphasizes this puzzle and proposes a model with different information structure: individuals learn their own labor income profile gradually over the life cycle by Bayesian updating. He claims the success of HIP model with learning, since it is able to match both the magnitude of increase and the shape of consumption inequality profile.

This paper takes another approach. I make a small modification to the standard model by only relaxing the assumption of homogeneous discount factors. After estimating the distribution of the discount factor, I use the model to study consumption inequality in both RIP and HIP models. To the author’s knowledge, this paper is the first one to investigate the role of discount rate heterogeneity in an otherwise standard incomplete market model. Perhaps the most related paper is by Badel and Huggett (2007), who study in a complete market setup how preference shocks can account for the life-cycle profile of both consumption and leisure inequality.

I study a class of calibrated quantitative models with discount rate heterogeneity. Unfortunately, there is no consensus in how to calibrate or estimate the distribution of discount rate. Since the discount rate is not observable, there is only some indirect evidence of discount rate heterogeneity either by experiments designed to reveal individual’s preference (Barsky et al., 1997) or by Euler equation estimation (Lawrance 1991, Alan and Browning 2003). For example, Alan and Browning (2003) report that the standard deviation of discount factor is 0.09. They contribute the difference in the unexplained intertemporal wedge of marginal consumption to discount rate heterogeneity, which may cause an upward bias for its true variance. In a quantitative life-cycle model, Hendricks (2007) estimate the distribution of discount rate hetero-

time consumption equality, though in Heathcote et al. (2005) and Heathcote et al. (2009), who only compute consumption inequality up to age 60, the consumption inequality increase less and the evidence of non-concavity is not clear. This may be due to the sample selection of longer periods of CEX data. In this paper, I follow the common view of "non-concavity" of consumption inequality.
3. Consumption Inequality and Discount Rate Heterogeneity

geneity using the wealth Gini coefficient. His estimation of the standard deviation of
discount factors is about 0.035. He attributes the wealth dispersion that the standard
model fail to explain to the effect of discount rate heterogeneity. However, the discount
rate heterogeneity is not the only resolution to the realistic wealth distribution in the
standard incomplete market models.\footnote{Either altering the income process (Castaneda et al. 2003) or introducing the entrepreneur behav-
ior and bequest motive (Cagetti and De Nardi 2005) can make the standard model match the wealth
inequality in the data.}

In the present paper, the additional moment I use is the fraction of agents holding
zero or negative wealth, which cannot be explained by a standard homogeneous dis-
count rate life-cycle model with realistic social security (Diamond and Housman 1984,
Huggett 1996). As consumption inequality is the focus of the paper, I estimate the
distribution of discount factors without taking any moments from consumption data.
Interestingly, the estimates in the benchmark models of this paper are close to that of
Hendricks (2007)’s estimates.

The main finding of this paper is that the quantitative model with discount rate
heterogeneity can be successful in matching the empirical age profile of consumption
inequality. A HIP model with discount rate heterogeneity matches the data well in
both the magnitude of increase and shape of consumption inequality profile, while RIP
model does not. If there exists partial insurance, however, a RIP model can also do
a good job in matching the consumption profile. Therefore, consumption data alone
is not sufficient for us to distinguish between HIP model with only self-insurance and
RIP model with partial insurance.

I find that borrowing constraints is more important in a model with discount rate
heterogeneity. It would not be a problem for the standard model to assume away
borrowing for simplicity, since the fact that in the model no one will be borrowing
constrained near retirement makes the borrowing constraints quantitatively less im-
portant for the consumption inequality of the old agents. If discount rate heterogeneity
is present, however, a very tight borrowing constraint may lower the with-in group con-
sumption inequality for the household group with low discount rate. The assumption
that there is no borrowing is not innocuous and therefore the borrowing constraint
should be chosen more carefully, if there does exist discount rate heterogeneity.

As consumption inequality can be caused directly by discount rate heterogeneity,
I also measure insurance by another measure designed by Blundell et. al (2008 BPP
thereafter). I find that the insurance in the model is up-ward biased by BPP’s measure
of consumption insurance.

Finally, I revisit a classical question which is asked by Keane and Wolpin (1997),
Storesletten et al. (2004) and Huggett et al. (2007): which is more important to the
3. Consumption Inequality and Discount Rate Heterogeneity

life-time welfare, initial condition or life-time risks? Presumably, the answer hinges on different model structures. While Keane and Wolpin (1997) and Huggett et al. (2007) find initial condition more important, Storesletten et al. (2004) finds life-time shocks are more important. I answer this question again in a model with heterogeneous discount rate and/or heterogeneous income profile. I find that the life-time inequality in welfare is more due to initial condition (96\%) than life-time shocks. Among those initial conditions, the discount rate heterogeneity is more important than heterogeneous income profile, which is in turn more important than the fixed effect of income.

The rest of the paper is organized as follows: in Section 2 sets up and calibrates an incomplete market model; Section 3 estimates the distribution of discount rate heterogeneity and reports the results on consumption inequality; Section 4 investigates the consumption insurance using BPP’s measure; Section 5 studies the welfare implication of the model and the relative importance of the initial condition and income shocks; Section 6 concludes.

3.2 Model

The economy is populated by $T$ overlapping generations, each of which consists of a continuum of agents. Each agent is born at age 1 and can live a maximum of $T$ periods. Agents face mortality risks. The probability of surviving between age $t - 1$ and age $t$ is denoted by $\xi_t$, with $\xi_1 = 1$ and $\xi_{T+1} = 0$. The unconditional probability of being alive at age $t$ is $s_t = \prod_{\tau=1}^{t} \xi_\tau$. The measure of the new born agents is denoted by $\mu_1$ and the population grows at a constant rate $n$, implying a stable population structure with $\mu_t = \mu_1 s_t (1 + n)^{1-t}$.

Agent $i$’s preference over stream of consumption is given by: $E \sum_{t=1}^{T} \beta_t^{1-1} s_t c_t^{1-\sigma} / (1 - \sigma)$, where $\beta_t$ is the time discount factor and $c_t$ is the consumption at age $t$, $\sigma$ is the relative risk aversion. For each individual $i$, her discount factor $\beta_t$ is drawn from a distribution $F(\beta)$ with mean $E(\beta)$ and variance $Var(\beta)$. In the standard model where the discount rate heterogeneity is absent, $Var(\beta) = 0$.

Agents enter the labor market at age 1 and the mandatory retirement age is $R$. At working age $t < R$, the agents supply inelastically one unit of labor, while they differ in the efficient unit of labor. The exogenous uninsurable labor income of agent $i$ is

$$y_t = (1 - \tau) w e_t,$$  

(3.1)

where $\tau$ is the pension tax, $w$ is the wage rate identical to all the agents in a given
3. Consumption Inequality and Discount Rate Heterogeneity

In the cross-section, the unit of labor which is assumed to follow

\[ \log e_i^t = \kappa_t + \alpha_i^t + \theta_i^t + Z_i^t + \epsilon_i^t, \]

where \( \rho \in (0, 1], \eta_i^t \sim N(0, \sigma_h^2), \epsilon_i^t \sim N(0, \sigma_e^2), Z_i^t = 0 \). The income can be decomposed into a predictable part and an idiosyncratic shocks part. \( \kappa_t \) is the income profile which is identical to all the agents of the same age. In the stationary equilibrium where there is only a cross-section of overlapping generations, there is no time effect and the concept of cohort and age coincide. The next two terms are heterogenous in agents: \( \alpha_i^t \) is the fixed effect which is predetermined before the agent enters the labor market; \( \theta_i^t \) is the heterogeneity in the growth rate of individual income. The idiosyncratic shocks part consists of a permanent (or AR(1) when \( \rho < 1 \)) and a transitory (i.i.d.) part \( \epsilon_i^t \). This income process nests both the Heterogenous Income Profile (HIP) process where \( \text{var}(\theta_i^t) > 0 \) and the Restricted Income Profile (RIP) process where \( \text{var}(\theta_i^t) = 0 \).

After retirement, the agent \( i \) receives pension \( B_{it} \) which is funded by a Pay-As-You-Go system through the pension tax \( \tau \). According to the U.S. Old Age pension system, the pension is a concave function of life-time average income. In computation, I will use the last period non-transitory income as a proxy of the average income to mimic the U.S. pension system.

The market is incomplete in the sense that agents can only have access to a risk-free bond which yields the net interest rate \( r \). In the benchmark setup, I assume that there exists perfect annuity markets for mortality risks, so that the return of asset is interest rate plus a survival premium. The agent \( i \)'s budget constraint is given by

\[ c_{i,t} + a_{i,t+1} \leq a_{i,t}(1 + r)/\xi_t + y_{i,t}, \]

where \( a_{i,t} \) is the asset or financial wealth. The agent can not leave negative asset at year \( T \) and faces a borrowing constraint \( a_{i,t+1} \geq z_{i,t+1} \), where \( z_{i,t+1} \) is an ad hoc borrowing constraint which can potentially a function of current state variables and can be set as low as the natural borrowing constraint. Each individual is endowed with initial wealth \( a_{i,0} \).

The interest rate is set to be exogenous, but it is straightforward to assume a production function to close the model. In that case, the interest rate is a function of the parameters of aggregate productivities.

For CRRA utility function, we can obtain the balanced growth path by dividing all the quantities by the accumulated productivity growth. Given constant \( r \) and \( w \), each agent’s decision problem can be written recursively as

\[ V(\alpha, \beta, \theta, a, \epsilon, z; t) = \max_{\alpha^t} \left\{ c^{1-\sigma}/(1 - \sigma) + \beta \xi_{t+1}(1 + g)^{1-\sigma}E[V(\alpha, \beta, \theta, a', \epsilon', z', t+1)|z] \right\} \]
subject to
\[ c + (1 + g)a' \leq a(1 + r)/\xi_t + \begin{cases} y_t & t < R \\ B_t & t \geq R \end{cases} \]
\[ a' \geq 0 \]
\[ a_{T+1} \geq 0 \]

The terminal period value function is set to \( V(\cdot; T + 1) = 0 \).

The decision rule is solved by backward induction using Euler equation. To speed up the algorithm, I use the endogenous grid method developed by Carrol (2006). I use linear interpolation with 71 grid points for positive asset and 50 grids for negative asset. Grids on positive asset are formed triple exponentially to make more grids where asset level is lower. I use two discrete states for each of the exogenous state variables \((\gamma, \theta, \varepsilon, \eta)\). 50,000 agents are used in the simulation. In the unit root income process, I use 41 state space for the permanent component. In the AR(1) income process, I discretize the persistent shock by a 41 state Markov chain using the method suggested by Tauchen (1986).

### 3.2.1 Calibration

#### Demography

The model period is 1 year. Agents begin to work at age 22, which coincides with age 1 in the model. Conditional on surviving, they then work for 45 years, retire at age 66 and die at age 100. Agents are interpreted as households in the data, and hence we chose the conditional surviving rate from the U.S. life table for females in 1989-1991. The annual population growth rate is set to \( n = 1.0\% \) per year. The interest rate \( r \) is set exogenously to be 4%.

#### Preference

The risk aversion is set to \( \sigma = 2 \). For the estimation of the distribution of \( \beta \), I will leave it to the next section.

#### Income process: RIP Vs HIP

So far, it is still an open question whether the restricted income process (RIP) or the heterogeneous income process (HIP) can better represent the household’s income process in economic research\(^1\). As one of the goals of this paper is to compare these two income processes, I will use both RIP and HIP as the income process in my quantitative

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\(^1\)See Guvenen (2009) for a good summary.

In the RIP model, I use Storesletten et al. (2004)’s estimation. The coefficient of auto-regression \( \rho \) is very close to 1, which is 0.98. And I will set it to 1 in computation and therefore the persistent shock is actually permanent. The variance of the fixed effect, persistent shock, transitory shocks are \( \sigma_\alpha^2 = 0.2105, \sigma_\eta^2 = 0.0161, \sigma_z^2 = 0.0630 \), respectively.

In the HIP model, I use the income process estimated by Guvenen (2007, 2009). The variance of the wage growth rate is \( \sigma_\theta^2 = 0.00038 \). The correlation coefficient between the idiosyncratic part of wage growth and the fixed effect is \( \text{cov} \theta \alpha = -0.002 \). The variance of the fixed effect, persistent shock and transitory shock is \( \sigma_\alpha^2 = 0.022, \sigma_\eta^2 = 0.029, \sigma_z^2 = 0.047 \), respectively. Unlike Guvenen (2007)’s learning story, I assume there is no prior uncertainty about the income process and thus the agent has complete information of her income profile when she enters the labor market.

The secular productivity growth rate is set to \( g = 1.5\% \) per year. The average age profile of income \( \kappa_t \) is chosen to match the average income in the U.S. Census 1990.

### Pension

The pension system in the benchmark model is designed to mimic the U.S. Old Age pension system as follows:

\[
B_t = \lambda \times \begin{cases} 
0.9y_{i,t,R-1} & \text{for } y_{i,t,R-1} < 0.3\bar{y}_{p,R-1} \\
0.27\bar{y}_{p,R-1} + 0.32(y_{i,t,R-1} - 0.3\bar{y}_{p,R-1}) & \text{for } \bar{y}_{i,t,R-1} \in (0.3\bar{y}_{p,R-1}, 2\bar{y}_{p,R-1}) \\
0.81\bar{y}_{p,R-1} + 0.15(y_{i,t,R-1} - 2\bar{y}_{p,R-1}) & \text{for } \bar{y}_{i,t,R-1} \in (2\bar{y}_{p,R-1}, 4.1\bar{y}_{p,R-1}) \\
1.1\bar{y}_{p,R-1} & \text{for } \bar{y}_{i,t,R-1} > 4.1\bar{y}_{p,R-1}
\end{cases}
\]

Different from Storesletten et al. (2004), I use the last working year income excluding the transitory part instead of life-time average income. Guvenen (2007) uses a similar expression where the last period income serves as the proxy for the average life-time income. I exclude the transitory part because the last period transitory part gives us little information of the average life-cycle income and the non-transitory part is still highly correlated with average life-cycle income. I rescale the pension system to make the replacement ratio of the model match that of the U.S. data, which is 0.48. It generates 0.92 in the benchmark RIP model and 0.82 in the HIP model. The pension tax \( \tau \) can be solved directly by the PAYG system, which is 0.1325. I will also discuss the extreme case when there is no pension.
Borrowing constraint

In the benchmark setup for both RIP and HIP models, the households are allowed to borrow up to the expected income of next year \( a_{t,t+1} = -E_t(y_{t,t+1}) \), which is the same as used in Storesletten et al. (2004). I also consider two other extreme cases: one is that all households are excluded from any borrowing, i.e. \( a = 0 \); the other is that no ad hoc borrowing constraints is imposed and I only impose the terminal condition that agents cannot die in debt at age \( T + 1 \). In other words, I set the borrowing constraint as low as the natural borrowing constraint which is not binding in household’s optimal solution with CRRA preference.

Initial Wealth

The initial wealth distribution is calibrated to mimic the wealth distribution of households under age 25 in SCF 1992 (Diaz-Gimenez et al. 1997). I approximate the initial wealth distribution by a log normal distribution whose mean is set to match the initial wealth/income ratio, which is 0.89, and then I calibrate its variance to match the wealth Gini for those young households, which is 0.87.

The above parameters for calibration are summarized in Table 1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relative Risk Aversion</td>
<td>( \sigma = 2 )</td>
</tr>
<tr>
<td>Mortality Risk</td>
<td>U.S. Female Life table 1991</td>
</tr>
<tr>
<td>Interest Rate</td>
<td>( r = 4% )</td>
</tr>
<tr>
<td>Population Growth</td>
<td>( n = 1% )</td>
</tr>
<tr>
<td>Secular Growth</td>
<td>( g = 1.5% )</td>
</tr>
<tr>
<td>Average Wage Growth</td>
<td>U.S. Census 1990</td>
</tr>
<tr>
<td>Restricted Income Process</td>
<td>( \rho = 1, \sigma_\gamma^2 = 0.2105, \sigma_\eta^2 = 0.0161, \sigma_\varepsilon^2 = 0.0630 )</td>
</tr>
<tr>
<td>Heterogeneous Income Process</td>
<td>( \rho = 0.821, \sigma_\gamma^2 = 0.022, \sigma_\eta^2 = 0.029, \sigma_\varepsilon^2 = 0.047 )</td>
</tr>
<tr>
<td></td>
<td>( \sigma_\theta^2 = 0.00038, \text{cov}\theta\gamma = -0.002 )</td>
</tr>
<tr>
<td>Pension Tax in PAYG</td>
<td>( \tau = 0.1325 )</td>
</tr>
<tr>
<td>Borrowing Constraints</td>
<td>( a_{t,t+1} = -E_t(y_{t+1,t,i}) )</td>
</tr>
</tbody>
</table>

3.2.2 Estimation of the distribution of \( \beta \)

Methodology

The distribution of \( \beta_i \) is crucial in the model. In a standard model without discount rate heterogeneity, the variance of \( \beta_i \) is restricted to be zero and the conventional
procedure of calibrating $\beta$ is to minimize the distance between the wealth/income ratio in the simulated model and that of the data. In the model with discount rate heterogeneity, however, we need more moments to identify the distribution of $\beta$.

It has been well noticed that the discount rate heterogeneity can help to generate a larger and more reasonable wealth inequality in the incomplete market model with income risks (Krusell and Smith 1998). In a quantitative life-cycle model, Hendricks (2007) estimates the distribution of discount factors using age profile of wealth Gini coefficient. Nevertheless, adding discount rate heterogeneity into the standard heterogeneous agent model is not necessary for a large and skewed wealth inequality, especially for the upper tail of the distribution. Either altering the income process (Castaneda et al. 2003) or introducing the entrepreneur behavior and bequest motive (Cagetti and De Nardi 2005) can make the standard model match the wealth inequality in the data.

This paper takes a different estimation approach. I focus on one particular moment in the wealth distribution: the fraction of old agents holding zero or negative wealth. Diamond and Housman (1984) argue that because of the life-cycle motive of saving, the standard life-cycle models have difficulty in generating significant amount of agents with zero or negative wealth when retirement is near, which is at odds with the data. Although in Huggett (1996), a life-cycle model with uninsurable income and life-time risks is able to generate a reasonable fraction of old agents holding zero or negative wealth, the results hinge on the existence of a very generous lump-sum pension and fairly loose borrowing constraints.

This model discrepancy with the data implies that there might be a significant amount of sufficiently impatient agent. Hence, it gives us useful information in calibrating the distribution of $\beta$. Specifically, I choose the fraction of agents holding zero or negative wealth from age 55 to 64 as another target for calibration. In the U.S. data from SCF 1992, this number is 8.9%. (Weicher 1997)\textsuperscript{5} The fraction of zero/negative wealth households in the model with estimated parameters are plot in Figure 1.

\textsuperscript{5}Diamond and Hausman (1984) calculate the fraction of individuals holding zero or negative wealth in a sample of men aged 45-59. Their result is 7%.
3. Consumption Inequality and Discount Rate Heterogeneity

I approximate the distribution of discount factor $F(\beta)$ by a discrete distribution with two values $\beta_l = \beta(1 - \Delta \beta)$ and $\beta_h = \beta(1 + \Delta \beta)$. The probability of being an relatively impatient agent is $p_l$, and the probability of being a relatively patient household is $p_h = 1 - p_l$. To reduce the parameters for calibrating, the middle value of the discount rate, $\bar{\beta}$, is assumed to be equal to the calibrated value of $\beta$ in a standard model without discount rate heterogeneity to match the wealth income ratio in the U.S. of 3.1 from SCF 1992 (Diaz-Gimenez et al. 1997). Alternatively, I will also consider the wealth/income ratio of 4.56, which is the average of SCF 1992 and 1998 and is used in Sun (2008). The two targets for estimating $\Delta \beta$ and $p_l$ are the wealth income ratio again and the average fraction of non-positive wealth household from age 55 to age 64. To save the time for computation, $\Delta \beta$ is assumed to lie on 30 grids, from 0.005 to 0.15, and $p_l$ is assumed to lie on 100 grids, from 1% to 99%. $\Delta \beta$ and $p_l$ are chosen to minimize the loss function which is the sum of the absolute value of the percentage deviation between data and model in these two targets.
3. Consumption Inequality and Discount Rate Heterogeneity

Estimation results

Table 2: Estimation of Distribution of Discount Rate

<table>
<thead>
<tr>
<th></th>
<th>RIP Model</th>
<th></th>
<th>HIP Model</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\bar{\beta}$</td>
<td>$\Delta \beta$</td>
<td>$p_l$</td>
<td>$\bar{\beta}$</td>
</tr>
<tr>
<td>Benchmark</td>
<td>0.9908</td>
<td>0.030</td>
<td>64%</td>
<td>0.9972</td>
</tr>
<tr>
<td>No pension</td>
<td>0.9559</td>
<td>0.085</td>
<td>81%</td>
<td>0.9566</td>
</tr>
<tr>
<td>No borrowing</td>
<td>0.9878</td>
<td>0.090</td>
<td>81%</td>
<td>0.9947</td>
</tr>
<tr>
<td>NBC</td>
<td>0.9918</td>
<td>0.020</td>
<td>56%</td>
<td>1.0120</td>
</tr>
<tr>
<td>High Wealth</td>
<td>1.0035</td>
<td>0.045</td>
<td>62%</td>
<td>1.0133</td>
</tr>
<tr>
<td>Alan and Browning (2003)</td>
<td>0.9359</td>
<td>0.090</td>
<td>50%</td>
<td>0.9447</td>
</tr>
<tr>
<td>Hendricks(2007)</td>
<td>0.9873</td>
<td>0.035</td>
<td>61%</td>
<td>0.9955</td>
</tr>
</tbody>
</table>

The estimations of $\beta_i$ are summarized in Table 2. I compute the mean and standard deviation of $\beta_i$, which are different, though not far from, $\bar{\beta}$ and $\Delta \beta$. The estimation results will be discussed in the next section.

$\Delta \beta$ is from 3% in benchmark HIP model to 9% in RIP model. $p_l$ varies 65% to 81%.

The estimates for RIP model is close to that from that in the HIP model. The middle value of discount factors, $\bar{\beta}$, is higher in HIP model than in RIP model, because less persistent shock reduces the precautionary motive for saving and thus lowers the wealth income ratio. To match the wealth income ratio, $\bar{\beta}$ has to be increased. When there is no pension or when there is no borrowing, the standard model without discount rate heterogeneity generate too few old agents holding zero or negative wealth. Increasing both the fraction and the value of relatively impatient agents helps to match this moment. $p_l$ is greater than 50% in all estimation, because discount rate heterogeneity will increase the mean wealth of the economy. To make $\bar{\beta}$ unaltered, the fraction of impatient agents has to be increased.

Another approach is to directly estimate $\beta_i$ using Euler equation (Alan and Browning 2003). However, this estimation attributes all intertemporal wedges into discount factors. For comparison, I will also consider the estimates of mean and variance of $\beta_i$ from the estimation of Hendricks(2007) and Alan and Browning (2003). Interestingly, although I have not used any information of wealth Gini coefficient, the estimation of distribution of $\beta_i$ in both RIP and HIP model are very close to the estimation by Hendrick (2007).

I take the estimation of $\beta$ in the no bequests model of Hendricks(2007 Table 2). Since he use 5 grids for $\beta_i$ and set $\beta$ to be 2% or 6% lower or higher than $\bar{\beta}$, I set $p_l$ to the fraction of the agents with 0.94 or 0.98 $\bar{\beta}$ plus half of the fraction of agents with $\bar{\beta}$, which gives $p_l = 61\%$. I set $\sigma$ to be half of the difference between weighted average of the lowe $\beta$s and the high $\beta$s, which is 0.035. For Alan and Browning (2003), I assume a symmetric distribution and take its coefficient of variance as $\sigma$, which is 0.09. In both cases, I recalculate the $\bar{\beta}$ to match the wealth income ratio.
3. Consumption Inequality and Discount Rate Heterogeneity

3.3 Results

3.3.1 Age profile of consumption inequality

Empirical evidence

The empirical evidence of the age profile of consumption inequality is drawn from the Consumer Expenditure Survey (CEX) data in U.S. by most authors. However, the estimates by different authors differ because of the choice of year effect or cohort effect and because of the sample selection. Deaton and Paxson (1994) assume away the time effect and get the consumption inequality increase by 0.25 log points from age 25 to age 65. Heathcote et al. (2005) argue that the time effect model is more plausible than the cohort effect model and they report the log consumption increase by only 0.02 log points from age 25 to age 60. They find that when including longer periods for CEX, the life-time increase of consumption inequality becomes less significant. Guvenen (2007)’s finding is somewhere in between these two former estimates. Heathcote et al. (2009) generates a even flatter shape consumption inequality.

Qualitatively, in all of these estimates, consumption inequality increases over the life cycle. The shape is approximately linear in Deaton and Paxson (1994) and Guvenen (2007), while it is not clear in Heathcote et al. (2005) and Heathcote et al. (2009), who only compute consumption inequality up to age 60. Quantitatively, the choice of data set can be crucial for the evaluation of models. While Storesletten et al (2004) claims a success of a RIP model based the estimates from Deaton and Paxson (1994), Guvenen (2007) claims the success of HIP models with learning based on his estimates. I will compare the age profile of consumption inequality in the model with the estimates from Deaton and Paxson (1994) and Guvenen (2007). In addition, I will focus more on Guvenen (2007)’s data which his learning model is supposed to match.

Model Vs data

I consider two versions of benchmark model where pension is included and borrowing limit is set to the next year’s expected income: one for RIP and one for HIP. Age profiles of consumption inequality in both RIP and HIP benchmark models from age 25 to age 65 are shown in Figure 2. Since we are interested in the increase of consumption inequality, not its levels, I normalize the consumption inequality for age 25 to be zero.
Figure 2: Age Profile of Consumption Inequality: Model Vs Data

Without discount rate heterogeneity, the consumption inequality in both the RIP model and HIP model increase over the life cycle and has a concave age profile. In RIP model, the life-time increase of consumption inequality is higher than data; in HIP model, the life-time increase of consumption is less significant and is lower than data. As Lucas(2005) and Storesletten et al.(2004) argued, this evidence of consumption inequality favors the HIP model in which the labor market income process is highly persistent.

With discount rate heterogeneity, the consumption inequality is tilted up, more so from age 50 on. In the RIP model, although the shape is not significantly linear, it does make the profile more convex in the last ten working years. There is no effect of the increase of consumption inequality in the RIP model.

In the HIP model, introducing discount rate heterogeneity makes the consumption inequality increase more. The profile with discount rate heterogeneity is approximately linear. HIP model with discount heterogeneity does a good job in accounting for the consumption inequality data, especially for Guvenen’s estimation. Adding discount heterogeneity to the RIP model makes the model even farther from the data. Unlike Guvenen (2007), the success of HIP model is not a result of Bayesian learning. Without changing the structure of the plain-vanelia life-cycle model, adding well-estimated discount rate heterogeneity can help to match the data in both magnitude and shape.
To single out the effect of discount rate heterogeneity, I decompose the consumption inequality into variance of with-in household group of same discount rate and between households groups with different discount rate. Figure 3 and 4 show that most of the
increase of consumption inequality of the last 10 working years is due to discount rate heterogeneity. Since the variance in benchmark model RIP is higher, the contribution of discount rate heterogeneity is higher than in HIP model.

Figure 5: Age Profile of Consumption Inequality : RIP Model

Figure 6: Age Profile of Consumption Inequality : HIP Model
3.3.2 Role of pension

Figure 5 and 6 show the age profile of consumption inequality without pension. The shape of consumption inequality become more convex when pension is excluded. In other words, including pension mitigates the effect of discount rate heterogeneity, especially for the old agents.

Excluding pension will make old household more willing to save for retirement. To match the model ratio of household holding zero or negative wealth to the data, it requires higher heterogeneity of discount rate and higher fraction of relatively impatient households. As we have seen in Table 2, it gives $\sigma = 0.080, 0.085$ and $p_t = 81\%, 84\%$ in RIP and HIP models, respectively. This drives up the consumption inequality between different household groups.

3.3.3 Role of borrowing constraint

As to the sensitivity of borrowing constraints, I consider two extreme cases: no borrowing and natural borrowing constraints (NBC). In Figure 5 and 6, we can see the effect of excluding borrowing is to tilt up the consumption inequality profile and is very close to the profile of no-pension economy for most part of the life cycle. This effect is the same as the effect of pension, which makes the households save more and thus requires higher degree of heterogeneity discount rate to match the data. However, the shape of profile in the no borrowing case differ from the profile without pension in that the former becomes concave in the last 10 working years.

Why tight borrowing constraints may cause the heterogeneous discount rate model give the opposite implication for consumption inequality for the old households? To see the mechanism clearly, I decompose in Figure 7 and 8 the consumption inequality into two with-in group inequality: one for the high $\beta$ households and one for the low $\beta$ households.

For the high $\beta$ households, the result is standard: the consumption inequality is flat in the RIP model and concave in the HIP model because of the increasing capability of self-insurance by the accumulation of life-cycle wealth. Actually, the shape is even more concave because the wealth-income ratio of the high $\beta$ household are higher than the average in the whole economy.
3. Consumption Inequality and Discount Rate Heterogeneity

![Figure 7: Consumption Inequality for Different Groups of Households: RIP Model](image)

**Figure 7:** Consumption Inequality for Different Groups of Households: RIP Model

![Figure 8: Consumption Inequality for Different Groups of Households: HIP Model](image)

**Figure 8:** Consumption Inequality for Different Groups of Households: HIP Model

For the low $\beta$ households, however, the consumption inequality decrease when approaching retirement. Although the decreasing of consumption inequality is a un-
expected feature in the standard life-cycle model, the reason is quite simple: for some of the low \( \beta \) households, they are too impatient to save, even when retirement is near. They are willing to borrow though they are not allowed to do so, which causes the borrowing constraints of those agents to be binding. The consumption of the borrowing constrained (no borrowing in this case) agents becomes:

\[
c_t = a(1 + r)/\xi_t + y_t, \quad \text{iff} \quad c(\gamma, \theta, a, \varepsilon, z; t) \geq a(1 + r)/\xi_t + y_t, \quad (3.4)
\]

where \( c(\gamma, \theta, a, \varepsilon, z; t) \) is the optimal consumption rule derived from Euler equation. Therefore, \( c_t \) for the borrowing constrained households is lower than that derived from the optimal consumption rule when the borrowing constraint is not binding. When approaching retirement, some of the previously borrowing constrained agents become unconstrained. Since those are the households with relatively lower consumption level, the with-in group consumption inequality decreases.

When there is only natural borrowing constraint, in the RIP model the result does not differ much from the benchmark case, because the borrowing constraint of expected next year’s income is hardly binding for any households. In the HIP model, some agents with lower slope of wage are more likely to hold zero or negative wealth and the borrowing constraints in the benchmark model are more often binding. Hence, the consumption inequality profile goes in the opposite direction as the no borrowing case. It is U-shaped, which is similar to the result in the complete market with no idiosyncratic shocks.

There is no consensus on the choice of borrowing constraints. It would not be a problem for the standard model without discount rate heterogeneity, since the fact that in the model no one will be borrowing constrained near retirement makes the borrowing constraints quantitatively less important for consumption inequality. Nevertheless, the assumption that there is no borrowing is not innocuous and therefore the borrowing constraint should be chosen more carefully, if there does exist discount rate heterogeneity.

### 3.3.4 Role of wealth-income ratio

As can be seen in Figure 5 and 6, although the choice of wealth-income ratio change the distribution of \( \beta \), it gives negligible effect on the consumption inequality profile. This result is different from the standard model. It is because higher wealth-income ratio has two effects. Normally it increases \( \bar{\beta} \) and thus makes the household holding more wealth for self-insurance, which lowers the consumption inequality. On the other hand, when \( \bar{\beta} \) is high, \( \Delta \beta \) has to be increased to match the fraction of zero or negative wealth agents, which increases the consumption inequality. The results show that these
two effects almost cancel each other out, and therefore the consumption inequality is not very sensitive to the choice of wealth-income ratio.

### 3.3.5 Consumption over the life cycle

Now we look at another dimension of the model prediction: the mean of consumption over the life cycle. Empirical evidence shows that consumption over the life cycle is hump-shaped. In a standard life-cycle incomplete model, the consumption over the life cycle can be hump-shaped when agents are impatient, save for precautionary motive when they are young, and save for life-cycle motive when they are old (Gourinchas and Parker 2002).

In Figure 9 and 10, I plot the mean consumption over the life cycle for household groups with different $\beta$ for the no-pension case, in which the life-cycle motive is the strongest when old. The average consumption of the economy is hump-shaped. However, the consumption of the relatively patient households are increasing linearly all the time while only the consumption of relatively impatient households decreases when approaching the retirement age.

![Graph of Mean Consumption of Different Household Groups](image)

Figure 9: Consumption over the Life Cycle: RIP Model
3. Consumption Inequality and Discount Rate Heterogeneity

![Chart: Mean Consumption of Different Household Groups](image)

Figure 10: Consumption over the Life Cycle: HIP Model

The between-group consumption inequality can be simply measured through the distance between these two profiles. It is U-shaped and increasing with age for the old households. As shown in the complete market model, even if there is no increase of with-group consumption inequality, the consumption inequality may still increase because the distance between the consumption profile of households with different \( \beta \) may increase over the life cycle.

If there exists some way to proxy discount rate by any observables, say, education, the difference of mean consumption profiles predicted by the model gives us a potential way of identifying the degree of discount rate heterogeneity.

### 3.4 Consumption Insurance

#### 3.4.1 BPP’s measure

We study consumption inequality over the life cycle because it is a good measure for the insurance against income shocks. When discount rate heterogeneity is present, however, using life-time increase of consumption inequality as a measure of consumption insurance is up-ward biased. To evaluate the consumption insurance over the life cycle in a model with discount rate heterogeneity, it is useful to also consult on other measure of consumption insurance.

BPP denote partial insurance as the degree of transmission of income shocks to
consumption growth and construct an empirical measure of it. They assume unexplained log income can be decomposed into a unit root permanent part and an i.i.d. transitory part and assume log consumption can be written as

$$\Delta \log c_t^i = \phi_t^i \eta_t^i + \psi_t^i \varepsilon_t^i + u_t^i, \quad (3.5)$$

Since the $\phi$ is the pass through of permanent income shocks to consumption change, it is a natural measure of consumption insurance. BPP’s main findings is that, in the whole sample, the estimate of $\phi$ and $\psi$ is 0.6423 and 0.0533, respectively. In other words, a 10 percent permanent (transitory) shock of disposable income induces a 6.4(0.5) percent change in household’s nondurable consumption. They also find a non-significant linear trend of decreasing partial insurance parameter for permanent shocks over the life cycle.

Kaplan and Violante (2010) and Sun (2010) measure consumption insurance in simulated life-cycle incomplete market models using BPP’s method and find that $\phi$ generated in the model is generally higher than that of the data, though the finding in the benchmark model of Sun (2010), which is 0.71, is much lower than the benchmark case in Kaplan and Violante (2010), which is 0.93, due to different model parameters and sample selection. Interestingly, they all find that the permanent pass-through for the old household is lower in the model than in the data, which is a flip side of the concave shape of consumption inequality age profile in life-cycle models. The reasons is similar: the old households simply accumulate too much wealth in the model. In short, in standard life-cycle models without discount rate heterogeneous, the consumption insurance measured by BPP’s methodology echoes with that from the measure of consumption inequality in both the magnitude and shape.

3.4.2 Consumption insurance: BPP

Table 3 shows the consumption insurance by BPP’s measure. In the simulated model, since shocks are known, it is straight forward to compute $\hat{\phi}_t$ and $\hat{\psi}_t$ by standard OLS approach. I also report the BPP’s estimation by instruments, when shocks are unknown to the econometrician:

$$\hat{\phi}_t^{Instr} = \frac{E(\Delta \hat{c}_t(\Delta \hat{y}_{t-1} + \Delta \hat{y}_t + \Delta \hat{y}_{t+1}))}{E(\Delta \hat{y}_t(\Delta \hat{y}_{t-1} + \Delta \hat{y}_t + \Delta \hat{y}_{t+1}))} \quad (3.6)$$

$$\hat{\psi}_t^{Instr} = \frac{E(\Delta \hat{c}_t \Delta \hat{y}_{t+1})}{E(\Delta \hat{y}_t \Delta \hat{y}_{t+1})} \quad (3.7)$$
### Table 3: Consumption Insurance

<table>
<thead>
<tr>
<th>Model</th>
<th>OLS Instrument</th>
<th>OLS Instrument</th>
<th>ΔVarlogC25,65</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data</td>
<td>BPP: 0.642</td>
<td>BPP: 0.053</td>
<td>Guvenen:0.21</td>
</tr>
<tr>
<td>RIP</td>
<td>Homo</td>
<td>0.78, 0.79</td>
<td>0.057, 0.057</td>
</tr>
<tr>
<td></td>
<td>Hetero</td>
<td>0.84, 0.87</td>
<td>0.098, 0.098</td>
</tr>
<tr>
<td>HIP</td>
<td>Homo</td>
<td>0.33, 0.34</td>
<td>0.089, 0.101</td>
</tr>
<tr>
<td></td>
<td>Hetero</td>
<td>0.41, 0.44</td>
<td>0.120, 0.146</td>
</tr>
<tr>
<td>RIP: Insurable</td>
<td>Homo: α = 0.20</td>
<td>0.64, 0.64</td>
<td>0.051, 0.051</td>
</tr>
<tr>
<td></td>
<td>Hetero: α = 0.30</td>
<td>0.64, 0.6</td>
<td>0.088, 0.088</td>
</tr>
</tbody>
</table>

Although they are in principle different, Sun (2008) proves that both Instrument and OLS estimator are consistent estimates of \( \phi \) and \( \psi \). By comparison, the increase of log consumption from age 25 to age 65 is also reported. The overall estimates of permanent pass-through is higher in models with heterogeneous discount rate. The implication from BPP’s measure is similar to that from consumption inequality. This shows the consumption insurance is upward biased if discount rate heterogeneity is present but assumed away. Notice that the transitory pass-through, \( \phi \), is much higher than the data in both HIP and RIP models with heterogenous discount rate. This is because the low \( \beta \) households on average have lower wealth and therefore have less ability of self-insurance even against the transitory shocks.

Figure 11 shows the pass-through of permanent shocks, \( \phi \), over the life cycle, using OLS estimator. The model with heterogeneous discount rate give a higher profile of \( \phi \), which implies a low profile of consumption insurance. The \( \phi \) in RIP is higher than data, except for the last 5 years, whereas the \( \phi \) in HIP is much lower then the data. There is a significant downward sloping trend for the RIP models, whereas the trend is not clear for the RIP models. Adding heterogeneous discount factor lowers the degree of insurance for the old households, which makes the model result for the old households closer to that of the data.
3. Consumption Inequality and Discount Rate Heterogeneity

![Pass-through of Permanent shocks over the Life Cycle](image)

Figure 11: Consumption Insurance over the Life Cycle

### 3.4.3 RIP model with partial insurance

The HIP model, whether or not discount rate heterogeneity is included, gives less consumption insurance than that of the data by both measures. This can be either because of less persistence of income shocks or because of the existence of partial insurance over and above self-insurance. If the latter is true, the highly persistent income process may not be rejected simply by consumption inequality. Note that the insurable part and predictable part of permanent income change cannot be identified through the consumption data only (Primiceri and Rens 2009).

To measure the degree of the maximum amount of partial insurance in the HIP model, I take a simple approach. Assume \( \pi \) fraction of the permanent shock is insurable. In the view of the households, exogenously including the insurable part is equivalent to reducing the variance of permanent shocks. Therefore, the "effective" income shock the household faces now becomes \((1 - \pi) \eta_i^t\). Then I ask: how much partial insurance does it require for RIP model to generate the same degree of consumption insurance as that of the data?

I first choose \( \pi \) such that the benchmark RIP model without discount rate heterogeneity generates the same insurance measured by BPP’s measure. Notice \( \overline{\beta} \) has to be recalibrated to match the income wealth ratio simultaneously. This gives \( \pi = 0.20 \) and \( \overline{\beta} = 0.9993 \) in the benchmark model, which means 20 percent of the income shocks...
3. Consumption Inequality and Discount Rate Heterogeneity

is insurable other than self-insurance. For the case of discount rate heterogeneity, I take the estimated \( \Delta \beta \) and \( p_t \) in the benchmark RIP model, in order to avoid using moments from consumption data which BPP’s measure is based on. I then redo the previous exercise and get the partial insurance parameter as 30\%. Figure 12 shows that when partial insurance is exogenously included, the RIP model can generate the similar predictions on consumption inequality as the HIP model. Therefore, consumption data is not sufficient to identify these two types of income shocks.

![Consumption Inequality over the Life Cycle](image)

Figure 12: Consumption Inequality over the life cycle: Model Vs Data

3.5 Initial Condition Vs Life-time Risk

There are possibly fours sources of inequality: fixed effect, individual income growth rate, discount rate and realization of income shocks. These four different dimensions of inequality contribute in different amount to the life-time inequality of consumption, wealth and ultimately, welfare.

Given the models with different income process and different implication for consumption inequality and consumption insurance, we are in a position to compare their different welfare implication.

The life-time money equivalent welfare is defined as

\[
M = u^{-1}\left[\frac{(1 - \beta^T_i)\sum_{t=1}^{T} \beta_i^{T-t} u(c_{it})}{1 - \beta_i}\right]
\]  

(3.8)
3. Consumption Inequality and Discount Rate Heterogeneity

Note that in the standard life-time welfare calculation, the welfare is higher for the household with higher $\beta$, even if the consumption stream are identical. To remove this direct effect, I discount the life-time welfare by $\frac{1-\beta^L}{1-\beta_t}$. The total variance of welfare can be decomposed into with-group and between-group variance as follows

$$Var M = E(Var(M|x)) + Var(E(M|x))$$  \hspace{1cm} (3.9)

where $x = \alpha, \beta, \theta$ or all of these three variables. $Var(E(M|x))/Var M$ can be interpreted as the contribution of the inequality in variable $x$ to the total inequality in life-time welfare. The results are summarized in Table 4.

<table>
<thead>
<tr>
<th></th>
<th>Initial Conditions</th>
<th>Income Shocks</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Only $\alpha$</td>
<td>Only $\theta$</td>
</tr>
<tr>
<td>HIP heterogeneous $\beta$</td>
<td>1%</td>
<td>62%</td>
</tr>
<tr>
<td>HIP homogeneous $\beta$</td>
<td>14%</td>
<td>54%</td>
</tr>
<tr>
<td>RIP heterogeneous $\beta$</td>
<td>16%</td>
<td>-</td>
</tr>
<tr>
<td>RIP homogeneous $\beta$</td>
<td>40%</td>
<td>-</td>
</tr>
</tbody>
</table>

The contribution of the fixed effect to the RIP model with homogeneous discount rate is 0.4, which is close to the estimate of 0.47 in Storesletten et al. (2004). When heterogeneous discount rate is present, the fixed effect can only account for 16% percent of total inequality and the total variance accounted for by initial condition is 96%, which is close to the estimate of 0.9 in Keane and Wolpin (1997). In the HIP model, the heterogeneous income profile alone can account for 62% for the life-time inequality and the heterogeneous discount factor alone can account for 80% of the total inequality. In short, the initial condition becomes more important when there is either inequality in discount rate or income profile. Among the initial conditions, discount rate heterogeneity is the most important, and the fixed effect of income is the least important.

3.6 Conclusion

This paper investigates consumption inequality over the life cycle in a class of incomplete market life-cycle models with discount rate heterogeneity. I find that the presence of discount rate heterogeneity has important implications for consumption inequality, consumption insurance, the identification of income process and life time welfare.

The major effect of the discount rate heterogeneity is to make the consumption inequality profile more convex, especially for the household when approaching retirement. Since in the U.S. data consumption inequality increases linearly with age and in
the standard model it increases concavely, the presence of discount rate heterogeneity drives the model prediction closer to the data.

When there is discount rate heterogeneity, a HIP model matches the data well in both the magnitude of increase and shape of consumption inequality profile, while RIP model does not. This result is consistent with the finding of Guvenen (2007), while the present model is much simpler. Just as the discount rate is unobservable, the income uncertainty and Bayesian learning is also unobservable. The result of the paper shows that the standard incomplete market models with small modification can account for the empirical age profile of consumption inequality. Different from Guvenen (2007), I find that if there exists partial insurance, RIP can also do a good job in matching the consumption profile. Therefore, consumption data alone is not sufficient for us to distinguish between HIP model with only self-insurance and RIP model with partial insurance. Note that the distribution of discount factors is not estimated by using any moments from consumption data. In particular, the model is consistent with both the evidence of the fraction of zero or negative wealth holding for the old households and consumption inequality over the life cycle.

When discount rate heterogeneity is present, the borrowing constraint and pension become more important for consumption inequality. Without taking into account discount rate heterogeneity, the insurance in the model is up-ward biased by BPP’s measure. These caveats must be noticed when studying consumption inequality or consumption insurance in standard models with homogenous preferences.

As to the normative question asked by Keane and Wolpin (1997), Storesletten (2004) and Huggett et al. (2007), I find that introducing discount rate heterogeneity as another dimension of initial condition will make both the fixed effect and life-time shocks less important for the life-time inequality. The life-time inequality in welfare is more due to initial condition (96%) than life-time shocks. Among those initial conditions, the discount rate heterogeneity is the most important and the fixed effect of income is the least important.

As Browning et al. (1999) noted, using micro data to calibrate the preference parameters in quantitative models may have potential flaws and has to be taken carefully, especially when there exists preference heterogeneity. This paper, among a few others, estimate the distribution of discount factors from the simulated model. It would be interesting to calibrate the distribution of discount factor from micro data if we can find new way of identification. For example, households who discount the future differently can potentially have very different consumption profiles: the consumption profile of the impatient households is hump-shaped and for the patient households it is increasing. It might be possible to test the discount rate heterogeneity by the shape of mean consumption profiles for different groups. I leave it for future research.
Bibliography


67


BIBLIOGRAPHY


Chapter 4

Complete Markets Strikes Back: Revisiting Risk Sharing Tests under Preference Heterogeneity

4.1 Introduction

How well do people insure against income risks? Recent risk sharing tests exploit the information from the micro data sets of both consumption and income distributions. One type of tests is to study the change of consumption dispersion: Deaton and Paxson (1994) study the consumption dispersion for each birth-cohort and find that the consumption dispersion increases significantly over the life cycle. As they point out, the fanning out over the life cycle of both consumption and income distributions indicates that, under the standard separable preferences, the individual’s labor income must include highly persistent and uninsurable risk. The other type of tests, which is theoretically more transparent while empirically more demanding, uses the co-movement of consumption and income. Attanasio and Davis (1996) study the relative movement of consumption of cohort-education groups as a response to a relative wage change. Blundell, Preston and Pistaferri (2008, BPP hereafter) generalize this idea and estimate the pass-through of individual income shocks to consumption growth using a constructed panel of both consumption and income. These authors find that individual or group consumption co-moves significantly with the income\(^1\).

All of the above risk sharing tests share the same conclusion: the hypothesis of complete markets is soundly rejected. In a standard complete market (CM) model,

\(^1\)The predecessors are Cochrane (1991), who rejects the hypothesis of complete markets for long illness and involuntary job loss, and Mace (1991), who rejects the complete market hypothesis with power utility.
there would be neither significant dispersion of consumption over the life cycle nor significant co-movement of individual or group consumption and income.

The striking evidence from these tests has a profound impact on the agenda of quantitative economists. Many dynamic quantitative model builders turn to another workhorse model: the class of standard incomplete market (SIM) models, as the SIM models outperform the CM models in all these tests. For example, Storesletten et al. (2004) find that the SIM model with highly persistent labor market shocks can account for the empirical increase of consumption dispersion over the life cycle in Deaton and Paxson (1994), whose estimates are based on the U.S. data of Consumption Expenditure Survey (CES) before early 1990s. To account for the recent empirical evidence from Heathcote et al. (2010a), who use the CES waves from 1980 to 2006, Heathcote et al. (2010b) introduce additional channels of insurance, such as labor supply, education choice and family formation, into the SIM model; their model can match the rise of the consumption dispersion both over time and over the life cycle. Moreover, in Sun (2010a) and Kaplan and Violante (2010), they find that the pass-through of individual income shocks to consumption growth in the SIM models is closer to the empirical estimate of BPP (2008) than its CM counterpart, although SIM models generate much less consumption insurance than that in the data and a counter-factual age trend of the degree of risk sharing.

In this paper, I revisit the implication of these risk sharing tests and re-evaluate the CM and SIM models, both qualitatively and quantitatively. I find that not only is the hypothesis of complete markets not rejected, but also it outperforms SIM in the sense that the CM model can match more closely the observed consumption dispersion and the co-movement of consumption and income.

To revive the hypothesis of complete markets, I extend an otherwise standard CM model to allow for positive correlation between heterogeneous discount factors and heterogeneous income growth rates. Why does it work? The intuition is simple: with discount rate heterogeneity, people with different discount factors have different profiles of consumption path. Even if markets are complete and consumption paths are deterministic, consumption dispersion will eventually increase over the life cycle when their consumption paths diverge. As an individual’s consumption growth rate is increasing in her discount factor when markets are complete, consumption will co-move with income if the discount factor is positively correlated with the income growth rate. As a result, with preference heterogeneity, the risk sharing tests arrive at the conclusion of imperfect risk sharing even if the markets are actually complete.

To deliver the key messages, I will first discuss the methodology of and evidence from the risk sharing tests. Then, I present two classes of complete market models. One with stochastic income and one with predictable income change. Both are

standard models except that the agents’ time preference is no longer restricted to be homogeneous. Simple as these models are, they deliver the key insight about how the presence of discount rate heterogeneity may change the age profile of consumption distribution and how its interaction with income profile heterogeneity may affect the co-movement of consumption and income. I demonstrate that it is not only theoretically possible, but also quantitatively admissible for the CM model to simultaneously account for both the observed increase of consumption dispersion over the life cycle and the co-movement of consumption and income. Finally, I evaluate the CM and SIM models by comparing the results with Sun (2010b).

The first new ingredient in my model is the discount rate heterogeneity, which is not directly observable. Nevertheless, there are some indirect evidence on the dispersion of discount factors by experiments designed to reveal individual’s preference (Barsky et al. 1997), by Euler equation estimation (Lawrance 1991, Alan and Browning 2003), and by analyzing the difference of retirement wealth with the same life-time income (Hendricks 2007b). After the seminal paper by Krusell and Smith (1998), it is well-known in the macroeconomic literature that discount rate heterogeneity can potentially play an important role in accounting for wealth distribution. Hendricks (2007a) follows this insight and estimates the distribution of discount factors in a calibrated life cycle model by matching the observed wealth inequality. In Sun (2010b), I estimate discount factors using the faction of zero/negative households at retirement age.

The second new ingredient is the Heterogeneous Income Profiles (HIP). This view of income process is emphasized by some authors (e.g. Lillard and Weiss 1979, Hause 1980 and Guvenen 2007, 2009), which is an alternative to the view of Restricted Income Profiles (RIP) with very persistent income shocks2. Without additional information or ad hoc model specification, it is difficult for the econometricians to tell apart the predictable income change, such as the heterogeneous income growth, from the unpredictable income change, such as the permanent income shocks. In a recent paper by Primiceri and Rens (2009), they estimate a HIP model and find that a significant part of income change is predictable. In my model, I will take a more general view of the HIP process, in which the RIP process is a special case.

The third new ingredient is the positive correlation between individual discount factors and individual income growth rates. I provide two justifications: a model of education and a health model where the discount factor incorporates the survival rate. Because both discount factors and income growth rates are unobservable, at least in the usual data set, I will choose this parameter to match the empirical co-movement of consumption and income.

This paper contributes to the large literature of the risk-sharing tests during the

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2See Guvenen (2009) for a summary.
past 20 years (e.g. Cochrane 1991, Mace 1991, Altonji, Hayashi and Kotlikoff 1992, Townsend 1994, Deaton and Paxson 1994, Attanasio and Davis 1996, Blundell and Preston 1998, BPP 2008, Heathcote et al. 2009). In particular, it contributes to a recent new strand of research which casts doubt on the risk sharing tests with the presence of preference heterogeneity (Dubois 2001, Mazzocco and Saini 2009, and Schulhofer-Wohl 2010). Different from this paper, these authors focus on the implication of the heterogeneity in individual risk aversion. In the complete market, the consumption paths of the less risk-averse agents are steeper because they would have lower consumption when aggregate resource is high and higher consumption when aggregate resource is low. If the less risk-averse agents choose the idiosyncratic income process with higher variance, then there would be positive co-movement between consumption and income.

Several papers have addressed the importance of preference on the consumption dispersion in a complete market setup. Deaton and Paxson (1994) mention that the nonseparable preferences between consumption and leisure could possibly generate an increase of consumption dispersion over the life cycle, even the markets are complete. Storesletten et al. (2001) take this theoretical possibility seriously and study a calibrated complete market model with a CES utility function of consumption and leisure. They find that, however, in order to generate the increase of consumption dispersion over the life cycle, it implies a counter-factual age profile of hours inequality. Different from their model, the increase of consumption dispersion in my model does not come from the assumption of non-separable preferences. Badel and Huggett (2010) study in a complete market setup how the preference shocks can account for the life-cycle profile of both consumption and hours dispersion. Their focus is the role of preference shifters, not time preferences as in this paper.

The rest of the paper is organized as follows: Section 2 presents the empirical risk sharing tests, their methodology and results. Section 3 presents two simple versions of complete market (CM) models with discount rate heterogeneity and heterogeneous income profiles, derives analytical results of the consumption dispersion and the co-movement of consumption and income. Section 4 matches the data by a simple quantitative excercise and evaluates the CM and SIM models by the consumption dispersion and the co-movement of consumption and income. Section 5 presents justifications for the positive correlation between individual’s discount factor and income growth rate. Section 6 concludes. Proofs are in the Appendix.

4.2 The Risk Sharing Tests

4.2.1 Methodology

The degree of risk sharing can be affected by the nature of market structure and/or the nature of income risks. To test risk sharing, a growing literature uses the information from the micro data sets of consumption and income distributions. In terms of the structure of the data they use, these tests can be classified as two categories: One is to use the data sets of consumption and income distributions separately, study the change of consumption distribution over the life cycle (Deaton and Paxson 1994) and compare it with the change of income distribution accordingly. The other is to exploit the joint-distribution of consumption and income. To do this, one has to construct either a synthetic panel (Attanasio and Davis 1996) or a combined panel data of consumption and income (BPP 2008). These two lines of risk sharing tests share the same methodology which can be illustrated by the following example:

Let us look at the textbook version of Permanent Income /Life Cycle Hypothesis (PILCH). Consider an individual saving problem where she is endowed with a stochastic income process $y_{it}$, lives for $T$ periods, and only has access to a risk-free bond with net interest rate $r$. Assume further that the period utility is linear quadratic with the discount factor $\beta = 1/(1 + r)$.

Assume the borrowing constraints are loose enough to make the first order conditions hold. Solving the model analytically yields that the consumption follows a martingale process. And the change of consumption is given by

$$\pi \Delta c_{i,t} = \frac{r}{1 + r} \sum_{s=0}^{T-t} \frac{(E_t - E_{t-1})y_{i,t+s}}{(1 + r)^s},$$

(4.1)

where $\pi \equiv 1 - \frac{1}{(1 + r)^{T-1}}$. The right hand side is the annuity value of the innovations to future income between $t - 1$ and $t$.

In the data, we observe the change of income distribution, both over time and over the life cycle. We can single out the idiosyncratic part of income, which is usually identified as "shocks" from the eyes of the econometricians. At this moment, I do not take any stand on the true empirical income process, but instead consider three extreme cases of the composition of the idiosyncratic income $y_{i,t}$.

Case A: Pure unit root income shock: $y_{i,t} = z_{i,t}$ and $z_{i,t} = z_{i,t-1} + \eta_{i,t}$, where $\eta_{i,t}$ is the permanent shock.

In this case, $\Delta y_{i,t} = \eta_{i,t}$. We can solve analytically to get $\Delta c_{i,t} = \eta_{i,t}$. The innovation of income passes one-to-one to the change of consumption. No risk sharing is attained for the permanent shocks.

Case B: Pure predictable and heterogeneous income profile: $y_{i,t} = \theta_i t$, where $\theta_i$ is the slope of individual income path.

Since all the income changes are predictable, the period by period risk-free bonds essentially complete the markets. Thus we have $\Delta y_{i,t} = \theta_i$ and $\Delta c_{i,t} = 0$. The individual consumption does not respond to income at all. Perfect risk sharing is attained.

Case C: Pure I.I.D. income shock: $y_{i,t} = \varepsilon_{i,t}$, where $\varepsilon_{it}$ is the transitory shock.

If shocks are transitory, we have $\Delta y_{i,t} = \varepsilon_{i,t} - \varepsilon_{i,t-1}$ and $\Delta c_{i,t} = \frac{r}{1+r} \pi^{-1} \varepsilon_{i,t}$. Notice that $\pi \approx 1$ if $T - t$ is a large number and $\frac{r}{1+r} \approx r$ if $r$ is a small number. Therefore, $\Delta c_t^i \approx r \varepsilon_{i,t}$ with large $T - t$ and small $r$. There is some pass-through from income to consumption, but the pass-through is small. A large degree of risk sharing of transitory shocks is attained.

The above three different income processes imply completely different consumption responses. By looking at the change of consumption distribution and income distribution, or look at their co-movement if possible, we can identify the nature of market structure and/or the nature of the income risks, and hence we can test the competing hypotheses of risk sharing.

4.2.2 Evidence

Consumption dispersion

Deaton and Paxson (1994) first construct the age profile of the variance of log consumption using Consumer Expenditure Survey (CEX) data before early 1990s. They find that the consumption dispersion of each cohort increases over the life cycle by 0.28 log points. Using longer time span from 1980 to 2006, Heathcote et al. (2010a) find that the log variance of consumption increases over the life cycle by 0.057 when controlling for year effects and by 0.13 when controlling for cohort effects. Their estimates are plotted in Figure 1. While Heathcote et al. (2010a) do not take a stand on which empirical strategy is better, Heathcote et al. (2005) suggest controlling for year effects3.

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3Recently, several authors argue that the diary survey in CEX is better designed than the interview survey. Attanasio et al. (2007) use the diary survey of the CEX and Attanasio et al. (2010) combine these two surveys. They find that the increase of consumption inequality from 1980 to 2006 rises twice as much as the result from Heathcote et al. (2010), who use the interview survey. But they did not report the increase of consumption dispersion over the life cycle.

![Graph: Consumption Inequality over the Life Cycle: Data](image)

**Figure 1: Consumption Inequality over the Life Cycle: Data**

The consensus is: consumption dispersion increases significantly over the life cycle. From the previous PILCH example under standard separable preferences, this result suggests that the individual’s labor income must include highly persistent and uninsurable risks.

**Co-movement of consumption and income**

**Between group inequality** Obviously, the joint distribution of consumption and income contains more information than separate data sets of consumption and income distributions. The reason why most researchers focus on the cross-sectional consumption distribution is mainly due to the lack of high quality longitude data of both consumption and income. To circumvent this data deficiency problem, Attanasio and Davis (1996) study the effect of relative wage movements among birth cohorts and education groups on the consumption distribution. They use cross-sectional consumption and income data sets to construct a synthetic panel of both consumption and income. They find that the relative wage movements among cohort-education groups of men drove large changes in the distribution of household consumption during 1980s, which they conclude as "strikingly sharp rejections of the consumption insurance hypothesis".

**Pass-through: BPP** To generalize Attanasio and Davis (1996)’s idea, BPP (2008) denote partial insurance as the degree of the transmission of income shocks to consumption growth. They assume that unexplained log income can be decomposed into
a unit root permanent part and an i.i.d. transitory part; they also assume the log consumption as

\[ \Delta \log c_{i,t} = \phi_{i,t} \eta_{i,t} + \psi_{i,t} \varepsilon_{i,t} + u_{i,t}, \quad (4.2) \]

where \( \eta_{i,t} \) is the innovation of the permanent shock and \( \varepsilon_{i,t} \) is the transitory shock, \( u_{i,t} \) is the error term. Since \( \phi_{i,t} \) is the pass-through of permanent income shocks to consumption change, it is a natural measure of (the lack of) consumption insurance.

BPP map food data into expenditure data using the estimates of a demand function for food that are present in both the Panel Study of Income Dynamics (PSID) and the CEX to create an unbalanced panel data series of consumption and income. BPP’s main finding is that, in the whole sample, the estimate of \( \phi \) and \( \psi \) is 0.6423 and 0.0533, respectively. In other words, a 10 percent permanent (transitory) shock of disposable income induces a 6.4(0.5) percent change in household’s nondurable consumption. In the complete market, \( \phi = 0 \), and therefore their "partial insurance" result rejects the hypothesis of complete markets.

Sun (2010a) and Kaplan and Violante (2010) measure consumption insurance in a class of simulated life-cycle incomplete market models using BPP’s method. In the simulated model, since shocks are known, it is straightforward to compute \( \hat{\phi}_t \) and \( \hat{\psi}_t \) from the definition equation (4.2) by standard OLS approach. To confront the model with data, however, we have to apply BPP’s estimation by instruments, since shocks are unknown to the econometrician:

\[ \hat{\phi}_{t}^{Instr} = \frac{E(\Delta \hat{c}_t (\Delta \hat{y}_{t-1} + \Delta \hat{y}_t + \Delta \hat{y}_{t+1}))}{E(\Delta \hat{y}_t (\Delta \hat{y}_{t-1} + \Delta \hat{y}_t + \Delta \hat{y}_{t+1}))}, \quad (4.3) \]

\[ \hat{\psi}_{t}^{Instr} = \frac{-E(\Delta \hat{c}_t \Delta \hat{y}_{t+1})}{E(\Delta \hat{y}_t \Delta \hat{y}_{t+1})}, \quad (4.4) \]

where \( \Delta \hat{c}_t \) and \( \Delta \hat{y}_t \) denotes log \( c_t \) and log \( y_t \), respectively. Although the direct OLS and the instrument method are in principle different, Sun (2010a) proves that both the instrument and OLS estimators are consistent estimators of \( \phi \) and \( \psi \), if the consumption model (4.2) is not mis-specified.

Message from the tests

The message from all the above risk sharing tests is: because in the data we all find that consumption distribution co-moves with income distribution, the hypothesis of perfect risk sharing is rejected. In other words, either (1) the hypothesis of complete markets must be rejected or (2) the hypothesis of fully predictable heterogeneous income profiles must rejected, or both. Notice that (2) is a special case of (1).

This message, however, is premature, if there exists preference heterogeneity. In the rest of the paper, I will revisit the implication of these tests by extending the standard complete and incomplete market models to include preference heterogeneity.
4.3 Complete Market Models

In this section, I will present two classes of complete market models. One with stochastic income and one with predictable income change. Both are standard models except that the agents’ time preferences are no longer restricted to be homogeneous. Simple as the models are, they deliver the key insight on how the presence of discount rate heterogeneity may change the age profile of consumption distribution and how its interaction with income profile heterogeneity may affect the co-movement of consumption and income.

4.3.1 Environment

**Demography** Consider an economy populated by \( T \) overlapping-generations, each of which consists of \( I \) agents. Each individual enters the labor market at age 1 and can live a maximum of \( T \) periods. Agents face mortality risks. The probability of surviving between age \( t-1 \) and age \( t \) is denoted by \( \xi_t \), with \( \xi_1 = 1 \) and \( \xi_{T+1} = 0 \). Define \( \xi_t = \prod_{\tau=1}^{t} \xi_\tau \) as the unconditional survival rate.

**Technology** For any \( t \geq 1 \), there is a stochastic event \( s_t \in S \). Let \( s^t \equiv [s_1, s_2, ..., s_t] \) denote the history of events up and until time \( t \). Each agent is endowed with a sequence of consumption good \( y_i(s^t) \).

Individual log income process is restricted to the following form:

\[
\log y_{i,t} = \log w + \alpha_i + \theta_i t + \underline{z_{i,t}} + \varepsilon_{i,t},
\]

\[
z_{i,t} = \rho z_{i,t-1} + \eta_{i,t},
\]

where \( \rho \in [0, 1] \), \( \eta_{i,t} \sim N(0, \sigma_\eta^2) \), \( \varepsilon_{i,t} \sim N(0, \sigma_\varepsilon^2) \), \( z_{i,1} = 0 \). \( w \) is the marginal return to aggregate labor supply and \( \alpha_i \) is the fixed effect which is predetermined before the agent enters the labor market. The next three terms captures the idiosyncratic income change. \( \theta_i \) is the heterogeneity in the growth rate of individual income. The idiosyncratic shocks part consists of a permanent (or AR(1) when \( \rho < 1 \)) part \( z_i^p \) and a transitory (i.i.d.) part \( \varepsilon_i^t \). Define \( \sigma_\theta \equiv \text{var}(\theta_i) \geq 0 \). This income process nests both the Heterogeneous Income Profiles (HIP) process where \( \sigma_\theta > 0 \) and the Restricted Income Profiles (RIP) process where \( \sigma_\theta = 0 \).

It can be the case that \( \eta_{i,t} \) and \( \varepsilon_{i,t} \) are "shocks" to the econometricians, but predictable to the individual \( i \). In one of the following models, I will allow \( \eta_{i,t} \) and \( \varepsilon_{i,t} \) to be predictable to the agent \( i \).

There is no aggregate uncertainty. For the analysis of allocations in the stationary equilibrium, we do not need to specify the production function and the supply of
capital, as long as we are given the equilibrium marginal product of capital net of depreciation $r$.

**Preference** Agent $i$’s preference over the stream of consumption is given by

$$u_i(c) = E \sum_{t=1}^{T} s_t \beta_i^{-t} \frac{c_i(s^t)^{1-\gamma}}{(1 - \gamma)},$$

where $\beta_i$ is the individual’s time discount factor, $\gamma$ is the relative risk aversion. Note that $\beta_i$ is individual specific. When each individual $i$ is born, her discount factor $\beta_i$ is drawn from a distribution $F_\beta$ with mean $\mu_\beta$ and variance $\sigma_\beta^2 \geq 0$. This nests the standard model where the discount rate heterogeneity is absent and $\sigma_\beta = 0$.

Then I will discuss models with different market structures and/or income shocks.

### 4.3.2 Complete markets with stochastic income

Suppose the markets are complete. The allocation of this economy can be characterized by the solution to the social planner’s problem of each birth-cohort, with Pareto weight $\lambda_i$ for each agent $i$:

$$\max \sum_i \lambda_i E \sum_{t=1}^{T} s_t \beta_i^{-t} \frac{c_i(s^t)^{1-\gamma}}{(1 - \gamma)}$$

$$s.t. \sum_{t,s^t} \sum_i c_{i,t}(s^t) - y_{i,t}(s^t) \leq 0.$$  \hspace{1cm} (4.6)

Combining the first order conditions for any two agent $i$ and $j$, we have

$$\frac{c_{i,t}(s^t)}{c_{j,t}(s^t)} = \left(\frac{\lambda_i}{\lambda_j}\right)^{\frac{1}{\gamma}} \left(\frac{\beta_i}{\beta_j}\right)^{\frac{t-1}{\gamma}}$$  \hspace{1cm} (4.7)

This simple necessary condition tells us two important messages: 1. The ratio of consumption of any two individuals does not depend on their own income history, therefore each consumer will get full insurance under this market structure. 2. If $\beta_i = \beta_j$, the consumption ratio is a constant which is determined at the beginning of the life cycle. Testing constant relative marginal utility of consumption is the key logic of the test of complete markets. If $\beta_i \neq \beta_j$, however, the ratio of $c_i$ and $c_j$ will change exponentially with age. For given $\{\lambda_i\}_{i=1}^I$, the log consumption can be solved analytically as

$$\log c_{i,t} = \frac{t - 1}{\gamma} \log \beta_i + \log \lambda_i^{1/\gamma} + \Gamma_t,$$  \hspace{1cm} (4.8)

where $\Gamma_t = \log \frac{(1+r)^{t-1} \sum_{s,t} y_{i,t}(s^t)}{\sum_{i} \lambda_i^{1/\gamma} \beta_i^{(t-1)/\gamma}}$. Notice that we can now write $c_{i,t}$ instead of $c_i(s_t)$, since consumption at time $t$ is not a function of the realization of $s^t$. This
equation tells us that the agent with higher $\beta$ has a steeper path of log consumption and the agent with higher Pareto weight is given a higher initial consumption level.

**Consumption dispersion**

The last term in equation (4.8) is not individual specific. So, only the variance of the first two terms can be positive. The variance of log consumption is:

$$\text{var} \log c_{i,t} = \frac{(t - 1)^2}{\gamma^2} \sigma_{\log \beta}^2 + \frac{2(t - 1)}{\gamma} \text{cov}(\log \beta_i, \log \lambda_i^{1/\gamma}) + \text{var}(\log \lambda_i^{1/\gamma}),$$  \hspace{1cm} (4.9)

where $\sigma_{\log \beta}^2 \equiv \text{var}(\log \beta_i)$.

Notice the last term is the initial consumption dispersion. Therefore, the variance of log consumption is a quadratic function of time $t$. Its shape in domain $[1, T]$ depends on $\text{cov}(\log \beta_i, \lambda_i^{1/\gamma})$. If $\text{cov}(\log \beta_i, \lambda_i^{1/\gamma})$ is not a big negative number, the variance of log consumption will eventually increase quadratically in the life cycle. If $\text{cov}(\log \beta_i, \log \lambda_i^{1/\gamma})$ is positive, the life cycle profile will be increasing from the beginning of the life cycle, otherwise, it is U-shaped. Formally, we have the following proposition:

**Proposition 4**

(1) If $T > t^* \equiv 1 - \gamma \text{cov}(\log \beta_i, \log \lambda_i^{1/\gamma})/\sigma_{\log \beta}^2$, then for $t > t^*$, the variance of log consumption in the complete market model increases convexly with age $t$. If $\text{cov}(\log \beta_i, \log \lambda_i^{1/\gamma}) < 0$, then the consumption dispersion age profile is U-shaped.

(2) If $\text{cov}(\log \beta_i, \log \lambda_i^{1/\gamma}) > \frac{(T - 1)\sigma_{\log \beta}^2}{2\gamma T}$, then the consumption dispersion increases over the life cycle. Therefore, the hypothesis of complete markets cannot be rejected by the observed increase of consumption dispersion over the life cycle.

(3) Consumption dispersion over the life cycle increases with $\text{cov}(\beta_i, \theta_i)$.

**Proof.** See Appendix. ■

This proposition says the dispersion of discount factors can be gradually transmitted into the dispersion of consumption over the life cycle. It should be noticed that the increase of consumption dispersion in this complete market model does not come from the assumption of non-separable preferences between consumption and leisure, as in Storesletten et al. (2001).

The last claim comes from the fact that the second term in (4.9) is a linear function of $t$ and the Pareto weight $\lambda_i$ is an increasing function of one’s life time income. Although we theoretically we do not need $\text{cov}(\beta_i, \theta_i) > 0$ to generate an increase of $\text{var}(\log c_{i,t})$, a positive correlation between $\beta_i$ and $\theta_i$ will generate a larger increase of consumption dispersion.

Co-movement of consumption and income

When markets are complete, individual’s consumption should not respond to income since there are no uninsurable income risks, even if there exists discount rate heterogeneity. In the OLS estimation by BPP’s measure in the model, all the permanent and transitory pass-throughs should be zero. However, the econometricians do not know individual’s true income process. The best thing they can do is to use the panel data of consumption and income. If they observe one’s income and one’s consumption are correlated, they conclude that there is uninsurable income "risk".

To see this, we derive the increase of log consumption from $t - 1$ to $t$,

$$\Delta_t \log c_{it} = \frac{\log \beta_i}{\gamma} - \Delta_t \Gamma_t. \quad (4.10)$$

This equation shows that the individual consumption growth is an increasing function of her discount factor. If individual’s income growth is also correlated with her discount factor, then BPP’s measure by the instrument estimation is no longer zero. This would be true if the individual income process follows heterogeneous income profile and the covariance between individual’s income growth rate $\theta_i$ and discount factor $\beta_i$ is positive. Formally, we have the following proposition:

**Proposition 5** (1) If markets are complete, the pass-through of permanent shocks $\phi_{t,CM}$ and transitory shocks $\psi_{t,CM}$ are zero for any $t$.

(2) The BPP’s estimators by instrument are 

$$\hat{\phi}_{t,CM} = \frac{\text{cov}(\log \beta_i, \theta_i)}{\gamma(\sigma^2 + \rho[(\rho - 1)(\rho^2 - 1) \sum_{r=1}^{\infty} \rho^{2(r-1)} + \rho(\rho - 1) + 1]\sigma_\theta^2/3)}$$

and 

$$\hat{\psi}_{t,CM} = \frac{\text{cov}(\log \beta_i, \theta_i)}{\gamma(\sigma^2_\theta - \rho^2 + \rho(\rho - 1)\sum_{r=1}^{\infty} \rho^{2(r-1)} + \rho(\rho - 1) + 1]\sigma_\theta^2/3)}.$$ 

(3) If $\text{cov}(\beta_i, \theta_i) > 0$, then $\hat{\phi}_{t,CM}$ is positive. Therefore, the hypothesis of complete markets cannot be rejected by the observed positive $\hat{\phi}_{t,CM}$ from BPP’s test.

**Proof.** See Appendix. ■

This proposition says that the covariance of discount factor and income growth rate can be transmitted into the co-movement of consumption and income. Therefore, it is not informative to test full risk-sharing by the co-movement of consumption and income.

When $\rho < 1$ or $\sigma^2_\theta > 0$, BPP’s coefficients are no longer consistent estimates of $\phi$ and $\psi$ because of model mis-specification. Two polar cases are worth noticing:

When $\rho = 1$,

$$\hat{\phi}_{t,CM} = \frac{\text{cov}(\theta_i, \beta_i)}{\gamma(\sigma^2 + \sigma^2_\theta/3)}; \hat{\psi}_{t,CM} = \frac{\text{cov}(\theta_i, \beta_i)}{\gamma(\sigma^2_\theta - \sigma^2_\eta)}.$$

When $\rho = 0$, $\eta_t$ is essentially transitory. It yields

$$\hat{\phi}_{t,CM} = \frac{\rho \theta_i \beta_i \sigma_\beta}{\gamma \sigma_\theta}; \hat{\psi}_{t,CM} = \frac{\text{cov}(\theta_i, \beta_i)}{\gamma(\sigma^2 + \sigma^2_\eta - \sigma^2_\theta)}.$$

4.3.3 Bond economy with predictable income change

Under this market structure, agents are assumed to have only access to a one-period risk-free bond with the gross interest \( R \) which clears the capital market in the stationary equilibrium. Assume perfect annuity markets. Each agent is endowed with an initial asset \( a_{i0} \). Only the natural borrowing constraints are imposed. Since the agent’s preference satisfies the Inada condition, the natural borrowing constraints will never be binding in the equilibrium. Suppose the idiosyncratic income change is fully predictable to the agent. Since all the consumption fluctuation can be smoothed by borrowing and saving, the markets are essentially complete.

The individual \( i \)'s optimization problem is:

\[
\max \sum_{t=1}^{T} \delta_t \beta_i^{t-1} \frac{c_{i,t}^{1-\gamma}}{1-\gamma}
\]

s.t. \( \sum_{t=1}^{T} \frac{c_{i,t}}{R^{t-1}/\delta_t} \leq \sum_{t=1}^{T} \frac{y_{i,t}}{R^{t-1}/\delta_t} + a_{i0} \).

Take \( R \) as given, the agent \( i \)'s log consumption can be thus solved as

\[
\log c_{i,t} = \frac{\log(\beta_i R)}{\gamma} t - \log \sum_{t=1}^{T} [\beta_i R^{(1-\gamma)}]^{\frac{t}{\gamma}} + \log \frac{W_i}{R},
\]

where \( W_i = \sum_{t=1}^{T} \frac{y_{i,t}}{R^{t-1}} + a_{i0} \) is the present value of life-time wealth.

Clearly, \( \log c_{i,t} \) is linear in age and its slope is increasing in \( \beta_i \). The log consumption of agent \( i \) is increasing (decreasing) over the life cycle if \( \beta_i R \) is greater (less) than 1. Consider two agents with the same life-time wealth and \( \beta_i > \beta_j \); the impatient agent will consume more in the beginning, but after some time the patient agent will catch up and consume more and more afterwards. Thus consumption dispersion is expected to decrease at the very beginning, reaches zero after some time and increases since then.

Consumption dispersion

The variance of log consumption can be computed as

\[
\text{var}(\log c_{i,t}) = \frac{t^2}{\gamma^2} \sigma_{\log \beta}^2 + \frac{2t}{\gamma} \text{cov}(\log \beta_i, \log W_i \Omega_i^{-1}) + \text{var}(\log W_i \Omega_i^{-1}),
\]

where \( \Omega_i = \sum_{t=1}^{T} [\beta_i R^{(1-\gamma)}]^{\frac{t}{\gamma}} \). Notice that the last term in the right hand side is constant over the life-cycle. The first term is increasing quadratically with age; the second

The term is a linear function of age whose slope is determined by the difference between \( \text{cov}(\log \beta_i, \log W_i) \) and \( \text{cov}(\log \beta_i, \log \Omega_i) \). Since the lifetime wealth \( W_i \) is increasing in \( \theta_i \), the higher the \( \text{cov}(\beta_i, \theta_i) \), the larger the slope. Generally, we have a U-shaped consumption dispersion profile. At the beginning of life, the consumption dispersion decreases with age; if \( t \) is sufficiently large, the consumption dispersion starts to increase convexly with age.

Notice the formal resemblance of equation (4.13) and (4.9). In fact, the bond economy with fully predictable heterogeneous income profile is a special case of the complete market. The advantage of this model is that we now can express all the conditions without any knowledge of \( \lambda_i \). Thus we have the following corollary:

**Corollary 6** In the bond economy with fully predictable heterogeneous income profile, the consumption dispersion is given by (4.13), which is a special case of that in a complete market model with stochastic income:

1. If \( T > t^* \equiv -\gamma \text{cov}(\log \beta_i, \log W_i/\Omega_i)/\sigma_{\log \beta}^2 \), then for \( t > t^* \), the variance of log consumption in the complete market model increases convexly with age \( t \). If \( 1 \leq t^* < T \), then the consumption dispersion age profile is U-shaped.

2. If \( \text{cov}(\log \beta_i, \log W_i/\Omega_i) > -\frac{T+1}{2\gamma} \sigma_{\log \beta}^2 \), then the consumption dispersion increases over the life cycle. Therefore, the hypothesis of complete markets cannot be rejected by the observed increase of consumption dispersion over the life cycle.

3. Consumption dispersion over the life cycle increases with \( \text{cov}(\beta_i, \theta_i) \).

**Co-movement of consumption and income**

In terms of BPP’s estimators, the bond economy with predictable income change coincides with the complete market with stochastic income. Therefore, predictability and insurability is observationally equivalent in BPP’s risk-sharing test.

**Corollary 7** In the bond economy with fully predictable heterogeneous income profile, the BPP’s estimators of the pass-through of both permanent and transitory shocks are the same as those in the complete market model with stochastic income.

**Proof.** See Appendix. □

**4.4 Matching the Data**

I have shown that it is theoretically possible for a complete market model to generate an increase of consumption dispersion over the life cycle and a positive co-movement of consumption and income by BPP’s measure. Therefore, the significance of the previous risk sharing tests should be questioned. To go one step further, we would like

It is not known if it is quantitatively admissible to match the observed increase of consumption dispersion, the co-movement of consumption and income, or even both simultaneously? While this question has to be answered in a more seriously calibrated model, here I first illustrate a tentative answer by a numerical example. It is convenient to do this in the bond economy with predictable income, a special case of complete markets, since we don’t need to solve the expression for the Pareto weights.

4.4.1 Identification

Empirical target

The empirical target for the consumption dispersion from Heathcote et al. (2010a). They report the age profile of consumption dispersion for average 5 years group, starting from age 27 (average of 25 to 30) to age 57 (average of age 55 to 60). They find that the variance of log consumption rises by 0.057 over the life cycle, controlling for year effects. To directly compare with their result, I set $T = 31$.

The empirical target for the co-movement of consumption and income is naturally the pass-through of permanent shocks estimated by BPP (2008), whose estimate is 0.642 and does not have an age trend. From Proposition 2 we know that the pass-through of permanent shocks in the complete market model varies with age if $0 < \rho < 1$. But a simple calculation shows that after the first several years, its variation is almost zero. Since BPP’s sample starts from age 30 and their first estimate is from age 33, we can therefore use any large $t$ to match BPP’s estimate and get the same result.

Parameters

Because this model is based on the assumption of fully predictable income process, which is different from previous empirical strategies for identifying income "shocks", it is not clear whose estimate is proper. To be consistent with time span of consumption data, I focus on the income dispersion over the life cycle in Heathcote et al. (2010a), where the variance of log earnings increases convexly by 0.30 over the life cycle. The increase of log variance of life time earnings is given by:

$$
\Delta_{u\nu} \text{var}(\log y_u) = \begin{cases} 
(T^2 - 1)\sigma_\theta^2 + \frac{1-\rho^2}{1-\rho^2} \sigma_\eta^2 & \text{for } \rho < 1 \\
(T^2 - 1)\sigma_\theta^2 + (T - 1)\sigma_\eta^2 & \text{for } \rho = 1 
\end{cases} 
$$

(4.14)

I take Gueven (2009)'s estimates of $\rho = 0.821$ and $\sigma_\eta^2 = 0.029$ from his estimation of the HIP model. However, his estimate of $\sigma_\theta = 0.0195$ generates much higher income dispersion than that of Heathcote (2010a). Instead of borrowing his estimate of $\sigma_\theta$, I will chose $\sigma_\theta$ to match the empirical income dispersion. Equation (4.14) implies $\sigma_\theta = 0.0169$. 

I set $R = 1.04$ and $\gamma = 0.5$. I assume that $\theta_i$ and $\log \beta_i$ follow joint Normal distribution with mean $\mu_\theta = \mu_{\log \beta} = 0$.

**Identification**

The parameters to estimate are $\sigma_{\log \beta}$ and $\rho_{\beta \theta}$, which we have two equations for the identification:

$$
\frac{T^2 - 1}{\gamma^2} \sigma_{\log \beta}^2 + \frac{2(T - 1)}{\gamma} \text{cov}(\log \beta_i, \log W_i \Omega_i^{-1}) = \Delta_1 T \text{var}(\log c_{it}). \tag{4.15}
$$

$$
\rho_{\beta \theta} \sigma_{\log \beta} \left( \frac{T-2}{\gamma} + \rho((\rho - 1)(\rho^2 - 1) \sum_{\tau=1}^{T-2} \rho^{\tau-1} + \rho(\rho - 1) + 1) \frac{\sigma_\gamma^2}{3} \right) = \hat{\phi}_{t,BPP}. \tag{4.16}
$$

From equation (4.16), we have a restriction for the product of $\rho_{\beta \theta}$ and $\sigma_{\log \beta}$. Then, $\rho_{\beta \theta}$ and $\sigma_{\log \beta}$ can be identified using equation (4.15) by numerical simulation of covariance. Figure 2 plots the restrictions on $\rho_{\beta \theta}$ and $\sigma_{\log \beta}$ implied by equation (4.15) and (4.16). The circled line matches the empirical increase of consumption dispersion over the life cycle and the solid line matches the empirical co-movement of consumption and income.

![Admissible Parameter Values](image)

**Figure 2:** Matching the data: admissible pairs of $\rho_{\beta \theta}$ and $\sigma_{\beta}$

I also plot the stared line, where the pairs of $\rho_{\beta \theta}$ and $\sigma_{\log \beta}$ generate zero consumption dispersion over the life cycle. Those pairs which lie in the southeast region to this line are admissible for generating a life-cycle increase of consumption dispersion. This implies: given $\sigma_{\beta}$, higher $\rho_{\beta \theta}$ generates higher consumption dispersion. Interestingly, given the $\rho_{\beta \theta}$, lowering the dispersion of $\beta$ will increase the dispersion of consumption! The lowest $\rho_{\beta \theta}$ for generating the observed consumption dispersion is 0.20. Therefore,
the positive correlation between discount factors and income growth rates is quantitatively necessary even in generating the increase of consumption dispersion.

The solid line (equation (4.15)) and the circled line (equation (4.16)) do intersect. So, the answer to the above question of quantitative admissibility is yes, and both. The intersection of the solid line and the circled line determines a unique fair of $\sigma_{\log \beta}$ and $\rho_{3\beta}$. This yields $\rho_{3\beta} = 0.63$ and $\sigma_{\log \beta} = 0.016$. In this parsimonious complete market model, a small amount of discount rate heterogeneity and a moderate correlation between discount factors and income growth rates is sufficient to simultaneously match both the empirical increase of consumption dispersion over the life cycle and the BPP’s estimate of the co-movement of consumption and income. Notice also that all the pairs on the solid line where $\rho_{3\beta}$ is higher than its intersection with the started line will cause the false rejection of the complete market model by the previous risk sharing tests.

4.4.2 Evaluating the models

I will compare quantitatively the complete market (CM) model with the standard incomplete market (SIM) models and evaluate these models by their performances in matching the data. In particular, I will present a class of calibrated "large scale" models, which is used in Sun (2010b) and is an extension of the class of standard incomplete market models with overlapping-generation. The layout of the SIM model is omitted here, which can be found in Sun (2010b), or Chapter 3.

I adopt the estimation strategy as in Sun (2010b) to match the fraction of zero/negative agents near retirement age. The estimation result of $\Delta \beta$ and $p_l$ in the complete market model shows that it requires much less dispersion of $\beta_i$ to generate the empirical observed wealth distribution. Only $\Delta \beta = 1.7\%$ and $p_l = 55\%$ is needed. It is because in the complete market people hold less wealth than in the incomplete market, since there is no precautionary saving motive. When the agents are old, they would not accumulate a large amount of life time wealth due to full risk sharing (as there is no risk at all).

Figure 3 plots the age profile of consumption dispersion in the CM models with $\text{cov}(\beta_i, \theta_i) = 0$ and $\text{cov}(\beta_i, \theta_i) > 0$. They are U-shaped as we expected. If $\text{cov}(\beta_i, \theta_i) = 0$, $\Delta \beta = 1.7\%$ and $p_l = 55\%$, the variance of log consumption starts to increase only after the middle age and does not generate an overall increase over the life cycle. This result echoes the results in the SIM models with $\sigma_\beta > 0$ in Sun (2010b), where its effect on the magnitude of increasing consumption dispersion is negligible. If $\text{cov}(\beta_i, \theta_i) > 0$, the variance of log consumption will increase more. As the first step, I will not estimate $\text{cov}(\beta_i, \theta_i)$, but instead ask: how much $\rho_{3\beta\theta}$ do we need to generate an empirical observed increase of consumption dispersion? In the benchmark CM model, to match
the data it requires $\rho_{\beta,\theta} = 0.20$.

![Graph showing Consumption Inequality over the Life Cycle: CM Models](image)

**Figure 3:** Consumption Inequality over the Life Cycle: CM Models

In short, with the presence of $\text{cov}(\beta_i, \theta_i) > 0$, the calibrated complete market model can generate an empirical plausible increase of consumption dispersion. Therefore, the risk sharing tests by consumption dispersion do not give us enough information to identify either the market completeness or the nature of income shocks.

Now let us look at the co-movement of consumption and income. In CM model we can either solve $\hat{\phi}_{t,CM}$ and $\hat{\psi}_{t,CM}$ from the analytical expression or from the result of the simulated model, both of which should give us the same result. The life cycle profile of $\hat{\phi}_{t,CM}$ in the model is in Figure 4, where $\sigma_\beta$ and $\rho_{\theta\beta}$ are given by the identifications. Although $\hat{\phi}_{t,CM}$ decreases with age in the CM model, we can see that after the first few years, the model $\hat{\phi}_{t,CM}$ is almost constant. Since BPP’s estimates starts from age 33, this is a feature consistent with the empirical finding, which the SIM models fail to match.

Figure 4: Pass-through of Permanent Shocks: CM

Compared with the quantitative results from Sun (2010b), or Chapter 3, we can see that SIM models and the CM model perform very differently. The key difference of their performance in matching the data can be summarized as follows:

1. The SIM models generate too much consumption dispersions over the life cycle and there is no way to lower it to its empirical counterpart⁴; the CM model can account for the observed consumption dispersion over the life cycle.

2. The SIM models generates a monotonic increase of consumption dispersion over the life cycle. In the data, it is U-shaped for the first 10 years. The CM model generates a U-shaped age profile of consumption dispersion.

3. The SIM models with RIP generate too much co-movement of consumption and income, while the SIM models with HIP generate too little. The CM model with positive covariance of discount factors and income growth rate can account for the observed co-movement of consumption and income.

4. The SIM models generate a down-ward sloping age trend of BPP’s permanent pass-throughs. There is no such trend in the data. The CM model generates an almost flat age profile of BPP’s permanent pass-throughs.

In all the above dimensions, the CM model outperforms the SIM models. It is therefore premature to reject CM model too quickly by the previous risk sharing

⁴Notice that the magnitude of increase in consumption dispersion in the SIM model is close to that in Deaton and Paxson (1994). In Sun (2010b), I use Deaton and Paxson (1994) simply because it is qualitatively correct and can be easily compared to Guevenen (2007). Nevertheless, we should use the recent data from Heathcote (2010a) for a serious quantitative inquiry of the increasing consumption dispersion.
tests. Due to its simplicity, the CM model provides us with a useful framework for risk sharing: by understanding the success and failure of the calibrated CM model in matching the data, we can learn more about the market structure, the income process and the preference heterogeneity.

4.5 Is \( \text{cov}(\beta_i, \theta_i) > 0 \) plausible?

For the complete market models to match the data, \( \text{cov}(\beta_i, \theta_i) > 0 \) is crucial. Is it empirical plausible? We cannot directly answer this question since both \( \beta \) and \( \theta \) are unobservable, at least from any normal data set. Nevertheless, there are plenty of indirect evidence suggesting the positive correlation between \( \beta \) and \( \theta \).

4.5.1 Education choice

Higher education generates higher \( \theta \) Empirical studies find that the group of higher education is associated with higher wage growth rate. (e.g. a recent study of PSID by Low, Meghir and Pistaferri (2010) ) This is also true in cross-country: for example, Barro (1991) finds that the difference in the schooling is positively correlated with income growth rate across countries. Bils and Klenow (2000) study this effect in a human capital model and find the effect is much smaller, but still positive.

Higher education implies higher \( \beta \) Cagetti (2003) estimates \( \beta \) for different education groups using the empirical age profile of wealth and he finds that the discount factor of the college graduate is higher than that of the high school graduate, which is itself higher than that of the individuals with no high school education. Since people with higher \( \beta \) will have a higher wealth-income ratio in a standard incomplete market model, his result comes from the empirical fact that the group of people with higher education have a higher wealth income ratio.

If the above two observations are true, that is: people with higher discount factor tend to have higher education and people with higher education are more likely to have a higher income growth rate, then we have \( \text{cov}(\beta_i, \theta_i) > 0 \).

The model of education choice Higher education, of course, is endogenous. Here I will present a simple extension of the previous complete market model, which can be the micro foundation of why higher education is positively correlated with higher discount factor.

Let \( V_i(\theta, \beta) \) denote the value function associated with indiviudal \( i \)'s optimization problem (4.11) in the previous complete market model. Insert the optimal consumption rule (4.12) into the life-time utility function, we get that \( V_i(\theta, \beta) \) is increasing in \( \theta \) and \( \beta \). Consider a schooling decision faced by each individudal at period 0, one period before
she enters the labor market. \( \theta \) consists of two values with \( \theta_H > \theta_L \). If she chooses \( E_i = 1 \) and goes to school, she has to pay a schooling cost in terms of utility \( \omega_i \), and get \( \theta_H \) when she starts working next period. If she chooses \( E_i = 0 \) and does not go to school, she will not pay any cost, and get \( \theta_L \) when she starts working next period. Here we can re-interpret \( T \) as the ratio of post-schooling working years to potential non-mandatory schooling years. Agents differ in utility cost of schooling \( \omega_i \), which can be due to the difference in various factors including innate cognitive or non-cognitive ability, social environment, and financial status, etc. The discount factor \( \beta_i \) and schooling cost \( \omega_i \) are drawn independently from distribution \( F_\beta \) and \( F_\omega \), respectively.

The individual \( i \)'s problem at period 0 can be solved as

\[
E_i = \begin{cases} 
1 & \text{if } \beta_i [V_i(\theta_H, \beta_i) - V_i(\theta_L, \beta_i)] > \omega_i \\
0 & \text{if } \beta_i [V_i(\theta_H, \beta_i) - V_i(\theta_L, \beta_i)] \leq \omega_i 
\end{cases}
\]  
(4.17)

There is a cut-off value of \( \omega_i^* \) for each individual \( i \). She will choose not to go to school if \( \omega_i \geq \omega_i^* \). As \( \beta_i [V_i(\theta_H, \beta_i) - V_i(\theta_L, \beta_i)] \) is increasing in \( \beta_i \), the cut-off value \( \omega_i^* \) is decreasing in \( \beta_i \). As a result, the fraction of individuals who choose schooling and thus get a higher income growth is higher in the groups of individuals with higher \( \beta \), which implies \( \text{cov}(\beta_i, \theta_i) > 0 \). \( \theta_i \) is not perfectly correlated with \( \beta_i \), because the random variable of individual schooling cost \( \omega_i \), which is assumed to be not correlated with \( \beta_i \), also plays a role in determining the schooling and income growth rate.

One caveat applies here. In the Mincer equation regression where we estimate the income process, education level is already controlled for. Therefore, the difference in \( \theta \) should be interpreted as income growth beyond the effect from formal schooling. In my model, there is no variance of \( \theta \), if schooling is controlled for. Nevertheless, we can interpret schooling in the above model as any human capital investment which takes time, such as informal education, self-education, or any learning activity which we cannot observe in the data. Higher unobservable human capital would also increase the individual productivity and the income growth rate. Thus we could study the investment of unobservable human capital using the above model and would still get \( \text{cov}(\theta, \beta) > 0 \), even if education level is controlled for.

4.5.2 Mortality and health

The discount factor is, by definition, subjective. This makes the empirical identification difficult. Alternatively, the model discount factor can be interpreted in such a way that it has some objective content. Suppose the survival probability is heterogeneous
in individuals. The period utility now becomes

\[ u_t(c) = E \sum_{t=1}^{T} \left( \prod_{\tau=1}^{t} \xi_\tau \right) S_t^{-1} \hat{\beta}_i^{t-1} \frac{[c_t(s'_t)]^{1-\gamma}}{(1-\gamma)}, \]

where \( S_t \) is the idiosyncratic conditional survival rate which is constant over the life cycle and \( E(S_t) = 1 \). This can be interpreted as an increasing function of one’s initial health level, \( H_t \), which can be either inherited or formed in early years. Redefine \( \hat{\beta}_i \equiv S_t \beta_i \), we get the same optimization problem as (4.11). Therefore, any factors correlated with \( S_t \) now correlated with \( \hat{\beta}_i \), the discount factor in the model. It is natural to think that the wage growth rate is positively correlated with labor productivity, which is itself positively correlated with one’s health: \( \theta'(H_t) > 0 \). Since \( S'(H_t) > 0 \), this implies \( \text{cov}(\beta_i, \theta_t) > 0 \). We can also use this interpretation in the previous schooling choice model, which will make the correlation between discount factor and income growth rate stronger.

### 4.6 Conclusion

This paper revisits the risk sharing tests which use micro data sets of consumption and income distributions. These tests reject the hypothesis of complete markets because they find that consumption dispersion increases over the life cycle and individual consumption co-moves with income, which cannot be reconciled with any standard complete market (CM) models.

I extend the standard CM model to allow positive correlation between heterogeneous discount factors and heterogeneous income growth rate, a feature consistent with the implications of an education choice model or a health model where the discount factor incorporates the survival rate. The main results are: it is not only theoretically possible, but also quantitatively admissible for a simple complete market model to account for the empirical evidence on both the consumption dispersion and the co-movement of consumption and income. Generally speaking, the CM model outperforms the standard incomplete market (SIM) models in the sense that the results from the CM model are closer to the data in both the magnitude and the shape of the life cycle profiles. Therefore, I conclude that the empirical findings from the previous risk sharing tests using micro data sets from consumption and income distributions are not sufficient for testing market completeness.

In the “heterogeneous agent” literature in macroeconomics, agents are ex ante homogeneous in most aspects. This paper shows that the incomplete market model with only ex post heterogeneity and the complete market model with ex ante heterogeneity can be observationally equivalent in the previous risk sharing tests. This result is the
first step towards future research in two directions: one is to design a more proper risk sharing test which does not merely rely on the information of either the consumption dispersion or the co-movement of consumption and income. The other is to estimate the ex ante heterogeneity to account for the empirical plausible consumption and income distributions, starting from a simple complete market framework, which could give us a better understanding of the relative importance of ex ante heterogeneity. In this sense, this paper is related to the literature of evaluating ex ante and ex post source of lifetime inequality as in Keane and Wolpin (1997) and Huggett et al. (2010).

When there is discount rate heterogeneity, the estimated degree of of consumption insurance is down-ward biased. Even if we do not take the extreme case of complete market, it suggests that there could exist more risk sharing, or more insurable risks, than we previously thought. In this sense, this paper is also related to models where there is more risk sharing than the SIM models and less risk sharing than the CM models, e.g., Kruger and Perri (2006), Attanasio and Pavoni (2009), Heathcote, Storesletten and Violante (2009), Ales and Maziero (2009), etc. For example, as similar discount factors could imply similar other individual characteristics, this paper presents a new mechanism of why the between-group consumption dispersion follows more closely to income inequality than the with-in group consumption dispersion does, which is complementary to Kruger and Perri (2006)’s limited enforcement explanation.

The complete market model highlights the importance of ex ante heterogeneity. If we can find new ways of identification, it is very useful to calibrate the joint-distribution of discount factors and income growth rates from micro data. Due to its simplicity, the CM model provides us with a useful framework for risk sharing: by understanding the success and failure of the calibrated CM model in matching the data, we can learn more about the market structure, the income process and the preference heterogeneity. In the future research, it would be interesting to estimate the discount rate heterogeneity, the income growth rate heterogeneity and their correlation simultaneously in one quantitative framework, using the data from income growth, consumption growth, education and health status.

4.7 Appendix

4.7.1 Proofs

Proof. (Proposition 4)

The first result comes straightforwardly from the property of quadratic function. The second result comes from checking $\Delta_{1,t} var \log c_{i,t} > 0$. ■

Proof. (Proposition 5)

The "true" pass-through of permanent shocks to consumption growth can be done by OLS: Since \( \text{cov}(\Delta \tilde{c}_t, \eta_t) = \text{cov}(\Delta \tilde{c}_t, \varepsilon_t) = 0 \) in complete market, \( \phi = \psi = 0 \). From the data, the econometricians can not observe \( \eta_t \) and \( \varepsilon_t \). BPP suggest instrument estimation to get the variance and covariance needed above. The variance of permanent shocks is identified as:

\[
E[\Delta \tilde{y}_t(\Delta \tilde{y}_{t-1} + \Delta \tilde{y}_t + \Delta \tilde{y}_{t+1})] \\
= E[(\theta_i + \rho(\rho - 1)z_{t-2} + (\rho - 1)\eta_{t-1} + \eta_t + \varepsilon_t - \varepsilon_{t-1}) \\
(3\theta_i + \eta_{t+1} + \rho\eta_t + \rho^2\eta_{t-1} + (\rho^3 - 1)z_{t-2} + \varepsilon_{t+1} - \varepsilon_{t-2})] \\
= 3\sigma_{\beta}^2 + \rho(\rho - 1)(\rho^3 - 1)\text{var}(z_{t-2}) + [\rho^2(\rho - 1) + \rho]\sigma_{\eta}^2 \\
= 3\sigma_{\beta}^2 + \rho(\rho - 1)(\rho^3 - 1)\sum_{t=1}^{T-2} \rho^{2(t-1)} + \rho(\rho - 1) + 1]\sigma_{\eta}^2
\]

The covariance of consumption growth and permanent shocks is identified as

\[
E(\Delta \tilde{c}_t(\Delta \tilde{y}_{t-1} + \Delta \tilde{y}_t + \Delta \tilde{y}_{t+1})) \\
= E\left(\frac{\gamma}{\gamma} + \Delta \Gamma_t(3\theta_i + \eta_{t+1} + \rho\eta_t + \rho^2\eta_{t-1} + (\rho^3 - 1)z_{t-2} + \varepsilon_{t+1} - \varepsilon_{t-2})\right) \\
= \frac{3}{\gamma} \text{cov}(\log \beta_i, \theta_i)
\]

The last step uses the fact that \( \Delta \Gamma_t \) is not individual \( i \) specific. Therefore,

\[
\hat{\phi}_{t,CM} = \frac{E(\Delta \tilde{c}_t(\Delta \tilde{y}_{t-1} + \Delta \tilde{y}_t + \Delta \tilde{y}_{t+1}))}{E(\Delta \tilde{y}_t(\Delta \tilde{y}_{t-1} + \Delta \tilde{y}_t + \Delta \tilde{y}_{t+1}))} \\
= \frac{\text{cov}(\log \beta_i, \theta_i)}{\gamma\{\sigma_{\beta}^2 + \rho(\rho - 1)(\rho^3 - 1)\sum_{t=1}^{T-2} \rho^{2(t-1)} + \rho(\rho - 1) + 1]\sigma_{\eta}^2/3\}}
\]

The BPP’s test of complete market comes from the fact that if \( \sigma_{\beta} = 0 \), we get \( \text{cov}(\log \beta_i, \theta_i) = 0 \) and thus \( \hat{\phi}_{t,CM} = 0 \). But the empirical finding of \( \hat{\phi}_{t,CM} > 0 \) can not reject the hypothesis of complete markets, if \( \text{cov}(\log \beta_i, \theta_i) > 0 \).

The variance of transitory shocks is identified as:

\[
E(\Delta \tilde{y}_t\Delta \tilde{y}_{t+1}) \\
= E[(\theta_i + (\rho - 1)z_{t-1} + \eta_t + \varepsilon_t - \varepsilon_{t-1}) \\
(\theta_i + \rho(\rho - 1)z_{t-1} + (\rho - 1)\eta_t + \eta_{t+1} + \varepsilon_{t+1} - \varepsilon_t)] \\
= \sigma_{\beta}^2 - \sigma_{\varepsilon}^2 + \rho(\rho - 1)^2\text{var}(z_{t-1}) + (\rho - 1)\sigma_{\eta}^2 \\
= \sigma_{\beta}^2 - \sigma_{\varepsilon}^2 + (\rho - 1)\{\rho(\rho - 1)\sum_{t=1}^{T-1} \rho^{2t} + 1\}\sigma_{\eta}^2
\]

The covariance of consumption growth and transitory shocks is identified as

\[
E(\Delta \hat{c}_t \Delta \hat{y}_{t+1}) = E[(\frac{\log \beta_i}{\gamma} - \Delta_t \Gamma_t)(\theta_i + (1 - \rho)z_t + \eta_{t+1} + \varepsilon_{t+1} - \varepsilon_t)] = \frac{1}{\gamma} \text{cov}(\log \beta_i, \theta_i)
\]

Therefore,

\[
\hat{\psi}_{t, CM} = -\frac{E(\Delta \hat{c}_t \Delta \hat{y}_{t+1})}{E(\Delta \hat{y}_t \Delta \hat{y}_{t+1})} = \frac{\text{cov}(\log \beta_i, \theta_i)}{\gamma \{\sigma_{\xi}^2 - \sigma_{\eta}^2 - (\rho - 1)[\rho(\rho - 1) \sum_{\tau=1}^{t-1} \rho^2 \tau + 1] \sigma_{\eta}^2\}}
\]

**Proof.** (Corollary 7)

In the incomplete market with predictable income change, the true pass-throughs are not well-defined because there are no shocks at all. The BPP instrument estimation of $\phi$ can still be calculated using $E[\Delta \hat{c}_t(\Delta \hat{y}_{t-1} + \Delta \hat{y}_t + \Delta \hat{y}_{t+1})]$, $E(\Delta \hat{c}_t \Delta \hat{y}_{t+1})$, $E[\Delta \hat{y}_{t+1}(\Delta \hat{y}_{t-1} + \Delta \hat{y}_t + \Delta \hat{y}_{t+1})]$ and $E(\Delta \hat{y}_t \Delta \hat{y}_{t+1})$. The last two terms are only functions for income process, so they are the same in both models. The consumption growth rate in this model is derived as

\[
\Delta \hat{c}_t = \frac{\log \beta_i + \log R}{\gamma}.
\]  

(4.18)

Recall in the complete markets case $\Delta \hat{c}_t = \frac{\log \beta_i}{\gamma} - \Delta_t \Gamma_t$. The difference of $\Delta \hat{c}_t$ in two models is only an additional aggregate term which is orthogonal to any individual income change. Therefore, $E[\Delta \hat{c}_t(\Delta \hat{y}_{t-1} + \Delta \hat{y}_t + \Delta \hat{y}_{t+1})]$ and $E(\Delta \hat{c}_t \Delta \hat{y}_{t+1})$ are the same in both models, which yield the same BPP’s estimators. ■
Bibliography


