# Mach's principle on rotating systems in general relativity

by

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***THESIS***

*for the degree of*

***MASTER OF SCIENCE***

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*June 2009*

# Preface

This thesis might be regarded as a review over a subject that has a history of active research of more than 100 years. However, it differs from many other reviews in certain respects. I have made efforts to not only present the results, but also how they were found. The hope is that this will be enough to get a deeper understanding of the results, and that it might expose ways to extend them. I also have made a serious effort to keep the mathematical level as simple as possible without the loss of precision that often is associated with such popularisations. My own contribution has mainly been to provide my own interpretations, examples and some suggestions where appropriate.

There are three sections I want to mention especially: The first two are those that cover two very recent results. One of those is the improved data analysis of the gravity probe B experiment detailed in 3.3.3. The other is Schmid's result on linear perturbations on FRW-universes that is presented in 4.1. Finally I would like to mention the section calculating dragging effects in a simple galaxy model 2.3. While I am ever present throughout this thesis in selecting, refining and commenting on works of others, this is the section where I truly feel that I am presenting work that is entirely my own.

This text is probably best used as an introduction to the field in question, or as a reading companion to the main articles presented in this thesis. It may also be read more lightly as a simple overview of the history of the more recent research on an engaging philosophical problem, or as a second point of view for those already familiar with the field.

This thesis is arranged partially historically and partially based on complexity. The first chapter is a simple introduction narrowing the focus of the rest of the thesis while providing some horizons for further study. The second chapter only examines the simplest deviations from special relativity theory. The third chapter extends on this, going to more complicated systems, but still keeping the Minkowski boundary. Finally in the fourth chapter the case of entire universes are treated. The last chapter is just a short wrapping up of the previous chapters.

Most of this text should be possible to enjoy for anyone having lower grade courses in basic mechanics and vector field theory. I also assume superficial familiarity with the main concepts of the general relativity theory like the metric tensor and the field equations. Full understanding will however demand some more advanced classical mechanics and familiarity with certain analytic methods. The exception is the section on galaxy rotation 2.3. Here some numerical methods and programming is used. This section is however not necessary for enjoying the rest of the thesis.

In order to be as useful as possible as a reading companion I have mostly preserved the notation of the sources formulas are based from. Exceptions are noted in the text. This will be explained in the relevant sections. I use a few common conventions I would like to mention here: I use Einstein's summation convention. $g_{\mu\nu}$ is the metric tensor. $T^{\mu\nu}$ is the energy-momentum tensor. The time like component is the 0-component of tensors. Greek indices represent all 4 dimensions, while Latin indices mark only the spatial components.

Of particular note is it that there are different conventions on the gravitational constant. Some use Newton's, while others use that of Einstein. In addition, it is quite common to use the convention that set the speed of light and the gravitational constant (Newton's) to unity.

I would like to thank my supervisor professor Øyvind Grøn for all his help, and my family for support and feedback. Also a big thank to all those books, articles and web pages that have served as inspiration and shaped my view of this amazing subject. Not nearly all of them did find their way to the bibliography, as they did not directly relate to any of the content.

# Contents

# Chapter 1

# Introduction

I will in this chapter give an introduction to the topic of this thesis, both historically and conceptually. This I will do by starting at the parts of the title and describing those in more detail, in addition to other possible approaches to the problem at hand.

## 1.1 Mach's principle

I will in this section give a short historical and philosophical introduction to how the term "Mach's principle" came to be, and give a short overview of possible meanings. In the later sections I will narrow down the scope of the rest of this thesis. This is necessary as Mach's principle itself is a far to broad concept for me to serve it justice to in the limited time and space of a master thesis. The historical treatment is primarily based on Norton [37] and Hoefer [23]

### 1.1.1 What Mach said

Mach's principle is the name given to a very loosely defined concept that is attributed to the physicist Ernst Mach. One of the key quotes from him that has lead to this concept being attributed to him is a critique of Newton's bucket experiment. In this experiment Newton considers a bucket filled with water, initially held at rest. He observes that the water has a flat surface. He then starts to rotate the bucket around its horizontal axis. After a little while the water is moving toward the edges, so that it is shallower in the middle than toward the sides. This he explains by referring to a centrifugal effect that arises when the water in the bucket start to rotate with respect to *absolute space*. Mach's answer to this is [33]:

> Newton's experiment with the rotating vessel of water simply informs us that the relative rotation of the water with respect to the sides of the vessel produces *no* noticeable centrifugal forces, but that such forces *are* produced by its relative rotation with respect to the mass of the Earth and the other celestial bodies. No one is competent to say how the experiment would turn out if the sides of the vessel increased in thickness and mass till they were ultimately several leagues thick. The one experiment only lies before us, and our business is, to bring it into accord with the other facts known to us, and not with the arbitrary fictions of our imagination.

This quote should be seen in the context that Mach in his text advocates the view that all observations is of how different bodies relate to each other. Hence it is problematic even to try to define a concept such as absolute space.

### 1.1.2 Interpretations of Mach

Exactly what Mach wanted to say with this quote has been up to some speculation. One possibility seems to be that it is an emphasis of the point that we can't know anything about situations we can't observe. In this case the main message of Mach seems to be a call for a redescription of the physics so that it only was described as how bodies move in relation to each other with no reference to absolute space. This may actually be done even within the framework of Newtonian physics under the simple assumption that the universe itself is not rotating with respect to such a real absolute space. This is for instance shown by Donald Lynden-Bell in [32].

A second way to read it is that he is proposing that there could be something other than *absolute space* that determines the outcome of Newton's bucket experiment. The problem is that if this is the case, he is giving little suggestions as to what and how, except that it should have something to do with how matter moves in relation to each other. One striking thing is that if this interpretation is right, then he is very vague about it compared with some of his contemporaries. For instance the brothers Imanuel and Benedict Friedlaender presented a paper in 1896 describing an experiment that would attempt to determine if the rotation of the Earth had any modifying effect on the law of inertia. They were however unable to find any deviations from Newton's mechanics, considering their error margin.

But why should there be any reason to search for factors that might change the outcome of Newton's bucket experiment? There are two important somewhat distinct lines of reasoning that classically seem to reach the same conclusion, but in later times have turned out to give quite different ways to approach the problem. The first is an argument concerning the aesthetics of causality: According to Newton's mechanics - If you know the relative distances and velocities of all bodies in the universe at some time, you know *almost* enough to determine how the system will evolve at all times. What is required to make the system completely determinable seems ridiculously little compared with the huge amount of information you have on the universe by then. One way is to put these bodies into a framework like that of Newton. Another way is simply stating that the universe is not rotating, or more general giving an axis and magnitude of rotation. It should be possible to determine this axis by observation by observing a few of the double-differentials of the relative positions of the matter. But even when this extra information is available, a theory where this it wouldn't be necessary would seem cleaner than Newton's.

The second line of reasoning is similar to that above, but stops before observing the double-derivatives. One should rather note that this extra needed information seems arbitrary. Why should it be so that a single axis of rotation should be so important for being able to completely describe nature? Could this rotation axis really be totally arbitrary, or is it possible that it is actually determined by the relative distances and velocities of the bodies in the universe?

There is one important observational fact that has been used to argue that it is unlikely that what has been called absolute space is independent of the masses of the universe: That such an absolute space seems to be unaccelerated with respect to the "fixed stars". Consider Newton's bucket experiment. When we are standing on the Earth, nearly at rest relative to the fixed stars, we observe the water climbing the edges while we are rotating the bucket. We are prone to argue that the reason for this is that the water in the bucket is rotating, and hence it experiences a centrifugal effect. If we on the other hand sit inside the bucket, we still see the water being shallower in the middle than farther out. But the water and the bucket is not moving relatively to us in this case. It is simple to claim that we are experiencing this because we are rotating ourselves, but how can we say? If you look up, maybe you can see the stars racing around the sky at high speed. Wouldn't it then be plausible from your point of view to claim that the reason for the water moving away from the centre actually is that the stars in the sky is rotating around it?

### 1.1.3 First usage of the term

Regardless of motivation, it is the last interpretation that has become the main idea of what is today called Mach's principle. When Mach was so little clear about this himself one might wonder how this principle came to bear his name? This is mostly attributed to Albert Einstein. He first used the term in his paper on general relativity from 1918 [18]:

> Das G-Feld ist *restlos* durch die Massen der Körper bestemmt. Da Masse und Energie nach den Ergebnissen der speziellen Relativitëtstheorie das Gleiche sind und die Energie formal durch den symmetrichen Energie-tensor ($T_\mu \nu$) beschrieben wird, so besagt dies, dass das G-Feld durch den Energietensor der Materie bedingt und bestimmt sei.

This definition is however not standing very strong. It seems like Einstein during the period 1912-1918 had some idea he attributed to Mach that he really wanted the theory he was working on to satisfy. But his actual formulation of this idea was changing over time. This definition doesn't stand much stronger when one considers that Einstein himself more or less gave up the entire idea the summer 1918. The background for this was the finding of the de Sitter space that was an empty-space solution with the cosmological constant. As it is hard to argue that the G-field is then caused by some matter distribution the general theory of relativity doesn't seem to fulfil the above given definition.

### 1.1.4 Present formulations

Even though Einstein's formulation of 1918 isn't very popular, the term "Mach's principle" has been much used in the literature with other meanings since then. But there has been no common consensus as to what the precise meaning of the term should be, and thus it has been used with quite a few different meanings depending on the writer. Common is that it somehow tries to grasp the ideas given by the second interpretation of the Mach quote. Several attempts have been made to collect the different uses of the term, for instance in [21], the index of [25] and in [7].

As several of these definitions fall outside the scope of this text I will here only list those formulations of Mach's principle I'll work with, for easy reference. Common for all of them is that it tells us something about how things far away have local effects.

- Formulation 1: The universe is spatially closed.

- Formulation 2: There is nothing that acts that is not acted upon.
- Formulation 3: In the rest frame of any body the total gravitational field on the body arising from all the other matter in the universe is zero.
- Formulation 4: Masses should somehow determine the inertial systems.
- Formulation 5/6: The inertial systems should be partially/completely determined by the masses of the universe.
- Formulation 7: The axes of inertial frames are perfectly dragged around by a weighted average of the motion of particles in the universe.

Finally I will add a formulation that I have not encountered anywhere, but that will be considered briefly later by me as it seems to be a possible interpretation.

Formulation x1: Mach's principle says that the boundary conditions are to be determined by local behaviour.

## 1.2 Alternatives to Rotation

In the previous section I considered Mach's principle in general. Most of this text will as the title suggests focus on rotational aspect of the principle, but I will devote this section to a short overview of some other possible approaches to Mach's principle that doesn't directly involve rotation.

### 1.2.1 Boundary conditions

When examining how things far away may affect local physics it may be interesting to examine the case where "far away" goes to the limit of infinity. In a theory governed by fields and differential field equations like the general theory of relativity this translates to boundary-conditions of the equations. According to [23] even Einstein himself tried this approach for some time in 1916-1917.

I can see major ways that the boundary-condition problem may be attempted related to Mach's principle. The first is to define Mach's principle as the boundary-conditions that give us the local behaviour we observe in this universe. The other is to begin with some other formulation of Mach's principle and see if that poses any limitations on what kind of boundary-conditions can be allowed. Neither of these approaches has proven very fruitful. I have found no examples of the suggested definition in the literature. I can see several possible reasons for that:

- It doesn't incorporate any relevance to things closer than infinity to Mach's principle, which breaks with the common idea attempted to put into Mach's principle.
- It has little or no physical significance as more than a self-fulfilling requirement to the boundary conditions.
- It is hard to do the calculations involved with it, and it may come in conflict with the desire of having continuity/convergence.

To find boundary conditions that fit an idea of Mach's principle has also proven most difficult or even impossible. A good illustration of how difficult this seems is that one of the main formulations of Mach's principle is that the space is spatially closed. This formulation dates back to Albert Einstein in 1917 [23]. In this case the need for boundary-conditions disappears. One major argument for this definition is this property. And in certain frameworks (most notably general relativity) this definition also turns out to directly lead to several effects that are considered Machian. And even in other definitions of Mach's principle it is tempting to have spatial closure as a requirement to avoid the boundary problems.

### 1.2.2 Requirement for determinability

In 1.1.2 it was argued that in Newton's theory we need to know all relative positions, velocities and *something else* at a given time in order to determine how the system evolves indefinitely. I also provided a sketch of why this *something else* was undesirable. To convert this notion to the general theory of relativity proves difficult as it operates with fields, not particles, and there are issues trying to define "a given time". It is thereby of interest to examine what information you need in order to be able to determine the configuration of the entire space-time.

One such formulation that can be considered important in relation to Mach's principle is the thin sandwich conjecture proposed in [3]. This considers the intrinsic geometries of two space-like surfaces close to each other (nearly alike). In this case the difference between these spaces behaves like a derivative. In the general theory of relativity it turns out that this should be enough to determine the geometry of the entire 4-space. This is very similar to the classically formulated wish that the physics should be determined by relative positions and their first differentials alone, without any extra factor.

Julian Barbour and Bruno Bertoni develops this idea further in [4]. This is nicely explained in [5]. Here it is not posed any compact definition of Mach's principle. The main difference from the above argument is however that the terminology is sharpened and generalized. The required knowledge should only be a point in a phase-space of geometries, and a direction. Appealing to the thin sandwich conjecture it is claimed that general relativity is completely Machian. One interesting idea that is proposed is that we only require the thin sandwich conjecture to be applied locally, at every point, not globally. This way it seems like one may avoid the problems related to boundary-conditions even in universes that isn't spatially closed.

### 1.2.3 Absolute elements

Another approach is to set the focus at the "absoluteness" of absolute space of Newtonian theory that Mach seems to protest against. This is done in some formality by Jürgen Ehlers in [17]. Here he attempts a definition of Mach's principle going something along the line "There is nothing that acts that is not acted upon". Newton's absolute space is such a thing that determines how things move, while nothing may change that space.

He then compares different theories with regard to what geometrical and physical properties of a system it takes into account and governs. He shows a general tendency that the general relativity theory has fewer "Absolute fields" than the special relativity theory, and that the special relativity theory in turn has fewer than Newton's theory. Those fields that are no longer absolute in the more general theories are found as dynamical fields that are intimately connected with the other fields of the theory. In particular this involves the metric and connection-fields, in addition to a conceivable "Ether field".

The definition of what may be considered a field in a theory, and how to determine/define absoluteness is however not very well explained here. In the discussion found in the proceedings after the paper [17], Karel Kuchar points out a possible absolute element in the underlying geometry of the general relativity theory. Ehlers acknowledges this, but says he feels there is a fundamental difference between this and the elements he has considered in his paper. He was however unable to formulate this difference. I have not found any more recent treatment of this approach.

One extension of this idea is also to look at the constants of a theory. Should these be considered fields of the theory? In this case, should they by

Mach's principle not be true constants, but somehow be determined by the physical state? This and similar considerations have been raised and led to several theories that claim to fit better with Mach's principle than general relativity. I will give these some treatment in the next section.

## 1.3 Alternatives to general relativity

There are lots of theories of gravitation that somehow addresses Mach's principle, and even the specific question of rotation related to it. Many of these are intimately related to the general relativity theory as an extension, generalization or restriction of it. I will in the remaining chapters only consider basic general relativity (and its standard lower order approximations). In order to narrow down and specify the scope of what I will here consider, and as I feel it deserves mentioning in a review regarding Mach's principle, I will here say a bit about some of the more profiled theories that I am not going to cover in the later chapters.

### 1.3.1 Restrictions of solutions to field equations

Einstein's field equations do have solutions that by some have been characterized as "un-Machian". I will get into some of these in later chapters. A way to deal with this could be to find some conditions that have to be applied in addition to the usual field equations that rule out such solutions. In particular this could be related to setting boundary-conditions as mentioned in the previous section.

Only allowing closed universes is also an example of this. As far as I know only the restriction to closed universes has been somewhat successful, and this has the major problem that it is an open question whether the universe actually is closed. Some of the problems are directly related to the lack of any strict definition of "Mach's principle" and hence it is hard to agree on exactly what solutions should be ruled out. Formulating boundary-conditions faces similar problems, but is also made difficult by the mathematical complexity involved.

I will in the remaining chapters use the full general relativity without restrictions. This way I will also be able to study some of the more dubious solutions seen from a Machian perspective and examine rotational effects in them.

### 1.3.2 Einstein-Cartan theory

Einstein-Cartan theory is the natural extension of general relativity to allow for spinning masses. The basics are given in a review article from 1976 [22]. The theory owes its name in part to Élie Cartan who in the first half of the 1920s made some basic work on differential geometry related to torsion. But as a full theory it was only developed later.

As a theory that allows for spin this theory could be highly interesting in the context of investigating rotational phenomena. The fact that there is an extension to general relativity allowing spinning masses shows that general relativity operates with non-spinning masses. This I will use to pose some qualitative suggestions on physical interpretation on some systems in 4.2.4. To give a proper analysis of spin-effects would however require this framework and hence fall outside the scope of this thesis.

### 1.3.3 Sciama

In his 1953 article [53] Sciama outlines a simplified theory that is based upon the quite common view that Mach's principle tells that inertia should be determined by matter. This is made more accurate in this quote:

> In the rest frame of any body the total gravitational field at the body arising from all the other matter in the universe is zero.

He then sets out to demonstrate a toy-theory that shows how this might get implemented. He assumes for simplicity that gravitation is governed by a vector field in a Minkowski space. He points out that the gravitational potential actually has to be a second rank tensor, and that this model thus only is illustrative.

The result is a model with some similarities with electromagnetism. A comparison between this and the gravitomagnetism described in the next chapter could be interesting, but falls outside the scope of this text. There is however one important result here, namely the relation:

$$G\rho\tau^2 \approx 1 \tag{1.1}$$

Where $G$ is the gravitational constant, $\rho$ is the density of the universe, and $\tau$ is the age of the universe. The approximation should be considered very "coarse" only meaning "in the order of".

In his paper he continuously refers to a "subsequent paper" where he is supposed to develop this theory in a much more realistic manner. I have however been unable to find this reference, or anyone referring to such an article. In 1964 Sciama seems to be working in the framework of general relativity, with possible extensions and restrictions [54]. The equation 1.1 still seemed to be central in his idea of Machianity then, however.

### 1.3.4 Brans-Dicke theory

The Brans-Dicke theory was first presented in a paper by Brans and Dike in 1961 [11]. This theory is based on the idea that the gravitational constant could indeed be different at different places determined by the mass distribution. They give two important motivations for the gravitational constant to be non-constant.

The first is the relation 1.1 somewhat rewritten: $GM/Rc^2 \approx 1$ where $M$ is the visible mass of the universe, $R$ is the radius of the visible universe and $c$ is the speed of light. This relation if solved with respect to $G$ gives an idea of how this quantity could be determined by the mass in the universe.

The second is the dimensionless number $m_e(G/\hbar c)$ where $m_e$ is the electron rest-mass. This has a size that is mathematically simply related to two seemingly unrelated observed and varying numbers: The age of the universe in atomic time units and the mass of the visible universe in proton masses. Wanting to keep $m_e$ $\hbar$ and $c$ constant the remaining factor that can be adjusted to take this into consideration is G.

They thereby constructed a theory formulated in similar terms as the general theory of relativity, but with a scalar field not present in the other. This theory is also determined by a parameter that has to be set by observation. This makes it hard to falsify, but there has been set rather strict constraints on the free parameter of the theory by the Casini-Hugens experiment [6].

# Chapter 2

# Gravitomagnetism

As said in the introduction, Mach's principle concerns how objects far away may affect certain experiments locally. One such example is Newton's bucket. In Newton's theory, if you have a situation where the stars are rotating in the universe around a bucket that stands still (relative to absolute space), then the water in it stays flat. There are no centrifugal, or "inertial" forces that give the result that the water moves up toward the wall. One may argue that this situation should be equivalent to the situation where you have an observer sitting inside a rotating bucket observing the universe. Hence we should look for some effect that makes the water in the bucket curve in all possible scenarios where the universe is rotating relative to it. Such an effect may actually be found in general relativity and is gravitomagnetism. This chapter will cover this phenomenon in simple local systems.

## 2.1 The fundamental formulas

I will in this section deduce the equations of gravitomagnetism from linearized general relativity. I will start by giving a simple argument from special relativity that should motivate that there is such an effect. After that I will go through the more detailed and accurate calculation of the equations for gravitomagnetism in linearized general relativity.

### 2.1.1 Simple motivation

I will here present an argument that may motivate the existence of a gravitational effect with similarity to electromagnetism in a relativistic theory. This is inspired by a description of electromagnetism attributed to E. M. Purcell as described in [52]. In the given reference one considers a particle moving along a wire carrying an electrical current, and argues that depending on the frame of reference the forces acting on the particle may be seen upon as an electric or a magnetic field. I will here simplify this to a less realistic system, but one that is simpler to relate to the gravitational case.

Consider a negatively charged particle initially at rest beside an infinite positively charged wire. In this case we know from classical electrostatics that there is an attractive force between the particle and the charged wire. If we however changes reference frame to one moving at a constant velocity relative to the rest frame of the particle, parallel to the wire, the particle is moving as an electrical current in the wire in the same direction as the initial velocity of the particle. According to classical electromagnetism there is then a magnetic force that pushes the particle away from the wire. As the particle has to behave similarly in both frames of reference one needs an effect that makes up for the effect of the magnetic force. Such an effect can be found in the special relativity theory. The length contraction of the wire in the moving reference system relative to the initial rest system of the particle makes the charge density higher. Thus we get a stronger electric force that cancels the effect of the magnetic force.

One can argue that this argument lacks several factors that may modify the relation between the magnetic and the electric forces like relativistic time dilation and mass increase. The key point that the length contraction makes a net increase in electric force is better founded in Purcell's original treatment as it is there demonstrated how one may go from a frame with no electrical, only magnetic forces, to a frame with no magnetic, only electrical forces by a simple velocity transition. I would also like to mention the paper [16] where an attempt is made to develop the entire electromagnetism in a similar way from only special relativity and electrostatics, even though I have been unable to verify whether this paper is trustworthy.

So, keeping in mind that Lorentz contraction may give frame dependent forces I turn the attention to a similar gravitational model as the electromagnetic case examined above. We now have an uncharged particle and a wire. In the rest-frame we know that there is a certain gravitational force between these. In a moving frame one may expect a stronger gravitational force as the mass-density of the wire increases due to length contraction. Opposite to the above case we then seek an effect that opposes this increased force in the frame, and one might be tempted to suggest that there is a gravitational counterpart to the magnetic field.

To make any formal calculations on this is however of little interest. There are several other effects that play into this picture. Most important is probably the special relativistic notion of increased inertial mass under high velocities that I suspect may be enough to give a complete explanation model of the presented case without having to refer to any kind of "gravitomagnetic" concept at all. In addition comes the question of how to formulate gravitation in a relativistic framework, which is exactly what general relativity does.

What I want to show in this section is however that it shouldn't be very surprising when it turns out that general relativity actually displays effects very similar to electromagnetism, and point out one idea that might give a understanding of how this difference from Newtonian physics might arise.

### 2.1.2 Linearized general relativity

The theory of gravity that we get by linearizing the general relativity theory may be traced back to Einstein's paper in 1916 according to for instance [21]. After that it has been treated in several works. I will here go through the main points in the derivation from general relativity following the approach given in [36].

Consider the situation where the metric may be written in the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{2.1}$$

where $\eta$ is diagonal with signature $-+++$, that is the metric of the Minkowski space in standard coordinates. We also assume for simplicity that $c = 1$. $h$ is a small perturbation of this metric, with small derivatives and second derivatives. This gives us a weak-field universe, that is without any high densities or relativistic velocities.

The connection coefficients may then be written:

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}(g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu}) \approx \frac{1}{2}\eta^{\mu\nu}(h_{\alpha\nu,\beta} + h_{\beta\nu,\alpha} - h_{\alpha\beta,\nu}) \tag{2.2}$$

In the approximation we have omitted products between the perturbation and it's derivatives, and used that $\eta$ is constant.

As we are at least close to a coordinate frame we have for the Ricci tensor:

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha} \tag{2.3}$$

In the approximation the two last terms are neglected as second order terms. The indices from 2.2 is raised using $\eta$ instead of using the full metric $g$. This is also done when calculating the Ricci scalar

$$R = g^{\mu\nu} R_{\mu\nu} \approx \eta^{\mu\nu} R_{\mu\nu} \tag{2.4}$$

It turns out that the field equations take a particularly nice form if we introduce $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$ where $h$ is the contraction of the corresponding tensor. Then we may impose on the system the following condition due to freedom of choice of coordinate system:

$$\bar{h}^{\mu\alpha}{}_{,\alpha} = 0 \tag{2.5}$$

Fixing coordinates like this is called to impose a gauge condition and this condition is analogous to the Lorenz gauge $A^{\alpha}{}_{,\alpha} = 0$ of electromagnetic theory. The field equations then become

$$-\Box \bar{h}_{\mu\nu} = 2\kappa T_{\mu\nu} \tag{2.6}$$

This equation along with the gauge and the expressions for the metric and $\bar{h}$ forms the basis for the linearized theory of relativity.

### 2.1.3 Gravitomagnetic equations

According to [35], Einstein suspected a relation between his field equations and Maxwell's equations for electrodynamics. It is claimed in this reference that Thirring did a paper on this in 1918, but I have unfortunately not been able to get hold of this reference to see how far this was done. In a footnote in the first article in this translation paper, he does however strongly suggest the correspondences described in this section. It is worth to mention that there are other approaches that give similar equations. In 1977 a general version of Maxwellian relations was found in [10] that was based on parameterized post-Newton formalism which is a formalism to describe a broad class of theories that include general relativity. However, this falls outside the scope of this text.

The approach I will take to show how one may relate the linearized equation with Maxwell's equation is inspired by [21], [56], [34] and [57].

In electromagnetism we know that $\Box A_{\nu} = \mu_0 J_{\nu}$ along with the Lorenz gauge, where $A_{\nu}$ is an electromagnetic four potential, gives us Maxwell's equations in standard form. I follow the same reasoning as in electrodynamics and restrict the attention to the $\bar{h}_{0\alpha}$ terms. This even give us directly the correct $c$ dependency. We can define

$$\vec{E_G}' = -c\nabla\bar{h}_{00} - \frac{d\bar{h}_{0i}}{dt} \tag{2.7}$$

$$\vec{B_G}' = \nabla \times \bar{h}_{0i} \tag{2.8}$$

where $\bar{h}_{0i}$ denotes the normal 3-vector corresponding to the usual vector potential.

The field equations then take the familiar Maxwell-equation form:

$$\nabla \cdot \vec{B_G}' = 0 \tag{2.9}$$

$$\nabla \cdot \vec{E_G}' = -c^2\kappa T_{00} \tag{2.10}$$

$$\nabla \times \vec{B_G}' = -2\kappa T_{0i} + \frac{1}{c^2}\frac{d\vec{E_G}'}{dt} \tag{2.11}$$

$$\nabla \times \vec{E_G}' = -\frac{d\vec{B_G}'}{dt} \tag{2.12}$$

Here $\mu_0 J_\nu$ is replaced by $-2\kappa T_{0\nu}$ from the standard expression.

We have here found some quantities related to general relativity that obey an equivalent of Maxwell's equations. However, apart from their counterparts in electrodynamics, $B'_G$ and $E'_G$ don't immediately have any simple physical interpretation. They are here simply defined so that they behave in the desired way. They are thus of little physical interest yet. The result above is thus only to be seen as a step in a calculation that will eventually lead to a physically interesting result.

We leave $B'_G$ and $E'_G$ for now and rather turn our attention to a simple physical system. Consider the case where $\bar{h}_{ij} = 0$, that is all non-zero elements of $\bar{h}$ can be found as $\bar{h}_{0\alpha}$. In this case we have from 2.6 that also $T_{ij} = 0$. This may be a reasonable model of a perfect fluid with no pressure and low velocities. In this case $T_{\mu\nu} = \rho u_\mu u_\nu$. With $u_0 \approx c$ we have $T_{00} \approx c^2\rho$ and $T_{0i} \approx \rho c u_i = c j_i$ where $\vec{j}$ is corresponding to classical matter flow. The products $u_i u_j$ are considered vanishing as both terms are small.

We now consider the movement by a particle having low velocity in this system. It will follow a geodetic curve given by

$$\frac{d^2x^\mu}{dt^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} \tag{2.13}$$

Ignoring second order spatial velocity terms, and using $\frac{dx^0}{dt} = 1$ and the symmetry in the lower indices of the connection coefficients allow us to simplify

2.13 to

$$\frac{d^2x^i}{dt^2} = -\Gamma^i_{00} - 2\Gamma^i_{0j}\frac{dx^j}{dt} \tag{2.14}$$

We are thus interested in finding these connection-coefficients.

In order to keep the equations simple I again introduce $c = 1$. By contracting the equation $h_{\mu\nu} = \bar{h}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}h$ we get in this case $h = h^\alpha_\alpha = \bar{h}_{00}$ which in turn gives $h_{\alpha\alpha} = \frac{1}{2}\bar{h}_{00}$ otherwise $h_{\alpha\beta} = \bar{h}_{\alpha\beta}$. Then we can use 2.2 to calculate the connection coefficients in terms of $\bar{h}$

$$\Gamma^i_{00} = \frac{1}{2}(2\bar{h}_{0i,0} - \frac{1}{2}\bar{h}_{00,i})$$

$$\Gamma^i_{0j} = \frac{1}{2}(\bar{h}_{0i,j} - \bar{h}_{0j,i} + \frac{\delta_{ij}}{2}\bar{h}_{00,0})$$

We now define the vector fields $\vec{B}_G$ and $\vec{E}_G$ by

$$\vec{E_G} = (\frac{\nabla\bar{h}_{00}}{4} - \frac{d\bar{h}_{0i}}{dt}) \tag{2.15}$$

$$\vec{B_G} = \nabla \times \bar{h}_{0i} \tag{2.16}$$

This gives us a movement equation of the form:

$$\vec{a} = \vec{E_G} + \vec{v} \times \vec{B_G} + a\vec{v} \tag{2.17}$$

where

$$\vec{a}^i = \frac{d\vec{v}^i}{dt} = \frac{d^2x^i}{dt^2}$$

$$a = \frac{1}{2}\bar{h}_{00,0}$$

We see that in this case $E_G$ and $B_G$ are the fields that play exactly the same role in the equations of motion in the case of gravitation as their electromagnetic counterparts. In addition, the definition of these fields are very similar to those of $E'_G$ and $B'_G$. The equivalent of the magnetic field is the same. However the $E_G$ term is not quite so nice. We see that the time variation of the vector-potential plays a smaller role compared to the scalar potential in determining the path of the particle than in the electromagnetic case. I will here restrict attention to the stationary case, that is $\bar{h}_{\mu\nu,0} = 0$. In this case, we get precisely:

$$E'_G = 4E_G$$

$$B'_G = B_G$$

Inserting this into the Maxwell equations 2.9-2.12 while ignoring time derivatives give us after insertion of c to make the units right:

$$\nabla \cdot \vec{B_G} = 0 \tag{2.18}$$

$$\nabla \cdot \vec{E_G} = -4\pi G\rho \tag{2.19}$$

$$\nabla \times \vec{B_G} = -\frac{16\pi G}{c}\vec{j} \tag{2.20}$$

$$\nabla \times \vec{E_G} = 0 \tag{2.21}$$

We see that the main differences from the stationary electromagnetic case is that the forces behave oppositely relative to the currents, and that the gravitomagnetic force that couples to movement is 4 times stronger than the gravitoelectric compared to the corresponding electromagnetic case.

In summary, I have here compared two approaches at combining linearized theory with classical electrodynamics. The first finds quantities in general relativity that behave according to Maxwell's equations. The second examines the movement of particles and try to make it in a form comparable to electrodynamics. There are some references where this inequivalence is poorly stated. This include [31], [56] and [1]. The first two do state that their Lorentz force law only holds in the stationary case, and the Wikipedia article seems to be based upon the first of these due to the reference list. I added this clarification to the Wikipedia article at the stated retrieval date.

## 2.2 Examples

In this section, I will give some examples of simple systems where we may use the above theory. I will also relate this to an idea of Mach's principle.

### 2.2.1 Classical laws

From the Maxwell equations, we may immediately deduce two laws that are important in stationary electrodynamical systems: Ampere's Law, and the law of Biot and Savart.

The equivalent of Ampere's law is gotten by using Stokes' theorem on 2.20. It becomes:

$$\oint \vec{B_G} \cdot d\vec{l} = -\frac{16\pi G}{c} I \tag{2.22}$$

where the integral is around a closed path and $I$ is the matter flow through any surface having the path as edge.

The equivalent of the law of Biot and Savart is trickier to deduce. It is done in [57] so I will simply set up the main result here:

$$\vec{B_G}(\vec{r}) = 4G \int \frac{\vec{r} - \vec{r'}}{\left|\vec{r} - \vec{r'}\right|^3} \times \frac{\vec{j}(\vec{r'})}{c} dV' \tag{2.23}$$

Here it is usual in electromagnetism to make the substitution $\vec{j}dV' = Idl$ where $I$ is the current through a line element of a wire $dl$. However, it is worth noting that such a one-dimensional reduction of the gravitational system is not without problems. The reason for this is the assumption of a weak field in the linearizing of the gravitational theory. This means that we need to have a limited mass-density, and current velocity. In this situation the mass current $I_M$ through the wire has to vanish in the limit of a one-dimensional wire.

As the wire-form of the law of Biot and Savart is very useful, I will show that it is a reasonable approximation if we are calculating the magnetic field far from the "wire". Consider a 3-dimensional wire divided into surfaces $S$ that is normal to $\vec{j}$. Assume further that $\vec{j}$ is constant on the surfaces and parallel to the wire. In this case 2.23 becomes:

$$\vec{B_G}(\vec{r}) = 4G \int \oint \frac{\vec{r} - \vec{r'}}{\left|\vec{r} - \vec{r'}\right|^3} \times \frac{\vec{j}(\vec{r'})}{c} dSdl \tag{2.24}$$

If the surfaces $S$ are relatively small and far from the point we are evaluating the magnetic field for we may assume $\vec{r} - \vec{r'}$ to be constant through the integration. If we then set $I_M d\vec{l} = dl \oint \vec{j} dS$, we get the familiar form of the law of Biot and Savart:

$$\vec{B_G}(\vec{r}) = \frac{4G}{c} \int \frac{(\vec{r} - \vec{r'}) \times I_M d\vec{l}}{\left|\vec{r} - \vec{r'}\right|^3} \tag{2.25}$$

I will add that the above argumentation may be used to calculate the fields far from a small concentration of mass with velocity $\vec{v}$, and total mass $M$:

$$\vec{B_G}(\vec{r}) = \frac{4G}{c} \frac{(\vec{r} - \vec{r'}) \times M\vec{v}/c}{\left|\vec{r} - \vec{r'}\right|^3} \tag{2.26}$$

It is also worth noting that 2.19 is the same as the formula for the gravitational field in Newton's theory of gravity, and hence we may use all the results we know from there.

### 2.2.2 Force strengths

I will here set up a model in order to try to compare the strength of the gravitomagnetic effect compared to that of the familiar gravitoelectric. Consider a small spherical gravitational source with mass $M$ and speed $v_M$. We then examine the behaviour of a test particle so far from this source that we may consider the distance a constant $r$. From 2.26 we can see that we get the strongest magnetic field if we assume that the test particle then is in the plane normal to the velocity direction of the mass-concentration. In this case the magnitude of the magnetic field become

$$B_G = \frac{4GMv_M}{r^2 c} \tag{2.27}$$

From 2.17 we see that the acceleration effect from the gravitomagnetic term becomes greatest if the test particle has velocity normal to the field. So we make this assumption, and set the speed to be $v$. Keep in mind that in 2.17 we have assumed $c = 1$ so that in general units we have to divide the velocity by $c$ in order to get the right units. Hence, the magnitude of the gravitomagnetic effect to the acceleration of the test particle is at most

$$a_B = \frac{4GMv_M v}{r^2 c^2} \tag{2.28}$$

We get the acceleration from the gravitoelectric term directly from Newtonian mechanics:

$$a_E = \frac{GM}{r^2} = \frac{c^2}{4v_M v} a_B \tag{2.29}$$

From these equations alone, it might seem like there is a possibility for the acceleration from the gravitomagnetic effect to become as large as 4 times that of the gravitoelectric. However, from the weak field approximation done in the linearizing we have that $v_M << c$, and from the deduction of 2.17 we also used $v << c$. So indeed the gravitomagnetic acceleration is smaller than traditional gravity in the second order of small velocities. Thus in most applications it seem like this effect is too small to be worth any attention. However, it leads to effects that is not found in Newtonian gravitational theories, and it may turn out to be important at a universal scale. Just like ordinary gravitation, it is a $r^{-2}$ law not "blocked" by anything and thus is long-range.

### 2.2.3 Gyroscopes

In the previous section, we saw that the gravitomagnetic effect of acceleration seems to be hard to detect. In this section, I will examine the behaviour of a gyroscope in a gravitomagnetic field. This is of particular interest, as we know that Newtonian gravitation does not affect the direction of a gyroscope. It turns out that the gravitomagnetic effect does. This may be used as a way to detect the effect without having to worry so much that the much stronger gravitoelectric effect will disturb the experiment.

Consider a right-handed Cartesian coordinate system with a gravitoelectric field in the positive z direction. At the origin, there is a gyroscope with angular momentum along the x-axis. We then see that in slightly positive z-direction it has a velocity in the negative y-direction. From 2.17 we can conclude that it thus experiences an acceleration/force in the negative x-direction. Similarly, in the slightly negative z-direction it experiences an acceleration/force in the positive x-direction. This adds up to a torque in the negative y-direction, and makes the angular momentum of the gyroscope turn toward the negative y-direction. A similar argument holds whenever the angular momentum is in the x-y plane, and we can conclude that the gyroscope is precessing around the z-axis. This is equivalent to the Larmor precession of electrodynamics.

The strength of the effect may be deduced from only 2.17 and classical rotational mechanics as presented in for instance [58]. Using Newton's second law, the torque-formula, and the relation $\vec{v} = \vec{\omega} \times \vec{r}$ we get that the total torque on the system becomes:

$$\vec{\tau} = \int \vec{r} \times \rho((\vec{\omega} \times \vec{r}) \times \vec{B_G})dV \tag{2.30}$$

where the integral is over any volume containing the entire rotating body. Using the Cartesian coordinates with $\vec{\omega} = (\omega, 0, 0)$, $\vec{B_G} = (0, 0, B_G)$ and $\vec{r} = (x, y, z)$ this evaluates to:

$$\vec{\tau} = \int \int \int \omega \rho B_G(0, -z^2, zy)dxdydz \tag{2.31}$$

We now apply the assumption that the gyroscope has its rotation-axis as a symmetry axis. As it is then symmetric upon changing signs of $z$ and $y$ we can conclude that the z-term of the torque cancels out under the integration. Going to cylindrical coordinates so that $r^2 = y^2 + z^2$ and $\cos\theta = \frac{z}{r}$ we get for the magnitude of the torque:

$$\tau = \omega B_G \int_0^R \int_0^{2\pi} \int \rho r^2 \cos^2\theta dx \ r \ d\theta \ dr \tag{2.32}$$

Using that $\rho$ and $r$ are independent of $\theta$ due to rotational symmetry, and that $\cos^2\theta$ is independent of $x$ and $r$ we may separate this integral into

$$\tau = \omega B_G \frac{I}{2\pi}\int_0^{2\pi}\cos^2\theta d\theta \tag{2.33}$$

where $I$ is the ordinary moment of inertia around the x-axis given by

$$I = \int_0^R \int_0^{2\pi} \int \rho r^2 dx\ r\ d\theta\ dr \tag{2.34}$$

The remaining integral in 2.33 is well known, and may be found in for instance [46]. It evaluates to $\pi$. As the system is rotating around a symmetry axis we have $\vec{\tau} = I\frac{d\vec{\omega}}{dt}$. Further, I will assume a perfect gyroscope. As we are working in a framework that depends on low velocities, the best way to implement this would be to use a spherically symmetric distribution. In this case the above argumentation holds at all times. The time derivative of the angular velocity vector is always of magnitude $\frac{B_G}{2}\omega$, orthogonal to the angular velocity itself and the z-axis. This means that the angular velocity vector is itself rotating around the z-axis with an angular velocity:

$$\Omega_G = \frac{B_G}{2c} \tag{2.35}$$

where the $c$ term is inserted to make the units right, and appears as c is assumed to be 1 in 2.17. One may note that this agrees with the result given in for instance [34] (up to a 2-factor due to different scaling of the gravitomagnetic field). Here the result is also generalized to the situation that the gyroscope having non-ortogonal angular momentum, with the result that it is still precessing around the axis of the magnetic field. It is of particular interest that this result is independent of $\omega$ and the mass-distribution, as long as the symmetry restrictions are satisfied.

### 2.2.4 Inside ring

I will here turn my attention to the situation at the centre of a rotating ring of radius $R$ and with a constant angular velocity $\omega$ relative to the background metric. We may choose cylindrical coordinates with z-axis orthogonal to the plane spanned by the ring, and origin at the centre of the ring. Due to symmetry we can conclude that there is no classical gravitational force at the centre of the ring; $\vec{E_G} = 0$. If we further assume that the cross-section of the ring vanishes compared to R we may use 2.25 to calculate the gravitomagnetic field:

$$\vec{B_G} = \frac{4G}{c} \int \frac{\vec{R} \times I_M d\vec{l}}{R^3} \tag{2.36}$$

We see that we only have non-zero z-components in this integral. We assume that $I_M$ is constant, where $I_M = \omega R A \rho$. Here $A$ is the area of constant $\theta$ cross-section of the ring and $\rho$ is the mass density, both assumed constant. As we are only working with orthogonal vectors, it is simple to calculate the magnitude of the magnetic field:

$$\left|\vec{B_G}\right| = \frac{8\pi G \omega A \rho}{c} \tag{2.37}$$

It is interesting to note that this expression is independent of the radius of the ring. This may seem like a deviation from the standard result in electromagnetism $B = \frac{\mu_0 I}{2R}$[58]. However, in the standard electromagnetic case it is practical to use the expressions for constant current $I$, while I here hold the angular velocity $\omega$ constant. This accounts for this difference.

If we now use 2.35 we see that in this case:

$$\Omega = \frac{4\pi G A \rho}{c^2}\omega \tag{2.38}$$

It is interesting to note that we get $\Omega_G = \omega$ when

$$A\rho = \frac{c^2}{4\pi G} = 10^{26} kg/m \approx \rho_U R_U^2 \tag{2.39}$$

where $\rho_U$ and $R_U^2$ are the measured mass-density and radius of the observational universe. As there are huge uncertainties on these two quantities the approximation is at best an "in the order of". (One may use for instance the critical mass density of the order of $10^{-29} g/cm^3$ and a radius of the order of 10 thousand million light years. These are in accord with [13])

Testing the direction of the precession, we find that it has the same sign as the angular velocity of the ring. Hence we have that if the condition 2.39 is satisfied a gyroscope at the centre of the ring will constantly point at the same point on the ring. For other values of $A\rho$ we still get that the gyroscope is precessing in the same direction as the ring rotates relative to the background. Thus, we say that the gyroscope is dragged by the ring. 2.39 is said to be a condition for this dragging to be perfect.

We may now turn our attention to freely moving particles. As mentioned above there is no gravitoelectric effect, so that we only have to pay attention to any gravitomagnetic effects. Particles moving parallel to the magnetic field will hence be unaccelerated, and locally move in a straight line. Particles moving in the plane of the ring with velocity $\vec{v}$ will experience an acceleration in the ring-plane orthogonal to the velocity with magnitude $B_G v$, where $v = \frac{|\vec{v}|}{c}$ is normalized to be dimensionless. Comparing with the argument in 2.2.3 we see that this means that if the particle had moved through a constant field it's velocity vector would rotate with a angular speed of $2\Omega_G$. This actually gives a nice connection between the movement of a gyroscope and the movement of the free particle. Consider a gyroscope pointing in the same direction as the initial velocity of the free particle. The initial position of the gyroscope is the same as that of the particle, but the gyroscope is at rest. During a short time $t$ we may assume the acceleration of the particle to be constant. In this case we find that the particle after a short time is at a distance $r = vct$, and has a deviation from the original gyroscope axis fixed to the background metric of $\frac{1}{2}B_G vt^2$. The gyroscope axis has however changed by an angle $\theta = \Omega_G t$. This means that the point that the gyroscope now points at, and that is a distance $r = vct$ from the gyroscope, has to be at a distance of $vct \sin \Omega_G t \approx vct^2 \Omega_G = \frac{1}{2}B_G vt^2$ from the original axis. This is the same point as we found the free particle to be at. We can conclude that the gyroscope is still pointing at the free particle.

From the above argument, we can conclude that in a local reference frame at the origin with axes fixed by gyroscopes free particles are moving along a straight line. This is the defining property of an inertial frame. It is here we get the connection with Mach's principle. Imagine a scientist living in a box at the centre of this rotating ring. Using gyroscopes and watching the motions of free particles close to him he finds that there is a certain frame in which the gyroscopes keep a fixed direction that is hard to change, and in which the particles move along a straight line. As he is unable to determine any cause for this, he is prone just to take it as a fact of nature that there is a "preferred" frame that happens to be as it is, and thus may be explained by means of an absolute space. Assume further that the equation for perfect dragging 2.39 is satisfied. If the walls of the box suddenly should become transparent so that the scientist could see the ring of dust around his laboratory, it should be easy to envision him wonder why this ring turns out to be at rest relative to his inertial frame. Above we have reasoned that this is no coincidence at all. No matter how the ring rotates (as long as it is within the weak field approximation), the scientist's frame would turn out to not rotate relative to it.

This raises the question, could we be in a similar situation? From the approach in this section, it would be natural to say that the result of the experiments the boxed scientist used to determine his inertial frame was, at least in part, caused by the properties of the surrounding ring. Mach's principle may be interpreted as a statement that it is this kind of explanation that is preferred, and even necessary. I am thus ready to formulate the main definition of Mach's principle I will concentrate most of the remaining treatment around:

> The inertial systems should be partially/completely determined by the masses of the universe.

### 2.2.5 Hollow infinite cylinder

I will here give a short presentation of a rotating hollow infinitely long cylinder. It might be an interesting system from a gravitomagnetic point of view, but I have found little use for it regarding Mach's principle. It will also later be used to demonstrate the limitations of the simplifications used to arrive at these equations for the gravitodynamics.

This situation may from a gravitomagnetic view be treated the same way as the magnetic field of a solenoid as described in [58]. In this case, we use Ampere's law on a rectangle with one side inside the cylinder parallel to the sides and the opposite side outside. The remaining sides are orthogonal to the sides of the cylinder. The simplified idea is that due to symmetry the magnetic field must be normal to the lines that pass through the cylinder. The line outside the cylinder experiences no magnetic field. One way to argue for this is that it may be as far away as we want showing that it at least can be set to zero. Personally, I am more fond of an argument regarding the magnetic field to be divergence less, hence its density must be the same inside and outside the cylinder; but outside is infinitely bigger. Anyway, we find that the only contribution to the path integral of Ampere's law is along the line inside the cylinder, and that the field is parallel to this. If we say that the length of this line is $L$ we get that 2.22 goes to

$$B_G L = -\frac{16\pi G}{c} L D \rho R \omega \tag{2.40}$$

where $D$ is the thickness of the cylinder, $\rho$ is the mass-density, $R$ is the radius of it, and $\omega$ is its angular velocity. $L$ may be cancelled at both sides. We find that we have a constant gravitomagnetic field inside the cylinder.

I have not found any treatment of the classical gravitation inside a cylinder, and in the electrodynamic case, the solenoid is usually considered neutral. The following argument should however show that there is indeed no gravitoelectric field inside the cylinder: Consider a closed finite cylinder inside the infinite cylinder. Its sides are parallel to that of the infinite one, and their centre axes coincide. As there are no gravitoelectric sources within it, the gravitoelectric flow should be zero. From symmetry, the gravitoelectric field lines should go the same way through the sides of the cylinder. Again due to symmetry we expect the top and bottom to not have any field lines through them, as there is as much mass above them as below them (with a theoretical extra infinitely far away from one of them, to account for the difference in position). Hence, we can conclude that there is no gravitoelectric field components normal to the cylinder at any place. As this holds for all cylinders inside, there cannot be any gravitoelectric field there.

I have been unable to confirm this directly either through other sources, or numerically. However, there is an exact result on this system that partially confirms this idea. It also clearly shows a flaw that could be considered major in the result that the inside of an infinite hollow cylinder can be treated as being an area with only a gravitomagnetic field. This drastically limits how useful this model is compared to that of a solenoid in electromagnetism. The detailed treatment of this is however better suited later in this text.

## 2.3 Rotating galaxy

. In this section, I will make a rough numerical study of a simple galaxy model within the framework presented earlier in this chapter. This section is not necessary for the understanding of any of the later parts of the thesis, so it may be skipped. It requires some knowledge of programming and numerical methods to appreciate fully.

There is a well-known problem that the visible mass distribution of galaxies does not provide explanation for their rotational pattern. The most usual solution to this is to introduce huge amounts of dark matter into the model of the galaxy in order to stabilize it. However, these calculations of the predicted movement pattern are based solely upon classical gravitational theory. The idea is that the speed and density of the galactic matter is not high enough for there to be any considerable relativistic effects.

In this section, I will try to determine how weak the relativistic effect of gravitomagnetism actually is in this system. We expect the rotation of the galaxy to set up a gravitomagnetic field orthogonal to the galaxy plane. The rotating matter have a velocity relative to this, thus we expect a radial gravitomagnetic force to work on the matter. The approximate strength of this will be found, and compared to that necessary to describe the rotational motion correctly.

I am not aware of anyone having done this before apart from rough order of magnitude estimates. It turns out that I won't do much better myself, but it stands as a computational confirmation on those order of magnitude calculations, and may be used as a base for further research.

### 2.3.1 Method

I will use the following model for the galaxy: A cylinder with constant height that is rotationally symmetric around the cylinder axis. Introducing cylindrical coordinates $r$, $z$ and $\phi$ with origin in the centre of the cylinder, I also demand that the system is independent of z-coordinate as long as it is inside the cylinder. The velocity field has no $r$ and $z$ component.

Now consider a vector $\vec{P}$ representing the position of a point with coordinates $r = R_0$, $z = 0$ and $\phi = 0$. The last of these components we can assume without loss of generality due to rotational symmetry. Now, the gravitomagnetic effect on this point as a result of the movement at a point with position vector $\vec{P'}$ is from the law of Biot and Savart of the form 2.26 given by:

$$\vec{B}_g = \frac{4G}{c} \frac{(\vec{P} - \vec{P'}) \times M\vec{v'}}{\left|\vec{P} - \vec{P'}\right|^3} \tag{2.41}$$

where $\vec{v'}$ is the velocity at the source point, and $M$ is the mass at that point. In order to find the total gravitomagnetic field we have to sum over all source points. Inserting the components into the equation, we then find that any non-z component of the gravitomagnetic field caused by a source is cancelled by that of the source with opposite z-coordinate. Thus the final gravitomagnetic field has only a z-component. As we are summing over infinitely many points, all with infinitely small mass, the sum turns to an integral. The magnitude of the z component may then be found to be:

$$B_z = \int_0^R \int_{-Z}^{Z} \int_0^{2\pi} \frac{4G}{c} \rho(r) v(r) \frac{r - R_0 \cos\phi}{\sqrt{r^2 - 2rR_0 * \cos\phi + R_0^2 + z^2}^3} r d\phi dz dr \tag{2.42}$$

Here $v(r)$ is the magnitude of the velocity field normalised to $c = 1$, and $\rho(r)$ is the density. $R$ is the radius of the galaxy, while $Z$ is half its height. From the data given in [13] I gather that $Z$ should be in the order of 0.1 kiloparcsec(kpc), while $R$ may be taken to be about $20kpc$ This uses rough data for the Milky way galaxy.

Be aware that I want to use this formula for $R_0 < R$. This may seem dubious by two reasons. The first is that in our linear field approximation assumed that we were far from the masses. Here we want to examine the case where the point we are measuring the field at is inside the mass distribution. This objection may be rejected by arguing that we are working inside relatively small densities, thus in effect there are no, or infinitesimally small mass close to the point we are calculating the field for. Thus, we are still calculating primarily the effect of masses that are far away. Then, the weak field approximation is still fulfilled.

The second problem is related to the first. As the point we are calculating for is inside our integration domain we get a singularity in our integrand at this point. However, from dimensional analysis we find that this singularity is only to the second order in distance, while we are integrating over three dimensions. From this it seem plausible that this singularity may be smoothed out so that the integral still converges.

In order to find the velocity and mass distribution, I use the results of a doctorate thesis from 1978 [8]. From these measurements it seem like the velocity of the arms is approximately constant some distance away from the galaxy core. As I will have primary interest of the situation in this area, the further simplification that the velocity is the same constant also in the core will not make a too big effect on these results. Thus I model $v(r) = v_0$. From the data of that thesis it seem like $v_0 \approx 2/3000$ is in the right order (remember $c = 1$). From this, he calculated the mass distribution needed for the classical gravitational force to balance the centrifugal force. The total mass seems to increase almost proportionally to the distance. Hence, we get for the mass distribution on the disk:

$$4\pi Z \int_0^{R'} \rho(r) r dr = AR' \tag{2.43}$$

Here $Z$ is still half the height of the disk, R' is the radius of the disk taking the total mass inside, and A is the proportionality factor of the total mass. This clearly gives us the solution:

$$\rho(r) = \frac{A}{4\pi Z r} \tag{2.44}$$

From the graphs of that paper I gather that $A$ is in the order of $10^{10}$ solar masses per kiloparsec.

When I insert this velocity and mass density into 2.42 I not get an integral I do not know how to solve. I also attempted to use the commercial program Mathematica to solve this exactly, but it was unable to do so. Therefore, I decided to try to solve it numerically. Thus, I made a Phyton program based upon Monte Carlo simulation. We will see that there will be some problems concerning the singularity in the integrand, thus making the result rather fuzzy. I made some tests to determine the extent and nature of this problem. This included testing two slightly different methods of setting up the integration, and looking at the sensitivity to changes in the number of points taken in the calculation.

The Monte Carlo method is based upon solving an integral by evaluating the integrand at random points and summing it up in the end. In cylindrical coordinates one have to pay heed to how to select these random points. If one chooses points by taking an independent uniform distribution of the $r$, $z$ and $\phi$ coordinates one will find that one obtains a higher density of points close to the centre than further out. This will affect the integral, and thus is undesirable. I tried two ways to counter this. The first is to change the probability distribution for the radial coordinate so that it is less likely to get low values for the radius rather than high. This makes sense as the further out you get the more points there is in the circle of that radius. The correct distribution of radial coordinates that gives an even distribution of points in the space is obtained by taking the square root of a numbers uniformly selected between 0 and $R^2$. The other way is to weight the selected points in such a way that points further out counts more in the final sum than points further in. This would be similar to choosing that each time you get a random point you actually add it a number of times to the sum depending on their distance from the centre. If the weight given to the point is equal to it's radius we find that when we use uniform distribution for $r$, we still get the same distribution of effective number of points at each radius as we had with distributing the points evenly in space.

The Monte Carlo method clearly depends on the number of random points taken. The more points, the more accurate we expect the result to get. The two ways to distribute the points described above are constructed so that they should give the same result in the limit where you have infinitely many points, but their behaviour at a finite number of points might differ. This could especially affect the stability properties of the solutions. I have also included a brief analysis of this. The commented Python source code may be found in the appendix A.

In order to interpret the strength of the gravitomagnetic field, I compare the acceleration induced by the mass moving through it, $a_{mg}$, with the total acceleration $a_{tot}$ we can find due to the particles of the galaxy moving in a circular orbit with radius $R_0$:

$$\frac{a_{mg}}{a_{tot}} = B_z * v_0 * c / \frac{(cv_0)^2}{R_0} = \frac{B_z}{c}\frac{R_0}{v_0} \tag{2.45}$$

Another interesting quantity to compare with is the acceleration we can attribute to the ordinary gravitation $a_{eg}$. This turn out to have a quite simple relation to $a_{tot}$ in our model:

$$a_{eg} = G\frac{M}{R_0^2} = G\frac{AR_0}{R_0^2} = \frac{GA}{v_0^2} * a_{tot} \approx 1.08 a_{tot} \tag{2.46}$$

The mass distribution was constructed in [8] to make these equal in the more advanced model used there. I find the fact that this relation is still somewhat conserved in the very simplified model studied here, as a sign that further results should at least be of the same order of magnitude as the corresponding values in the real world.

### 2.3.2 Results

I will here present the graphs resulting from of one running of the program. I have tested the program several times with different values for M and N. These test-runs have not provided any significant information other than that presented here, apart from confirming the general tendencies of the system. I work with 1000 points in the graph, which should be more than enough resolution. 10 to 100 points would have given the same general results, but with 1000 points, it become more statistically viable.

First the output from the Monte Carlo stability analysis is presented in figure 2.1. This graph illustrates nicely the general tendency I found the Monte Carlo simulation followed. As we would generally expect from Monte Carlo simulations the general trend of the graph is to swing around some ill-defined value. However, we see that in this case while it mostly moves rather smoothly after this number of simulations, it do make some jumps. These jumps I attribute to a random source point being selected very close to the point we are calculating the field for. This gives us a very small number in the denominator of our integrand resulting in a high contribution to the sum. In the true integral, we can expect the effects of nearby masses to cancel out as locally we are in a system where all particles are moving with roughly the same velocity and direction. However, if only one random point is taken in the local area then there is nothing to cancel the effect of this. Only by taking more points we can hope to get other points locally that sums up to cancel that effect. By the Monte Carlo method, we have no guarantee that there will be a distribution of points making the local contribution cancel out. This sensitivity to point distribution might have a physical interpretation as well, more on this in the conclusions.

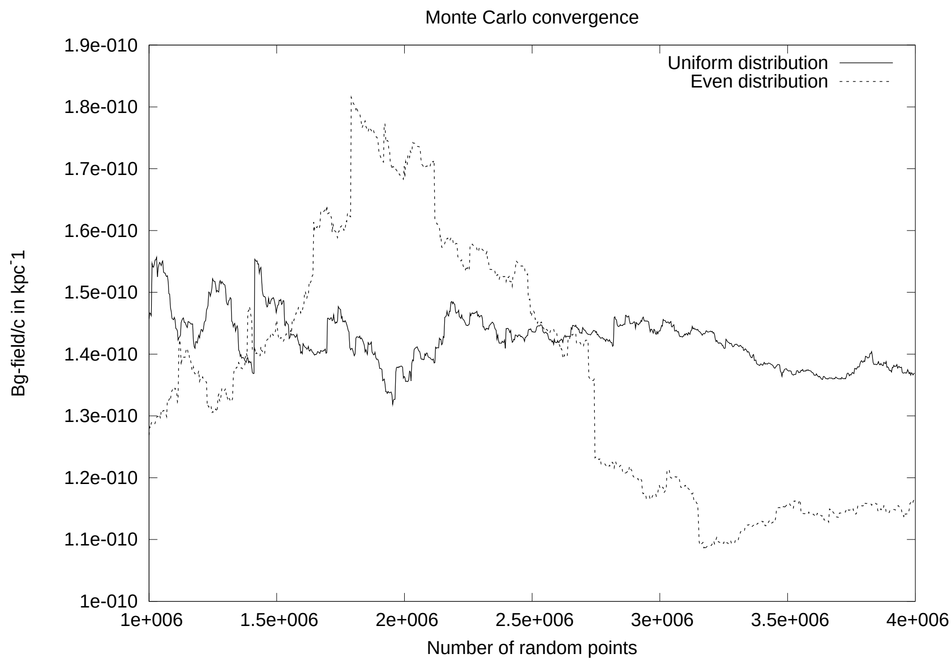


Figure 2.1: Monte Carlo stability

These jumps might also lead to worries regarding the numerical precision in the calculation. If the result from the integrand function is too high, the result might drown out other points due to the difference being too small for the computer to handle. A rough estimate on the size of the jumps when we know that Python floats is 8 bytes, indicate that this is not a major problem. This is also confirmed by observing that the graphs seem to have the same general behaviour after such jumps as before.

The graph presented here suggests that the uniform distribution is somewhat more stable than the even distribution. Other test runs have indicated that it might be a bit hasty to draw such a conclusion. Still I have seen no clear indication that the uniform distribution in general behaves worse than the even. I made the choice to settle with the uniform distribution due to one important reason: I do then not need to divide by the radius at any point. This I believe makes this method slightly faster than that of even distribution. It also might increase numerical stability, as there might become division by zero problems if the source point is chosen close to the centre of the galaxy.

These infrequent jumps in the graph also indicate that this method is not very sensitive to an increase in the number of simulation points. While the accuracy in general becomes better, the probability for getting points causing big jumps also increases. Still, at my computer from 2004 it turned out that memory usage was the main limitation for how high I could set N. I ended up using 3 million samples for the main calculations, which caused Python to consume about 400MB of memory. The total running time of the program was then a bit less than an hour.

The strength of the gravitomagnetic field is presented in figure 2.2 I have

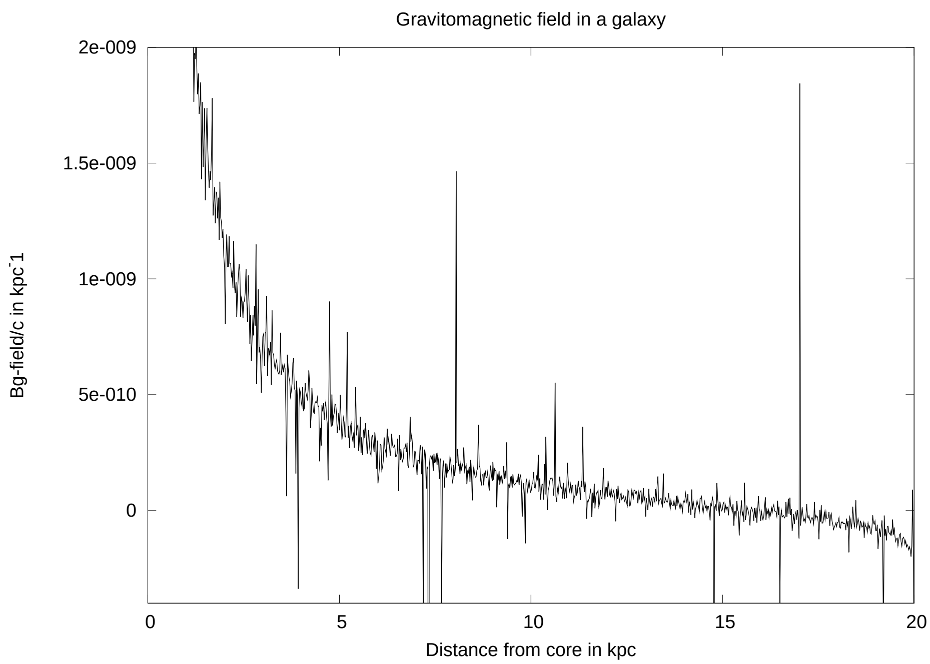


Figure 2.2: Gravitomagnetic field strength

here set the axes so that the magnetic field at the centre is not shown. This gravitomagnetic field may be if interest in other applications, but is of little interest here as this model of the galaxy is quite inaccurate in this area.

As might be expected from the instability of the Monte Carlo method, the graph is quite fuzzy. Still the general behaviour is quite obvious. Close to the core, there is a relatively strong gravitomagnetic field. This decreases the further from the core you come until it at about 15 out of 20 kpc turns negative. This change from positive to negative may be intuitively expected. Near the core most of the mass is swirling outside, hence we have a situation similar to that inside a current loop. However, as we get further out we get outside the rotating mass. Outside the galaxy, we have a situation similar to that outside a current loop, and in this case the gravitomagnetic field is opposite to that inside. At some point inside the galaxy, we would then expect those two effects to cancel out.

In order to interpret the strength of the field, I calculated how much this gravitomagnetic field accelerates the masses of the galaxy compared with the total acceleration. For this I used the formula 2.45. The result is illustrated by figure 2.3. In this figure I have set no restrictions on the axes of the graph

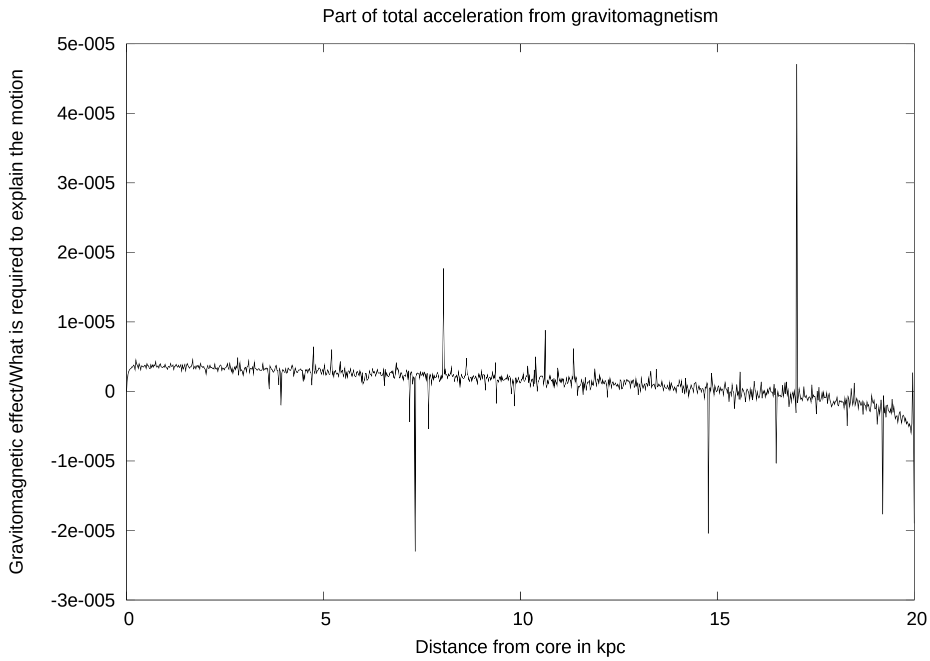


Figure 2.3: Gravitomagnetic effect, unconstrained axes

in order to show the magnitude of the spikes. Zooming in to get a better look at the most concentrated part of the graph we get figure 2.4.

From these figures, we easily see that the gravitomagnetic effect is in the order of a few parts of a millionth of the total acceleration on the galaxy matter. Even the highest spike in this data set doesn't get higher than $5 * 10^{-5}$. It is also here easier to see that the gravitomagnetic effect changes sign about 15kpc away from the core. For this case, the positive direction for the acceleration may be found to be toward the core.

### 2.3.3 Conclusions

The gravitomagnetic effect depends on the velocity both of the sources and the body acted upon. The gravitoelectric effect on the other hand does not

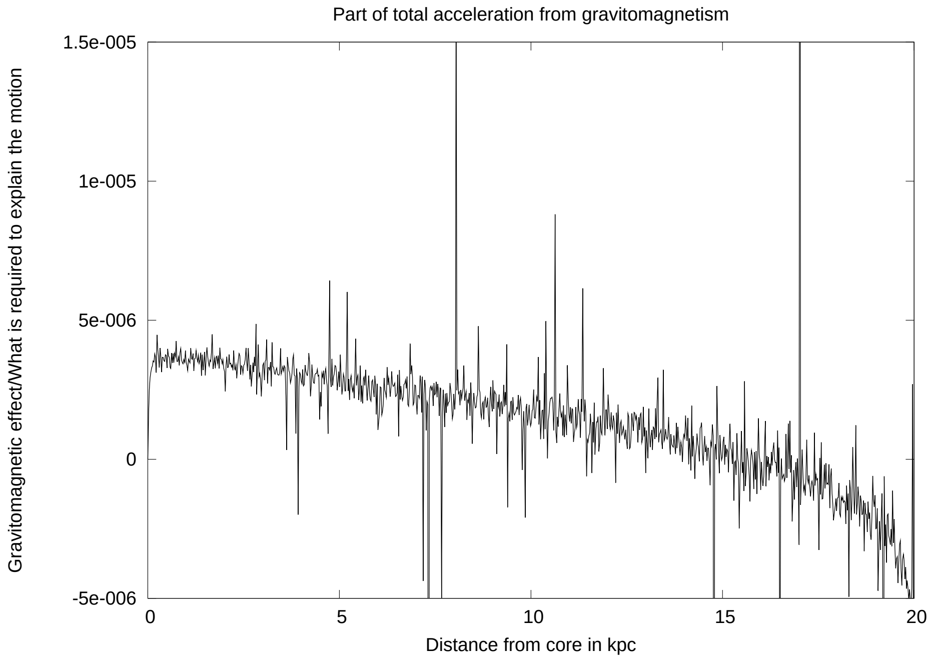


Figure 2.4: Gravitomagnetic effect, constrained axes

depend on velocity at all. One may then quickly make the assumption that the ratio between these will be in the order $(v/c)^2$. There are however other differences in the behaviour of gravitomagnetism that might separate it from the gravitoelectric. For instance while the gravitoelectric effect only depends on the masses inside the position it is calculated for, the gravitomagnetic effect depends on all the masses in the galaxy wherever this is calculated. The simulation performed here lend support to the idea that the first of these differences is the most important when it comes to approximating the ratio of the strength between those two fields. In this particular model, this even is true at the core where the second difference intuitively should have given zero gravitoelectric effect, with a non-zero gravitomagnetic. This I gather is due to the model having infinite mass density in the core.

From this, it is easy to conclude that for instance the gravitomagnetic effect is too weak to be used as an alternative solution to the dark matter problem. It is also probably not necessary to take into account when performing most theoretical calculations on galaxy models. For numerical simulations on the other hand, the magnitude of the effect calculated here is big enough to possibly make a difference. Even if the desired relative precision of the final result is less than $10^{-5}$ the error from omitting this part in a simulation that goes over several steps may quickly accumulate quite grave errors. This calls for an investigation of more precise methods of determining the actual gravitomagnetic field.

A thorough analysis of possible methods to get trustworthy values for the gravitomagnetic effect in numerical simulations will probably be better suited in a larger work on numerical methods on galaxy models. Thus, I will here restrict myself to give some ideas that might improve the method used here. However, I will first give an argument that shows that such a reduction of fuzziness might actually be undesirable.

As previously mentioned I attribute the spikes and fuzziness in the graph to the choice of random points in the Monte Carlo method. If our model were perfectly integrated, I would not expect any such effects. On the other hand, real galaxies have not mass perfectly evenly distributed, and local velocity differences are a matter of fact. This opens up the possible interpretation that the randomness in the Monte Carlo distribution actually may work as a model of these imperfections. In this case the fuzziness of the results actually may be interpreted as a measure of how sensitive the strength of the gravitomagnetism is to local behaviour. It may seem like this model has the property that local differences from the perfect model may have a stronger influence on the local gravitomagnetic field than the influence of the galaxy as a whole. In order to find exactly how much of the gravitomagnetic effect is determined by local behaviour we need knowledge of how big such variations in mass and velocity inside galaxies typically are, and preferably have a more realistic galaxy model. This is clearly outside the scope of this thesis. However, for many body simulations this problem may be completely removed as it is then natural to simply calculate the total gravitomagnetic effect of all bodies on each body.

With the above paragraph in mind, if we still want to reduce the noise in the gravitomagnetism, how may we do it? One obvious way is low-pass filtering. I tried a few simple low-pass filtering solutions myself without much effect, but in theory it should eventually smoothen out the curve. Another way is to either remove or cap the results for sources that get closer to the point we calculate the field for than a certain limit. The cap-method is relatively easy to implement, but this raises the questions of where to set the limit and how much this artificial change on the system will affect the results. A third way is to use points in a symmetric grid as sources instead of randomly chosen points. The grid should be made so that locally the effect of the sources mostly cancels each other, while the grid points are evenly distributed. There is a faint possibility that the grid chosen might have severe effects on the result, but this should be simple enough to detect and avoid.

# Chapter 3

# Asymptotically Minkowski spaces

Many of the general relativistic systems that have been studied are of the asymptotically Minkowski type. I will devote this chapter to examine some of these in regards to rotation and from a Machian perspective. Asymptotically Minkowski spaces are characterized by having a metric that goes to the Minkowski metric in spatial infinity. In technical terms this can be stated as $g_{\mu\nu} \to \eta_{\mu\nu}$ as $s \to \infty$. Here $s$ is the interval between the point where the metric tensor is evaluated, and the points of interest in the model. $\eta_{\mu\nu}$ is a flat (Minkowski) metric. The examples studied in the previous chapter were also asymptotically Minkowski. But in that chapter the focus was on the effects of the linearized theory of gravitomagnetism. In this chapter we will still keep these effects in mind, but only as a reference. The focus will be the models that have asymptotically flatness as an important common feature.

## 3.1 Minkowski universe

The most obvious universe that is asymptotically Minkowski is the Minkowski universe itself. This space is characterized by having a flat space metric; in standard coordinates

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \tag{3.1}$$

As this has zero curvature everywhere we can conclude from Einstein's field equations that the energy-momentum tensor also must vanish everywhere. Hence we have a universe with no matter-content. Free particles move along linear paths

$$x^\alpha(\lambda) = x_0^\alpha + \lambda v^\alpha \tag{3.2}$$

where $x^\alpha$ is the coordinates traced out by the particle, $\lambda$ is a free parameter, and $x_0^\alpha$ and $v^\alpha$ are constants characterizing the state of the particle.

### 3.1.1 Rotating observer

I will now examine how this space time may seem from the perspective of a rotating observer. Assume that we have an observer that is translatory at rest in the origin of the standard coordinate system of the Minkowski space. This observer has a standard clock. This he uses to give a time label to all points in the space time in the following manner: He sends light signals that are reflected in an event. He records the time on his standard clock of emission and reception of the light, and defines the time of the reflecting event to be the arithmetic mean of these two values. Analysing this from the framework of the standard coordinate-system it is simple to see that the time label he sets on each point coincides with the time coordinate of the standard coordinate-system; $t' = t$. Thus, he has sliced up the space-time in slices that he through experiment can verify that is spatial and flat, as these properties are independent of the observer.

Then the observer turns his attention to an object he has nearby. It is 3 sticks connected together in a common end-point at rest. Studying it he finds several fascinating properties of it: In any time slice it turns out that the sticks are orthogonal to each other, and geodesic. And light sent from their common edge reflecting of the other edges of the sticks return to their common edge at the same time. Fascinated by this instrument, the observer doesn't dare to touch it. Knowing that flat space has Euclidean geometry he concludes that it is excellent for making a complete Cartesian coordinate system on his time slices. He defines each of the coordinate axes as the extension of the geodesics of each of the sticks, and unit length along each axis as the length of the corresponding stick. Armed with this coordinate system, he sets out to map the behaviour of free test particles in it.

It is easy to verify that it is possible that the coordinates he found with this method might actually be the standard coordinate-system. Just let the common edge trace out the parameterized line $(t, 0, 0, 0)$, the x-axis stick trace out $(t, 1, 0, 0)$ etc. Also if the sticks are simply transported, changed orientation or given a constant velocity the method will yield the same metric in the new coordinate system (from special relativity). If on the other hand the sticks are rotating rigidly we get a different result. We can from symmetry assume that it rotates around the z-axis. The origin-edge and the z-stick edge still trace out the same path as in the non-rotating situation.

The x-stick do however trace out $(t, \cos(\omega t), \sin(\omega t), 0)$ The y-axis trace out $(t, -\sin(\omega t), \cos(\omega t), 0)$. Due to invariance the geodesics that make out the axes of the new coordinate system are the geodesics in Minkowski space, and thus we get a linear correspondence between the coordinates in the new system and the standard system:

$$t' = t \tag{3.3}$$
$$z' = z \tag{3.4}$$
$$x' = \cos(\omega t)x + \sin(\omega t)y \tag{3.5}$$
$$y' = \cos(\omega t)y - \sin(\omega t)x \tag{3.6}$$

Inserting the equations 3.2 into these expressions gives nothing new for the $t'$ and the $z'$ coordinates. However, the movement in the x-y plane takes the following form:

$$x' = \cos(\omega t')(x_0 + t'v^x) + \sin(\omega t')(y_0 + t'v^y) \tag{3.7}$$
$$y' = \cos(\omega t')(y_0 + t'v^y) - \sin(\omega t')(x_0 + t'v^x) \tag{3.8}$$

Here I have assumed $v^t \neq 0$ and used the freedom of parameterization to set $\lambda = t = t'$. Differentiating these equations once with respect to $t'$ gives us the following new equations:

$$\dot{x}' = \omega y' + \sin(\omega t')v^y + \cos(\omega t')v^x \tag{3.9}$$
$$\dot{y}' = \omega x' - \sin(\omega t')v^x + \cos(\omega t')v^y \tag{3.10}$$

Here the dots denote derivatives with respect to $t'$ Repeating we find the following nice expressions for the accelerations:

$$\ddot{x}' = \omega^2 x' + 2\omega \dot{y}' \tag{3.11}$$
$$\ddot{y}' = \omega^2 y' - 2\omega \dot{x}' \tag{3.12}$$

Comparing these equations with 2.17 we find that in this situation we have

$$\vec{E_G} = \omega^2 \vec{r} \tag{3.13}$$
$$\vec{B_G} = 2\omega \vec{e_z} \tag{3.14}$$

Where $\vec{r} = (x', y', 0)$ and $\vec{e_z} = (0, 0, 1)$. So the situation is that the observer can see that the universe around him behave as if there is a gravitoelectric field pointing away from him that becomes stronger the farther out he comes, and a constant gravitomagnetic field. The puzzling thing is that he cannot see any source that could give rise to such fields. One may imagine that this observer does indeed find that there is a simple coordinate-transformation that gives nice linear paths, but consider this little more than a mathematical trick. Seeking an explanation for the behaviour of the particles in the preferred frame of the marvellous sticks a search for hitherto unobserved sources for the gravitomagnetic field commences. As the Minkowski universe is open the search may continue forever, never reaching infinity. And as it can never be confirmed observationally, who are we to claim that there indeed isn't anything out there?

In the perfect Minkowski model there is indeed no such source. Thus the above reasoning is an indication of why this is a much used example to show how general relativity does not fulfil Mach's principle. As mentioned in the introduction, these problems may have been instrumental in Einstein himself abandoning the idea. The only real defence of Minkowski universe as fulfilling Mach's principle I have found is given in [5] and seem to take advantage of a variant of the infinity-argument sketched at the end of the last paragraph.

One question that naturally arises is if there actually may be a matter source that might give the fields 3.13-3.14? If this is not the case, then a search for such would surely be in vain. There are a few obvious problems that stand in the way from finding such solutions. The calculation of those fields were exact, and holds for any $\omega$. We also expect any sources to have to be far away to not disturb the local observed flatness. Thus the weak field approximation will at best be able to give indications of what kind of distributions to look for. Still there might be one strong clue to work from: The only observed systems that are approximately Minkowski do have a source that might turn out to be able to explain their internal behaviour. The one of the reasons that Minkowski and asymptotically Minkowski systems are interesting to study is that this is a good approximation for space far from gravitational sources; at least *in our universe*.

So if our observer from above crawled through space, found a veil of galactic proportions, dragged it aside and saw a copy of our universe swirling around his precious sticks, would he then be able to rest with the mysteries of the strangely behaving free particles settled once and for all? This question is deeply related to Mach's principle, and I will not try to directly answer it. It will however be a question that may be good to have in mind while proceeding.

## 3.2 Inside a hollow shell

In this section I will study some models of a mass shell in the limit that it is infinitely thin. This model is relatively easy to analyse, and still gives some interesting results. The historical approach I will be taking is based on [42] and [45] unless otherwise noted.

This model was introduced by Einstein in 1912. At this time he used it on a scalar approach to gravity. From this he calculated the approximate behaviour of free particles inside a rigidly rotating shell. He repeated this exercise in 1913 within the Entwurf theory, a tensor-theory that preceded the final general theory. But the first known to have made such calculations within the framework of the final gravitational theory was Hans Thirring. This result was published in 1918. A translation of this paper may be found in [35], along with the other papers by Thirring mentioned in this section. A little later, he published a paper on the effects outside a rigidly rotating sphere with Joseph Lense.

Later, all effects related to rotating bodies similar to those described in the 1918 papers has been referred to as Lense-Thirring effects, even though they are qualitatively very alike the results of Einstein in 1912, and parts of their results have been outdated, as will be shown in this section.

### 3.2.1 Thirring

I will here go through Thirring's treatment of the hollow sphere. I will not include the lengthy expressions he got during the calculation. I will rather focus on the approximations he use, and his results.

Equations 3.15-3.27 are all quotes from his article. Thus I will give a few general remarks on the notation he uses that differs from the one I use in this thesis: He denotes the time parameter as $x_4$, not $x_0$, and uses the formalism where it is imaginary $x_4 = it$. He further uses $\gamma'_{\mu\nu}$ for what in 2.1 were written as $\bar{h}_{\mu\nu}$. For the gravitational constant $\kappa$ he uses $\chi$.

His starting point is the linearized theory. He uses the following relation that is a consequence of 2.6:

$$\gamma'_{\mu\nu} = -\frac{\chi}{2\pi} \int \frac{T_{\mu\nu}(x, y, z, t-r)}{R} dV_0 \tag{3.15}$$

Here $x$, $y$, and $z$ are the coordinates of a point on the sphere. $r$ is stated to be the distance between the point under consideration and the centre of the sphere, and $R$ is the distance between this point and the integration element. The integration goes over the volume of the sphere. According to my understanding of the system, and [56] the $r$ in 3.15 should actually have been $R$ to get the retarded potential right (this is assuming $t$ is the time at the point the perturbation is evaluated for). As the system in question is stationary, it is simple to see that this doesn't matter anyway, and could be a typo.

He then neglects any stresses and sets the energy momentum tensor to

$$T_{\mu\nu} = T^{\mu\nu} = \rho_0 \frac{dx_\mu}{ds}\frac{dx_\nu}{ds} = \rho_0 \frac{dx_\mu}{dx_4}\frac{dx_\nu}{dx_4}(\frac{dx_4}{dx})^2 \tag{3.16}$$

That is that of perfect-fluid dust of density $\rho_0$. It later turned out that neglecting the stresses in this way actually gives rise to an error of the magnitude the calculation is done in. I will say more about this later. The first equality in 3.16 is justified by the linear approximation, and that he is using the imaginary time formalism.

He then goes to polar coordinates $a$, $\vartheta$, $\varphi$, with $a$ being the radius of the mass shell. He uses the following expressions for the rigidly rotating mass-shell with angular velocity $\omega$:

$$\frac{dx_1}{dx_4} = -i\frac{dx}{dt} = ia\omega \sin\vartheta \sin\phi \tag{3.17}$$

$$\frac{dx_2}{dx_4} = -i\frac{dy}{dt} = -ia\omega \sin\vartheta \cos\phi \tag{3.18}$$

$$\frac{dx_3}{dx_4} = 0 \tag{3.19}$$

He now for simplicity considers the case where the coordinate-system is chosen so that the point under consideration is situated in the Z-X plane. He derives an expression for $R^2$ in polar coordinates. When justifying the use of the linearized theory he stated that the test-point should be close to the centre of the sphere. Now he uses this to justify dropping terms of higher than second order in an expansion of $\frac{1}{R}$ in terms of $\frac{r}{a}$.

Then he sets out to examine terms of type $(\frac{dx_4}{ds})^3$ as he had this in every integral he now had managed to reduce 3.16 to. In an errata he explains that this should actually be $-i(\frac{dx_4}{ds})^2$ as he made a mistake regarding what kind of volume element should be used in the integration. However, this doesn't change the approach. He makes liberal use of series expansions and the approximation that he would ignore terms of higher order than $\omega^2 a^2$. From this he also argues that he could use an unperturbed expression for the interval as starting point for his calculation. He thus gives the following equations:

$$ds^2 = -dx_1{}^2 - dx_2{}^2 - dx_3{}^2 - dx_4{}^2 \tag{3.20}$$

$$\frac{ds^2}{dx_4{}^2} = -1 - \frac{dx_1{}^2 + dx_2{}^2 + dx_2{}^2}{dx_4{}^2} \tag{3.21}$$

$$= -1 + \omega^2 a^2 \sin^2 \vartheta \tag{3.22}$$

$$\frac{ds}{dx_4} = i(1 - \frac{\omega^2 a^2}{2} \sin^2 \vartheta) \tag{3.23}$$

$$(\frac{dx_4}{ds}^3) = i(1 + \frac{3}{2}\omega^2 a^2 \sin^2 \vartheta) \tag{3.24}$$

When this is done, the rest is straight forward integration to get the perturbation of the metric. The result he generalizes to the case where the point is not in the X-Z-plane rotating the coordinate system around the z-axis and finds the transformed metric tensor.

Then he uses the equivalent of 2.13 in a similar way that we did. This includes ignoring terms of second order in velocity. In his initial paper he made as mentioned above an error with regard to $dx_4 ds$ factors. In that context, he also made a minor mistake regarding the definition of mass. After correcting for these, he arrived at the following equations of motion:

$$\ddot{x} = -\frac{8kM}{3a}\omega\dot{y} + \frac{4kM}{15a}\omega^2 x \tag{3.25}$$

$$\ddot{y} = +\frac{8kM}{3a}\omega\dot{x} + \frac{4kM}{15a}\omega^2 y \tag{3.26}$$

$$\ddot{z} = -\frac{8kM}{15a}\omega^2 z \tag{3.27}$$

where $M = \int \rho_0 dV_0$, $k = \chi/8\pi$ and dots represent time derivatives.

Comparing these with 3.11-3.12 we see that the gravitoelectric field in the x-y plane here is only one fifth of what one would expect if the system inside should behave like a rotating Minkowski space compared to the gravitomagnetic field. The z component also shows this difference very clearly. As such it is hard to use this as an argument in any strong formulation of Mach's principle. At the time it was however the first calculation to clearly show that rotating masses indeed produced Coriolis and centrifugal-like forces. As such effects seem to be necessary in order to describe rotational phenomena in a Machian way, and such effects do not exist in Newton's theory, it may be seen as a step toward an understanding that might be in accord with Mach's principle. Thirring gives the increased effective mass in the equatorial plane as a result of having higher speed than the poles, and the Minkowski background as possible reasons for his result not giving the ordinary centrifugal force.

If we now consider the behaviour of gyroscopes from 2.35 We get, using $c = 1$

$$\Omega = \frac{4kM}{3a}\omega \tag{3.28}$$

So we see that gyroscopes are dragged along with the mass shell with a frequency that increases by higher mass or smaller radius of the shell. Unfortunately the weak field approximation is only valid if $M/a$ is small. If $a$ approaches the Schwartzschild radius of the mass $R_s = 2kM$ we clearly get a strong field, comparable to that of a black hole.

As mentioned early on Thirring's calculation was flawed by neglecting stresses. This caused his energy-momentum tensor to not obey the law of local conservation of energy-momentum; $T^{\mu}_{\nu;\mu} = 0$ A calculation that took this into account was done by Honl in a paper from 1956 [24]. The end result is equivalent to 3.25-3.27 with the exception that the "gravitoelectric" force is only half as strong. So apart from this model turning out to be even further from the ideal of fully describing our relatively Minkowski surroundings, there are nothing really new in this.

### 3.2.2 Brill-Cohen

The next major step in the treatment of this model is attributed to a paper from 1966 by Brill and Cohen [12]. They managed to find a solution for the rotating shell without using the linear approximation. Thus it is also valid for strong fields like we have if the radius of the shell approaches the Schwarzschild radius. Unfortunately they had to sacrifice second-order terms in the angular velocity of the shell in order to get this result.

The main trick they did to get their result was, as far as I can see, to make the educated guess that the metric can be written in the form

$$ds^2 = \psi^4[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta(d\phi - \Omega(r)dt)^2] - V^2 dt^2 \tag{3.29}$$

with $\psi$, $\Omega$ and $V$ functions of r. Initially they had studied the case where $\Omega(r) = 0$ as a static base metric for this perturbation. The metric then has standard Schwarzschild-form. Thus they argued that outside the shell the parameters should have the form

$$\psi = 1 + \alpha/r \tag{3.30}$$

$$V = (r - \alpha)/(r + \alpha) \tag{3.31}$$

Inside the static spherical shell, space-time should be flat, so these functions should be constant there. Requiring continuity, these constants should have the value of the functions at the shell radius. This means that the equations above should hold with the variable $r$ replaced by the constant shell radius $r_0$. Using units so that $G = c = 1$, $\alpha$ usually is interpreted as $m/2$, but they also gave an explicit expression for it from the field equation for $T^{00}$:

$$\alpha = 2\pi \int_0^\infty T^{00} r^2 \psi^5 dr \tag{3.32}$$

This equation helps giving a stringent definition of $\alpha$, but is hard to use to calculate it as $\psi$ depends on $\alpha$ itself. Hence we will quickly get a fifth order equation if we tried. It will however be used later to define the mass $m = 2\alpha$.

These expressions for $\psi$, $V$ and $\alpha$ is kept in the perturbed case with non-zero $\Omega$. They argue that due to rotational symmetry they can always rotate the coordinate system so that nothing is changed beside $\Omega(r)' = \Omega(r) - \Omega_0$ So that they can set $\Omega(\infty) = 0$. They don't mention that this requires $\Omega$ to converge, but I believe this is uncontroversial given their background before the perturbation.

They make their calculations within the natural Cartan orthonormal frame one gets from the metric. They then get by calculating the components of the Einstein-tensor, and using them in the field equations, that $T^{ii}$ (i=1,2,3) is independent of $\Omega$.

They then argue that as the components $T^{i0}$ should vanish in the rest frame of the shell, the stress-energy tensor must be of the form

$$T^{\mu\nu} = \rho u^\mu u^\nu + \sum_{i,j=1}^{3} t^{ij} v^\mu_{(i)} v^\nu_{(j)} \tag{3.33}$$

where $u$ is the four velocity, and $v_i$ are three orthogonal four-vectors orthogonal to the velocity. They then make a choice of $v_i$ so that the system becomes pretty simple. Due to symmetries they are then able to argue that the $t^{ij}$ matrix is diagonal. They also get $t^{ii} = T^{ii}$ to the first order in $\omega_s - \Omega$, where $\omega_s$ is the angular velocity of the rigidly rotating shell. I would like to observe that as long as $\Omega$ is between zero and $\omega_s$ this condition is weaker than limiting $\omega_s$ to first order. When they restrict themselves to first order like this, they get only 4 non-zero components of the stress-energy tensor, whereas only $T^{03}$ depend on $\omega_s - \Omega$ at all.

They then focus on the field equation for $T^{03}$. They use an expression for the Einstein-tensor they get from the metric. First they solve the field equations with regard to $\Omega$ using $\psi$ and $V$ from the base metric for the vacuum cases inside and outside the shell. They then get the interesting result that the only regular solution inside the shell is that $\Omega$ is constant. They then set out to determine the integration constants that appeared in their vacuum solutions by demanding $\Omega$ to be continuous across the shell, and integrate the field equation across the shell. They here explicitly use the approximation of an infinitely thin shell by using that a term in the integral of the Einstein-tensor vanished (compared to the other terms) in this limit, hence simplifying the integral.

This way they get a solution both for the interior and for the exterior shell. Here, the interior solution is the one of interest. The interior solution they got was:

$$\Omega = \frac{\omega_s}{1 + [3(r_0 - \alpha)/4m(1 + \beta_0)]} \tag{3.34}$$

where $\beta_0 = \alpha/(2(r_0 - \alpha))$.

Interpreting $\Omega$ in the inside of the shell may be done like this: Consider the change of coordinates to a frame rigidly rotating with respect to the original with angular velocity $\Omega$. That will be the coordinate-transformation $\phi' = \phi - \Omega t$. As $\Omega$ is constant in the interior the derivatives of this set into the metric 3.29 will give us the standard form of the flat metric in polar coordinates. Thus, experiments done locally inside the shell will be unable to discern between this space and a "true" Minkowski universe fixed to this coordinate system.

There are two interesting limits to this equation. The first is $r_0 >> \alpha = m/2$. In this case 3.34 may be simplified to $\Omega \approx \omega_s(4m/3r_0)$. This is in perfect agreement with Thirring's result 3.28 (remember we set $G = k = 1$). On the other hand if we let $r_0 = \alpha$ we get perfect dragging $\Omega = \omega_s$. This was interpreted in the paper as if the radius of the shell approached the Schwarzschild-radius, the inside metric was somehow shielded from the Minkowski background at infinity. Nevertheless, they stress that such an interpretation may be naive as the asymptotically Minkowski boundary condition did enter their calculations. They also claimed that such a shell with radius equal to its Schwarzschild radius often had been taken as an idealized model of our universe, but they doesn't give any references to this. Anyway, this might lend hand to the suggestion that our local inertial systems indeed have to be non-rotational with respect to the fixed stars.

While they through a combination of metric-guessing and solving field equations from the mass-energy tensor managed to find a combination of metric and mass-energy that perfectly fits any choice of $r_0$ and $m$, they did so by sacrificing second order terms in angular velocity (or more precisely

$\omega_s - \Omega$, but this only might matter for $\alpha$ near $r_0$). Thus, this result is not fit to make any strong arguments regarding centrifugal forces, as we have seen that this is a second order effect of angular velocity.

In the same paper, they also presents some results connected to collapsing shell of dust. This is a somewhat more realistic model. However, I do not find anything specific in there of much interest to this thesis.

### 3.2.3 Pfister and Braun

In 1985 Pfister and Braun released a paper [43] where they further analysed the model of a rotating shell. Their main idea was to find the conditions where you have flat interior inside the shell. A motivation for this may be just the situation previously studied with the observer that locally finds his surroundings to be flat, but possibly rotating, space, and set out to find what kind of mass distribution that might explain this situation.

They use the following form for the metric of the rotationally symmetric system:

$$ds^2 = -e^{2U}dt^2 + e^{2U}[e^{2K}(dr^2 + r^2 d\theta^2) + W^2(d\phi - \omega A dt)^2] \tag{3.35}$$

In order to fix the inside of the metric to be flat they demand $U, K$ and $A$ to be constant, and $W = e^K r \sin\theta$. The Minkowski boundary condition they set even stricter, by demanding that $U, K$ and $A$ is zero, $W$ having the same form as inside.

Having stated these basic properties of the system they will examine, they go forth and state the quite complicated exact expressions of the field equations when calculating the Einstein tensors from 3.35. By linear combinations of these, they get two new equations so that they are able to solve the system in a cascading and recursive way like this: Assume that the equations are solved up to a certain order, and we want to find the solution to a higher order. First, they only consider the exterior vacuum solution. There is an expression for $W$ alone that can be solved to the order one is striving for. This result, along with lower order results for $A$ may be put into a second equation that can then be solved to the correct order for $U$. With these known $K$ and $A$ each has an equation that can now be solved to the correct order. Having the exterior solution the rest is an exercise in fixing integration constants by matching it with an interior solution so that the metric is continuous and the boundary between these regions have an energy momentum tensor that represents a rotating shell of mass $M$, radius $R$ and angular velocity $\omega$.

Using this technique to zeroth order in $\omega$, that is for a static shell, they got the standard Schwartzschild solution with flat interior as expected. To first order they reconstruct the result from Brill and Cohen. However, the new result is that they become able to extend the analysis to second order. In order to do this they use a result argued for in [20]. This is that to second order in rotation $U$, $K$, $W/\sin\theta$ and $A$ only has $P_0$ and $P_2$ terms when expanded in $P_l(\cos\theta)$ Where $P_l$ is the l-th Legendre polynomial. Connected with this is an observation that these variables only depend on even orders of $\omega$, this because of the symmetry across the equatorial plane. Seeing that $A$ is multiplied with $\omega$ in the metric 3.35, we can conclude that there will be no new correction to the metric in second order rising from $A$.

They then perform the integral procedure as described above, stopping before solving for $K$. Five constants of integration were introduced. One was eliminated by requiring that $U$ had to fall off faster than $r^{-1}$ as $r \to \infty$ in order to make sure the total mass of the shell doesn't change. Two are eliminated by a previously unused field-equation.

Then it turns out that there is not enough freedom in the system to be able to make a continuous connection between the inside and the outside metric. However, they find that if they allow the shell to not be perfectly spherical, but rather have a $\theta$ dependent radius it will be possible. From their knowledge of the system, they attempt the following radius:

$$r_S = R(1 + \omega^2 f \sin^2\theta) \tag{3.36}$$

where $f$ is a parameter describing how far from a sphere the shell has to be. They are now able to derive equations that gives the remaining integration constants in terms of $K$ and $f$ only from the continuity conditions. Also from the continuity conditions they are able to now generate inhomogeneous linear differential equations for $K_0$ and $f$. It all turns out to be interconnected in a quite complicated way, so they only give the expression for $f$ in the end.

Using the abbreviation $x = R/\alpha$ they have that the value of $f/R^2$ that allows flat interior solutions of the mass shell to second order in $\omega$ is uniquely determined by $x$ is (quote):

$$\begin{aligned} \frac{f}{R^2} &= -\frac{16(x+1)^4(2x-1)^2}{3x^4(3x-1)^2} \\ &\quad \times \left( \frac{2x + (x^2+1)\log[(x-1)/(x+1)]}{2x(x^2+1) + (x^4 + \frac{2}{3}x^2 + 1)\log[(x-1)/(x+1)]} - \frac{3(x^2+6x+1)}{32x^2} \right) \end{aligned}$$

($\alpha$ is still given by 3.32)

In a paper one year later [44] they extend their work to the third order. As mentioned above $U$, $K$, $W/\sin\theta$ and $A$ all only has even order terms in $\omega$. Hence from the form of the metric, only the second order term of the $A$ parameter will give a third order contribution to the metric. Thus they only have to solve the equation for $A$. This still is quite complicated, as they now must use the second order results of the other variables in order to get the correct third order result in the metric. They use some pages to list through the integration steps they have to use to get the expressions they use. When the time comes to match their solution with the energy-momentum tensor of a rotating shell they stumble upon the problem that there is a $\theta$ dependence in their expressions for $A$ that is not compatible with a rigidly rotating shell. They solve this by still demanding that the body at all points has a purely axial rotation, but they then argue that the angular velocity has to have the form

$$\bar{\omega} = \omega(1 + \omega^2 R^2 e \sin^2\theta) \tag{3.37}$$

where $e$ is a parameter determined by the radius and mass of the shell (not to be confused with Euler's number). Due to the complexity of the equations involved they only give the solution for $e$ in terms of derivatives of two other complicated functions they have gotten explicit solutions for earlier, so I won't quote it here.

Their conclusion is that there in general may not be possible to find a rigidly rotating shell keeping the interior flat with given mass, mean radius and angular velocity of the interior with respect to the asymptotic infinity. However, they argue that given the restrictions they have set, there is to all (finite) orders one unique solution that gives a Minkowski background. For each new order, they have to add corrections to the shell geometry and rotation to the order they are going for. They show the form of these corrections and counts up that the free parameters in these are just enough to allow the system to be solvable. To actually carry out this integration would be very hard due to appearances of terms involving quadrates of logarithms in the differential equations.

As they have given up spherical symmetry and rigid rotation, it may be in order to review the restrictions they have set on the system. They do not do it themselves, but as far as I can see the critical parts are the following: A surface dividing space-time in two parts with axial and equatorial symmetry, spherical in the limit $\omega \to 0$. All velocities are parallel to the equatorial plane, and it is a stationary system.

In an article of 1989 [41] Pfister does some examination on easing up these restrictions. I will not go into dept in this as he does not get any definite results, except that for a certain small deviation from spherical shell in the static limit, there are no first order solutions with flat interior.

So, how does these results relate to Mach's principle? Thinking back to our thought experiment in 3.1.1, we see that we now have found a class of simple models where the masses of the universe can give the impression that one locally exist in a Minkowski frame at rest, even though one from the Minkowski infinity observes the frame as rotating. This is an argument for questioning the notion of any absolute rotation. That this solution is unique also serves as a demonstration that the local inertial frames really do depend in a real way on the masses.

The limit where the radius of the mass shell goes to the Schwarzschild radius of the mass also turns out to be highly interesting. While the system constants $f$ and $e$ above generally is respectively negative and positive, both of these goes to zero in this limit. Remembering back to the first order result that the dragging coefficient went to 1 in this limit as well, we see that at least to third order the rigidly rotating sphere completely screens away the effects of the outside Minkowski limit. This screening is such that it is impossible for an observer inside it to determine how it rotates only by observing the inside of the shell and the shell itself. Even though it is then tempting to argue that the inside metric is completely determined by the mass-shell, it may be worth keeping in mind that we here hasn't seen on the possibility of other boundary conditions than the Minkowski at infinity.

### 3.2.4 Revisiting the rotating cylinder

I will here briefly revisit the case of a rotating infinite cylinder from 2.2.5. It turns out that the interior of such a system has to be flat. This result was found by Davies and Caplan in [14]. They started out with a general form for the metric in a stationary rotating system with axial symmetry found by Levy and Robinson in [30]. They implement the rotating cylinder by demanding that the solution should be z-independent, and that the interior is vacuum. They then solved the field equations in the inside, demanding that there should be no infinite parameters there. Finally they presented a coordinate transformation from the initial coordinates to a new coordinate system where the metric got the standard form of a flat space in cylindrical coordinates.

This result is exact. Comparing with the result in 2.2.5 we see that the linear approximation taken in that section clearly falls short in this case. The approximations taken there lead us to leave out the "centrifugal force" that is a second order effect in the angular velocity. This could be considered a gravitoelectric force, and our conclusion from that section that there is no such in the system fails in the exact case. Thus we see that the Maxwellian approximation is not suited for study of centrifugal-like effects, and reminds us to not put too much faith in zero-results found in that framework.

Unfortunately, it is hard to apply this result in any Machian argumentation. One reason is that it does not model our universe very well. Another thing to be aware of is that this is not asymptotically Minkowski. This is most clearly seen as the boundary of the rotating cylinder stretches out to infinity, hence breaking the vacuum necessary for having Minkowski solution. However, the necessarily infinite total mass of the cylinder in order for it not to have zero mass-(surface)-density also makes the behaviour at infinity in other directions problematic. These things also make it hard to compare with Pfisters aproach. Another thing that makes this result of limited value is that it only shows that the interior is flat, but not anything about how for instance its rotational state is with respect to the masses making out the cylinder.

## 3.3 Outside rotating bodies

We have previously seen on the situation inside rotating shells. Here I will concentrate on what is going out on the outside. This is of relatively little interest to the question of how the universe at large affects us, as we are inside the universe. However, it turns out that effects critical to common interpretations of Mach's principle is easiest to test in systems outside rotating bodies. Most importantly because the universe at large is very "well behaved" while we are close to a certain easy to access rotating body: Earth.

### 3.3.1 Approximate solutions

The case of a field outside a rotating body was investigated to some extent by de Sitter as early as 1916 in the linearized theory [15]. Lense and Thirring extended upon this in a paper from 1921 that is also translated in [35]. They did this in a similar way as Thirring had used in the inside of the mass shell, but only going to first order in the rotation. They also used this to calculate the magnitude of these effects for some of the bodies of the solar system. The most important effect they found was the effect of the rotating planets on the rotation axes of their moons.

However, I will here concentrate on the work of Schiff from 1960. His main results are presented briefly in [48], and a more detailed treatment was given in [47]. The reason to empathize this is that he arrived at a form of the effects that lend it neatly to laboratory experiments. This is also the approximation that has been used as the basis for the work on the recent gravity probe B satellite experiment that I will come to later.

Schiff's approach is based upon a paper of Papapetrou from 1951 [40]. In that paper a method for finding the equations of motion for a certain kind of test particles were presented. This is based upon the continuity relation $T^{\mu\nu}_{;\nu} = 0$ alone. The kind of test particles considered have the properties that they do not themselves change the metric. Further, it is assumed that they are limited to a thin time-like tube in space-time. In order to track the position of the particle, they use a line inside the tube with coordinates $X^\mu$ in a way so that the space coordinates $X^i$ could be regarded a function of either $X^0$ or the proper time $s$ along it. The main characteristic of the particle is that $\int T^{\mu\nu} dv$ and $\int (x^i - X^i) T^{\mu\nu} dv$ is non-zero. Here the integrals is over the space slices with constant coordinate time; that is over the points $x^i$. Integrals with higher order products of the distance differences are zero. These particles are thus termed di-poles. Single-poles have only $\int T^{\mu\nu} dv$ non-zero, while higher-poles have non-zero integrals with the distances to higher orders.

Now one can write the continuity equations in terms of partial derivatives and Christoffel symbols instead of covariant derivative, and restrict our attention to the time-derivative. Then insert the Taylor-expansion of the Christoffel symbols around $X^\mu$. Now by integrating the equations over the space, and keeping in mind that it is only a dipole as defined above, all higher than first order derivatives of the Christoffel symbols disappear from the equations. From this it was possible to find equations of motion fully specifying the state of the particle, with an exception of three degrees of freedom. However, these seem to be due to freedom in exactly where in the tube the $X^\mu$ line is chosen to be, and thus may be chosen away quite simply through physical arguments on the system.

The general equation is quite complicated compared with that of the single pole case that is simply the geodesic equation. This is because the spin of the dipole particle also appears as an important property of the particle in addition to its position and velocity. The spin is defined by the tensor:

$$S^{\mu\nu} = \int (x^\nu - X^\nu) T^{\mu 0} dv - \int (x^\mu - X^\mu) T^{\nu 0} dv \tag{3.38}$$

I will remark that $(x^0 - X^0)$ here is zero as the integrals is over the constant coordinate-time slices.

Schiff essentially took this result, and applied it to the Schwarzschild metric modified by the off-diagonal elements found by de Sitter and Lense and Thirring for the outside of a rotating body in linear approximation. In order to get a nicely interpretable result he also made the following important and non-trivial coordinate transformation: Assume the test particle is a gyroscope moving around a rotating body. A perfect gyroscope will be an example of such a dipole particle. Then create the coordinate system of an observer that is moving with the gyroscope made by standard measuring rods at his position, but where the orientation of the axes still are parallel to those of the standard Schwarzschild Cartesian coordinates used when considering the system from the point of view of the central mass.

To simplify this coordinate transformation he takes advantage of the approximation that the distance to the massive object creating the field is large compared with it's Schwarzschild radius, so that he may work to first order in $m/r$. He also assumes that the ordinary space-velocity $v$ of the test particle relative to the central body is low compared to the speed of light, thus only working to second order in $v$. In this new frame, clearly the particle is not moving. It is also natural to let the points $X^\mu$ be so that $X^i$ trace out the space location of the centre of mass of the gyroscope. This condition completes the equations of motion. Given the symmetries of the system, it is simple to see from the definition that the components involving time of the spin tensor disappear. The only non-zero components then correspond to the classical spin vector in the following way:

$$\vec{S} = (S^{23}, S^{31}, S^{12}) \tag{3.39}$$

Thus he arrives at the following equations of motion:

$$(d\vec{S}/dt) = \vec{\Omega} \times \vec{S} \tag{3.40}$$

$$\vec{\Omega} = (3m/2r^3)(\vec{r} \times \vec{v}) + (I/r^3)[(3\vec{r}/r^2)(\vec{\omega} \cdot \vec{r}) - \vec{\omega}] \tag{3.41}$$

Here all vectors except $\vec{S}$ is as measured in the standard isotropic Schwarzschild coordinate frame with the source in the origin. $\vec{r}$ is made of the space-like components of the position coordinates, and $r$ is as usual the length of this. $\vec{v}$ is the space-part of the four-velocity of the test particle and $\vec{\omega}$ is the angular velocity of the central body (usually taken to be along the z-axis). $m$ is the mass of the central body. $I$ is the moment of inertia of the central body, for instance a homogenous sphere with radius $R$ $I = 2mR^2/5$. These equations also use the convention that sets the speed of light and the gravitational constant to unity.

Interpreting the first of these equations is quite simple, but still yields an important result: As the time derivative of the spin has to be orthogonal to the spin, so its magnitude will not change. This assures that the gyroscope is well behaved in a certain way, so that it may actually be used as a standard clock. The first term in the second equation is independent of the rotation of the central body. It is claimed that it can arise from an extension of the special relativity theory only incorporating the equivalence principle. It also is most usually found by a second order approximation of the theory. The second term on the other hand is a pure general relativistic effect. It clearly shows how the spin axis of the gyroscope is affected by the rotation of the central body.

Also pay attention to the fact that the last term is identical to the standard equation for a magnetic dipole with dipole momentum along $\vec{\omega}$. This clearly shows the relation between the Maxwellian analogy treated previously and this case. Another potential analogy is that this term shows that rotating masses somehow drag all other free particles along in their rotation. Thus, also the inertial frames are dragged in a certain way. This is most easily seen in the case above the poles. Here the term will become $2I\vec{\omega}/r^3$, thus dragging all gyroscopes in the same direction as the rotating body. On the other hand above equator it will become $-I\vec{\omega}/r^3$ thus making the inertial frames spanned out by the gyroscopes rotating in the opposite direction as the planet. This might at first glance seem to oppose the idea that frames are dragged along with the central body. This concern is addressed by pointing out that this is due to the reduction of the effect, as the distances grow larger.

A major part of the paper is also devoted to comparing two approaches to choose the three free variables of the system, and examining how such test particles behave when influenced by non-gravitational forces. This last would be important if attempting a laboratory experiment on the earth surface, as the forces keeping the experiment on the surface would have to be taken into account. The primary change found was that the equation 3.41 would have to be corrected by a term of $1/2(\vec{f} \times \vec{v})$. This result is of little interest for the current thesis so I will not delve further into this.

I will return to this equation when I come to the specific case of the gravity probe B experiment. More detailed approximations are made, for instance as a side effect by the calculations of Cohen and Brill, and by Pfister and Braun as presented earlier. None of these pays much attention to the external solution, and they are of limited value in this connection. This because the approximation given here is as good as one can test with today's technology, and that it is of little theoretical interest as one actually have an exact solution for this system as we are about to show now.

### 3.3.2 The Kerr metric

In 1963, Kerr presented a paper [26] in which he described a metric he had derived from certain mathematical properties. One of these was that it had to be a vacuum everywhere, except at any singularities. This original formulation is now mostly of historical value, as it has later been found formulations that make this easier to interpret correctly. Even though it was found only through mathematical considerations, he could see from the form that it probably could be the exterior solution of a rotating object.

A quite popular representation for the Kerr metric is called Boyer-Lindquist coordinates. This is named after Boyer and Lindquist who in a paper from 1967 [9] presented it as a "Schwarzschild like" form of the Kerr metric. That this form got to bear their name seem somewhat strange, as it was in their paper only a middle step for what they considered their main result of that paper. However, it has a few neat properties.

The metric is given as

$$ds^2 = \Sigma(dr^2/\Delta + d\theta^2) + (r^2 + a^2)\sin^2\theta d\phi^2 - dt^2 + 2mr/\Sigma(a\sin^2\theta d\phi + dt)^2 \tag{3.42}$$

where

$$\Delta = r^2 - 2mr + a^2 \tag{3.43}$$

$$\Sigma = r^2 + a^2\cos^2\theta \tag{3.44}$$

$a$ and $m$ is free parameters in the mathematical problem whose physical interpretation turns out to coincide with that of the rotation and mass properties of a central object.

One important property of this coordinate system is that it becomes the standard Schwarzschild coordinates when we set $a = 0$, and in this case it is easy to see that m represents the standard mass of the object. Lower order approximations of this solution also exhibits that the parameter $a$ makes the metric behave like the Thirring system where $a$ is corresponding to the angular momentum per unit mass along the $\theta = 0$ axis.

Further physical interpretation turns out to be quite complicated. While the Boyer-Lindquist coordinates have some common features with Schwarzschild and lower approximations of rotating bodies, it does not present any obvious way to build up the coordinate system from physical experiments. With this in mind, I will mention one property with the Kerr metric in Boyer-Lindquist coordinates that is often mentioned in that connection, and that can be related to frame dragging. This is partially following the treatment on the Kerr metric in the book by Grøn and Hervik [56]. Observe that

$$g_{tt} = 2mr/\Sigma - 1 \tag{3.45}$$

We find that this quantity becomes positive if

$$r^2 + a^2 \cos^2\theta - 2mr < 0 \tag{3.46}$$

Observe that the surface $\Delta = 0$ clearly has to be inside this region of space from $r^2 + a^2\cos^2\theta - 2mr = \Delta - a^2(1 - \cos^2\theta) <= 0$. The $\Delta = 0$ surface is significant as this gives an infinite $g_{rr}$ and thus plays the same role as the event horizon in the Schwarzschild metric. The $g_{tt} > 0$ region is however interesting as this marks the area where physical particles moving along timelike $ds < 0$ paths can have constant $r$, $\phi$ and $\theta$ coordinates. Examining the metric we find that the only way to get a negative interval is to have $d\phi$ negative. Thus one might say that this region plays the same role for frame dragging as the area inside the event horizon plays for ordinary gravitation. The area with this extreme dragging, outside the $\Delta = 0$ boundary is named the ergosphere.

Another way to see the effect of frame dragging in this coordinate system is to examine the path of a free particle initially at rest far from the source. This is relatively simple to analyze using Lagrangian formalism, but as this is somehow outside the scope of this thesis, I will not go into the details. The main idea is that the Lagrangian for the system becomes independent of $\phi$, hence there is a corresponding constant of motion $p_\phi$. One finds that a non-moving particle far from the source has approximately $p_\phi = 0$. This gives us the following result for the angular velocity of the particle in the coordinate system:

$$\frac{d\phi}{dt} = \frac{a(r^2 + a^2 - \Delta)}{(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta} \tag{3.47}$$

This isn't an easy expression, but inserting for $\Delta$ it is actually quite easy to see that if we ignore all but the first order products of $a$ we arrive at

$$\frac{d\phi}{dt} \approx \frac{2ma}{r^3} \tag{3.48}$$

Thus showing that for at least small angular velocities, we have a clear tendency that the particle is dragged along in the same direction as the central body is spinning, and that this effect becomes stronger as one get closer. I would like to mention that this approximation also demands that $r$ is large compared to a, as else one cannot justify keeping the $r^3$ term, and not higher order $a$ terms. The exact solution is not very complicated, but gives us little new qualitative information, except that the dragging effect is relatively somewhat weaker for high $a$.

The $\Delta = 0$ limit is also easy to calculate, but hard to interpret:

$$\frac{d\phi}{dt} = \frac{a}{r^2 + a^2} = \frac{a}{2mr} = \frac{a}{2m(m + \sqrt{m^2 - a^2})} \tag{3.49}$$

I have been unable to find any coordinate independent interpretation of these dragging-effects found here. Nevertheless, from the simple form of the metric it seems unlikely that one can be able to get completely rid of it. The Kerr-metric and its generalisations have been subject to much research, and have many interesting properties. However, I believe I have now briefly covered those results that are most interesting in regard to Machian rotation effects.

### 3.3.3 Gravity probe B

, This section is mainly based upon the NASA final report of the Gravity Probe B experiment [2]. Gravity probe B is a satellite experiment that has been under development at Stanford University since the 1960s. On 20 April 2004 the satellite were finally launched, and it produced data until 29th September 2005. However, the data-analysis has proven quite complicated, and it is still not completed. The results I base this section on were presented in the context of NASA no longer providing funds for the project.

The theoretical foundation for the experiment is the approximation found by Schiff as presented in 3.3.1. The idea was to send a satellite in an orbit over the poles with gyroscopes initially pointing toward a suitable heavenly body. This body should be so that when the satellite is over the equator the direction to that body from the gyroscope is either away of through the centre of the earth. This setup has several nice qualities. Looking back to the equation for the precession of gyroscopes 3.41 we see that in this case the first term in the equation will always be orthogonal to the plane the satellite move in, thus giving a pure North-South precession. This contribution to the precession will also thus be orthogonal to the spin-direction of the gyroscope, hence giving a maximal total displacement. The second term might be a bit more difficult, but integrating around the entire orbit it becomes clear from the symmetry that the direction of that precision contribution will sum up to be along the axis of rotation of the Earth. Hence, this precession will deflect the gyroscope in a purely East-West direction. Again, this deflection is as large as it can, as the precession is orthogonal to the spin.

Even though the experiment then in principle is easy, there were many practical and technical difficulties connected to doing this experiment. One thing is to find sufficiently accurate orbit information. Factors such as how oblate the Earth is had to be taken into account. This was necessary in order to obtain the right values for the position vector needed in the formulas. It was also necessary for some calibrating issues.

Finding a suitable heavenly body to use for reference was also important. It had to have a known, small velocity relative to the background of distant bodies, while being sufficiently strong to be possible to be tracked easily and be discerned from the surroundings. In addition comes the above-mentioned location requirement that it had to be above the equator. The choice fell on the star IM Pegasi.

For the required precision of this experiment, the telescope required for tracking IM Pegasi on board the satellite also had to severely push technological limits.

However, the requirements for the gyroscopes could almost be considered science fiction. In order for the drift rate of these to be as low as required there were several technical difficulties to overcome. One thing is that it needs to be almost perfectly spherical. However, it also needed to be very homogenous. This was in order to make sure the geometric and mass centre was as close as possible to each other. Even in space, external forces like for instance radiation pressure could have made making a sufficiently homogenous sphere all but impossible. Only by applying motor boosters to the satellite compensating for this drift was a sufficiently homogeneity within reach. In addition, they had to use superconducting coating and advanced coolers in order to be able to make measurements on the spin of the sphere. Magnetic shielding, being able to spin the gyroscope up and avoiding possible change of shape of the apparatus over time were also major concerns, all which were intimately connected by keeping it all cold. The final satellite was comprised of four gyroscopes, two rotating in one direction, and the other two in the opposite direction, thus doing the same experiment more or less independently of each other.

After the data were collected, one major problem showed up that drastically complicated the analysis of data. Simply put it turned out that electrical effects connected to the crystals of the material the spherical gyroscopes and the chasings they were made of were large enough to cause significant Newtonian disturbances to their data. These disturbances entered through the set up of a crucial calibration scheme, an associated torque, and another unforeseen resonance effect with the rotation of the satellite casing. The last essentially sometimes made the spin directions the gyroscopes to make a jump over some days independent of the others.

Fortunately, even though it was not intended, they actually had obtained data that could be used to map the critical electrical distribution inside the gyroscopes. Through this, they were able to drastically reduce the scattering of the results. A continuously greater understanding of the resonance effect also helped tremendously.

At the end of 2008, the main limitation on the results was that of computation power. Their results were based upon analysis of means over daily data, while they are striving for high-speed computational methods allowing analysis of data of every 2 seconds.

For the North-South direction, they calculated that the drift due to the movement around the earth would be 6606 milliarcseconds per year (marc-s/yr). In addition to the effect of the Earth, the effect of the motion around the sun corresponding to the first term in 3.41 and the effect of the motion through space of the star had to be taken into account when calculating the theoretical result of the experiment. Thus they arrived at a theoretical drift of 6571$\pm$1 marc-s/yr. Combining the result from all four gyroscopes they arrived at a drift of 6550$\pm$14.0 marc-s/yr. This they consider a very good confirmation to that effect.

The East-West effect of the rotation of the earth was calculated to be just 39 marc-s/yr. That is considerably less than that of the first effect, and explains why the need for such high precision on the experiment. Taking into account the other significant factors the expected measurement ended up to be 75$\pm$1marc-s/yr. The combined measurements yielded a result of 69.1$\pm$5.8 marc-s/yr. They stress however that these results is without systematic error or model sensitivity analysis included. Therefore, even though the theoretical result is outside their estimated error, they state that they consider the frame dragging effect to be confirmed with only 15% uncertainty. This may be intuitively justified by observing that the measured drift is closer to that of the theoretical result with frame dragging than that without.

So why is this experiment of interest to this thesis? This experiment stands as the best experimental confirmation of the effect that seem to be the main basis for arguments tying Mach's principle to general relativity, namely gravitomagnetism/frame dragging. Without such an effect it is hard to see how one may argue that far objects directly affect local systems. Even with this effect it is still not obvious that it is possible to find any relation that connects general relativity with any strong formulation of Mach's principle, but the possibility seem to be there.

# Chapter 4

# Universe models

Mach's principle concerns bodies far away. As such it makes sense that attempting to restrict attention to a small portion of the universe as is usually assumed in the asymptotically Minkowski case won't give us the full picture. All masses in the universe may play a role. Therefore, the need to turn to the field of cosmology in order to examine this fully seems to be evident. As this is a potentially huge subject, I will restrict my attention to two important ways cosmology has been seen in connection with Mach's principle. First I will present a recent result. This shows that the universe models that is most used for our universe - Friedmann-Robertson-Walker (FRW) universes, do have a very important "Machian" property. Secondly I will present a couple of universe models that I have often seen referred to as exploiting a lack of Machianity in general relativity, and some ideas as how one might understand them without having to let go of Mach's principle.

## 4.1 FRW/Schmid

In this section, I will present a recent result that can be considered quite important from a Machian point of view. It was found by Christoph Schmid, and is presented in detail for flat universes in [49], and expanded to curved universes in [50]. It states that for a linear perturbation of a FRW universe the orientation of the inertial frames is exactly dragged by a weighted mean of the rotation of the masses around them. Said in a different way it tells us that the rotational states of inertial frames are perfectly determined in a relatively simple way by the state of the universe. A compact and structured presentation of the path to the result is already available as notes from the proceedings of a presentation held in 43rd Rencontres de Moriond [51]. Thus, I will focus on the theoretical foundations not presented there.

### 4.1.1 FRW universes

The Friedmann-Robertson-Walker metric is named after three scientists who independently found important properties of the metric. Sometimes the name Lemaître is also included, and sometimes some of the names are excluded. The historical reasons for this lack of any strong naming-convention may be read from for instance Gravitation [36]. It turns out that Friedmann was the first to discover the metric in 1922, but it was independently discovered by Lemaître in 1927. It was however first when Robertson and Walker independently found that these universes are the only spatially homogenous and isotropic universes in general relativity in 1935 that the model got a real breakthrough. The assumptions that the universe at large can be sliced into spatial hypersurfaces so that where you are on it won't affect the observations (spatial homogeneity), and that you observe essentially the same whatever direction you observe in (isotropy) fits so well to our universe that they has been named the cosmological principles. Thus FRW universes are often one of the first universe models encountered in textbooks on cosmology, for instance that by Grøn and Hervik [56].

The metric of this model has a quite simple form. In Robertson-Walker form it becomes:

$$ds^2 = -dt^2 + a(t)^2\left(\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi)\right) \tag{4.1}$$

Here $K$ is a true constant determining the geometry of the space. It can be scaled by coordinate transformations, but never made to change sign. This coordinate transformation is essentially to draw the absolute value of $K$ into $a$. By that reason in theoretical applications a dimensionless parameter $k$ is introduced and is set to be $\pm 1$ or 0, each of these cases representing quite different geometries. If $K > 0$ the universe is said to be closed, and $k = 1$. If $K = 0$ it is flat and $k = 0$. Finally, if $K < 0$ it is open and $k = -1$. $a(t)$ is a time-dependent scale factor. Both $K$ and $a$ is to be determined by the matter-distribution through Einstein's field equations.

Some words on notation. Schmid uses $K$ in the same way as I here use $k$. I will stay with the standard notation. This avoids confusion with $K$ needing to have dimension that cancels $r^2$. If $r$ is fixed to be dimensionless by coordinate choice, $k$ might have substituted $K$. However, Schmid later will use $k$ for certain eigenvalues. I will here adopt the more common notation $\lambda$ for these eigenvalues.

Another common form for the metric that will prove useful in the sections to come are the following:

$$ds = -dt + a(t)^2(d\chi^2 + R(\chi)^2(d\theta^2 + sin^2\theta d\phi^2)) \tag{4.2}$$

Where $R(\chi) = \sin(\chi)$ if $k = 1$, $R(\chi) = \chi$ if $k = 0$ and $R(\chi) = \sinh(\chi)$ if $k = -1$.

An important quantity often met in the treatment of FRW-universes is the Hubble parameter defined by $H = \dot{a}/a$ where $\dot{a} = da/dt$.

It may also be interesting to note that if one set $k = 0$ and $a = 1$ one get the standard Minkowski metric. This shows that Minkowski is a special case of FRW. The result of Schmid will turn out to open up for an interesting interpretation of this that I strangely enough have not seen mentioned before.

It may also be worth to mention that these universes have the property that gyroscopes follow the matter flow, always pointing along the same line of matter. This may be intuitively confirmed from the isotropy condition, as any precession of the gyroscope relatively to the matter around it would make the precession axis stand out as a "favoured" direction.

### 4.1.2 Linear perturbation on FRW

Perturbation theory is the theory of what happens if you take a system and make small changes to it. The initial system is called the metric and is usually chosen in a way so that it has particularly simple or desirable properties. The new, changed system is called the perturbed system. As our universe seems to be well described as something not far from perfectly spatially homogenous and isotropic it makes sense to use FRW universe as a background when studying our universe.

One particular problem when it comes to all perturbations is the one of gauge freedom. A gauge is a relation that tells us what point in the perturbed system corresponds to what point in the unperturbed. To illustrate this, imagine a circle in the Euclidean plane. This plane is covered by a standard polar coordinate system with the origin in the centre of the circle. Use this as the background system. Then make a slight change/perturbation of it so that instead of being a circle we have an ellipse. Where should we put the origin, corresponding to the centre of the circle in this new system? Both foci, and the centre of those, present themselves as possibilities. Therefore, it is possible to introduce a standard coordinate system having any of these as centres mapping points in the background to the ellipse. In addition, one might want to keep a correspondence between the circle and elliptic curve. In that case, a kind of polar coordinate system where the radius-coordinate is constant for all points in the ellipse could be introduced. Knowing the metric of this coordinate system, it would still be evident that the perturbed system represented an ellipse. This freedom in mapping is referred to as gauge freedom. Thus, clearly defining gauges or working with quantities that is gauge invariant quantities is important, and I will come back to that issue later.

For the FRW-background there is an important result presented for instance by Kodoma and Sasaki in [28]. To explain it I want to introduce the notion of "pure" scalar, vector and tensor fields. Any scalar field is automatically "pure". A vector field may be decomposed into a scalar and a pure vector field where the purity of the vector field is defined by it being divergenceless. Similarly, any (symmetric) second rank tensor quantity may be decomposed into pure tensor, vector and scalar fields, where the pure tensor field is both traceless and divergenceless.

Consider an equation involving scalar, vector and/or tensor fields defined on the hypersurfaces of homogeneity in the FRW-universe with the following properties:

- It is covariant with respect to coordinate transformations in the hypersurface
- It is linear in unknown geometrical quantities
- If it is a differential equation it is at most of second order

It turns out that such an equation can then be decomposed into a group of equations where each only contains pure scalars, vector or tensor fields. The linear approximation to Einstein's field equations with FRW-background has these properties. Thus, the effect of any small perturbation may have its effects analysed independently in the scalar, vector and tensor sector. The pure scalar-field part of the perturbation is sometimes also referred to as irrotational or density perturbation. The pure vector part is sometimes referred to as rotational or vorticity perturbations. The pure tensor part is sometimes referred to as gravitational wave perturbation.

The possibility to make such decomposition is critical for the approach made by Schmid. Thus, it could be interesting to examine if it can be done for other universes than FRW as well. After the proof of this result, Kodoma and Sasaki stress that the background hypersurface having constant curvature is a critical part of the proof. One possible defining property of constant curvature is that

$$R_{\alpha\beta\gamma\delta} = K(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}) \tag{4.3}$$

where $K$ is constant. This poses a potentially severe restriction on the models for which this method may be used. FRW universes are the only universe models I have found to have been used as an example for this.

### 4.1.3 Eigenfields of Laplacian

In order to carry out the integrals required to arrive at his results, Schmid found a certain set of eigenfunctions for a Laplace operator. The Laplace operator in question Schmid refers to as the de Rham-Hodge Laplacian ($\Delta$) in order to separate it from what he refers to as the rough Laplacian. Strangely, I have found no standard naming convention for these, so I will stick with his terminology. The rough Laplacian is defined by $\nabla^a\nabla_a$. The de Rham-Hodge Laplacian is originally only defined on differential forms, but by going to the corresponding vector where necessary it may make sense to use it on vector fields as well. Instead of giving the full definition that would require a degree of mathematics than I don't want to assume in this thesis, I will simply state the two main properties that was necessary for Schmid to arrive at his final result: For scalar fields the rough and de Rham-Hodge Laplacian is equivalent, and for divergenceless (pure) vector fields $\Delta\vec{A} = curl(curl(\vec{A}))$.

He starts out by investigating the scalar eigenfields of the Laplacian. As the background is spherically symmetric, he can separate those into a radial and an angular part. The angular eigenfunctions of the laplacian is a set of well-known functions known as spherical harmonics. The standard notation for these are $Y_lm(\theta, \phi)$ where $l$ and $m$ are integers characterizing the function.

He then solved the radial part in terms of the coordinate system described by the metric 4.2. By demanding it to be regular in the origin, he arrived at the following function:

$$\tilde{J}_q^{(k)}l(\chi) = R^l(-\frac{1}{R}\frac{d}{qd\chi})^l(\frac{\sin q\chi}{qR}) \tag{4.4}$$

Remember that $R$ is a certain function of $\chi$. This is not marked explicit in this formula to avoid confusion with the parenthesises for the terms to be the multiplied. $l$ is the same as for the spherical harmonics, showing what radial functions can be used together with what angular eigenfunctions. In order to make a cleaner notation $q$ was introduced and is defined as $q^2 = \lambda^2 + k$. Here $\lambda$ is the eigenvalue corresponding to the radial function, and $k$ is the usual curvature-parameter. As mentioned earlier this notation differs from that used by Schmid. Instead of *lambda* he uses $k$, and he uses $K$ for what I here note as $k$.

$\tilde{J}$ is then recognized as a generalisation of another well-known family of functions: the Bessel functions. From knowledge of these he are quickly able to determine the eigenfunctions that is not regular at $\chi = 0$ as well. I would like to remark that the form of the posible functions $R$ was important in deriving the relatively simple expression 4.4. Thus expanding this result to other universes than FRW-universes may be problematic.

Then we may turn our attention to the vector fields. As there may be more than one vector field with a certain eigenvalue there may exist bases of vector fields that one can construct all other eigenvector fields from. There is a certain degree of freedom associated with the choice of this basis. This motivates trying to find vector fields that can be used as basis elements with particularly nice properties.

Schmid chooses to examine the following set of sets of fields spanning the three dimensions:

$$\vec{X}^{+}_{lm} = R\vec{\nabla}Y_{lm} \tag{4.5}$$

$$\vec{X}^{-}_{lm} = \vec{e}_{\chi} \times \vec{X}^{+}_{lm} \tag{4.6}$$

$$\vec{e}_{\chi}Y_{lm} \tag{4.7}$$

These fields have some quite nice properties. All of them are eigenfunctions of the total angular momentum operators $J^2$ and $J_z$ with values $l(l+1)$ and $m$ (we will later see why this is a good thing). If one examines the sign of the fields on changing sign on all coordinates one find the parity. The parity of all $X^-$ is $P = (-1)^{l+1}$. For all the other fields the party is $P = (-1)^l$. Finally, they are all orthogonal to surfaces with constant radius. This last property tells us that all vector fields can be decomposed uniquely into a sum of these fields at each shell of constant radius (that is, if these fields make a complete set, which I believe follows directly from them spanning three dimensions and $Y_{lm}$ being complete).

There is also the freedom of multiplying these fields by certain scalar radial functions. This is explored to some extent. Divergenceless fields are also constructed this way, except for the $X^+$ fields. In order to get these divergenceless a field of the $Y_{lm}e_{\chi}$ had to be added. However there are then still no mixing with the $X^-$ elements. Finally, it is shown that the following fields are eigenfields of the de Rham-Hodge Laplacian:

$$\tilde{J}^{(k)}_{ql}(\chi)\vec{X}^{-}_{lm}(\theta,\phi) \tag{4.8}$$

Here $\tilde{J}$ is the generalized Bessel function as before. The eigenvalue in this case is exactly $-q^2$. The other possible eigenfields with $X^i$ as angular part may be gotten by changing $\tilde{J}$ with one of the other previously mentioned scalar radial eigenfunctions of the Laplacian. It turns out that the eigenfields involving the other basis fields are of no interest in this context.

### 4.1.4 Perfect dragging in perturbed FRW

Now we are finally ready to have a brief look on the physics around the result that indicates perfect dragging in FRW-universes. Take a FRW-background with standard Robertson-Walker coordinates ($r$ as radius, not $\chi$). Then apply a pure vector perturbation on it, keeping the universe at infinity unperturbed. A result from perturbation theory is that in this case we may keep our old time coordinate without any gauge problems. This is because changing the time-structure between hypersurfaces of homogeneity would require scalar perturbation. Another interesting result is that the intrinsic geometry of each slice of constant time remains unchanged by the perturbation. It is then possible to chose a gauge so that the perturbed universe is covered by a coordinate-system in a way so that the metric is identical to the background metric, with the exception of the components $g_{0i} = \beta_i$. And as the perturbation is purely vector, the vector field $\vec{\beta}$ must be divergenceless.

In this universe, consider the following setup: At each point is an observer moving so that his coordinates remain constant. Each observer has constructed a local orthonormal frame. These frames have their orientation fixed so that they are part of geodesics between the observer and constant heavenly bodies at infinity at the same time coordinate. As the intrinsic geometry of the surfaces is unchanged and there is no perturbation at infinity, these directions are well-defined, as they are well-defined in the background. Each of these observers has a set of gyroscopes. By observing the movement and precession of these, they are able to observationally define gravitomagnetic and electric fields in their orthonormal frame.

Now the question we would like to ask is, how do the gyroscopes precess? The precession of the gyroscopes defines the orientation of the local inertial frames. We want to examine how this orientation is affected by the flow of the masses of the universe. Thus, we are only interested in the gravitomagnetic field.

This turn out to be very similar to what was done earlier in this thesis for the Minkowski background. And as one might expect also in this case one actually gets the equivalent of 2.15-2.16 with $\bar{h}_{00}$ constant, and $\bar{h}_{0i} = \beta_i$.

The field equations do become similar as well, but with a very important difference.

Schmid is using Cartan's equations to arrive at the field equations for the perturbed metric. This involves working on locally orthonormal frames as opposed to the coordinate frames. He keeps it to first order in $\vec{\beta}$. In this case he can restrict attention to the $\hat{0}\hat{i}$ components of the equations. Here $\hat{\alpha}$ is used to empathise that we are working with the components in an orthonormal frame, not those in the coordinate basis. The equations become:

$$(-\delta + \mu^2)\vec{\beta} = -16\pi G\vec{J}_\epsilon \tag{4.9}$$

where G is Newton's gravitational constant, and $\vec{J}_\epsilon$ is given by the components $J_\epsilon^{\hat{i}} = T^{\hat{0}\hat{i}}$. That the energy-momentum tensor is given in an orthonormal frame is important as that means that this quantity can be measured by local observers without any knowledge of any overall coordinate metric. Apart from this, it is identical to the field equation we had for Minkowski-perturbation with constant $g_0 0$, with the exception of the $\mu^2$ term. $\mu$ is defined by $(\mu/2)^2 = -(dH/dt)$. We can confirm that this term disappears as one could expect in the non-expanding Minkowski case.

Now we are only interested in the precession of a gyroscope at one point, let that be the centre of our coordinate system. Now the precession turns out to be a rotation that has to have total angular momentum and parity given by $J^P = 1^+$. The only vector fields of those presented in equations 4.5 - 4.7 is actually $\vec{X}_{1m}^-$; For these eigenfunctions of $J$ $l$ has to be 1, and we saw that in this case $\vec{X}^-$ was the only one that could have positive parity.

We actually get the huge simplification that the only components we need to be concerned about of $\vec{A}_g$ are those that are products between radial scalar functions and $X_{1m}^-$. It even follows from angular momentum and parity properties of the rotation that neither scalar nor tensor perturbation can affect it, as none of those can generate the right kind of field. Thus, the restriction to vector perturbations turn out to be no real restriction at all.

Turning our attention back to 4.9, we se that if the right hand side is zero it actually becomes an eigenfield-equation for the Laplace-operator. And we know the eigenfields for the Laplace-operation for fields of the form $X_{1m}^-$. This invites use of the method of Green functions. This method essentially is based on first dividing the space into surfaces, and then to solve the equation for the case that the right hand side is zero everywhere except at one of the surfaces. Finally, we are to sum up the result. Such summation methods usually do not work for the exact field equations in general relativity due to their non-linearity. Thanks to the linear approximation, this field equation has a form where this method actually works.

In this case, it is natural to choose the surfaces to use in the method spheres given by the surfaces of constant radius. Now the orthonormality of the vector fields given in 4.5 - 4.7 over these spheres is useful. This allows us to for each sphere decompose $\vec{J}_\epsilon$ so that we only have to mind the $X^-_{1m}$ component of this vector field as well. With $l = 1$ $m$ may only have the values 0 and 1. Examining the properties of these vector fields one finds that the sum of those on a surface represent rigidly rotating shells, and that any such shell may be made from it. As $m = 0$ represent rotation around the z-axis it is possible to only find the solution for this case, and correct for the direction differences later.

This can be solved with the help of knowledge of the relevant eigenfunctions. For $k = 0$ and $k = -1$ the radial eigenfunction used outside the shell is determined by the openness into infinity. Summing up and analysing the resulting precession on the gyroscope one finds the main result of his paper:

$$\vec{\Omega}_{\text{gyro}} = \int_0^\infty dr \vec{\Omega}^{\text{equiv}}_{\text{matter}}(r) W(r) \tag{4.10}$$

$$W(r) = \frac{1}{3} 16\pi G(\rho + p) R^3 Y_\mu(r) \tag{4.11}$$

$$Y_\mu(r) = \frac{-d}{dr}\left[\frac{1}{r}\exp(-\mu r)\right] \tag{4.12}$$

Here $\vec{\Omega}_{\text{gyro}}$ is the precession observed by the local observer of the gyroscope. $\vec{\Omega}^{\text{equiv}}_{\text{matter}}(r)$ is the angular velocity of the rigidly rotating shell-portion of the matter flow at distance $r$. $\rho$ and $p$ is the mass density and pressure in the background.

The first of these equations has the form of a weighted average. But in order for it to actually be such $W$ must be normalized to 1. Schmid examined whether this was the case, and concluded that it was.

For $k = 1$ a slightly different radial eigenfunction had to be used outside the shell taking into account the finite size of the closed universe. The result was exactly the same as the one presented above with the exception that $\exp(-\mu\chi)$ had to be replaced by $\sinh^- 1(\mu\pi)\sinh(\mu(\pi - \chi))$.

### 4.1.5 Summary and conclusions

I will before leaving this result tie it to Mach's principle and make some comments on possible extensions. The main result here tells us that there is no unknown ad-hock factor needed to understand why the inertial frames behave as it does. They do it only because of how the matter of the universe around them behaves. It also depends on it in a maybe surprisingly simple way. The angular velocity of the inertial frame is perfectly decided by the angular momentum of all rigidly rotating shells around it. All other motions of bodies that are not part of the rigidly rotating component simply are chaotic fluctuations that cancel each other. Thanks to the exponential cut-off in the $Y_\mu$ factor we also do not have to worry too much about things extremely far out. This is especially nice when having to worry about the event-horizon. Thus it seems like at least for our universe Mach's principle is very well, maybe even perfectly, satisfied.

As promised, I will say some word about the Minkowski case. As mentioned before this is a special case of the FRW-universe. Unfortunately, it may seem like the result found by Schmid cannot be directly applied to this case. This may be seen from the $(p+\rho)$ factor in the weight function, giving a zero contribution of all perturbations. This makes sense as all vector perturbations here would not have any masses to move, and creation of masses would be a scalar perturbation. The normalization of the weight function might however still be defended by observing that in this case the integral over $Y_\mu(r)$ diverge, as $\mu = 0$, and thus there is no exponential cut of. However, as this result claims validity for all linear perturbations, this result might act as a support for another theory regarding Minkowski spaces: That a (FRW-kind of) Minkowski universe is unstable in a way so that if you put any mass in such an universe it will collapse in a way so that for instance all gyroscopes pointing at it will keep pointing at it. There is no mass outside to keep it Minkowski at infinity.

While it turns out that the result may be hard to interpret for universes with $p + \rho = 0$, is there any conceivable way to extend it? In particular, is there any other universe models than the FRW-ones that may be treated in a similar way? Unfortunately, this seems to me to be quite unlikely. I have already mentioned the property of constant curvature that is critical to the ability to be able to restrict attention to the vorticity sector. The gauge simplification, and complete disappearance of higher than first order tensors would be hard to do without. In addition, all of the work on the vector basis fields in the eigenfield section was based upon spherical symmetry. Without this, the entire argumentation allowing us to reduce attention to only two of these would fall apart. Again, it would be hard to imagine reproducing this result without use of these symmetry properties. Examples of other nice properties with the FRW-background that one may not take for granted in other universes are:

- Having a nice background of matter at infinity to point observer's axes toward
- Having the gyroscopes nicely following the initial matter flow
- Being able to slice the space time into spatial slices
- Giving such a nice eigenfield-like equation

All of these are properties that somehow enter into the process of arriving at this result.

While it seems to be hard to find other suitable universe models to apply this method to, what about going to higher order than linear? The answer is that this is maybe harder than finding other universe models. Also in this case, the decomposition into scalar, vector and tensor perturbations break down. This as it had as a requirement that the unknowns in the equation to be decomposed were only linearly dependent. This will naturally not be the case in higher orders. The field equation will also probably no longer be of a form where any form for Green function method may be used, as this also depends on linearity of the system. The vector field results should however still hold, and thus maybe be used in other approaches.

An extension that Schmid himself states that he is working at is to extend his result to the movement and acceleration properties of inertial systems. This absolutely is interesting from a Machian point of view, but falls outside the scope of this thesis as it does not relate to rotation.

## 4.2 Rotating universes

Previously we saw how FRW universes have the property that all inertial axes follow the matter flow. Even in the case of linear perturbation, we saw that there still was a close connection between the flow of matter and the gyroscope axes. The connection simply being a certain weighted average. I will in this section present a couple of universe models where there seem to be no such connection. In these, we will find that gyroscopes everywhere are rotating with respect to the flow of the nearby matter. Such universes are referred to as rotating. I will also tie these to the question of Mach's principle. It might seem at first glance like they are defying this principle, but there are some suggestions to how even these might be interpreted in a Machian way.

### 4.2.1 Goedel Universe

In an article from 1949 [19] Kurt Gödel presented a universe model that were surprisingly simple, but still had quite a few important qualities. The metric is given by:

$$ds^2 = a^2(-dx_0^2 + dx_1^2 - (e^{2x_1}/2)dx_2^2 + dx_3^2 + 2e^{x_1}dx_0dx_2) \tag{4.13}$$

This metric represent a dust-filled universe where the dust is moving along the curves with constant $x_i$. In addition, there is a cosmological constant. Thus, the energy-momentum tensor becomes:

$$T_{\mu\nu} = 8\pi\kappa\rho u_\mu u_\nu + \lambda g_{\mu\nu} \tag{4.14}$$

where $u_\mu$ is the components of the velocity of the dust particles, $\rho$ is the mass density and $\lambda$ is the cosmological constant. In this coordinate system only the 0-component of $u_\mu$ is non-zero. Solving the field equations give us $\lambda = 1/2a^2 = 4\pi\kappa\rho$.

One interesting property of this solution is that it is completely homogenous. That is that every coordinate-independent result found for one point will automatically be satisfied at every other point. With this in mind the most important result in our context of this metric is that one may show that the inertial systems have to rotate with an angular velocity of $2\sqrt{\pi\kappa\rho}$ with respect to this coordinate system. This rotation has constant sign and direction along the third coordinate axis. As the matter is at rest in the original coordinate system one may conclude that if one change coordinates to an inertial system one will find that the matter is rotating, at least locally, with respect to this frame.

Extending this result from a local perspective to a global is far from trivial. One of Gödel's stated motivations for studying this model was that it is impossible to slice the space globally into spatial slices that is separated by a timelike distance. This property may be intuitively understood from the probably most quoted property of this universe: It has closed timelike curves (CTLs). CTLs are curves that start a place, moves through space, always in positive time direction, but still end up at the same place as it started. The existence of CTLs is easily seen from the metric if one makes a coordinate-change to a certain set of cylindrical-like coordinates:

$$ds^2 = 4a^2(-dt^2 + dr^2 + dy^2 - (\sinh^4 r - \sinh^2 r)d\phi^2 - 2\sqrt{2}\sinh^2 r d\phi dt) \tag{4.15}$$

Here we clearly see that for $\sinh^4 r > \sinh^2 r$ a particle moving along a path with all coordinates constant except for $\phi$ will always have a timelike movement. This can be seen, as the square of the interval change is always negative. However this coordinate transformation is constructed so that $\phi$ is cyclical with a period of $2\pi$. Thus the particle will end up at the same point as it started once it has had a total change of $\phi$ equal to this.

The existence of CTLs clearly shows that no global slicing of the space-time into surfaces of constant time in an ordinary way is possible. As the space is completely homogenous, there is neither any natural way to slice it into hypersurfaces of homogeneity - all surfaces would do. I have not found any simple form for the geodesics not simply being those of the $x_4$ coordinate lines. This makes the physical interpretation and prediction of large-scale observations also quite difficult. However, there is nothing that suggests that there should be any effect working at long range that could be observed and interpreted from an inertial frame as matter at a distance rotating in an opposite direction than the matter locally. In addition, the fact stands that there is a natural coordinate system with no movement of matter where the gyroscopes is rotating. Seen from this coordinate system, it seems impossible to explain this motion from the properties of the masses. Thus, several ideas concerning Mach's principle is put to a serious test.

There have been some objections to the Gödel universe that might be used to weaken its position as a counter-argument to some formulations of Mach's principle. One of them is that it is open. As the universe is open the possibility of something further out than observed, or at infinity, may be interfering is present. This is somewhat similar to the Minkowski solution of demanding the view that there has to be some big masses at infinity to explain the phenomena in that universe. However, there seems to be no extra reason for wanting to introduce such infinity condition in the Gödel universe than as an ad-hook solution to the Machian problem. In the Minkowski universe, we had the argument that the observed Minkowski-like universe has masses far away. In addition as briefly mentioned in the introduction there are some theories regarding a solution needing to have a certain matter content that can be used for justifying introducing extra-masses in the Minkowski universe, I have not seen any similar arguments for the Gödel case.

Another objection to the Gödel argument against Mach's principle is that the Gödel universe is unphysical. This is due to it having CTLs. It seem however that whether CTLs should be allowed in physically significant models is still a matter of taste, and that there are still being done some research on that field. However, we shall see that there has been found a model that both is closed and has no CTLs, but still poses the same problems to Mach's principle as the Gödel universe. Thus, it seems like one should be searching for a solution that covers that universe as well.

As a side note: While working on this thesis I examined the possibility of finding a connection between Schmid's result and the Gödel universe through a certain parameterized family connecting the FRW and Gödel universe [29]. I ended this pursuit as I found that this approach probably break down due to the exponential dependency on spatial coordinates of the Gödel metric. This makes even a small deviation from the FRW-case impossible to interpret as a linear perturbation unperturbed at infinity.

### 4.2.2 Ozsváth and Schücking

In 1962 Ozsváth and Schücking presented a metric with some similar properties as the Gödel metric, but being closed and without CTLs [39]. More recently Ozsváth did some more examinations on it, and at the same time presented the metric in a slightly more compact form than in the original paper [38]. However, this last metric is with respect to differential forms, and thus is more difficult to interpret than the original one that is with respect to standard coordinate differences. I will thus here present the metric as in the original to keep the mathematics somewhat simple, even though this form should be considered slightly outdated:

$$ds^2 = dt^2 + R^2k^2\alpha e^3_i dx^i dt + e^a_i \gamma_{ab} e^b_j dx^i dx^j \tag{4.16}$$

Here $e^a_i$ and $\gamma_{ab}$ were given by their matrix representations:

$$e^a_i = \begin{pmatrix} -\sin x^3 & \sin x^1 \cos x^3 & 0 \\ \cos x^3 & \sin x^1 \sin x^3 & 0 \\ 0 & \cos x^1 & 1 \end{pmatrix} \tag{4.17}$$

$$\gamma_{ab} = (\frac{R}{2})^2 \begin{pmatrix} -(1-k\cos\alpha t) & k\sin\alpha t & 0 \\ k\sin\alpha t & -(1+k\cos\alpha t) & 0 \\ 0 & 0 & -(1+2k^2) \end{pmatrix} \tag{4.18}$$

Here $\alpha$, $R$ and $k$ are constants that determine the solution, and have the following constraints. $R > 0$, $|k| < 1/2$ and $\alpha = \frac{2}{R}\sqrt{\frac{2}{1-4k^2}}$. It turns out that this metric describes a dust-filled universe with cosmological constant where the motion of the dust is given by it having constant $x^i$ coordinates. Thus, the coordinate system is commoving with the dust. The cosmological constant $\Lambda$ and density $\rho$ are related to $R$ and $k$ by:

$$\Lambda = \frac{1}{R^2(1-k^2)} \tag{4.19}$$

$$\frac{\kappa\rho}{2\Lambda} = 1 - 4k^2 \tag{4.20}$$

It turns out that in this system the inertial frames defined by gyroscopes also have a certain angular velocity $\omega$ with respect to the matter motion given by:

$$\omega = \frac{\alpha k^2}{\sqrt{1-k^2}} \tag{4.21}$$

As the metric is quite complicated, there is not any obvious global interpretation of this system. I estimate trying to find such an interpretation would consume more time than I have available, and still be of little or no use due to the complexity of the problem. If someone would like to pursue this matter further however, I believe a good starting point would be an article from 1969 by Ozsváth and Schücking. I have not gotten hold of this article myself, but it is referred to as holding more details about the system in [38].

Anyway, the main importance of this metric is that it serves as an example of a spatially closed universe where the inertial frames are rotating with respect to the (local) matter flow. This universe also is not prone to the objections given for the Gödel universe, so other approaches need to be considered if one is to try to save certain interpretations of Mach's principle inside the general framework of the relativity theory.

### 4.2.3 Gravitational waves solution

It seems to me like the most common opinion is that the Ozsváth-Schücking universe truly is an example that general relativity does admit solutions that is incompatible with Mach's principle. However, there are some paths that might turn this around if studied more closely. I will cover two of them here. The first is one taking into account gravitational waves. The other is a brief sketch of an idea of my own that I strangely enough have not found anyone mention in the literature.

The first idea is that somehow matter represented by the standard energy-momentum tensor isn't the only quantity that has to be taken into account when discussing Machian ideas. Another candidate is that of gravitational waves. I will illustrate this path by a summary of the treatment of King in [27].

The main idea of King is to introduce an average background metric $g^{(B)}_{\mu\nu}$ that is spatially homogenous and isotropic. It is worth noting that this background metric is chosen so that it does not need to fulfil the field equations.

Thus other possibilities than the FRW-backgrounds are still present. In this background metric there is then certain Killing-vector fields $\xi^i$ that represent rotational symmetry. He further introduces a set of coordinates on this background metric that makes each hypersurface of homogeneity labelled by a time coordinate and the metric being diagonal with $g_{00} = -1$

He introduces the following notation representing a kind of average of a scalar field:

$$< A >= \frac{1}{V} \int_V A dV \tag{4.22}$$

where the first $V$ represents the total volume of the hypersurface at a given time, the second $V$ represents an integral over this volume, and $dV$ is a volume element on the surface. The total volume makes sense as the universe is assumed to be spatially closed with finite volume.

"Ordinary" angular momentum of a stress-energy field $T^{\mu\nu}$ may then be defined as:

$$L_p(t) = \int_V T^{0i} \xi_i dV = -V < T_{0i} \xi^i > \tag{4.23}$$

Here the integration is taken over a hypersurface of homogeneity in the background metric. We also may make use of the assumption that the universe is spatially closed, so that we have a finite volume to integrate over. The last identity takes advantage of the form of the chosen coordinate system.

His main result is that he finds a tensor that may represent gravitational waves $T_{0i}^{(G)}$ and where he can prove that

$$< (T_{0i}^{(M)} + T_{0i}^{(G)}) \xi^i >= 0. \tag{4.24}$$

Here $T_{0i}^{(M)}$ is the ordinary energy-momentum tensor of matter.

To understand the definition of $T_{0i}^{(G)}$ we first have to introduce $h_{\mu\nu} = g_{\mu\nu} - g_{\mu\nu}^{(B)}$. Then we expand the Einstein tensor of the real metric $G_{\mu\nu}$ in a power series in $h_{\mu\nu}$, that is

$$G_{\mu\nu} = G_{\mu\nu}^{(B)} + G_{\mu\nu}^{(1)} + G_{\mu\nu}^{(2)} + \cdots \tag{4.25}$$

where $G_{\mu\nu}^{(B)}$ is the Einstein tensor of the background metric, that must coincide with the zero-order part of the real metric. This requirement on the background metric was not explicitly mentioned by King, but it is possible that it follows from the other restrictions he sets on the background metric that I will come to later.

Now the tensor $T_{\mu\nu}^{(G)}$ is defined by

$$T_{\mu\nu}^{(G)} = \frac{1}{8\pi} (G_{\mu\nu}^{(2)} + G_{\mu\nu}^{(3)} + \cdots) \tag{4.26}$$

Here, and in the rest of the section we assume the gravitational constant to be 1. How this might be seen upon as a kind of energy-momentum tensor for gravitational waves may be seen from the following form of Einstein's field equation:

$$G^{(1)}_{\mu\nu} = 8\pi(T^{(M)}_{\mu\nu} + T^{(G)}_{\mu\nu} - T^{(B)}_{\mu\nu}) \tag{4.27}$$

Here $T^{(B)}_{\mu\nu}$ is the energy momentum tensor that would have been required for the background metric to satisfy the ordinary form of Einstein's field equations, that is $T^{(B)}_{\mu\nu} = G^{(B)}_{\mu\nu}/8\pi$. Thus we see that $T^{(G)}_{\mu\nu}$ plays a similar role as the ordinary energy momentum tensor in this formulation of the field equations. King states that this form of the field equation is called the field theory approach to gravity, and that it usually had been used in the context of a Minkowski background.

Now, the issue of an "average metric" has to be addressed. Near the beginning of his treatment King points out that finding a good such average is an unsolved problem. He avoids this problem by only requiring a few conditions on the background, not determining it completely. He then argues that there has to exist some backgrounds that satisfy this by giving a rough outline for constructing such. The conditions are that the background and real metric must have the following relations to each other (formulas given by components in the given coordinate system):

- Measure the same proper time on average $< g_{00} - g^{(B)}_{00} >= 0$
- Measure the same spatial distances on average $< g^k_k - g^{(B)k}_k >= 0$
- Have no relative translation or rotation $< (g_{0i} - g^{(B)}_{0i})\xi^i >= 0$

From these relations, and the observation that in the given coordinate system $T^B_{0i} = 0$ King claimed to be able to derive 4.24 for closed universes. He refers to his doctoral thesis for the full proof, which I have not found important enough to try to obtain.

This result seems neat. It removes the problem of the matter rotating with respect to the gyroscopes in the Ozsváth-Schücking universe by taking into account a rotation of gravitational waves that goes in the opposite direction cancelling the effect of the matter as seen from the inertial frames. This serves as an explanation for the rotation of the inertial axes with respect to the pure matter-field. However, I am not able to feel completely convinced by this argumentation. The definition of the energy-momentum tensor for the gravitational waves seems a kind of ad-hock. It is hard to find any good physical interpretation for this tensor. It serves as a source term in an Einstein-like field equation. However, the left hand-side of this equation is not the same geometrical term as in Einstein's field equations, and hence the physical meaning of the right hand-side is not a perfect analogy to the ordinary energy-momentum with its usual physical interpretations.

I would like to focus on what makes this result different from simply introducing any arbitrary tensor field with the property that its derived angular momentum cancels that of the ordinary angular momentum. It must be so that a certain set of fields used in connection with work on the general relativity theory turns out to be a subset of those fields that has the wanted property. The existence of this overlap might seem to be too good to be a coincidence. Thus, it may work as a strong argument for the idea that it is the field theory approach to gravity that is the most natural framework for formulating a version of Mach's principle that may hold.

However, the physical interpretation of this is still not clear. It is hard to say whether this really is a physical result, or simply a well hidden mathematical consequence of the form of the field equation 4.27. The comments on the reference I have used also sow doubt about the physical content of this approach, and I have not found any further work on this. Nevertheless, it still stands as an example of a way to approach the Machian problem of the rotating universes.

### 4.2.4 Spinning particles solution

The other approach I will only present briefly is the interpretation of spinning particles. As mentioned in the introduction, plain general relativity is working with non-spinning particles. If one introduces spinning particles, one has to use the Einstein-Cartan theory. However, this begs the question - with respect to what is the particles non-spinning? I have been unable to find any sources that address this question. I assume finding such would require diving into more details of the Einstein-Cartan theory, and this was outside the scope of this thesis. I will still give a quite simplified thought experiment involving rotating particles.

Assume that dust particles in general relativity have to be rotationally at rest in their inertial frame. This does not necessarily contradict a matter flow that is not in rotational rest in the inertial frames. In a great scale, we may regard each grain of dust as a point particle, and thus should the rotational state of these would appear as spin, and not matter in rotation in this perspective. This assumption may thus be regarded as a possible alternative of the correct spin-free particle. Another alternative might be that the rotational state of the dust particles has to follow the general matter-current in the region, but I have found no sources that explicitly favour this interpretation.

Now regard the Gödel universe in its standard coordinate system. Here all the dust particles are at rest with respect to the coordinates, but the inertial systems are rotating. This means that at every point, even though the matter current is zero, there are particles rotating with respect to that coordinate frame. If we now analyze the gravitomagnetic field in this coordinate frame, we get a situation analogue to the situation inside a magnetized object. There are many small spins that by being oriented the same way together form a considerable magnetic field. Thus as we have several point-masses rotating around the same axis in our coordinate frame, we can expect to experience a significant gravitomagnetic field. As the masses are at rest in our frame, this field will not affect the movement of these, but it is clear that it may explain the rotation of the inertial frames!

To determine if the expected gravitomagnetic effect required to account for the rotation of the inertial frames in the Gödel universe actually coincides with that generated by particles rotationally at rest in this is however not straight forward. The approach in 2 may not be used, as obviously neither the Gödel nor the Ozsvát-Schücking universe is well approximated as a linear perturbation of the Minkowski space. In addition, an approximate solution would not be expected to arrive at the possible identity in this exact solution.

Further study of this approach should probably be done with the Einstein-Cartan framework in mind. A good starting point for this may be an article by Smalley [55]. Here Smalley presents some work on the Gödel universe within this extended theory.

# Chapter 5

# Concluding remarks

I have presented several results to some detail in the previous chapters. It is now time to take a few steps back and look at the big picture. We see that through the past hundred years, more and more accurate calculations have been done with regard to systems that could shed light upon the status of Mach's principle in the general theory of relativity. We saw that the rotating shell model confirmed frame-dragging effects to progressively higher precision inside the theoretical framework of general relativity. Still, in order to observe this effect we had to go to the exterior solutions of rotating bodies. Only very recently was this effect confirmed observationally to some extent, by the gravity probe B experiment.

At cosmological scales, it is striking that there are relatively simple connections between the rotational state of inertial frames and that of the content of the universe in two huge classes of universe models. For closed universes, it enters through non-rotation with respect to matter and a form for gravitational waves. For linearly perturbed FRW-universes, the connection is that of a weighted average. Both of these owe to the concept of frame dragging as described earlier for simpler systems. Historically the closed universe solution has been the favoured in regard to Mach's principle. The recent result that FRW universes also have very Machian qualities might however be used as an argument for shifting that balance. While there are little indications that the universe is closed, its FRW-like nature is mostly uncontroversial.

We also have seen a couple of examples of universes that might be considered non-Machian in a certain way. Both of these may be solved by restricting the validity of Mach's principle to closed universes, and taking into account gravitational wave effects. However, neither of these is very "FRW-like". This means that asserting that universes should be "FRW-like" would exclude these as well. As our universe seems to be "FRW-like", this assumption seems to be more practical than the assumption that it is closed. This do however rise a host of new questions: Exactly how may one define "FRW-like"? What properties must the universe have if it is rotational properties are defined exactly as a kind of weighted average, and not only through linear perturbations? Is it possible to find a simpler and more precisely defined principle than Mach's principle that would clearly disallow universes of dubious Machian nature? All of these questions seem like possible avenues for further work.

And even if it should turn out that our universe doesn't obey Mach's principle perfectly it seem pretty clear that it may still serve a purpose. It shares one important property with the absolute space it is said to be in direct opposition to: It may be a useful tool. With the aid of the conceptually simple and philosophically appealing principle we may quickly predict and get a kind of intuitive feeling for some quite complicated systems in our universe. This may range from frame dragging and light-shifting effects of rotating black holes, to appreciation of the close connection between our inertial frames and that of the heavenly bodies far away.

# Appendix A

# Source code for galaxy model

```
from scitools.all import *

# Initialising global parameters
G=4.786e-17  #Newton's gravitational constant/c^2 in kpc/solar mass
v0=2./3000   #dimensionless
A=1.e10      #solar masses/kpc
Z=0.1        #kpc
R=20         #kpc
Rmin=0       #kpc, minimum radius to integrate over.

constR0=10   #kpc, the value for R0 to use when examining stability

Nmax=4000000 #the maximum tested number of simulations
Nmin=1000000 #the least number of simulations before plotting results

N=3000000    #the number of random points for each Monte Carlo simulation
M=1000       #the number of points in the plots

volume=2*Z*2*pi*R**2     #The volume of the galaxy in kpc^3

# define function for our integrand
def integrand(R0, r, z, phi) :
    """
    returns r times the variable contribution to the gravitomagnetic field at
    the point (R0, 0, 0) made by the matter at the point with
    cylindrical coordinates (r, z, phi). This coinsides with the integrand
    in our integral due to the r d\phi factor in cylindrical coordinates.
```

```
    Due to optimizing reasons 4*G*v0*(A/(4*pi*Z)) should be multiplied to
    this result after calling this function.
    The dimension of the return value is distance^-2
    """
    return (r-cos(phi)*R0)/((sqrt(r**2-2*r*R0*cos(phi)+R0**2+z**2)**3))

"""
first compare the result for the gravitomagnetic field for
two different ways of handling the distribution properties
for cylindrical coordinates, with a given R0
"""

#Draw random numbers
Rbase=random.uniform(Rmin, R**2, size=Nmax)
z=random.uniform(-Z, Z, size=Nmax)
phi=random.uniform(0, 2*pi, size=Nmax)

#Calculate the gravitomagnetic fields

#weighted for uniform distribution of radial coordinates
r1=Rbase/R
BUniform=4*G*v0*(A/(4*pi*Z))*integrand(constR0, r1, z, phi)
#for even distribution of points in the cylinder
r2=sqrt(Rbase)
BEvenDisp=4*G*v0*(A/(4*pi*Z))*integrand(constR0, r2, z, phi)/r2

# Performing the integral-summation and plotting.
# The integral is the mean of the contribution per volume times the volume
# For uniform distribution, remember that we have weighted values
Bmean=zeros(Nmax)
Bsum=0.
Rsum=0.
for i in range(Nmax):
    Bsum+=BUniform[i]
    Rsum+=r1[i]
    Bmean[i]=Bsum/Rsum

points= range(Nmin, Nmax, (Nmax-Nmin)/M)

plot(points, Bmean[points]*volume)
legend("Uniform distribution")
```

```
#for the even distribution no special consideration needs to be taken
hold('on')
Bmean=zeros(Nmax)
Bsum=0
for i in range (Nmax):
    Bsum+=BEvenDisp[i]
    Bmean[i]=Bsum/(i+1)

plot(points,Bmean[points]*volume)
legend("Even distribution")
title("Monte Carlo convergence")
xlabel("Number of random points")
ylabel("Bg-field/c in kpc^-1")
hardcopy("Galaxy1.eps")
hold('off')

dummy=raw_input("please press enter")

"""
Drawing the gravitomagnetic field as function of distance from
the galaxy core. Using the uniform distribution method as I believe
it to be slightly faster.
"""

#prepearing for going through the points from the centre
R0s=linspace(Rmin, R, M)
Bfield=zeros(M)
i=0

for R0 in R0s:
    #get new random coordinates
    r=random.uniform(Rmin, R, size=N)
    z=random.uniform(-Z, Z, size=N)
    phi=random.uniform(0, 2*pi, size=N)

    #calculating the raw data for the Bfield at distance R0
    #I wait with multiplying in constants in order to speed up the program
    Bcore=integrand(R0, r, z, phi)
    Bfield[i]=Bcore.sum()/r.sum()
    i+=1
```

```
#multiplying in the constants to the raw data
Bfield*=4*G*v0*(A/(4*pi*Z))*volume

#plot the gravitomagnetic field
plot(R0s, Bfield, \
     title="Gravitomagnetic field in a galaxy", \
     xlabel="Distance from core in kpc", \
     ylabel="Bg-field/c in kpc^-1", \
     axis=[0, R, -4e-10, 2e-9], \
     hardcopy="Galaxy2.eps")

dummy=raw_input("please press enter")

#plot the fraction of the acceleration that is given by gravitomagnetism
#first without axis restrictions to get the extremes
plot(R0s, Bfield*R0s/v0, \
     title="Part of total acceleration from gravitomagnetism", \
     xlabel="Distance from core in kpc", \
     ylabel="Gravitomagnetic effect/" + \
      "What is required to explain the motion", \
     hardcopy="Galaxy3.eps")

dummy=raw_input("please press enter")

#then focus on the part where there are most measurements
plot(R0s, Bfield*R0s/v0, \
     title="Part of total acceleration from gravitomagnetism", \
     xlabel="Distance from core in kpc", \
     ylabel="Gravitomagnetic effect/" + \
      "What is required to explain the motion", \
     axis=[0, R, -5e-6, 1.5e-5], \
     hardcopy="Galaxy4.eps")
```

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